On Dynamics of Double-Diffusive Magnetoconvection in a Non-Newtonian Fluid Layer

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July 25, 2022

Abstract

This article concerns the dynamic transitions of a non-Newtonian horizontal fluid layer with thermal and solute diffusion and in the presence of vertical magnetic field. First, a linear stability analysis is done by deriving the principle of exchange of stability condition, which shows the system loses stability when thermal Rayleigh number exceeds a critical threshold. Second, we considered the transition induced by real eigenvalues and complex eigenvalues, respectively, and two nonlinear transition theorems along with several transition numbers determining the transition types are obtained via the method of center manifold reduction. Finally, rigorous numerical computations are performed to offer examples of possible transitions, as well as the stable convection patterns. Our results show that when the diffusivities from big to small are thermal, solute concentration and magnetic diffusion, both continuous and jump transitions can occur for certain parameters; and if the diffusivities from big to small is the inverse of the previous case, only continuous transition induced by real eigenvalues are observed, which indicate a stationary convection.

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Abstract

This article concerns the dynamic transitions of a non-Newtonian horizontal fluid layer with thermal and solute diffusion and in the presence of vertical magnetic field. First, a linear stability analysis is done by deriving the principle of exchange of stability condition, which shows the system loses stability when the thermal Rayleigh number Rt exceeds a critical threshold. Second, we considered the transition induced by real eigenvalues and complex eigenvalues, respectively, and two nonlinear transition theorems along with several transition numbers determining the transition types are obtained via the method of center manifold reduction. Finally, rigorous numerical computations are performed to offer examples of possible transitions, as well as the stable convection patterns. Our results show that when the diffusivities from big to small are thermal, solute concentration and magnetic diffusion, both continuous and jump transitions can occur for certain parameters; and if the diffusivities from big to small is the inverse of the previous case, only continuous transition induced by real eigenvalues are observed, which indicate a stationary convection.

Keywords: Dynamic transition, Magnetoconvection, Double-Diffusive, Non-Newtonian fluids, Center manifold reducution, Numerical computation.
2010 MSC:

1. Introduction

In 1901, the experimental work of Bénard [1] discovers a convection phenomenon of a horizontal fluid layer heated uniformly from below. And the instability problem followed by that discovery was draw attention to Rayleigh

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[2], and other researchers later on [3, 4, 5]. Nowadays, the thermal convection phenomenon, known as Rayleigh-Bénard convection, is appearing in varies contexts [6, 7, 8, 9, 10]. Hence, the stability and bifurcation of Rayleigh-Bénard convection coupled with other effects are of our interest.

The double diffusive convection is the name given to a convective motion when there are two molecular diffusivities which cause opposing effect on vertical density gradients. The competing of two stratifying effects can give rise to instability and vertical mixing. The best known double diffusive convection comes from oceanography [11], which occurs due to distinct diffusivities of heat and solute concentration. Besides oceanography, double diffusive convection is also found in diverse fields such as astrophysics, chemistry, geophysics and so on [12, 13]. As this phenomenon is widespread in nature, both theoretical and experimental work have been carried out to address this issue [14, 15, 16, 17, 18, 19]. Moreover, recent articles [20, 21] by Hsia et al. discuss the phenomenon from the perspective of the theory of phase transition dynamics, which was developed in monographs [22, 23]. They offered a comprehensive analysis of bifurcation and stability for the Boussinesq equations with diffusion of temperature and salinity in two- and three-dimensional spaces.

The thermal convection in electrically conducting media is known as magneto-convection, which has been extensively studied due to its commonly presence in earth's outer core and other late-type stars like sun. In monograph [3] of Chandrasekhar, linear instability of magnetoconvection is thoroughly examined. We also recommend the review of Proctor and Weiss [24] and the references within for an overall understanding of the linear theory and some new results in the non-linear regime. Regarding phase transition dynamics of magneto-convection, Wang and Sengul [25] studied the dynamic transition of the incompressible MHD equations in a rectangular domain with a large magnetic Prandtl number, and diverse convection patterns is obtained depending on the choice of parameters; More recently, Li et al.[26] established a complete stability and bifurcation analysis for a rotating electrically conducting fluid layer in the presence of an external magnetic field based on the Boussinesq approximation.

However, the fluid media may be non-Newtonian among the aforementioned circumstances, which is a general setting in industrial practice. In particular, the couple stress fluid is a typical case that stimulated widespread interest of many researchers. And the modeling work of Stokes [27] is the most recognized and simplest, which allows for polar effects such as the presence of couple stresses, body couples and a non-symmetric stress tensor. As for stability problems, Sunil et al. [28] shows the equivalence of linear and non-linear stability of a couple-stress fluid. Nonetheless, there are few work concerning solely non-Newtonian fluids in thermal convection, various other effects are often addressed simultaneously. Sharma and Thakur [29] investigated an electrically conducting couple-stress fluid layer heated from below in porous medium and in presence of magnetic field. They have reported that couple-stress and magnetic field postpone the convection while the medium permeability hasten the onset of convection. Linear and non-linear stability analysis for Non-Newtonian double diffusive convection studied by Malashetty et al. [30] and Gaikwad et al. [31]. Shivakumara and Kumar [19] considered linear and weakly nonlinear stability of a triply diffusive fluid layer. Convective instability of a doubly diffusive incompressible couple stress fluid layer was investigated by Kumar et al. [32]. On account of phase transition, Pan et al. [33] considered stability and transition of a couple stress fluid in saturated porous media.

Owing to the importance of the combining effect of thermal/solute concentration diffusion and magnetic field on non-Newtonian fluid, our main goal of this article is to produce stability and bifurcation analysis of such model purposed by Kumar et al. [34] via the phase transition theory established by Ma and Wang [22, 23]. In the framework of phase transition, the dynamic transitions of all dissipative systems are classified into three types: continuous, jump and mixed. The continuous transition indicates a gradual change from the base state to the bifurcated state as the control parameter of the system crossing the critical value, while a jump transition suggests a drastic change from base state as the control parameter crosses the critical threshold, which is also called a "catastrophic" transition. A mixed transition means there are two non-empty open sets near the base state, the perturbation in one sets give rise to a continuous transition and in another leads to a jump transition. The mixed transition is also called a "random" transition since it became a random event whether the system will undergo a gradual change or a drastic one.

This paper is three-fold. First, we verify the principle of exchange of stability (PES) condition by analyzing the linear stability via Fourier transform. Second, by reducing the system into ODEs, we analyze dynamic behavior of the transition from first real eigenvalue and first complex eigenvalue. Finally, by numerical investigation under selected parameter ranges, we demarcated the parameter ranges into different regions where each region corresponds to different transition types. Also, some convection patterns are drawn.

This article is organized as follows. The basic governing equations are given in section 2. section 3 is devoted to prove the principle of exchange of stability. section 4 and section 5 states and proves the main conclusions on the transitions from a pair of real eigenvalues and two pairs of complex eigenvalues, respectively. section 6 gives several numerical examples and convection patterns. We end the article with a summary of our main results in section 7. And the appendix A completes the center manifold approximation formula used in section 5.

2. Mathematical Settings

In this article, we consider the double-diffusive magnetoconvection of coupled stress fluid layer, which is described by the dimensionless model [34]:

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \mathrm{Pr}\Delta + \mathrm{Pr}\ \Lambda_{\mathrm{c}}\Delta^2\right)\Delta\psi = &- \mathrm{Pr}\ \mathrm{Rt}\frac{\partial T}{\partial x} + \mathrm{Pr}\ \mathrm{Rs}\frac{\partial C}{\partial x} + \mathrm{Pr}\ \mathrm{Rm}\Delta\left(\frac{\partial A}{\partial z}\right) \\
&- J(\psi, \Delta\psi) - \mathrm{Pr}\ \mathrm{Rm}J(A, \Delta A), \\
\left(\frac{\partial}{\partial t} - \Delta\right)T = &- \frac{\partial \psi}{\partial x} - J(\psi, T), \\
\left(\frac{\partial}{\partial t} - \tau_1\Delta\right)C = &- \frac{\partial \psi}{\partial x} - J(\psi, C), \\
\left(\frac{\partial}{\partial t} - \tau_2\Delta\right)A = &\frac{\partial \psi}{\partial z} - J(\psi, A),
\end{aligned}
\tag{2.1}
$$

where $\psi(x,z)$ is the stream function, $T(x,z)$ is the temperature field, $C(x,z)$ is the solute concentration, $A(x,z)$ is the magnetic flux function and $(x,z) \in \Omega = \mathbb{R} \times [0,1]$. And, $J(f,g) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$ is the Jacobian operator with respect to x and z for arbitrary functions f and g. The dimensionless numbers $\mathrm{Pr}, \Lambda_{\mathrm{c}}, \mathrm{Rt}$ and Rs are Prandtl number, couple stress parameter, thermal Rayleigh number, solute Rayleigh number. τ_1 and τ_2 are the ratios of diffusivities, and Rm is Chandrasekhar-Rayleigh number, which is named in [34].

The definitions of these dimensionless numbers are as follows

$$
\begin{aligned}
&\mathrm{Pr} = \frac{\nu}{\kappa_t}, \qquad \Lambda_{\mathrm{c}} = \frac{\mu_c}{\mu d^2}, \qquad \tau_1 = \frac{\kappa_c}{\kappa_t}, \qquad \tau_2 = \frac{\nu_m}{\kappa_t}, \\
&\mathrm{Rm} = \frac{\mu H_0^2 d^2}{\rho_0 \nu \kappa_t}, \quad \mathrm{Rt} = \frac{\beta_t g \Delta T d^3}{\nu \kappa_t}, \quad \mathrm{Rs} = \frac{\beta_c g \Delta C d^3}{\nu \kappa_t},
\end{aligned}
\tag{2.2}
$$

where symbols in above definitions are: d is the thickness of fluid layer; H_0 is the strength of the vertical magnetic field; ΔT and ΔC are the temperature difference and solute concentration difference between upper and lower boundary of the fluid layer; β_t is the thermal expansion coefficient, β_c is the solute analog of β_t; ρ_0 is the reference density in Boussinesq approximation; κ_t is the thermal diffusivity, κ_c is the solute diffusivity; ν and ν_m are kinematic viscosity and magnetic viscosity, respectively; μ is material viscosity, while μ_c is material constants having dimensions of momentum; g is the constant of gravity.

The boundaries are assumed to be stress-free with vanishing couple stress and perfect conductors of heat and solute concentration. Hence, the appropriate boundary conditions in z-direction is given by

$$
\psi = \frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^4 \psi}{\partial z^4} = T = C = \frac{\partial A}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{2.3}
$$

With notation $\Psi(x,z) = (\psi(x,z), T(x,z), C(x,z), A(x,z))^t$, we assume periodic boundary condition in x direction:

$$\Psi(x,z) = \Psi\left(x + \frac{2\pi}{\alpha}, z\right), \tag{2.4}$$

where the parameter $\alpha > 0$ is the inverse of the period. Furthermore, it is natural to require

$$\int_0^1 \int_0^{\frac{2\pi}{\alpha}} A \, \mathrm{d}x \, \mathrm{d}z = 0. \tag{2.5}$$

In what follows, we simplify the equations (2.1) by using an abstract functional setting that is standard in the framework of dynamic transitions. Denoting $H^k(\Omega)$ and $L^2(\Omega)$ to be usually Sobolev spaces, we define the following function spaces:

$$\begin{aligned}
\mathbf{H}_{-1} &= \Big\{\Psi \in L^2(\Omega) \times \big(L^2(\Omega)\big)^3 | \text{ satisfying (2.4)- (2.5) } \Big\} \\
\mathbf{H}_0 &= \Big\{\Psi \in H^2(\Omega) \times \big(L^2(\Omega)\big)^3 | \text{ satisfying } \psi|_{z=0,1} = 0, \text{ (2.4)-(2.5) } \Big\} \\
\mathbf{H}_1 &= \big\{H^6(\Omega) \times (H^2(\Omega))^3 | \text{ satisfying (2.3)- (2.5) } \big\}
\end{aligned} \tag{2.6}$$

which are endowed with their natural inner products. Then the linear operators

$$\mathcal{A} = \begin{pmatrix} \Delta \\ id \\ id \\ id \end{pmatrix}, \quad L = \begin{pmatrix} \mathrm{Pr}\Delta^2 - \mathrm{Pr}\Lambda_{\mathrm{c}}\Delta^3 & -\mathrm{PrRt}\frac{\partial}{\partial x} & \mathrm{PrRs}\frac{\partial}{\partial x} & \mathrm{PrRm}\Delta\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial x} & \Delta & & \\ -\frac{\partial}{\partial x} & & \tau_1\Delta & \\ \frac{\partial}{\partial z} & & & \tau_2\Delta \end{pmatrix},$$

have the following properties

$$L : \mathbf{H}_1 \to \mathbf{H}_{-1}, \quad \mathcal{A} : \mathbf{H}_0 \to \mathbf{H}_{-1}, \tag{2.7}$$

and gives the abstract form of equation (2.1):

$$\frac{\partial \mathcal{A}\Psi}{\partial t} = L\Psi + G(\Psi, \Psi), \tag{2.8}$$

where the bilinear form $G(\Psi, \widetilde{\Psi})$ is defined by

$$G(\Psi, \widetilde{\Psi}) = \begin{pmatrix} -J(\psi, \Delta\widetilde{\psi}) - \mathrm{PrRm}J(A, \Delta\widetilde{A}) \\ -J(\psi, \widetilde{T}) \\ -J(\psi, \widetilde{C}) \\ -J(\psi, \widetilde{A}) \end{pmatrix} \tag{2.9}$$

in which $\widetilde{\Psi} = (\widetilde{\psi}, \widetilde{T}, \widetilde{C}, \widetilde{A})$. We will also use the abbreviation: $G(\Psi) = G(\Psi, \Psi)$.

Since $\mathcal{A}$ is an isomorphism, it is natural to apply the following abstract form,

$$\frac{\partial \Psi}{\partial t} = \mathcal{L}\Psi + \mathcal{G}(\Psi, \Psi), \tag{2.10}$$

where the linear operator is defined by

$$\mathcal{L} = \mathcal{A}^{-1} \circ L : \mathbf{H}_1 \to \mathbf{H}_0, \tag{2.11}$$

and bilinear form is given by

$$\mathcal{G}(f, g) = \mathcal{A}^{-1} \circ G(f, g) : \mathbf{H}_1 \times \mathbf{H}_1 \to \mathbf{H}_0. \tag{2.12}$$

The abbreviation $\mathcal{G}(\Psi) = \mathcal{G}(\Psi, \Psi)$ is frequently applied in the remaining article.

