CMMSE: Maximal regularity and two-sided estimates of the approximation numbers of the nonlinear Sturm-Liouville equation solutions with rapidly oscillating coefficients in \$L_{2} (R)\$

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Abstract

A theorem on the maximum regularity of solutions of the nonlinear Sturm-Liouville equation with greatly growing and rapidly oscillating potential in the space \$L_2(R)\\,(R=(-\\infty,\\infty))\$ is proved in this paper. Two-sided estimates of the Kolmogorov widths of the sets associated with solutions of the nonlinear Sturm-Liouville equation are also obtained. As is known, the obtained estimates given the opportunity to choose approximation apparatus that guarantees the maximum possible error.

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CMMSE: Maximal regularity and two-sided estimates of the approximation numbers of the nonlinear Sturm-Liouville equation solutions with rapidly oscillating coefficients in $L_2(R)$[†]

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Summary

A theorem on the maximum regularity of solutions of the nonlinear Sturm-Liouville equation with greatly growing and rapidly oscillating potential in the space $L_2(R)\,(R = (-\infty, \infty))$ is proved in this paper. Two-sided estimates of the Kolmogorov widths of the sets associated with solutions of the nonlinear Sturm-Liouville equation are also obtained. As is known, the obtained estimates given the opportunity to choose approximation apparatus that guarantees the maximum possible error.

KEYWORDS:
nonlinear Sturm-Liouville equation, maximal regularity, approximation numbers, Kolmogorov widths, oscillating coefficients, greatly growing coefficients

INTRODUCTION

In this paper we study the nonlinear Sturm-Liouville equation

$$Ly = -y'' + q(x, y)y = f(x) \in L_2(R),\;\; R = (-\infty, \infty).$$

The existence and the smoothness of nonlinear elliptic equations solutions in a bounded domain have been studied quite well. A very comprehensive bibliography is contained, for example, in [1-6] and the works cited there.

However, nonlinear equations in an unbounded domain with greatly increasing and rapidly oscillating coefficients arise in applications. For example, the nonlinear Sturm-Liouville equation, which is especially interesting for quantum mechanics.

Here we are interested in the question:

A) to find out the conditions on the potential function $q(x, y)$ which provide $y'' \in L_2(R)$, when $y(x)$ is a solution of the nonlinear equation $Ly = f \in L_2(R)$.

We note that the linear case is well studied and reviews are available in [7–12].

It is known that eigenvalues $\lambda_n\,(n = 1, 2, ...)$ of the self-adjoint positive completely continuous operator A in the Hilbert space H are numbered according to their decreasing magnitude and observing their multiplicities have the following approximative properties

a) $\lambda_n = \min\limits_{k\in l_n} \|A - K\|$, where l_n is the set of all finite-dimensional operators with dimension no greater than n;

b) $\lambda_n \to 0$, when $n \to \infty$, wherein the faster convergence to zero, the operator A better approximated by finite rank operators.

It will be natural to explore a similar issues for a nonlinear Sturm-Liouville operator, i.e. to study the question

B) Is it possible for a given non-linear operator to specify a numerical sequence that characterizes properties a)–b)?

This paper is devoted to the study of the issues A) and B) for the nonlinear Sturm-Liouville equation.

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1 | FORMULATION OF THE MAIN RESULTS. EXAMPLE

We will make some notation and definitions for the statement of results.

The set of integrable functions with respect to the square of the module in each strictly internal subdomain $\Omega \subset R$ is denoted by $L_{2,loc}(R)$.

The set of functions from $L_{2,loc}(R)$ having generalized first-order derivatives (from $L_{2,loc}(R)$) will be denoted by $W^1_{2,loc}(R)$. We denote the subset of $W^1_{2,loc}(R)$ by $W^1_2(R)$, which elements together with the first generalized derivatives belong to $L_2(R)$. By $W^2_{2,loc}(R)$ we denote the set of all functions $u \in L_{2,loc}(R)$ which with their generalized derivatives up to and including the second order belong to $L_{2,loc}(R)$.

$\|\cdot\|_2$ is the norm of an element in $L_2(R)$, $\|\cdot\|_{2,1}$ is the norm of an element in $W^1_2(R)$, $\|\cdot\|_{2,loc}$ is the norm of an element in $L_{2,loc}(R)$.

