# Generating functions for nth Collatz iteration

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## Generating Function for $n^{th}$ Collatz Iteration

We can consider the generating function for the Collatz map applied to the positive integers. Define

$$C(n) = \begin{cases} n/2 & n \bmod 2 = 0 \\ 3n+1 & n \bmod 2 = 1 \end{cases} \tag{1}$$

and define the $m^{th}$ composition of the function as $C_m(n)$, such that $C_0 = n$ and $C_1(n) = C(n)$ and $C_2(n) = C(C(n))$.

The generating function for positive integers is

$$G_0(x) = \frac{x}{(1-x)^2} \tag{2}$$

for the first iteration we have numbers $4, 1, 10, 2, 16, 3, ...$

$$G_1(x) = \frac{x}{(1-x^2)^2}(4 + x + 2x^2) \tag{3}$$

for the second iteration giving $2, 4, 5, 1, 8, 10, 11, 2, 14, ...$ we have

$$G_2(x) = \frac{x}{(1-x^4)^2}(2 + 4x + 5x^2 + x^3 + 4x^4 + 2x^5 + x^6) \tag{4}$$

the next iteration is

$$G_3(x) = \frac{P_3(x)}{(1-x^8)^2} \tag{5}$$

in general this gives

$$G_n(x) = \frac{P_n(x)}{(1-x^{2^n})^2} \tag{6}$$

for a polynomial of which seems to be order $2^{n+1}-1$ these polynomials appear to be related to the current iteration sequence by the following relationship

$$P_n(x) = \left(\sum_{k=1}^{2^n} C_n(k)x^k\right) + \left(\sum_{k=2^n+1}^{2^{n+1}-1} (C_n(k) - 2C_n(k-2^n))x^k\right) \tag{7}$$

which we can write as

$$P_n(x) = \left(\sum_{k=1}^{2^{n+1}-1} C_n(k)x^k\right) - 2\left(\sum_{k=2^n+1}^{2^{n+1}-1} C_n(k-2^n)x^k\right) \tag{8}$$

We could consider the Cauchy product of this and the simple series

$$\frac{1}{(1-x^{2^n})^2} = 1 + 2x^{2^n} + 3x^{2\cdot 2^n} + 4x^{3\cdot 2^n} + \cdots \tag{9}$$

what does this mean? This means that for any level of iteration, we can describe the coefficient for any number, however large, using the first few function evaluations and a composition. However the expressions rapidly become complicated, with $2^{n+1}$ terms.

What conditions would then be required for a coefficient to be 1? For a given iteration this will depend on the number of ways to write a target number $t$, as the sum of an integer in the range $[1, 2^n - 1]$ and any of $[0, 2^n, 2\cdot 2^n, 3\cdot 2^n, \cdots]$, for one iteration that's combinations in $[1,2,3] + [0,2,4,6,8,\cdots]$ which can make $[1,2,3]$,$[3,4,5]$,$[5,6,7]$ and so on indicating there are multiple ways to make $3,5,7,\cdots$.

All of the coefficient terms are positive which is nice. The only way a coefficient can be 1 in this iteration is if it is 1 in the polynomial, and multiplied by the 1 in the expanded series.

This means we can look at a subset of the polynomial, namely $P_n(x)$. We can then ask, how can a coefficient become 1 in $P_n(x)$? We can see that $C_n(k) > 2C_n(k-2^n)$ for $k \in [2^n+1, 2^{n+1}-1]$ to keep the terms positive and non-zero.