2.1. Eigenvalue Problems

Consider the following eigenvalue problem:

$$\mathcal{L}\Psi = \beta\Psi, \quad \Psi \in \mathbf{H}_1, \tag{2.13}$$

which is explicitly reads

$$\begin{aligned}
\mathrm{Pr}\Delta^{-1}\left[\Delta^2\psi - \Lambda_c\Delta^3\psi - \mathrm{Rt}\frac{\partial T}{\partial x} + \mathrm{Rs}\frac{\partial C}{\partial x} + \mathrm{Rm}\Delta\left(\frac{\partial A}{\partial z}\right)\right] &= \beta\psi, \\
\Delta T - \frac{\partial \psi}{\partial x} &= \beta T, \\
\tau_1 \Delta C - \frac{\partial \psi}{\partial x} &= \beta C, \\
\tau_2 \Delta A + \frac{\partial \psi}{\partial z} &= \beta A.
\end{aligned} \tag{2.14}$$

According to boundary conditions (2.3) and (2.4), the eigenfunctions have the following form in complexified space:

$$\begin{pmatrix} \psi(z) \\ T(z) \\ C(z) \\ A(z) \end{pmatrix} e^{ik\alpha x}, \quad k \in \mathbb{Z}. \tag{2.15}$$

Inferring from eqs. (2.3) and (2.14), we have

$$\frac{\partial^{2n}\psi}{\partial z^{2n}} = \frac{\partial^{2n}T}{\partial z^{2n}} = \frac{\partial^{2n}C}{\partial z^{2n}} = \frac{\partial^{2n+1}A}{\partial z^{2n+1}} = 0, \quad \forall n \in \mathbb{Z}_+ \text{ at } z = 0, 1, \tag{2.16}$$

which further suggesting the eigenfunctions are of following form

$$\begin{pmatrix} \psi \sin(l\pi z) \\ T \sin(l\pi z) \\ C \sin(l\pi z) \\ A \cos(l\pi z) \end{pmatrix} e^{ik\alpha x}, \quad (k, l) \in I_0 \coloneqq \mathbb{Z} \times \mathbb{N} \setminus (0, 0). \tag{2.17}$$

Plugging (2.17) into (2.14), we arrive at the eigenvalue problem for each fixed wave number:

$$\begin{pmatrix} -\gamma_J^2\mathrm{Pr}(\gamma_J^2\Lambda_\mathrm{c}+1) & \frac{ik\alpha\mathrm{PrRt}}{\gamma_J^2} & -\frac{ik\alpha\mathrm{PrRs}}{\gamma_J^2} & -l\pi\mathrm{PrRm} \\ -i\alpha k & -\gamma_J^2 & 0 & 0 \\ -i\alpha k & 0 & -\gamma_J^2\tau_1 & 0 \\ \pi l & 0 & 0 & -\gamma_J^2\tau_2 \end{pmatrix} \begin{pmatrix} \psi \\ T \\ C \\ A \end{pmatrix} = \beta \begin{pmatrix} \psi \\ T \\ C \\ A \end{pmatrix}, \tag{2.18}$$

where $\gamma_J^2 = l^2\pi^2 + k^2\alpha^2$, $J = (k,l)$. From above equations, we can obtain the characteristic polynomial

$$\begin{aligned} f(\beta) = {} & \gamma_J^2(\beta+\gamma_J^2)(\beta+\gamma_J^2\tau_1)\left[\pi^2 l^2\mathrm{PrRm} + (\beta+\gamma_J^2\tau_2)\big(\beta+\gamma_J^2\mathrm{Pr}(\gamma_J^2\Lambda_\mathrm{c}+1)\big)\right] \\ & + \alpha^2k^2\mathrm{Pr}(\beta+\gamma_J^2\tau_2)\big[\mathrm{Rs}(\beta+\gamma_J^2) - \mathrm{Rt}(\beta+\gamma_J^2\tau_1)\big]. \end{aligned} \tag{2.19}$$

Regarding the characteristic polynomial as a polynomial of β, we also have following equivalent expression:

$$f(\beta) = a_4\beta^4 + a_3\beta^3 + a_2\beta^2 + a_1\beta + a_0 \tag{2.20}$$

where the coefficients are

$$\begin{aligned} a_0 &= \gamma_J^4\mathrm{Pr}\left(\gamma_J^6\tau_1\tau_2\left(\gamma_J^2\Lambda_c+1\right) + k^2\alpha^2\tau_2(\mathrm{Rs}-\mathrm{Rt}\tau_1) + l^2\pi^2\gamma_J^2\mathrm{Rm}\tau_1\right), \\ a_1 &= k^2\alpha^2\gamma_J^2\mathrm{Pr}(\mathrm{Rs}(\tau_2+1) - \mathrm{Rt}(\tau_1+\tau_2)) + l^2\pi^2\gamma_J^4\mathrm{PrRm}(\tau_1+1) \\ &\qquad + \gamma_J^8\left(\mathrm{Pr}\left(\gamma_J^2\Lambda_c+1\right)(\tau_1\tau_2+\tau_1+\tau_2) + \tau_1\tau_2\right), \\ a_2 &= k^2\alpha^2\mathrm{Pr}(\mathrm{Rs}-\mathrm{Rt}) + l^2\pi^2\gamma_J^2\mathrm{PrRm} \\ &\qquad + \gamma_J^6\left(\mathrm{Pr}\left(\gamma_J^2\Lambda_c+1\right)(\tau_1+\tau_2+1) + \tau_1\tau_2+\tau_1+\tau_2\right), \\ a_3 &= \gamma_J^4\left(\mathrm{Pr}(\gamma_J^2\Lambda_c+1) + \tau_1+\tau_2+1\right), \\ a_4 &= \gamma_J^2. \end{aligned} \tag{2.21}$$

For the sake of convenience, we discuss the eigenvalue problem in three cases,

$$\begin{aligned} I_1 &= \{(k,l)\in I_0 | k,l\neq 0\}, \\ I_2 &= \{(k,l)\in I_0 | k=0\}, \\ I_3 &= \{(k,l)\in I_0 | l=0\}. \end{aligned} \tag{2.22}$$

Note that $I_0 = I_1\cup I_2\cup I_3$. We define the following notions for simplicity:

$$\xi_J = e^{ik\alpha x}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad \zeta_J = \begin{pmatrix}\sin(l\pi z)\\ \sin(l\pi z)\\ \sin(l\pi z)\\ \cos(l\pi z)\end{pmatrix}, \quad \kappa_J^s = \begin{pmatrix}1\\ K_0(J,s)\\ K_1(J,s)\\ K_2(J,s)\end{pmatrix} \tag{2.23}$$

where the complex coefficients K_i are:

$$K_0(J,s) = -\frac{k\alpha}{\beta_J^s+\gamma_J^2}i, \quad K_1(J,s) = -\frac{k\alpha}{\beta_J^s+\tau_1\gamma_J^2}i, \quad K_2(J,s) = \frac{l\pi}{\beta_J^s+\tau_2\gamma_J^2}.$$

And we use $*$ to denote the Hadamard product of matrices, $\overline{J}$ to denote the conjugate index of $J=(k,l)$, i.e. $\overline{J}=(-k,l)$.

2.2. Eigenvalues and eigenvectors of index Sets I_i

For index set I_1, i.e. $k,l\neq 0$, there are four eigenvalues $\beta_J^s,(s=1,2,3,4)$. We make the assumption that $\Re\beta_J^1\geq\Re\beta_J^2\geq\Re\beta_J^3\geq\Re\beta_J^4$. And the corresponding eigenvectors are

$$\psi_J^s=\kappa_J^s*\xi_J*\zeta_J. \tag{2.24}$$

Since for real eigenvalue β_J^s, the conjugate relations $\beta_J^s=\overline{\beta_{\overline{J}}^s}$, $\kappa_J^s=\overline{\kappa_{\overline{J}}^s}$ and $\psi_J^s=\overline{\psi_{\overline{J}}^s}$ are valid. We intend to let the relations also hold for complex eigenvalues. Hence, we make further assumption that when β_J^s and β_J^{s+1} is a pair of conjugate eigenvalues, $\beta_J^s=\beta_{\overline{J}}^{s+1}$ and $\beta_J^{s+1}=\beta_{\overline{J}}^s$ holds. This ensures the conjugate relations are valid for both real and complex eigenvalues.

For index set I_2, i.e. $k=0,l\neq 0$, the two roots of characteristic polynomial (2.19) are

$$\beta_J^1=-l^2\pi^2,\quad \beta_J^2=-\tau_1 l^2\pi^2,\quad J\in I_2. \tag{2.25}$$

The other two roots $\beta_J^s,(s=3,4,J\in I_2)$ are zeros of following quadratic equation

$$\left(\beta+\pi^2l^2\tau_2\right)\left(\beta+\pi^2l^2\mathrm{Pr}\left(\pi^2\Lambda_c l^2+1\right)\right)+\pi^2l^2\mathrm{PrRm}=0, \tag{2.26}$$

which are located on the left of imaginary axis due to Vieta's theorem. Denoting

$$\chi_J^1=\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad \chi_J^2=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad \chi_J^3=\chi_J^4=\begin{pmatrix}1\\0\\0\\1\end{pmatrix},\quad \text{for}\quad J\in I_2. \tag{2.27}$$

The corresponding eigenvector, which can be calculated via (2.18), are

$$\begin{aligned}\psi_J^s&=\chi_J^s*\zeta_J &&\text{for}\quad s=1,2,\\ \psi_J^s&=\kappa_J^s*\chi_J^s*\zeta_J &&\text{for}\quad s=3,4.\end{aligned} \tag{2.28}$$

We remark that there is no critical-crossing eigenvalue with index J in set I_2.

For index set I_3, i.e. $l=0,k\neq 0$, there is an explicitly calculated eigenvalue $\beta_J^1=-\tau_2k^2\alpha^2$ which corresponds to eigenvectors ψ_J^1

$$\psi_J^1=\chi_J^1*\xi_J:=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}*\xi_J\quad\text{for}\quad J\in I_3. \tag{2.29}$$

And the other eigenvectors corresponds to the remaining eigenvalues of (2.19) are excluded due to the boundary condition (2.3).

Since $\mathcal{L}$ is a completely continuous field (see [23]), we have the following decomposition of $\mathbf{H}_0$ by collecting all the eigenvalues,

$$\mathbf{H}_0 = \mathbf{E_1} \oplus \mathbf{E_2} \oplus \mathbf{E_3} =: \underset{\substack{J\in I_1 \\ s=1,2,3,4}}{\operatorname{span}} \psi_J^s \oplus \underset{\substack{J\in I_2 \\ s=1,2,3,4}}{\operatorname{span}} \psi_J^s \oplus \underset{\substack{J\in I_3 \\ s=1}}{\operatorname{span}} \psi_J^s. \tag{2.30}$$

And for each J, $\operatorname{span}_s \psi_J^s$ is an invariant subspace of $\mathcal{L}$.

To sum up, by the settings of κ_J^s for $J \in I_1, I_2, I_3$, the conjugate relations

$$\beta_J^s = \overline{\beta_{\overline{J}}^s}, \quad \kappa_J^s = \overline{\kappa_{\overline{J}}^s} \quad \text{and} \quad \psi_J^s = \overline{\psi_{\overline{J}}^s} \tag{2.31}$$

hold for all (J, s). Moreover, for a pair of conjugate eigenvalues β_J^s and β_J^{s+1}, we have

$$\begin{aligned} K_m(J,s) &= -\overline{K_m(J,s+1)} \quad \text{for} \quad m = 0,1, \\ K_2(J,s) &= \overline{K_2(J,s+1)}. \end{aligned} \tag{2.32}$$

The above conjugate relations are useful for calculations in latter sections.

2.3. Adjoint Eigenvectors

To facilitate the center manifold reduction, the adjoint eigenvectors $\psi_J^{s,*} \in \mathbf{H_0}$ satisfying

$$\begin{aligned} &\left\langle \psi_J^s, \psi_J^{s,*} \right\rangle \neq 0, \quad \text{and} \quad \left\langle \psi_J^s, \psi_{J'}^{s',*} \right\rangle = 0, \quad \text{for} \quad (J,s) \neq (J', s') \\ &\mathbf{H}_0 = \underset{\substack{J\in I_1 \\ s=1,2,3,4}}{\operatorname{span}} \psi_J^{s,*} \oplus \underset{\substack{J\in I_2 \\ s=1,2,3,4}}{\operatorname{span}} \psi_J^{s,*} \oplus \underset{\substack{J\in I_3 \\ s=1}}{\operatorname{span}} \psi_J^{s,*} \end{aligned} \tag{2.33}$$

is of our interest. We note that the inner product is taken in space $\mathbf{H_0}$ through this article if there is no explicit clarification.

By setting

$$\begin{aligned} \psi_J^{s,*} &= \kappa_J^{s,*} * \xi_J * \zeta_J, && \text{for} \quad J \in I_1, \\ \psi_J^{s,*} &= \kappa_J^{s,*} * \zeta_J && \text{for} \quad J \in I_2, \\ \psi_J^{1,*} &= \kappa_J^{1,*} * \xi_J && \text{for} \quad J \in I_3, \end{aligned} \tag{2.34}$$

our aim is to determine the coefficients

$$(\kappa_J^{s,*} = \left(K_{-1}^*(J,s), K_0^*(J,s), K_1^*(J,s), K_2^*(J,s)\right)^t)$$

by making use of (2.33). We notice that, for different indexes J and J', $\left\langle \psi_J^s, \psi_{J'}^{s',*} \right\rangle = 0$ is immediately valid. Hence, the calculation is reduced to each invariant subspace $\operatorname{span}_s \psi_J^s$ of $\mathcal{L}$.

First, for $J \in I_3$, the invariant subspace is one-dimensional. Thus, we can easily determine $\kappa_J^{1,*} = (0,0,0,1)^t$ for $J \in I_3$, i.e. $\psi_J^1 = \psi_J^{1,*}$.