Consider the nonlinear Sturm-Liouville equation

$$Ly = -y'' + q(x, y)\, y = f(x) \in L_2(R), \quad R = (-\infty, \infty). \tag{1}$$

Suppose that $q(x, y)$ satisfies the conditions:

$i)$ $q(x, y)$ is a continuous mapping $R \times C$ in $[\delta,\ \infty)$, $\delta > 0$, C is a set of complex numbers;

$ii)$ $\sup\limits_{|x-\eta|\le 1} \sup\limits_{|c_1-c_2|\le A} \frac{q(x,c_1)}{q(\eta,c_2)} \le \mu(A) < \infty$, where A is a finite value, $\mu(A)$ is a continuous function from A.

Definition 1.1. The function $y \in L_2(R)$ is called a solution of the equation (1) if there exist a sequence $\{y_n\}_{n=1}^{\infty} \subset W^1_2(R)$ such that $\{y_n\}_{n=1}^{\infty} \subset W^2_{2,loc}(R)$, $\|y_n - y\|_{L_{2,loc}} \to 0$ $\|Ly_n - f\|_{L_{2,loc}} \to 0$ as $n \to \infty$.

Definition 1.2. Following [13-15], we say that the solution $y(x) \in L_2(R)$ of the equation (1) called the maximal regular in $L_2(R)$ if $q(x, y)\, y \in L_2(R)$, $y'' \in L_2(R)$.

Theorem 1.1. Let the conditions $i) - ii)$ be fulfilled. Then there is the most regular solution to the equation (1).

The condition $ii)$, imposed in Theorem 1.1 and in [16], limits the potential oscillations. This condition is removed in the following theorem. In order to formulate the theorem, we introduce the following condition:

$i_0)$ $\sup\limits_{x\in R} \sup\limits_{|c_1-c_2|} \frac{q(x,c_1)}{Q^2(x,c_2)} < \infty$, $Q(x, c_2)$ is a special averaging of the function $q(x, c_1)$ [11], i.e.

$$Q(x, c_2) = \inf_{d>0} \left(d^{-1} + \int\limits_{x-\frac{d}{2}}^{x+\frac{d}{2}} q(t, c_2)\, dt \right),$$

where A is a finite value.

Theorem 1.2. Let the conditions $i) - i_0)$ be fulfilled. Then there exist the maximal regular solution to the equation (1).

Example 1.1. Let $q(x, y) = e^{|x|} \cdot \sin^2 e^{|x|} + e^{|y|}$. Then it is not difficult to verify that all conditions of Theorem 1.2 are satisfied for the equation

$$Ly = -y'' + \left(e^{|x|} \cdot \sin^2 e^{|x|} + e^{|y|}\right) y = f(x).$$

Therefore, there exists a solution $y(x)$ for the equation such that $y''(x) \in L_2(R)$.

This shows that Theorem 1.2 holds for a very wide class of nonlinear equations, including equations with potentials that are rapidly oscillating at infinity.

Now, consider the question B), i.e. finding such sequences of numbers that have approximative properties of the type a)-b). To do this, we study the behavior of the Kolmogorov k-widths of the set

$$M = \left\{ u \in W^1_2(R) : \|-y'' + q(x, y)\, y\|_2^2 \le T \right\}.$$

By definition [17], the Kolmogorov k-width of the set M is called the quantity

$$d_k(M, L_2) = d_k = \inf_{\{\ell_k\}} \sup_{u\in M} \inf_{v\in\ell_k} \|u - v\|_2,$$

where ℓ_k is a subspace of dimension k.

Note that the Kolmogorov widths of a compact set have the following properties: 1) $d_0 \geq d_1 \geq d_2 \geq ... \geq d_k \geq ...$, 2) $d_k \to 0$ as $k \to \infty$.

By $L_2^2(R, q(x,0))$ we denote the space obtained by completing $C_0^\infty(R)$ with respect to the norm

$$\left\| y \cdot L_2^2(R, q(x,0)) \right\|_2 = \left(\int_{-\infty}^{\infty} \left(\left| y'' \right|^2 + q(x,0) |y|^2 \right) dx \right)^{1/2}$$

Theorem 1.3. Let the conditions i)-ii) be fulfilled. Then any bounded set is compact in $L_2^2(R, q(x,0))$ if and only if

$$\lim_{|x|\to\infty} q(x,0) = \infty. \qquad (*)$$

We introduce the following counting function $N(\lambda) = \sum_{d_k > \lambda} 1$ of those widths d_k greater than $\lambda > 0$.

