Analyses of solutions of Riemann-Liouville fractional oscillatory differential equations with pure delay

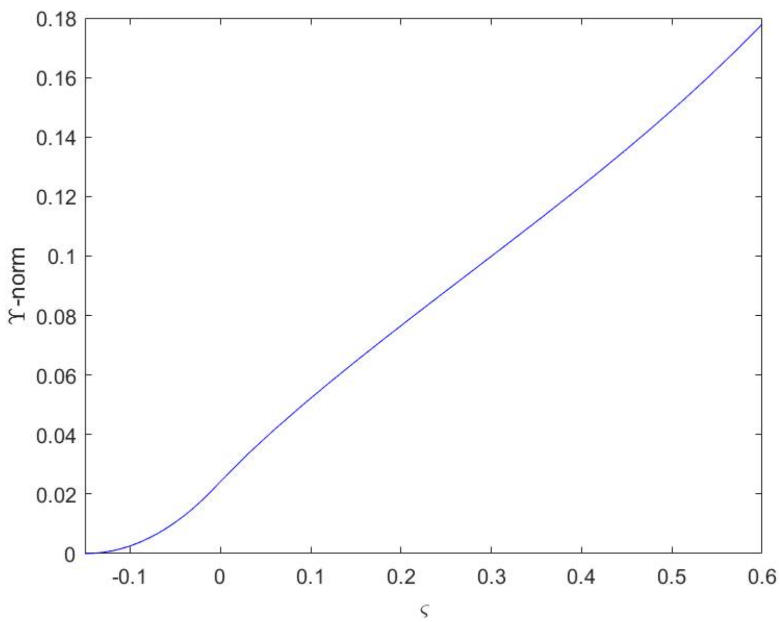
Zhenbin Fan[1] and Renjie Pan[1]

[1]Yangzhou University College of Mathematical Sciences

September 21, 2022

Abstract

This paper firstly finishes off the exact solutions of Riemann-Liouville fractional differential time-delay oscillatory system of order ρ[?](1 ,2) by using two newly defined delayed perturbations of Mittag-Leffler matrix functions and constant variation method. In the light of the exact solutions, we explore the finite time stability of the nonhomogeneous fractional oscillatory differential equations with pure delay. Ultimately, an example is cited to verify the rationality of the results. Through our method, the public problems left by Mahmudov in 2022 were partially solved.



Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

RESEARCH ARTICLE

Analyses of solutions of Riemann-Liouville fractional oscillatory differential equations with pure delay

Renjie Pan | Zhenbin Fan*

School of Mathematical Sciences, Yangzhou University, Yangzhou , China

Correspondence
*Zhenbin Fan, School of Mathematical Sciences, Yangzhou University, Yangzhou, Jiangsu 225002, China.
Email: zbfan@yzu.edu.cn

Abstract

This paper firstly finishes off the exact solutions of Riemann-Liouville fractional differential time-delay oscillatory system of order $\rho \in (1, 2)$ by using two newly defined delayed perturbations of Mittag-Leffler matrix functions and constant variation method. In the light of the exact solutions, we explore the finite time stability of the nonhomogeneous fractional oscillatory differential equations with pure delay. Ultimately, an example is cited to verify the rationality of the results. Through our method, the public problems left by Mahmudov in 2022 were partially solved.

KEYWORDS:
Riemann-Liouville fractional derivative; Finite time stability; Delayed Mittag-Leffler matrix function

1 | INTRODUCTION

Fractional oscillatory differential equation is used to describe many processes in mechanical and technical systems. These systems are usually used to simulate phenomena in practical problems. As is known to all, fractional oscillatory differential equations have a wide range of applications in many fields, such as electronic science, polymer, viscoelastic material model, signal and image processing, etc. Thanks to fractional oscillatory differential equations have important theoretical and practical significances for research in different fields, there is currently a significant amount of theoretical researches being conducted on fractional oscillatory differential equations, mainly focusing on initial value problems, boundary value problems, stability, controllability. We can refer to [1]-[5] and the references they contain.

It is of far-reaching significance to study the stability of fractional oscillatory differential equations. Different from other stabilities (such as Lyapunov stability, asymptotical stability and so on), finite time stability (FTS) studies the conduct of the trajectory of a system in a limited time interval. It is worth noting that the finite time stability problem needs to give the required time interval ahead. Lately, the FTS of fractional oscillatory differential equations have been widely studied [6]-[8] and the references they contain.