Second, for $J \in I_1$, the coefficients are determined by following equations,

$$\left\langle \kappa_J^{s,*}, \kappa_J^{s'} \right\rangle + \gamma_J^2 \left\langle K_{-1}^*(J,s), 1 \right\rangle = 0, \quad \text{for} \quad s \neq s', \tag{2.35}$$

in which the inner product is the natural inner product in $\mathbb{C}^4$. Hence, by setting $K_{-1}^*(J,s) = 1$, we arrive at

$$\begin{aligned} K_0^*(J,s) &= \frac{\left(\gamma_J^2 + 1\right) \prod_{n=1,n\neq s}^4 \left(\beta_J^n + \gamma_J^2\right)}{k\alpha\gamma_J^4(\tau_1 - 1)(\tau_2 - 1)} i, \\ K_1^*(J,s) &= \frac{\left(\gamma_J^2 + 1\right) \prod_{n=1,n\neq s}^4 \left(\beta_J^n + \tau_1\gamma_J^2\right)}{k\alpha\gamma_J^4(\tau_1 - 1)(\tau_1 - \tau_2)} i, \\ K_2^*(J,s) &= \frac{\left(\gamma_J^2 + 1\right) \prod_{n=1,n\neq s}^4 \left(\beta_J^n + \tau_2\gamma_J^2\right)}{l\pi\gamma_J^4(\tau_2 - 1)(\tau_1 - \tau_2)}. \end{aligned} \tag{2.36}$$

Last, for $J \in I_2$, we obtain

$$\begin{aligned} &K_0^*(J,s) = K_1^*(J,s) = 0, \\ &K_2^*(J,s) = -\frac{\left(\gamma_J^2 + 1\right) \prod_{n=3,4,n\neq s}^4 \left(\beta_J^n + \gamma_J^2\tau_2\right)}{\pi l}, \end{aligned} \tag{2.37}$$

when $s = 3,4$. For $s = 1,2$ and $J \in I_2$, direct calculation shows $\kappa_J^{1,*} = (0,1,0,0)^t, \kappa_J^{2,*} = (0,0,1,0)^t$.

3. Principle of Exchange Stablities

In order to study the dynamical transitions of (2.1), we need to verify PES condition. From the preceding discussion, the real parts of all eigenvalues in index sets I_2 and I_3 are negative. Thus, the critical indexes can only belong to index set I_1. Furthermore, when $\mathrm{Rt} = 0$, according to **Routh-Hurwitz Test**, we know the characteristic polynomial (2.19) has four negative real-part roots. Therefore, to verify PES condition, we only consider $\mathrm{Rt} > 0$.

Note that when $a_0 = 0$, (2.19) has a zero root. Now, the expression of one critical value is as follows:

$$\mathrm{Rt}_{c_1} = \min_{J \in I_1} h(J), \tag{3.1}$$

where $h(J) = \frac{\tau_1 l^2\pi^2\gamma_J^2\mathrm{Rm} + \tau_2 k^2\alpha^2\mathrm{Rs} + \tau_1\tau_2\gamma_J^6 + \Lambda_c\tau_1\tau_2\gamma_J^8}{\tau_1\tau_2 k^2\alpha^2}$.

Besides, according to the relationship between roots and coefficients, when $a_1^2a_4 - a_1a_2a_3 + a_0a_3^2 = 0$, (2.19) has a pair of conjugate pure complex roots. Thus, we have the equation as follows:

$$c_2(J)Rt^2 + c_1(J)Rt + c_0(J) = 0, \tag{3.2}$$

where

$$\begin{aligned}
c_2(J) =& -k^4\alpha^4Pr^2\gamma_J^6(\tau_1+\tau_2)(\Lambda_c Pr\gamma_J^2 + Pr + 1) < 0,\\
c_1(J) =& k^2\alpha^2Pr\gamma_J^6(k^2\alpha^2PrRs(Pr(\Lambda_c\gamma_J^2+1)(\tau_1+2\tau_2+1)\\
&+\tau_1(\tau_1+\tau_2)+\tau_2+1) + l^2\pi^2Pr\gamma_J^2Rm(Pr(\Lambda_c\gamma_J^2+1)(2\tau_1\\
&+\tau_2+1)+\tau_2(\tau_1+\tau_2)+\tau_1+1) + \gamma_J^6(\tau_1+\tau_2)(\Lambda_cPr\gamma_J^2\\
&+Pr+1)(Pr(\Lambda_c\gamma_J^2+1)(\tau_1+\tau_2+2)+\tau_1^2+\tau_1+\tau_2^2+\tau_2)) > 0, \quad (3.3)\\
c_0(J) =& -\gamma_J^6(k^2\alpha^2PrRs(\tau_2+1) + l^2\pi^2Pr\gamma_J^2Rm(\tau_1+1)\\
&+\Lambda_cPr\gamma_J^8(\tau_1+1)(\tau_2+1) + (Pr+1)\gamma_J^6(\tau_1+1)(\tau_2+1))\\
&((\Lambda_cPr\gamma_J^2+Pr+\tau_1)(k^2\alpha^2PrRs + \gamma_J^6(\tau_1+\tau_2)(\Lambda_cPr\gamma_J^2\\
&+Pr+\tau_2)) + l^2\pi^2Pr\gamma_J^2Rm(\Lambda_cPr\gamma_J^2+Pr+\tau_2)) < 0.
\end{aligned}$$

By computation,

$$\begin{aligned}
\delta = c_1(J)^2 - 4c_0(J)c_2(J) =& k^4\alpha^4Pr^2\gamma_J^{12}(\Lambda_cPr\gamma_J^2+Pr+\tau_1+\tau_2+1)^2\\
&(2l^2\pi^2Pr\gamma_J^2Rm(\tau_2-1)(k^2\alpha^2PrRs(\tau_1-1) - \gamma_J^6(\tau_1-\tau_2)\\
&(\tau_1+\tau_2)(\Lambda_cPr\gamma_J^2+\mathrm{Pr}+1)) + (k^2\alpha^2PrRs(\tau_1-1)\\
&+\gamma_J^6(\tau_1-\tau_2)(\tau_1+\tau_2)(\Lambda_cPr\gamma_J^2+Pr+1))^2\\
&+l^4\pi^4Pr^2\gamma_J^4Rm^2(\tau_2-1)^2).
\end{aligned} \tag{3.4}$$

Note that when $1 < \tau_1 < \tau_2$ or $\tau_2 < \tau_1 < 1$, $\delta > 0$. Thus, (3.2) always has two positive roots as follows:

$$0 < Rt_{c_{21}} = \frac{-c_1(J)+\sqrt{\delta}}{2c_2(J)} < \frac{-c_1(J)-\sqrt{\delta}}{2c_2(J)} = Rt_{c_{22}}. \tag{3.5}$$

Therefore, we obtain another critical value as follows:

$$\mathrm{Rt}_{c_2} = \min_{J\in I_1}\{Rt_{c_{21}}\}. \tag{3.6}$$

Then, the exact value of the threshold Rt_c for the system (2.1) can be intuitively derived by Rt_{c_1} and Rt_{c_2}, given by

$$\mathrm{Rt}_c = \min\{\mathrm{Rt}_{c_1}, \mathrm{Rt}_{c_2}\}. \tag{3.7}$$

In what follows, let us verify the PES condition. To this end, we introduce a critical index set defined by

$$X = \{J \in I_1 | \mathrm{Rt}_c = \min\{h(J), Rt_{c_{21}}(J)\}\}. \tag{3.8}$$

Note that Card(X) is finite, since when $|J| \to +\infty$, both $h(J)$ and $Rt_{c_{21}}(J)$ go to positive infinity. We thus have the following PES condition:

Lemma 3.1. *For the system* (2.1)*, we have the following assertions:*

(1) When $\mathrm{Rt}_c = \mathrm{Rt}_{c_1} < \mathrm{Rt}_{c_2}$*, we have*

$$\begin{aligned} &\beta_J^1 \begin{cases} < 0, & \mathrm{Rt} < \mathrm{Rt}_{c_1}, \\ = 0, & \mathrm{Rt} = \mathrm{Rt}_{c_1}, \qquad J \in X, \\ > 0, & \mathrm{Rt} > \mathrm{Rt}_{c_1}, \end{cases} \\ &\Re\beta_J^s(\mathrm{Rt}_c) < 0 \quad \forall (J, s) \notin X \times \{1\} \ (s = 1, 2, 3, 4). \end{aligned} \tag{3.9}$$

(2)When $\mathrm{Rt}_c = \mathrm{Rt}_{c_2} < \mathrm{Rt}_{c_1}$*,* $1 < \tau_1 < \tau_2$ *or* $\tau_2 < \tau_1 < 1$*, we have*

$$\begin{aligned} &\Re\beta_J^1 = \Re\beta_J^2 \begin{cases} < 0, & \mathrm{Rt} < \mathrm{Rt}_{c_2}, \\ = 0, & \mathrm{Rt} = \mathrm{Rt}_{c_2}, \qquad J \in X, \\ > 0, & \mathrm{Rt} > \mathrm{Rt}_{c_2}, \end{cases} \\ &\Re\beta_J^s(\mathrm{Rt}_c) < 0 \quad \forall (J, s) \notin X \times \{1, 2\} \ (s = 1, 2, 3, 4). \end{aligned} \tag{3.10}$$

Proof. For $\mathrm{Rt} = 0$, by computation, we know

$$\begin{aligned} &a_0 > 0, \quad a_3 > 0, \quad a_2 a_3 - a_4 a_1 > 0, \\ &a_4 > 0, \quad a_1(a_2 a_3 - a_4 a_1) - a_3^2 a_0 > 0. \end{aligned}$$

Thus, by **Routh-Hurwitz Test**, for $J \in I_1$, the real parts of all eigenvalues of (2.19) are negative.

(1)For $\mathrm{Rt}_{c_1} < \mathrm{Rt}_{c_2}$, which means Rt firstly arrives at Rt_{c_1} when Rt increases from 0. Then according to the definition of Rt_{c_1} and the continuous dependence of β_J^s $(s = 1, 2, 3, 4)$ on Rt, the first conclusion holds.

(2)For $\mathrm{Rt}_{c_2} < \mathrm{Rt}_{c_1}$, which means Rt firstly arrives at Rt_{c_2} when Rt increases from 0. According to the definition of Rt_{c_1} and the continuous dependence of β_J^s $(s = 1, 2, 3, 4)$ on Rt, when Rt is in the neighborhood of Rt_{c_2}, for $J \notin X$ and $J \in I_1$, $\Re\beta_J^s < 0$. In order to verify that the first eigenvalues $\beta_J^1 = \overline{\beta_J^2} = \sigma + i\rho$ $(J \in X)$ will change the signs of their real parts when Rt passes through Rt_{c_2}, we need to show $\sigma'(\mathrm{Rt}_{c_2}) > 0$. We differentiate the (2.19) with respect to Rt, and we find that when $\tau_1, \ \tau_2 > 1$ and $\tau_1 < \tau_2$,

$$\begin{aligned} \sigma'(\mathrm{Rt}_{c_2}) = {} & [[\sqrt{\delta}(k^2\alpha^2 PrRs(\tau_1 - 1) + l^2\pi^2 Pr\gamma_J^2 Rm(\tau_2 - 1) + \gamma_J^6(\tau_1^2 \\ & + \tau_2^2)(\Lambda_c Pr\gamma_J^2 + Pr + 1))][\Lambda_c Pr\gamma_J^2 + Pr + \tau_1 + \tau_2 + 1]^{-1} \\ & + k^2\alpha^2 Pr(2l^2\pi^2 Pr\gamma_J^2 Rm(\tau_2 - 1)(k^2\alpha^2 PrRs(\tau_1 - 1) \\ & - \gamma_J^6(\tau_1 - \tau_2)(\tau_1 + \tau_2)(\Lambda_c Pr\gamma_J^2 + Pr + 1)) + (k^2\alpha^2 PrRs \\ & (\tau_1 - 1) + \gamma_J^6(\tau_1 - \tau_2)(\tau_1 + \tau_2)(\Lambda_c Pr\gamma_J^2 + Pr + 1))^2 \\ & + l^4\pi^4 Pr^2\gamma_J^4 Rm^2(\tau_2 - 1)^2)][\Lambda_c \mathrm{Pr}\gamma_J^2 + \mathrm{Pr} + 1]^{-1} > 0. \end{aligned} \tag{3.11}$$

Therefore, the second conclusion is valid.

□

When the parameters $\Lambda_c = 1$, $\mathrm{Rs} = 80000$ and $\mathrm{Pr} = 7$, we compare the values of Rt_{c_1} and Rt_{c_2} as a function of Rm in the following cases: (1) $\tau_1 = 0.72,\ \tau_2 = 0.5$; (2) $\tau_1 = 1.2,\ \tau_2 = 1.5$, see Figure 3.1. Note that Rt_{c_1} and Rt_{c_2} depend on Rm almost linearly. A mathematical interpretation is that Rs is far larger than Rm, thus, the expression (3.6) is taken as a linear function of Rm.

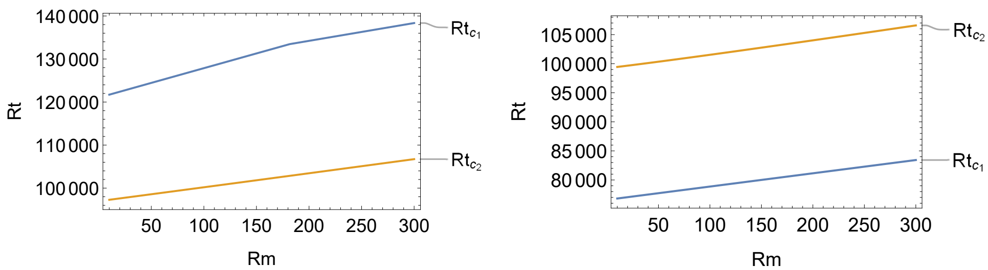


Figure 3.1: Values of Rt_{c_1} and Rt_{c_2} with $\mathrm{Rm} \in [10, 300]$ (left: $\tau_1 = 0.72,\ \tau_2 = 0.5$) and (right: $\tau_1 = 1.2,\ \tau_2 = 1.5$).

Rt is considered to be the control parameter of the model (2.1) in this article. Therefore, we remark that the linear operator defined by (2.11) is depending on Rt, and we denote the operator by $\mathcal{L}_{\mathrm{Rt}}$. Similarly, the eigenvalues $\beta_J^s(\mathrm{Rt})$ and eigenfunctions $\psi_J^s(\mathrm{Rt})$ are also depend on Rt. However, we may not specify the dependence throughout the rest of the article for sake of simplicity.