Theorem 1.4. Let the conditions i)-ii) be fulfilled. Then the estimates

$$c^{-1}\lambda^{-1/2} mes\left(x \in R : q(x,0) \leq c^{-1}\lambda^{-1}\right) \leq$$
$$\leq N(\lambda) \leq c\lambda^{-1/2} mes\left(x \in R : q(x,0) \leq c\lambda^{-1}\right),$$

hold, where $c > 0$ is a constant depending, generally speaking, on T.

Example 1.2. Let us take $q(x) = |x| + |y| + 1$. Then, by virtue of Theorem 1.4, the estimate $c^{-1}\lambda^{-3/2} \leq N(\lambda) \leq c\lambda^{-3/2}$ holds for the distribution function of the widths of the set $M = \left\{ y \in W_2^1(R) : \ \|-y'' + q(x,y)\, y\|_2^2 \leq 1 \right\}$, where $c > 0$ is a constant. Since $N(\lambda)$ is a monotone function then we have $d_k \sim \frac{c}{k^{2/3}}, \ \ k = 1, 2, 3, ...$

2 | ON THE EXISTENCE OF SOLUTIONS OF THE NONLINEAR STURM-LIOUVILLE EQUATION

In this section we prove a lemma on the existence of solutions.

Lemma 2.1. Let the condition i) be fulfilled. Then there exists a solution of the equation (1) in the space $W_2^1(R)$ for any $f \in L_2(R)$.

To prove this lemma we need some auxiliary assertions.

Consider the following problem

$$L_{n,v} y = -y''(x) + q(x,v)\, y = f \cdot \chi_n, \qquad (2)$$

$$y(-n) = y(n) = 0, \qquad (3)$$

where χ_n is the characteristic function of the segment $[-n, n]$, $n = 1, 2, ...$, $v(x) \in C[-n, n]$, $C[-n, n]$ is a space of continuous functions.

Lemma 2.2. Let the condition i) be fulfilled and let $v \in C[-n, n]$. Then there exists a unique solution of the problem (2)-(3) for any $f \cdot \chi_n \in L_2(-n, n)$ such that

$$\|y\|_{W_2^1[-n,n]} \leq c_0 \|f\|_2, \qquad (4)$$

$$\|y\|_{C[-n,n]} \leq c \|f\|_2. \qquad (5)$$

where $c_0 > 0$ and $c > 0$ are constant numbers.

Proof. Assume $q(x) = q(x, v)$. Then the problem (2)-(3) takes the form

$$L_{n,v} y = -y''(x) + q(x)\, y = f \cdot \chi_n, \qquad (6)$$

$$y(-n) = y(n) = 0. \qquad (7)$$

From the general theory of boundary value problems [7] it follows that the problem (6)-(7) has a unique solution $W_2^2(-n, n)$ such that

$$\|y\|_{W_2^1[-n,n]} \leq c_0(\delta) \left\| f\chi_n \right\|_2 \leq c_0(\delta) \|f\|_2. \qquad (8)$$

It is known that $W_2^1(-n, n)$ is completely continuously embedded in the space $C[-n, n]$. Therefore, the following estimate

$$\|y\|_{C[-n,n]} \leq c_1 \|y\|_{W_2^1(-n,n)}, \qquad (9)$$

holds, where $c_1 > 0$ is the constant of the embedding theorem.

So the problem (6)-(7) has a unique solution

$$y_{n,\upsilon} = L_{n,\upsilon}^{-1}\left(f\chi_n\right), \tag{10}$$

where $L_{n,\upsilon}^{-1}$ is the inverse operator of the operator $L_{n,\upsilon}$ corresponding to the problem (6)-(7). And

$$\left\|L_{n,\upsilon}^{-1}\right\|_{C[-n,n]} \leq c, \tag{11}$$

where $c = c_1 \cdot c_0(\delta)$. □

Lemma 2.3. Let the condition *i*) be fulfilled. Then the operator $L_{n,\upsilon}^{-1}$ maps the ball $\bar{s}$ into itself, where $\bar{s} = \{\upsilon \in C[-n,n] : \|\upsilon\|_{C[-n,n]} \leq A\}$ is a ball in the space $C[-n,n]$ and A is an arbitrary positive number.

