Quaternion Based Metrics in Relativity

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1 Abstract

By introducing a new form of metric tensor the same derivation for the electromanetic tensor $F_{\mu\nu}$ from potentials A_μ leads to the dual space (Hodge Dual) of the regular $F_{\mu\nu}$ tensor. There are additional components in the i, j, k planes, however if after the derivation only the real part is considered a physically consistent electromagnetic theory is recovered with a relabelling of $\vec{E}$ fields to $\vec{B}$ fields and vice versa.

2 Introduction

A prominent feature in relativistic physics is the Minkowski metric tensor

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (1)$$

On probing where this comes from it was postulated that the matrix could be the 'Real' (non-quaternion) part of the outer product of two unit quaternions $Q = 1 + i + j + k$,

$$\eta_{\mu\nu} = Q \otimes Q = Re\left(\begin{bmatrix} 1 & i & j & k \\ i & -1 & k & -j \\ j & -k & -1 & i \\ k & j & -i & -1 \end{bmatrix}\right) \quad (2)$$

The implications of carrying through the physics made with this tensor without taking the real part were considered. The creation of an electromagnetic tensor is considered.

When being used for a metric in the form

$$\mathrm{ds}^2 = \begin{bmatrix} dt & dx & dy & dz \end{bmatrix} \begin{bmatrix} 1 & i & j & k \\ i & -1 & k & -j \\ j & -k & -1 & i \\ k & j & -i & -1 \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} \quad (3)$$

Then from explicit calculation it can be shown that an equaivalent matrix gives

$$\mathrm{ds}^2 = \begin{bmatrix} dt & dx & dy & dz \end{bmatrix} \begin{bmatrix} 1 & i & j & k \\ i & -1 & 0 & 0 \\ j & 0 & -1 & 0 \\ k & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} dt \\ dx \\ dy \\ dz \end{bmatrix} (4)$$

This would be equaivalent to a new type of number with the rules

$\mathrm{i} \cdot i = -1$
$j \cdot j = -1$
$k \cdot k = -1$

$\mathrm{i} \cdot j = j \cdot i = 0$
$i \cdot k = k \cdot i = 0$
$j \cdot k = k \cdot j = 0 (5)$

This is similar to having 3 independant imaginary numbers or basis vectors. If this were an inner product then they are orthoganal but antiparallell with themselves. They form a NON-ASSOCIATIVE 'group' under a product with elements $0, 1, i, j, k, -1, -i, -j, -k$, this is an Abelian relationship as the non-Abelian properties of the quaternions was removed with the cross interactions. For example,

$(\mathrm{i} \cdot i) \cdot j = -1 \cdot j = -j$
$i \cdot (i \cdot j) = i \cdot 0 = 0$
$(a \cdot b) \cdot c \neq a \cdot (b \cdot c) (6)$

3 Electromagnetism

This formulation would require

$$\eta^{\mu\nu} = \frac{1}{1+i^2+j^2+k^2} \begin{bmatrix} 1 & i & j & k \\ i & -1 & -k & j \\ j & k & -1 & -i \\ k & -j & i & -1 \end{bmatrix} (7)$$

This would require perhaps a normalisation of $\frac{1}{\sqrt{2}}$ on each matrix.

For the potential 4 vector

$\mathrm{A}^\mu = (\varphi/c, A_x, A_y, A_z) (8)$

Then

$\mathrm{A}_\mu = \eta_{\mu\nu} A^\nu (9)$

Which gives

$$A_\mu = \begin{bmatrix} \frac{\varphi}{c} + A_x i + A_y j + A_z k \\ \frac{\varphi i}{c} - A_x + A_y k - A_z j \\ \frac{\varphi j}{c} - A_x k - A_y + A_z i \\ \frac{\varphi k}{c} + A_x j + A_y i - A_z \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\varphi}{c} \\ -A_x \\ -A_y \\ -A_z \end{bmatrix} + \begin{bmatrix} A_x \\ \frac{\varphi}{c} \\ A_z \\ -A_y \end{bmatrix} i + \begin{bmatrix} A_y \\ -A_z \\ \frac{\varphi}{c} \\ A_x \end{bmatrix} j + \begin{bmatrix} A_z \\ A_y \\ -A_x \\ \frac{\varphi}{c} \end{bmatrix} k (10)$$

There are now hypercomplex four potentials from the real potential such that $A_\mu \to A_\mu + B_\mu i + \Gamma_\mu j + \Delta_\mu k$. Now, as usually defined we can create the tensor

$$H_{\mu\nu} = (dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$
(11)

But with the hypercomplex components it is clear that $H_{\mu\nu} = F_{\mu\nu} + I_{\mu\nu} i + J_{\mu\nu} j + K_{\mu\nu} k$ where

$$F_{\mu\nu} = Re[H_{\mu\nu}] = \partial_\mu A_\nu - \partial_\nu A_\mu$$
$$I_{\mu\nu} = Im_i[H_{\mu\nu}] = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$J_{\mu\nu} = Im_j[H_{\mu\nu}] = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu$$
$$K_{\mu\nu} = Im_k[H_{\mu\nu}] = \partial_\mu \Delta_\nu - \partial_\nu \Delta_\mu (12)$$

This means that the real component $F_{\mu\nu}$ is equal to th regular EM tensor

$$F_{\mu\nu} = \frac{1}{c} \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -cB_z & cB_y \\ -E_y & cB_z & 0 & -cB_x \\ -Ez & -cB_y & cB_x & 0 \end{bmatrix} (13)$$

However,

$$F^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\delta} F_{\gamma\delta} (14)$$

Through explicit calculation this results in

$$Re[F^{\mu\nu}] = \frac{1}{2c} \begin{bmatrix} 0 & cB_x & cB_y & cB_z \\ -cB_x & 0 & E_z & -E_y \\ -cB_y & -E_z & 0 & E_x \\ -cB_z & E_y & -E_x & 0 \end{bmatrix}$$
$$= \frac{1}{2} G_{\gamma\delta} = \frac{1}{4} \varepsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta}$$
(15)

4 Maxwell's Equations

In SI units we have the Faraday-Gauss Law

$\partial_\alpha F^{\alpha\beta} = \mu_0 J_e^\beta (16)$

The Ampere-Gauss Law

$\partial_\alpha \star F^{\alpha\beta} = \frac{\mu_0}{c} J_m^\beta (17)$

and the Lorentz force law

$\mathrm{dp}_\alpha \overline{{}_{d\tau = q_e F_{\alpha\beta} v^\beta + q_m \star F_{\alpha\beta} v^\beta}(18)}$

But we now have the different form for the Hodge Dual $\star F$