4. Transition Induced by Real Eigenvalues

In this section, we consider the transition of the system (2.1) from a pair of real eigenvalue $\beta^1_{J_r} = \beta^1_{\overline{J}_r}$ where $J_r = (k_r, 1), \overline{J}_r = (-k_r, 1)$ are in index set I_1. The transition can be characterized by introducing a non-dimensional real number $\Gamma(\mathrm{Rt})$ given by

$$\begin{aligned}\Gamma(\mathrm{Rt}) =& \frac{i\pi^2 k_r}{\left\langle \psi^1_{J_r}, \psi^{1,*}_{J_r} \right\rangle}\Bigg[2h^1_{J_{r_3}}\left(\frac{\gamma^2_{J_r}+1}{\gamma^2_{J_r}}\mathrm{Pr}\mathrm{Rm}\left(\gamma^2_{J_r} - \gamma^2_{J_{r_3}}\right)K_2(\overline{J}_r, 1)\right.\\ &\left. + K_2^*(\overline{J}_r, 1)\right) + h^1_{J_{r_2}}K_0^*(\overline{J}_r, 1) + h^2_{J_{r_2}}K_1^*(\overline{J}_r, 1)\Bigg]\end{aligned} \tag{4.1}$$

in which $h^1_{J_{r_2}}$, $h^2_{J_{r_2}}$ and $h^1_{J_{r_3}}$ are given in (4.14).

Remark 1. *We will use abbreviations Γ for $\Gamma(\mathrm{Rt})$.*

Theorem 4.1. *For the system* (2.1), *assume that the first eigenvalue is real and of algebraic multiplicity 2, we have the following conclusions:*

(i) *If* $\Gamma(\mathrm{Rt}_{c_1}) < 0$*, the system undergoes a continuous transition from* $(\Psi, \mathrm{Rt}) = (0, \mathrm{Rt}_{c_1})$*, and bifurcates on* $\mathrm{Rt} > \mathrm{Rt}_{c_1}$ *to a local attractor* Σ_{Rt}*, which is homeomorphic to the one-dimensional sphere* S^1*, as shown in Figure 4.1. Furthermore,* Σ_{Rt} *consisting of steady state solutions of the problem. The approximated solutions are given by*

$$\Psi = 2\sqrt{-\frac{\beta^1_{J_r}(\mathrm{Rt})}{\Gamma(\mathrm{Rt})}}\Re\big(\exp(-i\theta)\psi^1_{J_r}\big) + o\Big(\big|\beta^1_{J_r}\big|^{\frac{1}{2}}\Big), \quad \textit{for} \quad \theta \in [0, 2\pi]. \tag{4.2}$$

(ii) *If* $\Gamma(\mathrm{Rt}_{c_1}) > 0$*, the system undergoes a jump transition from* $(\Psi, \mathrm{Rt}) = (0, \mathrm{Rt}_{c_1})$*, and bifurcates on* $\mathrm{Rt} < \mathrm{Rt}_{c_1}$ *to a repeller consisting of steady state solutions of the problem.*

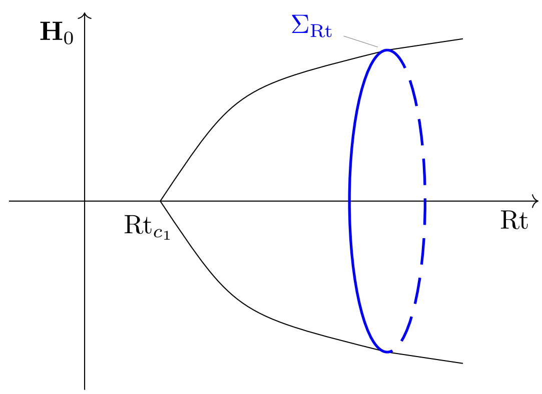


Figure 4.1: If the conditions in Theorem 4.1 (i) holds, the system bifurcates from $(0, \mathrm{Rt}_{c_1})$ to an attractor $\Sigma_{\mathrm{Rt}} \in \mathbf{H}_0$, which consists of stationary states.

Proof. Due to foregoing arguments of decomposition of space $\mathbf{H}_0$, there exists the decomposition

$$\mathbf{H}_0 = \mathbf{E}_0 \oplus \mathbf{E}_h \tag{4.3}$$

where the space $\mathbf{E}_0$ is spanned by the critical eigenvectors, i.e.

$$\mathbf{E}_0 = \mathrm{span}\left\{\psi^1_{J_r}, \psi^1_{\overline{J}_r}\right\}.$$

And $\mathbf{E}_h$ is the orthogonal complement of $\mathbf{E}_0$.

We denote P_0 and P_h the orthogonal projection onto $\mathbf{E}_0$ and $\mathbf{E}_h$, respectively. Thus, according to center manifold theorem, for any $\Psi \in \mathbf{H}_0$ the solution of the model (2.1), the following decomposition is valid

$$\Psi = P_0\Psi + P_h\Psi := \psi_{J_r} + \Phi(\psi_{J_r}) \tag{4.4}$$

where $\psi_{J_r} = s\psi^1_{J_r} + \overline{s}\psi^1_{\overline{J}_r} \in \mathbf{E}_0$ for $s \in \mathbb{C}$ and $\Phi : \mathbf{E}_0 \to \mathbf{E}_h$ is the center manifold function. By center manifold reduction, the ODE

$$P_0 \frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\mathrm{d}P_0\Psi}{\mathrm{d}t} = \frac{\mathrm{d}\psi_{J_r}}{\mathrm{d}t} = P_0\mathcal{L}\Psi + P_0\mathcal{G}(\Psi) = \mathcal{L}\psi_{J_r} + P_0\mathcal{G}(\Psi) \tag{4.5}$$

governs the local dynamics of the system. That is,

$$\frac{\mathrm{d}s}{\mathrm{d}t}\left\langle \psi^1_{J_r}, \psi^{1,*}_{J_r}\right\rangle \psi^1_{J_r} + c.c. = \left(\left\langle \mathcal{L}\psi_{J_r}, \psi^{1,*}_{J_r}\right\rangle + \left\langle \mathcal{G}(\Psi), \psi^{1,*}_{J_r}\right\rangle\right)\psi^1_{J_r} + c.c. \tag{4.6}$$

where $c.c.$ denotes the complex conjugate. The equivalent ODE for s reads

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{1}{\left\langle \psi^1_{J_r}, \psi^{1,*}_{J_r}\right\rangle}\left(\left\langle \mathcal{L}\psi_{J_r}, \psi^{1,*}_{J_r}\right\rangle + \left\langle \mathcal{G}(\Psi), \psi^{1,*}_{J_r}\right\rangle\right), \\ \frac{\mathrm{d}\overline{s}}{\mathrm{d}t} &= \text{ complex conjugate of above expression.} \end{aligned} \tag{4.7}$$

Hereinafter, we will calculate the approximation of above ODE. To do so, the lower order terms of Φ is necessary. Due to the center manifold theorem, the center manifold function have the following expansion

$$\Phi = \Phi_2 + \mathcal{O}\left(|s|^3\right) \tag{4.8}$$

where Φ_2 is of order $|s|^2$ and can be direct calculated by the formula purposed in the appendix of [22]

$$\Phi = (-\mathcal{L})^{-1}P_h\mathcal{G}(\psi_{J_r}) + o(|s|^2) + \mathcal{O}\left(\left|\Re\beta^1_{J_r}\right||s|^2\right). \tag{4.9}$$

Thus, Φ_2 is the first term on the right-hand side of the equation. A direct calculation leads to

$$\mathcal{G}(\psi_{J_r}) = \begin{pmatrix} 0 \\ -2\pi k_r\alpha|s|^2\Im(K_0(J_r,1))\sin(2\pi z) \\ -2\pi k_r\alpha|s|^2\Im(K_1(J_r,1))\sin(2\pi z) \\ -2\pi k_r\alpha|s|^2\Im(K_2(J_r,1))\cos(2\pi z) + \left[i\pi k_r\alpha s^2 K_2(J_r,1)e^{2ik_r\alpha x} + c.c.\right] \end{pmatrix} \tag{4.10}$$

Furthermore, considering the fact that $K_2(J_r,1) \in \mathbb{R}$, we can denote $\mathcal{G}(\psi_{J_r})$ in following summation of eigenvectors:

$$\mathcal{G}(\psi_{J_r}) = \left(g^1_{J_{r_2}}\psi^1_{J_{r_2}} + g^2_{J_{r_2}}\psi^2_{J_{r_2}}\right)|s|^2 + \left[g^1_{J_{r_3}}\psi^1_{J_{r_3}}s^2 + c.c.\right] \tag{4.11}$$

where $J_{r_2} = (0,2) \in I_2$, $J_{r_3} = (2k_r, 0) \in I_3$ and the coefficients are:

$$\begin{aligned} g^1_{J_{r_2}} &= -2\pi k_r\alpha\Im K_0(J_r,1), \\ g^2_{J_{r_2}} &= -2\pi k_r\alpha\Im K_1(J_r,1), \\ g^1_{J_{r_3}} &= i\pi k_r\alpha K_2(J_r,1). \end{aligned} \tag{4.12}$$

And we also notice that $\mathcal{G}(\psi_{J_r}) \in \mathbf{E}_h$, since direct calculation shows $P_0\mathcal{G}(\psi_{J_r}) = 0$. Therefore, $P_h\mathcal{G}(\psi_{J_r}) = \mathcal{G}(\psi_{J_r})$, and the left-hand side of equation (4.9) (dropping the higher order terms) can be denoted as:

$$\Phi_2 = \left(h^1_{J_{r_2}}\psi^1_{J_{r_2}} + h^2_{J_{r_2}}\psi^2_{J_{r_2}}\right)|s|^2 + \left[h^1_{J_{r_3}}\psi^1_{J_{r_3}}s^2 + c.c.\right] \tag{4.13}$$

where the coefficients are

$$h^i_I = -\frac{g^i_I}{\beta^i_I}, \quad \text{for} \quad (I,i) = (J_{r_2}, n), n = 1,2 \quad \text{and} \quad (I,i) = (J_{r_3}, 1). \tag{4.14}$$

With the approximated center manifold function at our disposal, the next step to get the approximated ODE (4.5) is to calculate the lower order terms of $P_0G(\psi_{J_r} + \Phi_2)$. That is

$$\begin{aligned} P_0\mathcal{G}(\psi_{J_r} + \Phi_2) &= P_0\mathcal{G}(\psi_{J_r}) + P_0\mathcal{G}(\psi_{J_r}, \Phi_2) + P_0\mathcal{G}(\Phi_2, \psi_{J_r}) + o(|s|^3) \\ &= P_0\mathcal{G}(\psi_{J_r}, \Phi_2) + P_0\mathcal{G}(\Phi_2, \psi_{J_r}) + o(|s|^3). \end{aligned} \tag{4.15}$$

For this reason, a direct calculation show that

$$P_0\mathcal{G}(\psi_{J_r}, \Phi_2) = \begin{pmatrix} \frac{-8i\pi\alpha^3k_r^3}{\gamma^2_{J_r}}\mathrm{PrRm}s|s|^2h^1_{J_{r_3}}K_2(\overline{J}_r, 1)e^{ik_r\alpha x}\sin(\pi z) + c.c. \\ i\pi\alpha k_r|s|^2\sin(\pi z)h^1_{J_{r_2}}\left(se^{i\alpha k_0 x} - \overline{s}e^{-i\alpha k_0 x}\right) \\ i\pi\alpha k_r|s|^2\sin(\pi z)h^2_{J_{r_2}}\left(se^{i\alpha k_0 x} - \overline{s}e^{-i\alpha k_0 x}\right) \\ 2i\pi\alpha k_r s|s|^2h^1_{J_{r_3}}\cos(\pi z)e^{ik_r\alpha x} + c.c. \end{pmatrix} \tag{4.16}$$

and

$$P_0\mathcal{G}(\Phi_2, \psi_{J_r}) = \begin{pmatrix} 2i\pi k_r\alpha\mathrm{PrRm}s|s|^2h^1_{J_{r_3}}K_2(\overline{J}_r, 1)e^{ik_r\alpha x}\sin(\pi z) + c.c. \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.17}$$

Next, we require $\left\langle \mathcal{G}(\Psi), \psi^{1,*}_{J_r}\right\rangle$ for reduced ODE (4.6).

$$\begin{aligned} \left\langle \mathcal{G}(\Psi), \psi^{1,*}_{J_r}\right\rangle =& \left\langle \mathcal{G}(\psi_{J_r}, \Phi_2), \psi^{1,*}_{J_r}\right\rangle + \left\langle \mathcal{G}(\psi_{J_r}, \Phi_2), \psi^{1,*}_{J_r}\right\rangle + o(|s|^3) \\ =& i\pi^2k_r|s|^2s\Bigg[2h^1_{J_{r_3}}\left(K^*_2(\overline{J}_r, 1) - \frac{\gamma^2_{J_{r_3}}\left(\gamma^2_{J_r}+1\right)\mathrm{PrRm}K_2(\overline{J}_r, 1)}{\gamma^2_{J_r}}\right) \\ &+ h^1_{J_{r_2}}K^*_0(\overline{J}_r, 1) + h^2_{J_{r_2}}K^*_1(\overline{J}_r, 1)\Bigg] \\ &+ 2\pi^2i\left(\gamma^2_{J_r}+1\right)k_r\mathrm{PrRm}s|s|^2h^1_{J_{r_3}}K_2(\overline{J_r}, 1) + o(|s|^3) \\ =& \Gamma\left\langle \psi^1_{J_r}, \psi^{1,*}_{J_r}\right\rangle s|s|^2 + o(|s|^3) \end{aligned} \tag{4.18}$$

In last equation, a bifurcation parameter Γ is defined, i.e.

$$\begin{aligned}\Gamma =&\frac{i\pi^2 k_r}{\left\langle \psi^1_{J_r}, \psi^{1,*}_{J_r}\right\rangle}\left[2h^1_{J_{r_3}}\left(\frac{\gamma^2_{J_r}+1}{\gamma^2_{J_r}}\mathrm{PrRm}\left(\gamma^2_{J_r}-\gamma^2_{J_{r_3}}\right)K_2(\overline{J}_r,1)\right.\right.\\&\left.\left.+K_2^*(\overline{J}_r,1)\right)+h^1_{J_{r_2}}K_0^*(\overline{J}_r,1)+h^2_{J_{r_2}}K_1^*(\overline{J}_r,1)\right]\end{aligned} \tag{4.19}$$

By the definition of eigenvectors and coefficients, we find that

$$\begin{aligned}&\left\langle \psi^1_{J_r}, \psi^{1,*}_{J_r}\right\rangle, \gamma_{J_r}, \gamma_{J_{r_3}}, K_2(\overline{J}_r,1), K_2^*(\overline{J}_r,1), h^1_{J_{r_2}}, h^2_{J_{r_2}} \in \mathbb{R},\\&h^1_{J_{r_3}}, K_0^*(\overline{J}_r,1), K_1^*(\overline{J}_r,1)\in\mathbb{C},\end{aligned} \tag{4.20}$$

which suggests that $\Gamma\in\mathbb{R}$. Therefore, ignoring the higher order terms $o(|s|^3)$, the reduced equation (4.7) reads

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}&=\beta^1_{J_r}s+\Gamma|s|^2 s,\\\frac{\mathrm{d}\overline{s}}{\mathrm{d}t}&=\beta^1_{J_r}\overline{s}+\Gamma|s|^2\overline{s}.\end{aligned} \tag{4.21}$$

By setting $s=r(t)e^{i\theta(t)}, (r(t)\geqslant 0)$, we obtain the reduced ODE in polar coordinates

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t}&=\beta^1_{J_r}r+\Gamma r^3\\\frac{\mathrm{d}\theta}{\mathrm{d}t}&=0\end{aligned} \tag{4.22}$$

Finally, analyzing dynamics of (4.22) near $\mathrm{Rt}=\mathrm{Rt}_{c_1}$ gives the theorem.