Proof. If the radius A of the ball $\bar{s}$ is equal to the right side of the inequality (5), i.e. $A = c\|f\|_2$, then Lemma 2.2 implies that the operator $L_{n,\upsilon}^{-1}$ maps the set $\bar{s}$ into itself. Lemma 2.3 is proved.

Let $K = \left\{y_{n,\upsilon} \in C[-n,n] : y_{n,\upsilon} = L_{n,\upsilon}^{-1}\left(f\chi_n\right), \upsilon \in \bar{s}, f \in L_2(R)\right\}$ is the image of the ball $\bar{s}$ under the mapping $L_{n,\upsilon}^{-1}$. □

Lemma 2.4. Let the condition *i*) be fulfilled. Then the set K is compact in the space $C[-n,n]$.

Proof. Lemma 2.2 implies that the inequality

$$\left\|y_{n,\upsilon}\right\|_{W_2^1(-n,n)} \leq c_0\|f\|,$$

holds for any function $y_{n,\upsilon}(x)$ from K, where $c_0 > 0$ is a constant.

This and the embedding theorem imply that the set K is compact in $C[-n,n]$. Lemma 2.4 is proved. □

Lemma 2.5. Let the condition *i*) be fulfilled. Then the operator $L_{n,\upsilon}^{-1}$ is continuous.

Proof. Let $f(x) \in L_2(R)$ and let the sequence $\left\{\upsilon_k\right\}_{k=1}^{\infty}$ converge to the element $\upsilon(x)$ of the ball $\bar{s}$ in the norm of the space $C[-n,n]$ and

$$L_{n,\upsilon_k} y_{n,\upsilon_k} = f(x)\cdot\chi_n, \tag{12}$$

$$L_{n,\upsilon} y_{n,\upsilon} = f(x)\cdot\chi_n, \tag{13}$$

From the equality (12)-(13) we find that

$$-\left(y_{n,\upsilon_k} - y_{n,\upsilon}\right)'' + q\left(x,\upsilon_k\right)\left(y_{n,\upsilon_k} - y_{n,\upsilon}\right) + \left(q\left(x,\upsilon_k\right) - q(x,\upsilon)\right)y_{n,\upsilon} = 0.$$

Hence

$$L_{n,\upsilon_k}\left(y_{n,\upsilon_k} - y_{n,\upsilon}\right) = \left(q(x,\upsilon) - q\left(x,\upsilon_k\right)\right)y_{n,\upsilon}. \tag{14}$$

It is easy to verify that the coefficients of the operator L_{n,υ_k} satisfy the conditions of Lemma 2.2, therefore there exist an inverse operator L_{n,υ_k}^{-1} and the equality

$$y_{n,\upsilon_k} - y_{n,\upsilon} = L_{n,\upsilon_k}^{-1}\left(q(x,\upsilon) - q\left(x,\upsilon_k\right)\right)y_{n,\upsilon}$$

holds.

From this and the inequalities (4)-(5) and (9)-(11) we obtain that

$$\begin{aligned}\left\|y_{n,\upsilon_k} - y_{n,\upsilon}\right\|_{C[-n,n]} &= \left\|L_{n,\upsilon_k}^{-1}\left(q(x,\upsilon) - q\left(x,\upsilon_k\right)\right)y_{n,\upsilon}\right\|_{C[-n,n]} \leq \\ &\leq \left\|L_{n,\upsilon_k}^{-1}\right\|_{C[-n,n]} \cdot \left\|\left(q(x,\upsilon) - q\left(x,\upsilon_k\right)\right)y_{n,\upsilon}\right\|_{C[-n,n]} \leq \\ &\leq c \cdot \sup_{x\in[-n,n]}\left|q(x,\upsilon) - q\left(x,\upsilon_k\right)\right| \cdot \left\|y_{n,\upsilon}\right\|_{L_2(-n,n)}.\end{aligned}$$

From this and from the inequality (4) we have

$$\begin{aligned}\left\|y_{n,\upsilon_k} - y_{n,\upsilon}\right\|_{C[-n,n]} &\leq c \cdot \sup_{x\in[-n,n]}\left|q(x,\upsilon) - q\left(x,\upsilon_k\right)\right| \cdot A_0 \cdot \|f\|_2 = \\ &= c_1 \cdot \sup_{x\in[-n,n]}\left|q(x,\upsilon) - q\left(x,\upsilon_k\right)\right| \cdot \|f\|_2,\end{aligned} \tag{15}$$

where $c_1 = c \cdot c_0$.

Since $\|v_k - v\|_{C[-n,n]} \to 0$ for $k \to \infty$ then we obtain from (15) that

$$\lim_{k\to\infty} \left\| y_{n,v_k} - y_{n,v} \right\|_{C[-n,n]} \le c_0 \cdot \lim_{k\to\infty} \sup_{x\in[-n,n]} \left| q(x,v) - q\left(x, v_k\right) \right| \cdot \|f\|_2 \to 0$$

The last relation shows that the operator L^{-1}_{n,v_k} is continuous. Lemma 2.5 is proved. □

Now, consider the following nonlinear problem

$$L_n y_n \equiv -y_n'' + q\left(x, y_n\right) y_n = f \cdot \chi_n \tag{16}$$

$$y_n(-n) = y_n(n) = 0. \tag{17}$$

Lemma 2.6. Let the condition *i*) be fulfilled. Then there exist a solution of the problem (16)-(17) for any $f \in L_2(R)$ such that

$$\|y_n\|_{C[-n,n]} + \|y_n\|_{W_2^1(-n,n)} \le c \cdot \|f\|_2 , \tag{18}$$

where $c > 0$ is a constant.

Proof. The function $y_{n,v} = L^{-1}_{n,v}\left(f\chi_n\right)$ belongs to the domain $D\left(L_n\right)$ of the operator L_n for each function $v \in C[-n,n]$ corresponding to the problem (16)-(17). Therefore, the existence of a solution to problem (16)-(17) is equivalent to the existence of a fixed point of the operator $L^{-1}_{n,v}$ in the space $C[-n,n]$, i.e., to the existence of a function $y_n \in C[-n,n]$ such that $y_n = L^{-1}_{n,y_n} f \cdot x_n$. Thus $y_n \in D\left(L_n\right)$, since $L^{-1}_{n,v}\left(f\chi_n\right) \in D\left(L_n\right)$ for any $v(x)$ from $C[-n,n]$.

To find a fixed point, it remains to show that the operator $L^{-1}_{n,v}$ maps the convex set into itself and it is completely continuous. The proof of this assertion follows from Lemmas 2.2-2.5. Lemma 2.6 is proved. □

Proof of Lemma 2.1. Each y_n is continued by zero outside $[-n,n]$ and the continuation is denoted by $\tilde{y}_n$. As you know, we get the elements $W_2^1(R)$ with such a continuation and (18) implies that their norm is bounded

$$\|\tilde{y}_n\|_{W_2^1(R)} \le c \cdot \|f\|_{L_2(R)} . \tag{19}$$

Therefore, from the sequence $\left\{\tilde{y}_n\right\}$ one can select a subsequence $\tilde{y}_{n_k}$ such that

$$\tilde{y}_{n_k} \to y \ \text{ weakly in } \ W_2^1(R), \tag{20}$$

$$\tilde{y}_{n_k} \to y \ \text{ strongly in } \ L_{2,loc}, \tag{21}$$

and the estimate

$$\|y\|_{W_2^1(R)} \le c \cdot \|f\|_2 \tag{22}$$

holds. The last estimate follows from (19) and (20).

Let $[-\alpha,\alpha]$ is an arbitrary fixed segment in R, where $\alpha > 0$ is any number. Then there exists a number N for any $\varepsilon > 0$ such that

$$\left\| L\tilde{y}_{n_k} - f \right\|_{L_2(-\alpha,\alpha)} \to 0 \text{ for } n_k \to \infty \tag{23}$$

for $n \ge N$ $[-\alpha,\alpha] \subset \operatorname{supp} \tilde{y}_{n_k}$ and by virtue of (16).