□

5. Transition Induced by Complex Eigenvalues

In this section, we consider the transition of the system (2.1) from two pairs of complex eigenvalue $\beta^1_{J_c}=\beta^2_{\overline{J}_c}=\overline{\beta^2_{J_c}}=\overline{\beta^1_{\overline{J}_c}}$ where $J_c=(k_c,l_c), \overline{J}_c=(-k_c,l_c)$ are in index set I_1. The transition is determined by three transition numbers $\delta_0(\mathrm{Rt}), \delta_1(\mathrm{Rt})$ and $\delta_2(\mathrm{Rt})$ and the quadratic form

$$\Re(A_1)r_1^4+\Re(B_6)r_2^4+\Re(A_4+B_3)r_1^2r_2^2, \tag{5.1}$$

the definition of the coefficients are given in (5.41) and (5.44).

Theorem 5.1. *For the system (2.1), assume that the first eigenvalue is a pair of complex numbers with algebraic multiplicity 2, we have the following conclusions:*

(i) *If* (5.1) *is negative definite, then the system* (2.1) *undergoes a continuous transition from* $(\Psi, \mathrm{Rt}) = (0, \mathrm{Rt}_{c_2})$*, and bifurcates on* $\mathrm{Rt} > \mathrm{Rt}_{c_2}$ *to a local attractor* Σ_{Rt}*. Furthermore, if* $\delta_0\delta_1 > 0$ *and* $\delta_0\delta_2 > 0$ *hold at* $\mathrm{Rt} = \mathrm{Rt}_{c_2}$*, the attractor have three double periodic solutions, denote by* $\Psi_j, (j = 0, 1, 2)$*. And there is only one solution, say* Ψ_0*, is stable, while the other two is unstable, (see Figure 5.1b). The approximated formula for the stable solution is*

$$\Psi_0 = 2\sqrt{\frac{\delta_1}{\delta_0}\Re(\beta^1_{J_c})}\Re\big(e^{i\theta_1 t}\psi^1_{J_c}\big) + 2\sqrt{\frac{\delta_2}{\delta_0}\Re(\beta^1_{J_c})}\Re\big(e^{i\theta_2 t}\psi^2_{J_c}\big), \tag{5.2}$$

where

$$\begin{aligned} \theta_1 &= \Im(\beta^1_{J_c}) + \Im(A_1)\frac{\delta_1}{\delta_0}\Re(\beta^1_{J_c}) + \Im(A_4)\frac{\delta_2}{\delta_0}\Re(\beta^1_{J_c}), \\ \theta_2 &= -\Im(\beta^1_{J_c}) + \Im(B_3)\frac{\delta_1}{\delta_0}\Re(\beta^1_{J_c}) + \Im(B_6)\frac{\delta_2}{\delta_0}\Re(\beta^1_{J_c}). \end{aligned} \tag{5.3}$$

(ii) *If* (5.1) *is positive definite, then the system* (2.1) *undergoes a jump transition from* $(\Psi, \mathrm{Rt}) = (0, \mathrm{Rt}_{c_2})$*, and bifurcates on* $\mathrm{Rt} < \mathrm{Rt}_{c_2}$ *to a repeller* Σ_{Rt}*.*

The proof of foregoing theorem requires the approximation of a system of reduced ODEs and the related dynamic analysis. Thus, we first investigate the reduced ODEs, which gives us the following lemma.

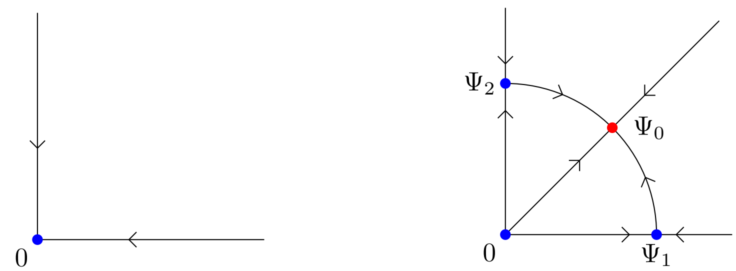
Lemma 5.2. *In the vicinity of* $\mathrm{Rt} = \mathrm{Rt}_{c_2}$*, the stability and transition of the system* (2.1) *for any small initial condition is equivalent to these of the following ODEs:*

$$\begin{aligned} \frac{\mathrm{d}s_1}{\mathrm{d}t} &= \beta^1_{J_c}s_1 + A_1|s_1|^2 s_1 + A_4 s_1|s_2|^2, \\ \frac{\mathrm{d}s_2}{\mathrm{d}t} &= \overline{\beta^1_{J_c}}s_2 + B_3|s_1|^2 s_2 + B_6|s_2|^2 s_2, \\ \frac{\mathrm{d}\overline{s}_1}{\mathrm{d}t} &= \overline{\beta^1_{J_c}}\overline{s}_1 + \overline{A}_1|s_1|^2\overline{s}_1 + \overline{A}_4\overline{s}_1|s_2|^2, \\ \frac{\mathrm{d}\overline{s}_2}{\mathrm{d}t} &= \beta^1_{J_c}\overline{s}_2 + \overline{B}_3|s_1|^2\overline{s}_2 + \overline{B}_6|s_2|^2\overline{s}_2, \end{aligned} \tag{5.4}$$

where the coefficients are given in (5.41)*. By setting* $s_1 = r_1(t)e^{i\theta_1(t)}, s_2 = r_2(t)e^{i\theta_2(t)}, (r_1(t), r_2(t) \geqslant 0)$*, the reduced ODEs* (5.4) *in polar coordinates read*

$$\begin{aligned} \frac{\mathrm{d}r_1}{\mathrm{d}t} &= \Re(\beta^1_{J_c})r_1 + \Re(A_1)r_1^3 + \Re(A_4)r_1r_2^2, \\ \frac{\mathrm{d}r_2}{\mathrm{d}t} &= \Re(\beta^1_{J_c})r_2 + \Re(B_3)r_1^2r_2 + \Re(B_6)r_2^3, \end{aligned} \tag{5.5a}$$

$$\begin{aligned} \frac{\mathrm{d}\theta_1}{\mathrm{d}t} &= \Im(\beta^1_{J_c}) + \Im(A_1)r_1^2 + \Im(A_4)r_2^2, \\ \frac{\mathrm{d}\theta_2}{\mathrm{d}t} &= -\Im(\beta^1_{J_c}) + \Im(B_3)r_1^2 + \Im(B_6)r_2^2, \end{aligned} \tag{5.5b}$$



(a) $\mathrm{Rt} < \mathrm{Rt}_{c_2}$

(b) $\mathrm{Rt} > \mathrm{Rt}_{c_2}$

Figure 5.1: Topological structure of continuous transition described in Theorem 5.1 (i), the bifurcated attractor in (b) has two saddle points and a stable node.

Proof. Similar to the real case, the following decomposition of space is valid

$$\begin{aligned} \mathbf{H}_0 &= \mathbf{E}_0 \oplus \mathbf{E}_h \\ \mathbf{E}_0 &= \operatorname{span}\left\{\psi^1_{J_c}, \psi^1_{\overline{J}_c}, \psi^2_{J_c}, \psi^2_{\overline{J}_c}\right\}. \end{aligned} \tag{5.6}$$

And for any $\Psi \in \mathbf{H}_0$ the solution of the model (2.1), $\Psi \in \mathbf{H}_0$ has the following decomposition

$$\Psi = P_0\Psi + P_h\Psi \coloneqq \psi_{J_c} + \Phi(\psi_{J_c}) \tag{5.7}$$

where $\psi_{J_c} = s_1\psi^1_{J_c} + \overline{s}_1\psi^1_{\overline{J}_c} + s_2\psi^2_{J_c} + \overline{s}_2\psi^2_{\overline{J}_c} \in \mathbf{E}_0$ for $s_1, s_2 \in \mathbb{C}$ and $\Phi : \mathbf{E}_0 \to \mathbf{E}_h$ is the center manifold function.

Then accordingly, the reduced ODEs are

$$\begin{aligned} \frac{\mathrm{d}s_i}{\mathrm{d}t} &= \frac{1}{\left\langle \psi^i_{J_c}, \psi^{i,*}_{J_c} \right\rangle}\left(\left\langle \mathcal{L}\psi_{J_c}, \psi^{i,*}_{J_c}\right\rangle + \left\langle \mathcal{G}(\Psi), \psi^{i,*}_{J_c}\right\rangle\right), \\ \frac{\mathrm{d}\overline{s}_i}{\mathrm{d}t} &= \text{ complex conjugate of above expression,} \end{aligned} \tag{5.8}$$

for $i = 1, 2$. The center manifold function also has the expansion

$$\Phi = \Phi_2 + \mathcal{O}\left(|s|^3\right), \tag{5.9}$$

where $|s|^2 = s_1^2 + s_2^2$ is different from the squared modulus in real case. And the center manifold function Φ is determined by following (see A):

$$\begin{aligned} \Phi = & -2\Im\beta^1_{J_c} i(-\mathcal{L})^{-1}\left((2\Im\beta^1_{J_c} i - \mathcal{L})^{-1}\hat{\mathcal{G}} - (-2\Im\beta^1_{J_c} i - \mathcal{L})^{-1}\widetilde{\mathcal{G}}\right) \\ & + (-\mathcal{L})^{-1}P_h\mathcal{G}(\psi_{J_c}) + o(|s|^2) + \mathcal{O}\left(\left|\Re\beta^1_{J_c}\right||s|^2\right) \end{aligned} \tag{5.10}$$

where $\hat{\mathcal{G}} = P_h\mathcal{G}(s_1\psi^1_{J_c} + \overline{s}_2\psi^2_{\overline{J}_c})$ and $\widetilde{\mathcal{G}} = P_h\mathcal{G}(\overline{s}_1\psi^1_{\overline{J}_c} + s_2\psi^2_{J_c}) = \overline{\hat{\mathcal{G}}}$. Next, we calculate the right-hand side of the formula term by term. By direct calculation,

$$\mathcal{G}(\psi_{J_c}) = \left(\begin{array}{c} 0 \\ -2\pi\alpha k_c l_c \sin(2\pi l_c z)\Im\left(\left(|s_1|^2+|s_2|^2+2s_1\overline{s}_2\right)K_0(J_c,1)\right) \\ -2\pi\alpha k_c l_c \sin(2\pi l_c z)\Im\left(\left(|s_1|^2+|s_2|^2+2s_1\overline{s}_2\right)K_1(J_c,1)\right) \\ \hdashline 2\pi\alpha k_c l_c\left(|s_2|^2-|s_1|^2\right)\cos(2\pi l_c z)\Im K_2(J_c,1) \\ +i\pi\alpha k_c l_c(s_1+s_2)e^{2i\alpha k_c x}\left(s_2\overline{K_2(J_c,1)}+s_1K_2(J_c,1)\right)+c.c., \end{array}\right) \tag{5.11}$$

which has the eigenfunction expansion

$$\mathcal{G}(\psi_{J_c}) = \sum_{n=1}^{4} g^n_{J_{c_2}}\psi^n_{J_{c_2}} + \left[g^1_{J_{c_3}}\psi^1_{J_{c_3}} + c.c.\right] \tag{5.12}$$

where $J_{c_2} = (0, 2l_c) \in I_2$, $J_{c_3} = (2k_c, 0) \in I_3$ and the coefficients are:

$$\begin{aligned} g^1_{J_{c_2}} &= -g^{1s}_{J_{c_2}}\Im\left(\left(|s_1|^2+|s_2|^2+2s_1\overline{s}_2\right)K_0(J_c,1)\right), \\ g^2_{J_{c_2}} &= -g^{2s}_{J_{c_2}}\Im\left(\left(|s_1|^2+|s_2|^2+2s_1\overline{s}_2\right)K_1(J_c,1)\right), \\ g^1_{J_{c_3}} &= g^{1s}_{J_{c_3}}(s_1+s_2)\left(s_2\overline{K_2(J_c,1)}+s_1K_2(J_c,1)\right), \end{aligned} \tag{5.13}$$

in which

$$g^{1s}_{J_{c_2}} = g^{2s}_{J_{c_2}} = 2\pi\alpha k_c l_c, \quad g^{1s}_{J_{c_3}} = i\pi\alpha k_c l_c. \tag{5.14}$$

And the coefficients $g^j_{J_{c_2}}, (j = 3, 4)$ are solved from

$$\begin{aligned} g^3_{J_{c_2}} + g^4_{J_{c_2}} &= 0, \\ g^3_{J_{c_2}}K_2(J_c,3) + g^4_{J_{c_2}}K_2(J_c,4) &= 2\pi\alpha k_c l_c\left(|s_2|^2-|s_1|^2\right)\Im K_2(J_c,1). \end{aligned} \tag{5.15}$$

And for latter calculation, we define $g^{3s}_{J_{c_2}}$ and $g^{4s}_{J_{c_2}}$ by

$$\left(|s_2|^2-|s_1|^2\right)g^{3s}_{J_{c_2}} = g^3_{J_{c_2}}, \quad \left(|s_2|^2-|s_1|^2\right)g^{4s}_{J_{c_2}} = g^4_{J_{c_2}}. \tag{5.16}$$

We remark that $P_0\mathcal{G}(\psi_{J_c}) = 0$. Next, we calculate $\hat{\mathcal{G}}$ and $\widetilde{\mathcal{G}}$:

$$\hat{\mathcal{G}} = \begin{pmatrix} 0 \\ 2i\pi\alpha k_c l_c s_1\overline{s}_2\sin(2\pi l_c z)K_0(J_c,1) \\ 2i\pi\alpha k_c l_c s_1\overline{s}_2\sin(2\pi l_c z)K_1(J_c,1) \\ i\pi\alpha k_c l_c s_1^2 e^{2i\alpha k_c x}K_2(J_c,1) - i\pi\alpha k_c l_c\overline{s}_2^2 e^{-2i\alpha k_c x}K_2(J_c,1) \end{pmatrix} \tag{5.17}$$