(21) and (23) imply that $y(x)$ is a solution to the equation (1). Lemma 2.1 is proved. □

3 | ON SMOOTHNESS OF SOLUTIONS

Proof of Theorem 1.1. Let $|x-\eta| \le 1$, then by Lemma 2.1 and from the inequality (22) we have

$$|y(x) - y(\eta)| = \left| \int_\eta^x y'(t)\,dt \right| \le \sqrt{x-\eta} \cdot c\,\|f\|_2 \le c\,\|f\|_2 .$$

Now supposing $y(x) = c$, $\ y(\eta) = c_2$ $\ A = c\,\|f\|_2$ we obtain that

$$\sup_{|x-\eta|\le 1} \frac{q(x, y(x))}{q(\eta, y(\eta))} \le \sup_{|x-\eta|\le 1} \ \sup_{\left|c_1-c_2\right|\le A} \frac{q\left(x, c_1\right)}{q\left(\eta, c_2\right)} \le \mu(A) < \infty.$$

Hence, according to Theorem 3 in [11] y'', $\ q(x,y)\,y$ belongs to $L_2(R)$. Theorem 1.1 is proved. □

Proof of Theorem 1.2. By Lemma 2.1, there exist a solution $y(x)$ for the equation (1) such that $y(x) \in W_2^1(R)$. Consequently, by the Sobolev embedding theorem $y(x) \in C(R)$. The norm in the space $C(R)$ is defined by the formula

$$\|y\|_{C(R)} = \sup_{x \in R} |y(x)|.$$

Then, according to the condition $i)$ $q(x, y(x)) \in C_{loc}(R)$. Further, the inequality

$$|y(x) - y(\eta)| \leq |c_1 - c_2| \leq A$$

holds, where $y(x) = c_1, \;\; y(\eta) = c_2$.
Hence, we have:

$$\sup_{x \in R} \frac{q(x, y(x))}{Q^2(x, y(x))} \leq \sup_{x \in R} \sup_{|c_1| \leq A} \frac{q(x, c_1)}{Q_A^2(x, c_1)} \leq \sup_{x \in R} \sup_{|c_1 - c_2| \leq A} \frac{q(x, c_1)}{Q^2(x, c_2)}$$

From the last inequality according to the condition $i)$ we find that

$$\sup_{x \in R} \frac{q(x, y(x))}{Q^2(x, y(x))} < \sup_{x \in R} \sup_{|c_1 - c_2| \leq A} \frac{q(x, c_1)}{Q^2(x, c_2)} < \infty.$$

It follows that all the conditions of Theorem 4 of [11] are fulfilled. Consequently, $q(x, y)\, y(x), y'' \in L_2(R)$. Theorem 1.2 is proved. $\square$

4 | TWO-SIDED ESTIMATES OF THE APPROXIMATION NUMBERS OF SOLUTIONS OF THE NONLINEAR STURM-LIOUVILLE EQUATION

As is known for a compact set, especially, when it contains solutions of a differential equation, the problem of the asymptotics of their widths is meaningful. The Kolmogorov widths estimation of the set M can be used to determine for the equation $Ly = f$ the convergence rate of approximate solutions to the exact one.

In order to prove Theorem 1.3, first we prove several lemmas.

Lemma 4.1. Let the conditions $i) - ii)$ be fulfilled. Then there exist a number $K(T)$ such that

$$\tilde{M} \subseteq M \subseteq \tilde{\tilde{M}},$$

where $\tilde{\tilde{M}} = \left\{ y \in L_2(R) : \|-y''\|_2^2 + \|q(x, y)\, y\|_2^2 \leq K(T) \right\}$, $\tilde{M} = \left\{ y \in L_2(R) : \; \|-y''\|_2^2 + \|q(x, y)\, y\|_2^2 \leq \frac{T}{2} \right\}$.

Proof. Let $y \in \tilde{M}$. Then, using the triangle inequality, we get

$$\left\|-y'' + q(x, y)\, y\right\|_2^2 \leq 2\left(\left\|-y''\right\|_2^2 + \|q(x, y)\, y\|_2^2\right) \leq 2 \cdot \frac{T}{2} \leq T.$$

It follows that $y \in M$, i.e. $\tilde{M} \subseteq M$.

Let $y \in M$. Then, by virtue of Lemma 2.1 and the estimate (22) and the embedding theorem $W_2^1(R)$ in the space $C(R)$ we have

$$\|y\|_{C(R)} \leq c \left\|-y'' + q(x, y)\, y\right\|_2,$$

where c is independent of $y(x)$ $q(x, y)$.