The eigenvector expansion of above term is:

$$\hat{\mathcal{G}} = \hat{g}^1_{J_{c_2}}\psi^1_{J_{c_2}} + \hat{g}^2_{J_{c_2}}\psi^2_{J_{c_2}} + \hat{g}^1_{J_{c_3}}\psi^1_{J_{c_3}} + \hat{g}^1_{\overline{J}_{c_3}}\psi^1_{\overline{J}_{c_3}} \tag{5.18}$$

where the coefficients are

$$\begin{aligned} \hat{g}^1_{J_{c_2}} &= \hat{g}^{1s}_{J_{c_2}} s_1 \overline{s}_2, \quad & \hat{g}^2_{J_{c_2}} &= \hat{g}^{2s}_{J_{c_2}} s_1 \overline{s}_2, \\ \hat{g}^1_{J_{c_3}} &= \hat{g}^{1s}_{J_{c_3}} s_1^2, & \hat{g}^1_{\overline{J_{c_3}}} &= \hat{g}^{1s}_{\overline{J_{c_3}}} \overline{s}_2^2, \end{aligned} \tag{5.19}$$

in which

$$\begin{aligned} \hat{g}^{1s}_{J_{c_2}} &= 2i\pi\alpha k_c l_c K_0(J_c, 1), \qquad & \hat{g}^{2s}_{J_{c_2}} &= 2i\pi\alpha k_c l_c K_1(J_c, 1), \\ \hat{g}^{1s}_{J_{c_3}} &= i\pi\alpha k_c l_c K_2(J_c, 1), & \hat{g}^{1s}_{\overline{J_{c_3}}} &= -i\pi\alpha k_c l_c K_2(J_c, 1). \end{aligned} \tag{5.20}$$

With above results at our disposal, we next calculate the second order term of center manifold function. Decomposing Φ_2 into three parts:

$$\Phi_2 = \Phi_2^1 + \Phi_2^2 + \Phi_2^3 \tag{5.21}$$

where

$$\begin{aligned} \Phi_2^1 &= (-\mathcal{L})^{-1} P_h \mathcal{G}(\psi_{J_c}), \\ \Phi_2^2 &= -2\Im\beta^1_{J_c} i(-\mathcal{L})^{-1}\big(2\Im\beta^1_{J_c} i - \mathcal{L}\big)^{-1} P_h \hat{\mathcal{G}}, \\ \Phi_2^3 &= 2\Im\beta^1_{J_c} i(-\mathcal{L})^{-1}\big(-2\Im\beta^1_{J_c} i - \mathcal{L}\big)^{-1} P_h \widetilde{\mathcal{G}}. \end{aligned} \tag{5.22}$$

For Φ_2^1, calculation leads to

$$\Phi_2^1 = \sum_{n=1}^{4} h^n_{J_{c_2}} \psi^n_{J_{c_2}} + \Big[h^1_{J_{c_3}} \psi^1_{J_{c_3}} + c.c.\Big], \tag{5.23}$$

in which the coefficients are

$$h^j_J = -\frac{g^j_J}{\beta^j_J}, \quad \text{for} \quad (J, j) = (J_{c_2}, n), n = 1, 2, 3, 4 \quad \text{and} \quad (J, j) = (J_{c_3}, 1). \tag{5.24}$$

For latter calculation, we also define

$$h^{js}_J = -\frac{g^{js}_J}{\beta^j_J}. \tag{5.25}$$

As for Φ_2^2 and Φ_2^3, we have

$$\begin{aligned} \Phi_2^2 &= \hat{h}^1_{J_{c_2}} \psi^1_{J_{c_2}} + \hat{h}^2_{J_{c_2}} \psi^2_{J_{c_2}} + \Big[\hat{h}^1_{J_{c_3}} \psi^1_{J_{c_3}} + c.c.\Big] \\ \Phi_2^3 &= \overline{\Phi_2^2} \end{aligned} \tag{5.26}$$

in which the coefficients are

$$\hat{h}^j_J = B^j_J \hat{g}^j_J, \quad \text{for} \quad (J, j) = (J_{c_2}, n), n = 1, 2, 3, 4 \quad \text{and} \quad (J, j) = (J_{c_3}, 1), \tag{5.27}$$

where

$$B_J^j = \frac{2\Im(\beta_{J_c}^1)i}{\beta_J^j(2\Im(\beta_{J_c}^1)i - \beta_J^j)}. \tag{5.28}$$

For latter calculation, let us also define

$$\hat{h}_J^{js} = B_J^j \hat{g}_J^{js}. \tag{5.29}$$

To sum up, Φ_2 have the following form

$$\Phi_2 = \sum_{n=1}^{4} f_{J_{c_2}}^n \psi_{J_{c_2}}^n + \Big[f_{J_{c_3}}^1 \psi_{J_{c_3}}^1 + c.c. \Big], \tag{5.30}$$

in which

$$\begin{aligned} f_{J_{c_2}}^1 =& h_{J_{c_2}}^1 + \Big[\hat{h}_{J_{c_2}}^1 + c.c. \Big], \\ f_{J_{c_2}}^2 =& h_{J_{c_2}}^2 + \Big[\hat{h}_{J_{c_2}}^2 + c.c. \Big], \\ f_{J_{c_2}}^m =& h_{J_{c_2}}^m, \quad \text{for} \quad m = 3, 4, \\ f_{J_{c_3}}^1 =& h_{J_{c_3}}^1 + \hat{h}_{J_{c_3}}^1 + \overline{\hat{h}_{\overline{J}_{c_3}}^1}. \end{aligned} \tag{5.31}$$

Next, we calculate the lower order term of $\left\langle \mathcal{G}(\Psi), \psi_{J_c}^{j,*} \right\rangle$, for $j = 1, 2$.

$$\left\langle \mathcal{G}(\psi_{J_c} + \Phi_2), \psi_{J_c}^{j,*} \right\rangle = \left\langle \mathcal{G}(\psi_{J_c}, \Phi_2) + \mathcal{G}(\Phi_2, \psi_{J_c}), \psi_{J_c}^{j,*} \right\rangle + o(|s|^2) \tag{5.32}$$

For convenience, we renumber the variables and eigenfunctions

$$\begin{aligned} & s_3 = \overline{s}_1, \quad s_4 = \overline{s}_2, \quad f_n = f_{J_{c_2}}^n, \quad f_5 = f_{J_{c_3}}^1, \quad f_6 = \overline{f}_5, \\ & \sigma_1 = s_1 \psi_{J_c}^1, \quad \sigma_2 = s_2 \psi_{J_c}^2, \quad \sigma_3 = s_3 \psi_{\overline{J}_c}^1, \quad \sigma_4 = s_4 \psi_{\overline{J}_c}^2, \\ & \rho_n = f_n \psi_{J_{c_2}}^n, \quad \rho_5 = f_5 \psi_{J_{c_3}}^1 = \overline{\rho}_6, \quad \text{for} \quad n = 1, 2, 3, 4. \end{aligned} \tag{5.33}$$

Using above notion, we define

$$s_n f_m G_{nm}^j = \left\langle \mathcal{G}(\sigma_n, \rho_m) + \mathcal{G}(\rho_m, \sigma_n), \psi_{J_c}^{j,*} \right\rangle, \tag{5.34}$$

which means

$$\left\langle \mathcal{G}(\psi_{J_c} + \Phi_2), \psi_{J_c}^{j,*} \right\rangle = \sum_{\substack{n=1,2,3,4 \\ m=1,2,3,4,5,6}} s_n f_m G_{nm}^j + o(|s|^2). \tag{5.35}$$

For $j=1,2$, the total 24 terms in that summation are shown below: for $q=1,2$

$$
\begin{aligned}
G^j_{q1} &= i\pi^2 k_c l_c K_0^*(\overline{J}_c, j),\\
G^j_{q2} &= i\pi^2 k_c l_c K_1^*(\overline{J}_c, j),\\
G^j_{qp} &= i\pi^2 k_c l_c \Big[\Big(K_2(J_{c_2}, p)K_2^*(\overline{J}_c, j)\\
&\quad + \frac{(\gamma^2_{J_c}+1)\gamma^2_{J_{c_2}}}{\gamma^2_{J_c}}(\mathrm{PrRm}K_2(J_{c_2}, p)K_2(J_c, q)+1)\Big)\\
&\quad - (K_0(J_c, q)K_0^*(\overline{J}_c, j) + K_1(J_c, q)K_1^*(\overline{J}_c, j) - K_2(J_c, q)K_2^*(\overline{J}_c, j),\\
&\quad + (\gamma^2_{J_c}+1)(\mathrm{PrRm}K_2(J_{c_2}, p)K_2(J_c, q)+1))\Big], \quad \text{for} \quad p=3,4\\
G^j_{q5} &= G^j_{q6} = 0.
\end{aligned}
\tag{5.36}
$$

When $q=3,4$,

$$
\begin{aligned}
G^j_{qp} &= 0, \quad \text{for} \quad p=1,2,3,4,6,\\
G^j_{q5} &= 2i\pi^2 k_c l_c \left(\frac{(\gamma^2_{J_c}+1)\mathrm{PrRm}(\gamma^2_{J_c}-\gamma^2_{J_{c_3}})}{\gamma^2_{J_c}}K_2(\overline{J}_c, q-2) + K_2^*(\overline{J}_c, j)\right).
\end{aligned}
\tag{5.37}
$$

Thus, by omitting the higher order terms, the reduced equation reads

$$
\begin{aligned}
\frac{\mathrm{d}s_1}{\mathrm{d}t} &= \beta^1_{J_c} s_1 + A_1|s_1|^2 s_1 + A_2 s_1^2\overline{s}_2 + A_3|s_1|^2 s_2 + A_4 s_1|s_2|^2 + A_5\overline{s}_1 s_2^2 + A_6|s_2|^2 s_2\\
\frac{\mathrm{d}s_2}{\mathrm{d}t} &= \overline{\beta^1_{J_c}} s_2 + B_1|s_1|^2 s_1 + B_2 s_1^2\overline{s}_2 + B_3|s_1|^2 s_2 + B_4 s_1|s_2|^2 + B_5\overline{s}_1 s_2^2 + B_6|s_2|^2 s_2\\
\frac{\mathrm{d}\overline{s}_j}{\mathrm{d}t} &= \text{ complex conjugate of above expressions, for } j=1,2
\end{aligned}
\tag{5.38}
$$

By normal form theory (see [35]), the above system can be further reduced by following change of variables

$$
\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \end{pmatrix} + \begin{pmatrix} \frac{A_2}{2\beta^1_{J_c}}\hat{s}_1^2\overline{\hat{s}}_2 + \frac{A_3}{2\overline{\beta^1_{J_c}}}|\hat{s}_1|^2\hat{s}_2 + \frac{A_5}{3\overline{\beta^1_{J_c}}-\beta^1_{J_c}}\overline{\hat{s}}_1\hat{s}_2^2 + \frac{A_6}{2\overline{\beta^1_{J_c}}}|\hat{s}_2|^2\hat{s}_2 \\ \frac{B_1}{2\beta^1_{J_c}}|\hat{s}_1|^2\hat{s}_1 + \frac{B_2}{3\beta^1_{J_c}-\overline{\beta^1_{J_c}}}\hat{s}_1^2\overline{\hat{s}}_2 + \frac{B_4}{2\beta^1_{J_c}}\hat{s}_1|\hat{s}_2|^2 + \frac{B_5}{2\overline{\beta^1_{J_c}}}\overline{\hat{s}}_1\hat{s}_2^2. \end{pmatrix}
\tag{5.39}
$$

By dropping the hat, we arrive at the normal form:

$$
\begin{aligned}
\frac{\mathrm{d}s_1}{\mathrm{d}t} &= \beta^1_{J_c} s_1 + A_1|s_1|^2 s_1 + A_4 s_1|s_2|^2,\\
\frac{\mathrm{d}s_2}{\mathrm{d}t} &= \overline{\beta^1_{J_c}} s_2 + B_3|s_1|^2 s_2 + B_6|s_2|^2 s_2,\\
\frac{\mathrm{d}\overline{s}_1}{\mathrm{d}t} &= \overline{\beta^1_{J_c}}\,\overline{s}_1 + \overline{A}_1|s_1|^2\overline{s}_1 + \overline{A}_4\overline{s}_1|s_2|^2,\\
\frac{\mathrm{d}\overline{s}_2}{\mathrm{d}t} &= \beta^1_{J_c}\overline{s}_2 + \overline{B}_3|s_1|^2\overline{s}_2 + \overline{B}_6|s_2|^2\overline{s}_2,
\end{aligned}
\tag{5.40}
$$

in which the coefficients are

$$
\begin{aligned}
A_1 = & -h^{1s}_{J_{c_2}}\Im(K_0(J_c,1))G^1_{11} - h^{2s}_{J_{c_2}}\Im(K_1(J_c,1))G^1_{12} \\
& - h^{3s}_{J_{c_2}}G^1_{13} - h^{4s}_{J_{c_2}}G^1_{14} + \left(\hat{h}^{1s}_{J_{c_3}} + h^{1s}_{J_{c_3}}K_2(J_c,1)\right)G^1_{35} \\
A_4 = & \, 2h^{1s}_{J_{c_3}}\Re(K_2(J_c,1))G^1_{45} \\
& + \left(-h^{1s}_{J_{c_2}}\Im(K_0(J_c,1)) - h^{1s}_{J_{c_2}}\Re(K_0(J_c,1)) + \hat{h}^{1s}_{J_{c_2}}\right)G^1_{21} \\
& + \left(-h^{2s}_{J_{c_2}}\Im(K_1(J_c,1)) - h^{2s}_{J_{c_2}}\Re(K_1(J_c,1)) + \hat{h}^{2s}_{J_{c_2}}\right)G^1_{22} \\
& - h^{1s}_{J_{c_2}}\Im(K_0(J_c,1))G^1_{11} - h^{2s}_{J_{c_2}}\Im(K_1(J_c,1))G^1_{12} \\
& + h^{3s}_{J_{c_2}}G^1_{13} + h^{4s}_{J_{c_2}}G^1_{14} \\
B_3 = & \, 2h^{1s}_{J_{c_3}}\Re(K_2(J_c,1))G^2_{35} \\
& + \left(-h^{1s}_{J_{c_2}}\Im(K_0(J_c,1)) + h^{1s}_{J_{c_2}}\Re(K_0(J_c,1)) + \overline{\hat{h}^{1s}_{J_{c_2}}}\right)G^2_{11} \\
& + \left(-h^{2s}_{J_{c_2}}\Im(K_1(J_c,1)) + h^{2s}_{J_{c_2}}\Re(K_1(J_c,1)) + \overline{\hat{h}^{2s}_{J_{c_2}}}\right)G^2_{12} \\
& - h^{1s}_{J_{c_2}}\Im(K_0(J_c,1))G^2_{21} - h^{2s}_{J_{c_2}}\Im(K_1(J_c,1))G^2_{22} \\
& - h^{3s}_{J_{c_2}}G^2_{23} - h^{4s}_{J_{c_2}}G^2_{24} \\
B_6 = & -h^{1s}_{J_{c_2}}\Im(K_0(J_c,1))G^2_{21} - h^{2s}_{J_{c_2}}\Im(K_1(J_c,1))G^2_{22} \\
& + h^{3s}_{J_{c_2}}G^2_{23} + h^{4s}_{J_{c_2}}G^2_{24} + \left(\overline{\hat{h}^{1s}_{\overline{J}_{c_3}}} + h^{1s}_{J_{c_3}}K_2(\overline{J}_c,1)\right)G^2_{45}
\end{aligned}
\tag{5.41}
$$

□

We next prove Theorem 5.1 by analyze the dynamics of the reduced equation.