It follows that

$$\sup_{y \in M} \|y(x)\|_{C(R)} \leq c \cdot T^{1/2} \tag{24}$$

On the other hand, using the estimate (22), we have

$$|y(x) - y(\eta)| \leq c \left\|-y'' + q(x, y)\, y\right\| \leq c \cdot T^{1/2} \tag{25}$$

for any $y \in M$, where $c > 0$ is a constant independent of $y(x)$.

Now, supposing $y(x) = c_1, \;\; y(\eta) = c_2, \;\; A = c \cdot T^{1/2}$ from (25) we obtain that $|c_1 - c_2| \leq A$.

Let $y_0(x) \in M$ and suppose $q_0(x) = q\left(x, y_0(x)\right)$. Denote by L the closure in the norm of $L_2(R)$ of the operator defined on $C_0^\infty(R)$ by the equality

$$L_0 y = -y''(x) + q_0(x)\, y.$$

It is easy to verify that the operator L is self-adjoint, positive definite and $y_0(x) \in D(L)$, wherein the estimate

$$\left\|-y_0''\right\|_2 \leq \mu(A)\left\|-y_0 + q\left(x, y_0\right) y\right\|_2 \tag{26}$$

holds. The estimate (26) follows from Theorem 1.1

This shows that the inequality

$$\left\|-y''\right\|_2 \leq \mu(A) T^{1/2}. \tag{27}$$

holds for all $y \in M$.

From the inequality (27) we have

$$\begin{aligned} \|q(x, y) y\|_2 = \left\|-y'' + q(x, y) y + y''\right\|_2 \leq \left\|y''\right\|_{,2} + \left\|-y'' + q(x, y) y\right\|_2 \leq \\ \leq \mu(A) \cdot T^{1/2} + T^{1/2} \leq 2\mu(A) \cdot T^{1/2} \end{aligned} \tag{28}$$

for any $y \in M$. Here we take into account that the condition ii) implies that $\mu(A) \geq 1$.

From the inequalities (27) and (28) we find

$$\left\|-y''\right\|_2^2 + \|q(x, y) y\|_2^2 \leq \mu^2(A) \cdot T + 4\mu^2(A) \cdot T \leq K(T) \tag{29}$$

for any $y \in M$, where $K(T) = 5\mu^2(A) \cdot T$. The estimate (29) proves Lemma 4.1. $\square$

Lemma 4.2. Let the conditions $i) - ii)$ be fulfilled. Then $\tilde{\tilde{M}} \subseteq \tilde{\tilde{B}}$, where

$$\tilde{\tilde{B}} = \left\{u \in L_2(R) : \left\|-y''\right\|_2^2 + \|q(x, 0) y\|_2^2 \leq K_1(T)\right\}.$$

Proof. By the embedding theorems, we have

$$\|y\|_{C(R)} \leq c\left(\left\|-y''\right\|_2^2 + \|q(x, y) y\|_2^2\right)^{1/2} \leq c \cdot K(T) \tag{30}$$

for any $y(x) \in \tilde{\tilde{M}}$, where $c > 0$ is the constant of the embedding theorem.

Hence, using the computations and arguments used in the proof of (29), we obtain that

$$y(x) = c_1, \ y(\eta) = c_2, \ \left|c_1 - c_2\right| \leq A, \ A = 2c \cdot K^{1/2}(T). \tag{31}$$

Hence, using the conditions of ii) for all $y(x) \in \tilde{\tilde{M}}$, we have

$$\mu^{-1}(A) q(x, 0) \leq q(x, y(x)) \leq \mu(A) q(x, 0), \tag{32}$$

where $A = 2c \cdot K^{1/2}(T), \ \mu(A) = \mu\left(2cK^{1/2}(T)\right)$.
From (32) we have

$$\begin{aligned} \left\|-y''\right\|_2^2 + \|q(x, 0) y\|_2^2 \leq \left\|-y''\right\|_2^2 + \mu^2(A)\|q(x, y) y\|_2^2 \leq \mu^2(A)\left(\left\|-y''\right\|_{,2}^2 + \right. \\ \left. + \|q(x, y) y\|_2^2\right) \leq \mu^2(A) \cdot K(T) \leq K_1(T), \ K_1(T) = \mu^2\left(2cK^{1/2}\right) \cdot K(T) \end{aligned}$$

for any $y(x) \in \tilde{\tilde{M}}$. This implies $\tilde{\tilde{M}} \subseteq \tilde{\tilde{B}}$. $\square$

Lemma 4.3. Let the conditions $i) - ii)$ be fulfilled. Then $\tilde{B} \subseteq \tilde{M}$, where

$$\tilde{B} = \left\{u \in L_2(R) : \left\|-y''\right\|_2^2 + \|q(x, 0) y\|_2^2 \leq K_2(T)\right\},$$

$K_2(T)$ is a positive number depending on T, such that $K_2(T) \leq \frac{T}{2}$.

Proof. Let $u \in \tilde{B}$. Then, using the embedding theorem, we have

$$\|y\|_{C(R)} \leq c \cdot K_2(T) \leq c \cdot \frac{T}{2}, \tag{33}$$

$c > 0$ is the constant of the embedding theorem from $W_2^2(R)$ to $C(R)$.

Now, using the condition ii), we obtain from (33) that for all $u \in \tilde{B}$

$$\mu^{-1}\left(c \cdot \frac{T}{2}\right) q(x, 0) \leq q(x, y(x)) \leq \mu\left(c \cdot \frac{T}{2}\right) q(x, 0). \tag{34}$$

Hence, we find

$$\|-y''\|_2^2 + \|q(x,y)\,y\|_2^2 \le \|-y''\|_2^2 + \mu^2\left(c\cdot\frac{T}{2}\right)\cdot\|q(x,0)\,y\|_2^2 \le$$
$$\le \mu^2\left(c\cdot\frac{T}{2}\right)\left(\|-y\|_{,2}^2 + \|q(x,0)\,y\|_2^2\right) \le \mu^2\left(c\cdot\frac{T}{2}\right)K_2(T)$$

for any $y \in \tilde{B}$.

If we assume $K_2(T) = \frac{\frac{T}{2}}{\mu^2\left(c\cdot\frac{T}{2}\right)}$ then the inequality $\|-y''\|_2^2 + \|q(x,y)\,y\|_2^2 \le \frac{T}{2}$ holds for all $y \in \tilde{B}$. Therefore $\tilde{B} \subseteq \tilde{M}$. Lemma 4.3 is proved. □

Lemma 4.4. Let the conditions $i) - ii)$ be fulfilled. Then the estimates

$$c^{-1}d_k \le \tilde{d}_k \le c d_k,\ \ k = 1, 2, ...$$

hold, where $c > 0$ depends only on T, $\tilde{d}_k, d_k$ k are the Kolmogorov widths of the sets M and B, respectively, where $B = \left\{ y \in L_2(R) : \|-y''\|_2^2 + \|q(x,0)\,y\|_2^2 \le 1 \right\}$.

Lemma 4.4 is proved in the same way as Lemma 4.3 in [18].

Lemma 4.5. Let the conditions $i) - ii)$ be fulfilled. Then the estimates

$$N(c\lambda) \le \tilde{N}(\lambda) \le N\left(c^{-1}\lambda\right)$$

hold, where $N(\lambda) = \sum_{d_k > \lambda} 1$ is the counting function of those d_k greater than $\lambda > 0$, $\tilde{N}(\lambda) = \sum_{\tilde{d}_k > \lambda} 1$ is the counting function of those $\tilde{d}_k$ greater than $\lambda > 0$, $c > 0$ is a constant.

The proof of Lemma 4.5 follows from Lemma 4.4.

Proof of Theorem 1.3. Repeating the computations and arguments used in the proof of Theorems 1.2-1.3 from [18] we obtain the proof of Theorem 1.3 □

Proof of Theorem 1.4. Using Lemmas 4.4-4.5 and the proofs of Theorems 1.1–1.4 from [18] and the results from [19], we obtain the proof of Theorem 1.4 □

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Author contributions

Madi Muratbekov: supervision, methodology, investigation, writing and editing original manuscript, funding acquisition, project administration.
Mussakan Muratbekov: conceptualization, formulation of the problem, methodology, investigation, writing original draft.
Serik Altynbek: investigation, some computations, writing and editing original manuscript, project administration, bibliography.

Conflict of interest

The authors declare no potential conflict of interests.

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