Proof. To discuss the dynamics of the system, we only need to consider radius equations (5.5a).

First, according to Theorem 5.2 of [23], the system (2.10) bifurcates to an attractor when the origin is locally asymptotically stable for $\mathrm{Rt} = \mathrm{Rt}_{c_2}$. To this end, we define the energy $E = r_1^2 + r_2^2$. Hence,

$$
\frac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}t} = \Re(\beta^1_{J_c})E + \Re(A_1)r_1^4 + \Re(B_6)r_2^4 + \Re(A_4 + B_3)r_1^2r_2^2 \tag{5.42}
$$

Therefore, the origin is locally asymptotically stable at $\mathrm{Rt} = \mathrm{Rt}_{c_2}$ when (5.1) is negative definite. And if the quadratic form (5.1) is positive definite, we obtain the subcritical bifurcation.

Second, we prove the statements concerning the double periodic solutions in the attractor. By solving $\frac{\mathrm{d}r_j}{\mathrm{d}t} = 0$ for $j = 1, 2$, we have three possible non-trivial solutions

$$\begin{aligned}
\mathcal{R}_0 &= \left(\sqrt{\frac{\delta_1}{\delta_0}\Re(\beta^1_{J_c})}, \sqrt{\frac{\delta_2}{\delta_0}\Re(\beta^1_{J_c})}\right), \\
\mathcal{R}_1 &= \left(\sqrt{-\frac{\Re(\beta^1_{J_c})}{\Re(A_1)}}, 0\right), \mathcal{R}_2 = \left(0, \sqrt{-\frac{\Re(\beta^1_{J_c})}{\Re(B_6)}}\right)
\end{aligned} \tag{5.43}$$

in which

$$\begin{aligned}
\delta_0 &= \Re(A_4)\Re(B_3) - \Re(A_1)\Re(B_6), \\
\delta_1 &= \Re(B_6) - \Re(A_4), \\
\delta_2 &= \Re(A_1) - \Re(B_3).
\end{aligned} \tag{5.44}$$

When $\mathrm{Rt} > \mathrm{Rt}_{c_2}$, i.e. $\Re(\beta^1_{J_c}) > 0$, then each solution $\mathcal{R}_j$ exists under following conditions

$$\begin{aligned}
&\mathcal{R}_0 : \delta_0\delta_1 > 0 \quad \text{and} \quad \delta_0\delta_2 > 0, \\
&\mathcal{R}_1 : \Re(A_1) < 0, \quad \mathcal{R}_2 : \Re(B_6) < 0.
\end{aligned} \tag{5.45}$$

Third, we investigate the stability of above three solutions. By solving the standard eigenvalue problem of the linearized equation for $\mathcal{R}_0$, the eigenvalues μ_{01} and μ_{02} are

$$\mu_{01} = -2\Re(\beta^1_{J_c}), \quad \mu_{02} = \frac{2\Re(\beta^1_{J_c})\delta_1\delta_2}{\delta_0}. \tag{5.46}$$

Hence, $\mathcal{R}_0$ is stable when $\delta_0 < 0$. Similarly, for $\mathcal{R}_1$, we obtain the eigenvalues

$$\mu_{11} = -2\Re(\beta^1_{J_c}), \quad \mu_{12} = \frac{\Re(\beta^1_{J_c})}{\Re(A_1)}\delta_2, \tag{5.47}$$

which indicates the stability when $\delta_2 > 0$. And for $\mathcal{R}_2$, the eigenvalues are

$$\mu_{21} = -2\Re(\beta^1_{J_c}), \quad \mu_{22} = \frac{\Re(\beta^1_{J_c})}{\Re(B_6)}\delta_1, \tag{5.48}$$

which indicates the stability when $\delta_1 > 0$.

Finally, if (5.1) is negative definite, then $\Re(A_1), \Re(B_6) < 0$ is valid, which ensures the existence of $\mathcal{R}_1$ and $\mathcal{R}_2$. And $\mathcal{R}_0$ exists by the conditions $\delta_0\delta_1 > 0$ and $\delta_0\delta_2 > 0$. The conditions also indicate that δ_j have same signs for $j = 0, 1, 2$, which suggest the stability of $\mathcal{R}_j$ for $j = 0, 1, 2$. Precisely, since $\delta_0 < 0$ can be deduced from the fact that (5.1) is negative definite, thus $\mathcal{R}_0$ is stable while $\mathcal{R}_1$ and $\mathcal{R}_2$ are unstable.

□

6. Numerical Investigations

We have investigated the transition of the system (2.1) theoretically in preceding sections, and it is clarified that the type of transition can be determined by several non-dimensional numbers. To determine the transition and convection types of the system (2.1), we analyzed the effect of parameters $(\mathrm{Pr}, \mathrm{Rm}, \mathrm{Rs}, \Lambda_\mathrm{c}, \alpha)$ on the critical values of Rt_{c_1}, Rt_{c_2} and transition types.

The effect of parameters on the critical values and the transition types is numerically determined through several specimens listed in the Table 1, shown in Figure 6.1-Figure 6.3. The parameters in region I, II and Σ_2 correspond

Specified parameters	variational parameters
$(\mathrm{Pr},\ \alpha,\ \mathrm{Rs}) = (1,\ 1,\ 20000)$	$\Lambda_c \in [0.1, 3],\ \mathrm{Rm} \in [10, 300]$
$(\mathrm{Pr},\ \alpha,\ \mathrm{Rs}) = (1,\ 3.9,\ 20000)$	$\Lambda_c \in [0.1, 3],\ \mathrm{Rm} \in [10, 300]$
$(\mathrm{Pr},\ \alpha,\ \mathrm{Rm}) = (1,\ 1,\ 150)$	$\Lambda_c \in [0.1, 3],\ \mathrm{Rs} \in [2000, 80000]$
$(\mathrm{Pr},\ \alpha,\ \mathrm{Rm}) = (1,\ 3.9,\ 150)$	$\Lambda_c \in [0.1, 3],\ \mathrm{Rs} \in [2000, 80000]$
$(\mathrm{Pr},\ \Lambda_c,\ \alpha) = (1,\ 1,\ 1)$	$\mathrm{Rs} \in [2000, 80000],\ \mathrm{Rm} \in [10, 300]$
$(\mathrm{Pr},\ \Lambda_c,\ \alpha) = (1,\ 1,\ 3.9)$	$\mathrm{Rs} \in [2000, 80000],\ \mathrm{Rm} \in [10, 300]$

Table 1: Parameters with fixed $\tau_1 = 0.72$ and $\tau_2 = 0.5$).

to the jump transition from real-eigenvalues, continuous transition from real-eigenvalues, continuous transition from complex-eigenvalues, respectively. And the parameters in region Σ_1 corresponds to transition from complex-eigenvalues. Besides, the choice of parameter ranges is followed from [34], which is $\mathrm{Pr} \in [1, 7]$, $\alpha \in [0.1, 4]$, $\mathrm{Rm} \in [10, 300]$, $\mathrm{Rs} \in [2000, 80000]$ and $\Lambda_c \in [0.1, 3]$.

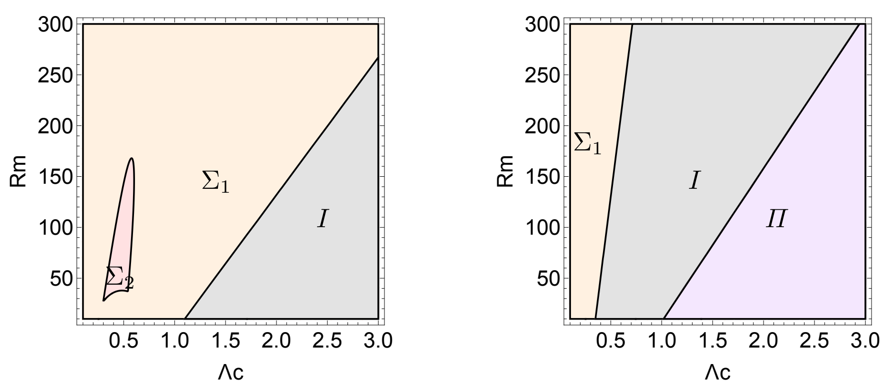


Figure 6.1: Parameter regions: $(\mathrm{Pr}, \alpha, \mathrm{Rs})$=$(1, 1, 20000)$(left); $(\mathrm{Pr}, \alpha, \mathrm{Rs})$=$(1, 3.9, 20000)$(right)

By observing Figure 6.1-Figure 6.3, we can obtain the that for $\tau_1 = 0.72, \tau_2 = 0.5$, there are total four cases are shown in the parameter ranges: jump transition

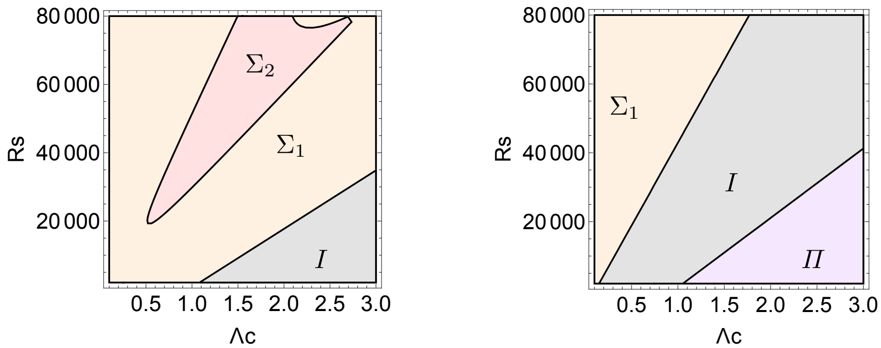


Figure 6.2: Parameter regions: $(\mathrm{Pr}, \alpha, \mathrm{Rm}) = (1, 1, 150)$(left); $(\mathrm{Pr}, \alpha, \mathrm{Rm}) = (1, 3.9, 150)$(right).

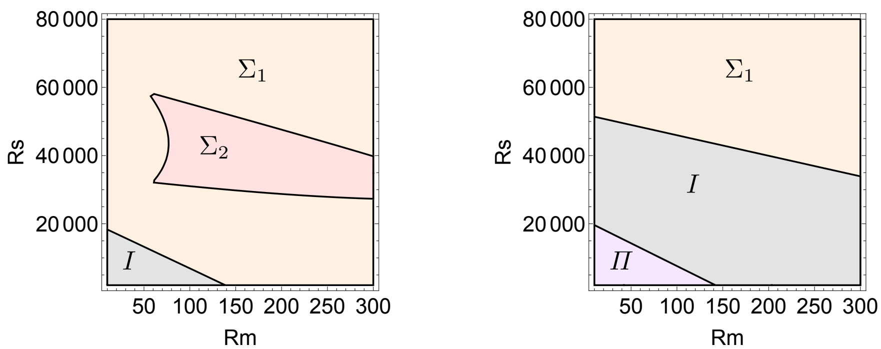
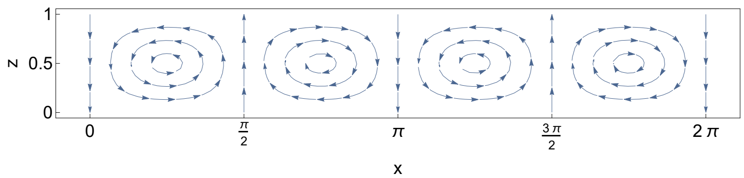


Figure 6.3: the variational parameter regions: $(\mathrm{Pr}, \Lambda_c, \alpha) = (1, 1, 1)$(left); $(\mathrm{Pr}, \Lambda_c, \alpha) = (1, 1, 3.9)$(right).

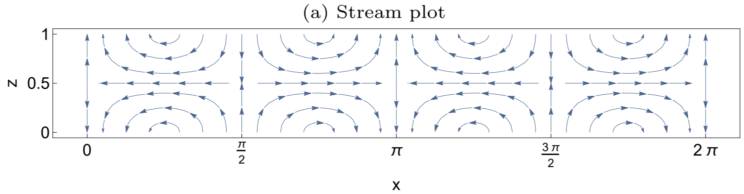
from real-eigenvalues, continuous transition from real-eigenvalues, continuous transition from complex-eigenvalue, complex-eigenvalue.

We also performed numerical simulations for these parameters in Table 1 with $1 < \tau_1 = 1.2 < 1.5 = \tau_2$. In that case, only continuous transition induced by real-eigenvalue was found in parameter ranges. In what follows, we give some specific examples to show our results.

Example 1. *Let* $(\tau_1, \tau_2, \mathrm{Pr}, \alpha, \mathrm{Rm}, \Lambda_c, \mathrm{Rs}) = (1.2, 1.5, 1, 1, 150, 1.5, 40000)$*, by computation, the critical indexes are* $J_r = (\pm 2, 1)$*, and the critical values are* $\mathrm{Rt}_{c_1} = 51299.2745 < \mathrm{Rt}_{c_2} = 77359.5917$*. Then, the first eigenvalues are real. Besides, by computation,* $\Gamma(\mathrm{Rt}_{c_1}) = -2.17481 < 0$*. The system* (2.1) *will undergo a continuous transition, and a steady convection occurs. Let* $\mathrm{Rt} = 51310 > \mathrm{Rt}_{c_1}$*, by computation,* $\Gamma = -2.17471$*,* $K_0(J_r, 1) = -0.14413i$*,* $K_1(J_r, 1) = -0.120118i$ *and* $K_2(J_r, 1) = 0.150957$*. Let* $\theta = 0$ *in the first conclusion of the theorem* (4.1)*, then we plot the streamline* $(\frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x})$*, magnetic field* $(\frac{\partial A}{\partial z}, -\frac{\partial A}{\partial x})$*, temperature* T *and solute concentration* C*, see Figure 6.4 and Figure 6.5.*

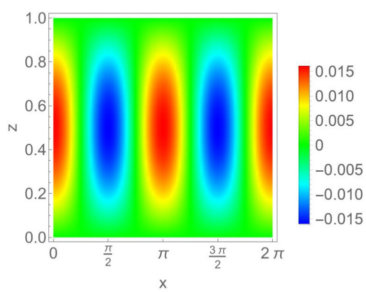


(a) Stream plot

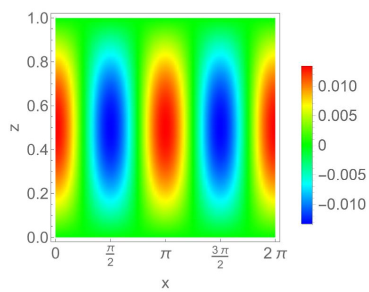


(b) Magnetic field plot

Figure 6.4: Stream and magnetic field plot of the steady state solution with parameter choices declared in Example 1. The steady state is a typical convective roll.



(a) Temperature plot



(b) Solute Concentration plot

Figure 6.5: Temperature and solute concentration plot of the steady state solution with parameter choices declared in Example 1. The plots exhibit positive correlation between temperature and solute concentration in stable convection.

Example 2. *Let* $(\tau_1, \tau_2, \mathrm{Pr}, \alpha, \mathrm{Rm}, \Lambda_c, \mathrm{Rs}) = (0.72, 0.5, 1, 1, 150, 2.5, 20000)$*, by computation, the critical indexes are* $J_r = (\pm 2, 1)$*, and the critical values are* $\mathrm{Rt}_{c_1} = 61839.2567 < \mathrm{Rt}_{c_2} = 63799.6544$*. Then, the first eigenvalues are real. Besides, by computation,* $\Gamma(Rt_{c_1}) = 64.037 > 0$*. The system* (2.1) *will undergo a jump transition according to Theorem 4.1.*

Example 3. *Let* $(\tau_1, \tau_2, \mathrm{Pr}, \alpha, \mathrm{Rm}, \Lambda_c, \mathrm{Rs}) = (0.72, 0.5, 1, 1, 150, 1.5, 50000)$*, by computation, the critical indexes are* $J_c = (\pm 2, 1)$*, and the critical values are* $\mathrm{Rt}_{c_1} = 94254.761 > 79566.405 = \mathrm{Rt}_{c_2}$*. Then, the first eigenvalues are complex. Besides, by computation,* $\delta_0 = -1.46 \times 10^6$*,* $\delta_1 = -2943.32$*,* $\delta_2 = -1132.57$*,* $\Re A_1 = \Re B_6 = -1176.26 < 0$ *and* $\Re(A_4 + B_3) = 1723.38$*. Thus,* $\delta_0 \delta_1 > 0$*,* $\delta_0 \delta_2 > 0$ *and* $4(\Re A_1)^2 - (\Re(A_4 + B_3))^2 > 0$*. According to Theorem 5.1, the system* (2.1) *will undergo a continuous transition, and and a oscillatory convection*

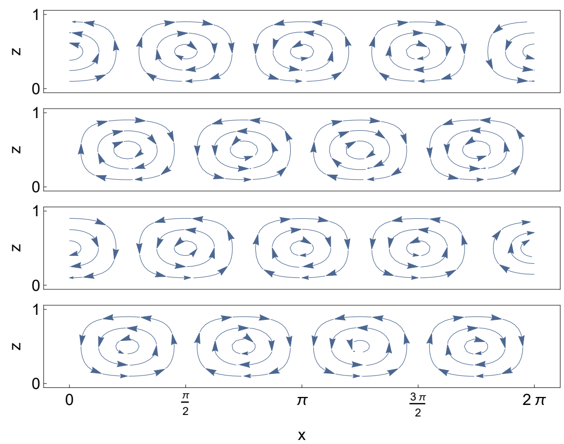


Figure 6.6: Snapshots of the stream plot for the stable oscillatory convection in a period, taken at $t = 0$, $t = \frac{1}{4}p$, $t = \frac{1}{2}p$ and $t = \frac{3}{4}p$ from top to bottom, where $p = 0.581776$ is the period of the solution. A traveling roll convection is observed.

occurs. Letting $\mathrm{Rt} = 79600 > \mathrm{Rt}_{c_2}$*, by computation* $\delta_0 = -1.44107 \times 10^6$, $\delta_1 = -2958.44$, $\delta_2 = -1142.57$, $\Re A_1 = \Re B_6 = -1175.63$, $A_4 = 1782.8 - 5809.68i$, $B_3 = -33.0642 + 1787.5i, \theta_1 = -\theta_2 = 10.8$ *and the period is* $p = 0.581776$*, we plot the stream* $(\frac{\partial \psi}{\partial z}, -\frac{\partial \psi}{\partial x})$ *and magnetic field* $(\frac{\partial A}{\partial z}, -\frac{\partial A}{\partial x})$ *in 4 different moments during one period. See Figure 6.6 and Figure 6.7.*

7. Conclusion

We have investigated the bifurcation and dynamic transition of a double-diffusive magnetoconvection of non-Newtonian fluid layer based on the model purposed in [34] by a hybrid analysis-computation approach. It is shown that PES condition holds at a critical control parameter Rt_c, and the system (2.1) undergoes a dynamic transition at that point. Two transition theorems are established by reducing the original PDE to a finite dimensional ODE, which have equivalent dynamical behavior near the critical value owing to the center manifold theorem. According to the transition theorems, the transition type is determined by several explicit calculated transition numbers.

By some careful numerical investigations based on the transition theorems, there are three main conclusions we can draw: (i) If the diffusivities from big to small are thermal, solute concentration and magnetic diffusion, i.e. $1 > \tau_1 > \tau_2$, then the system can not only undergo a continuous transition, but also a jump one in some parameter configurations, and both stationary and oscillatory convection can occur. This is differs from the classical Rayleigh Bénard convection; (ii) If the diffusivities from large to small are magnetic,

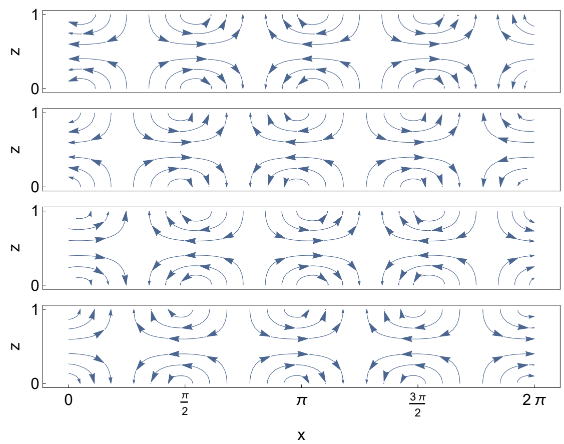


Figure 6.7: Snapshots of the magnetic field plot for the stable oscillatory convection in a period, taken at $t = 0$, $t = \frac{1}{4}p$, $t = \frac{1}{2}p$ and $t = \frac{3}{4}p$ from top to bottom, where $p = 0.581776$ is the period of the solution.

solute concentration and thermal diffusion, i.e. $1 < \tau_1 < \tau_2$, the system seems to prefer a continuous transition induced by real eigenvalues, which leads to a stationary convection that is consistent with the Rayleigh Bénard convection; (iii) For the case of (i), if the horizontal period is large (small α), then the transition of the system is more likely to induced by complex eigenvalues and less likely to be induced by real eigenvalues compare to the case when the horizontal period is relatively smaller. Moreover, it is more likely to observe an oscillatory convection when the period in x-direction is larger.

There are several directions in this article that we can develop further in future work. We only consider two of six orders of the diffusivities, the dynamic transition of the system under other four orders of diffusivities are also considerable. And we would like to prove that all transition are continuous and induced by real eigenvalues when the thermal diffusivity is the smallest and the magnetic diffusivity is largest, since we did not find the numerical exceptions.

Acknowledgement

The work of Y. Fan, L. Li and Q. Wang are supported by the National Science Foundation of China (NSFC), grant No.11901408.

A. Appendix

In the appendix, we will prove the center manifold formula (5.10).

Theorem A.1. *If the critical eigenvalues is two pairs of complex conjugate numbers, the center manifold has the following approximation formula near the critical threshold:*

$$\begin{aligned}\Phi = & -2\Im\beta^1_{J_c} i(-\mathcal{L})^{-1}\Big((2\Im\beta^1_{J_c} i-\mathcal{L})^{-1}\hat{\mathcal{G}} - (-2\Im\beta^1_{J_c} i-\mathcal{L})^{-1}\widetilde{\mathcal{G}}\Big) \\ & + (-\mathcal{L})^{-1}P_h\mathcal{G}(\psi_{J_c}) + o(|s|^2) + \mathcal{O}\Big(\big|\Re\beta^1_{J_c}\big||s|^2\Big)\end{aligned} \tag{A.1}$$

where $\hat{\mathcal{G}} = P_h\mathcal{G}(s_1\psi^1_{J_c} + \overline{s}_2\psi^2_{\overline{J}_c})$ *and* $\widetilde{\mathcal{G}} = P_h\mathcal{G}(\overline{s}_1\psi^1_{\overline{J}_c} + s_2\psi^2_{J_c}) = \overline{\hat{\mathcal{G}}}$.

Proof. According to Theorem A.1.1 of [22], the center manifold have the following approximation formula:

$$\Phi(\psi, \mathrm{Rt}) = \int_{-\infty}^{0} e^{-\tau\mathcal{L}_{\mathrm{Rt}}}\rho_\varepsilon P_h\mathcal{G}(e^{\tau\mathcal{L}_{\mathrm{Rt}}}\psi)\,\mathrm{d}\tau + o(\|\psi\|^2) \tag{A.2}$$

where $\rho_\varepsilon : \mathbf{E}_0 \to [0,1]$ is a C^∞ cut-off function defined by

$$\rho_\varepsilon(x) = \begin{cases} 1, & \text{if } \|x\| < \varepsilon, \\ 0, & \text{if } \|x\| > 2\varepsilon. \end{cases} \tag{A.3}$$

And $\psi = s_1\psi^1_{J_c} + \overline{s}_1\psi^1_{\overline{J}_c} + s_2\psi^2_{J_c} + \overline{s}_2\psi^2_{\overline{J}_c} \in \mathbf{E}_0$. Since $\psi \in \mathbf{E}_0$, then the application of an operator $e^{\tau\mathcal{L}_{\mathrm{Rt}}}$ on ψ can be replaced by multiplication of $e^{\beta^1_{J_c}}$ (or $e^{\overline{\beta^1_{J_c}}}$). And if Rt is near Rt_{c_2}, then $\Re(\beta^1_{J_c})$ is considered to be small, which leads to following derivation (together with integration by parts):

$$\begin{aligned}\Phi(\psi, \mathrm{Rt}) =& (-\mathcal{L}_{\mathrm{Rt}})^{-1}\int_{-\infty}^{0} e^{-\tau\mathcal{L}_{\mathrm{Rt}}}\rho_\varepsilon P_h\mathcal{G}(e^{\tau\mathcal{L}_{\mathrm{Rt}}}\psi)\,\mathrm{d}(-\tau\mathcal{L}_{\mathrm{Rt}}) + o(\|\psi\|^2) \\ =& (-\mathcal{L}_{\mathrm{Rt}})^{-1}\int_{-\infty}^{0} \rho_\varepsilon P_h\mathcal{G}(e^{\tau\mathcal{L}_{\mathrm{Rt}}}\psi)\,\mathrm{d}e^{-\tau\mathcal{L}_{\mathrm{Rt}}} + o(\|\psi\|^2) \\ =& (-\mathcal{L}_{\mathrm{Rt}})^{-1}e^{-\tau\mathcal{L}_{\mathrm{Rt}}}P_h\mathcal{G}(e^{\tau\mathcal{L}_{\mathrm{Rt}}}\psi)|^{0}_{-\infty} \\ & - \int_{-\infty}^{0} e^{-\tau\mathcal{L}_{\mathrm{Rt}}}\rho_\varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau}\big(P_h\mathcal{G}(e^{\tau\mathcal{L}_{\mathrm{Rt}}}\psi)\big)\,\mathrm{d}\tau + o(\|\psi\|^2) \\ =& -\int_{-\infty}^{0} e^{-\tau\mathcal{L}_{\mathrm{Rt}}}\rho_\varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau}\Big(e^{2\Im(\beta^1_{J_c})i\tau}\hat{\mathcal{G}} + e^{-2\Im(\beta^1_{J_c})i\tau}\widetilde{\mathcal{G}}\Big)\,\mathrm{d}\tau \\ & + (-\mathcal{L}_{\mathrm{Rt}})^{-1}P_h\mathcal{G}(\psi) + o(|s|^2) + \mathcal{O}\Big(\big|\Re\beta^1_{J_c}\big||s|^2\Big) \\ =& -\int_{-\infty}^{0} e^{-\tau\mathcal{L}_{\mathrm{Rt}}}\rho_\varepsilon(2\Im(\beta^1_{J_c})ie^{2\Im(\beta^1_{J_c})i\tau}\hat{\mathcal{G}} - 2\Im(\beta^1_{J_c})ie^{-2\Im(\beta^1_{J_c})i\tau}\widetilde{\mathcal{G}})\,\mathrm{d}\tau \\ & + (-\mathcal{L}_{\mathrm{Rt}})^{-1}P_h\mathcal{G}(\psi) + o(|s|^2) + \mathcal{O}\Big(\big|\Re\beta^1_{J_c}\big||s|^2\Big) \\ =& -2\Im\beta^1_{J_c} i(-\mathcal{L}_{\mathrm{Rt}})^{-1}\Big((2\Im\beta^1_{J_c} i-\mathcal{L}_{\mathrm{Rt}})^{-1}\hat{\mathcal{G}} - (-2\Im\beta^1_{J_c} i-\mathcal{L}_{\mathrm{Rt}})^{-1}\widetilde{\mathcal{G}}\Big) \\ & + (-\mathcal{L}_{\mathrm{Rt}})^{-1}P_h\mathcal{G}(\psi) + o(|s|^2) + \mathcal{O}\Big(\big|\Re\beta^1_{J_c}\big||s|^2\Big).\end{aligned} \tag{A.4}$$

Hence, (5.10) is proved. □

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