Not long ago, Khusainov and Shuklin [9] firstly put forward a delayed exponential function e_τ^{Bt}, which can be used to explore an explicit solution of a first-order linear delay differential equation. It opened a new era for exploring the explicit solutions of time-delay differential equations. Based on it, Li and Wang [10],[11] made further promotions. They constructed a new delayed Mittag-Leffler type matrix function $E_{\tau,\beta}^{Bx^\alpha}$ to gain the representations of Caputo fractional differential equations with order $\rho \in (0, 1)$. In order to get expressions of Caputo fractional differential equations with order $\rho \in (1, 2)$, Liu, Dong and Li [12] constructed two fundamental matrices C_α^τ and S_α^τ. Inspired by these papers, Elshenhab and Wang [13],[14] expanded on the previous results. They introduced a new fundamental solution $\mathcal{H}_{h,\alpha}(Ax^\alpha)$ and two other functions stated in Liu, Dong and Li [12] to gain the representations of Caputo fractional single-delay differential equations with order $\rho \in (1, 2)$. With a view to exploring the expressions of Caputo fractional multiple-delay differential equations where the linear parts are given by permutation or nonpermutation matrixes, they adopted Laplace transformation.

How about the Riemann-Liouville fractional time-delay differential equations? Li and Wang [15],[16] solved the explicit solutions of fractional differential equations with order $\rho \in (0,1)$ by using the delayed Mittag-Leffler function $Z_{\tau,\beta}^{Bx^{\alpha}}$ previously constructed in Caputo. Mahmudov [17] recently solved the linear nonhomogeneous fractional multi-delay differential equations of order $\rho \in (l-1, l]$. Through reading Mahmudov's paper, we have the following findings: Firstly, in Mahmudov's, the lower limit of integral in the definition of fractional derivative starts at 0 rather than $-\iota$. This leads to a slight difference between Mahmudov's model and our research. Secondly, the multivariate determining matrix function Q_{k+1} is obtained by iterative method, and then the fundamental solution $X_{h,\alpha.\beta}^{A,B}$ is constructed by Q_{k+1}, that is, the basic solution of the equation is given by using the method of twice construction. The explicit solution obtained by this method is rather complex, which is not conducive to the subsequent finite-time stability analysis. Through the same way, Mustafa Aydin and Mahmudov [18],[19] got the explicit solutions of Caputo fractional single-delayed and multi-delayed differential equations with order $\rho \in (0,1)$. Thirdly, the conclusions of Mahmudov's article were more general and expanded the problem to higher order, which provides further research direction for our follow-up research. At the end of the article, Mahmudov made clear the deficiencies in the research and put forward some open questions. It is worth affirming that Mahmudov made a significant step forward in the research of fractional delay differential equations. However, we only use two newly defined delayed Mittag-Leffler functions constructed once to obtain the explicit solution, which can be well characterized on every subinterval in this paper. The advantage of this method is that it is convenient for the subsequent finite-time stability analysis, and it partly solves the remaining public problems.

Generally speaking, constructing the perturbation matrix functions of delayed Mittag-Leffler type or adopting the method of Mahmudov's twice constructions are used to derive the fundamental solutions. On the basis of the fundamental solutions, we can get the exact solution by Laplace transform or the constant variation method. The approach we adopted to in this paper is the constant variation method (See Remark 3).

In this paper, fractional differential time-delay oscillatory system we investigated has the form

$$\begin{cases} {}^{R}\mathcal{D}_{-\iota^{+}}^{\rho} Y(\varsigma) = \varpi Y(\varsigma - \iota) + f(\varsigma), & \varsigma \in (0, \mathcal{T}], \iota > 0, \\ Y(\varsigma) = \varphi(\varsigma), & -\iota < \varsigma \leq 0, \\ \mathcal{J}_{-\iota^{+}}^{2-\rho} Y(-\iota^{+}) = \mathtt{a}, \quad {}^{R}\mathcal{D}_{-\iota^{+}}^{\rho-1} Y(-\iota^{+}) = \mathtt{b}, \end{cases} \tag{1}$$

where ${}^{R}\mathcal{D}_{-\iota^{+}}^{\rho}$ denotes the Riemann-Liouville fractional derivative of order $\rho \in (1,2)$, $\mathcal{J}_{-\iota^{+}}^{2-\rho}$ denotes the Riemann-Liouville fractional integral of order $2-\rho$, $\mathcal{J}_{-\iota^{+}}^{2-\rho} Y \in AC^{2}((-\iota, 0], \mathbb{R}^{n})$, $f : (0, \mathcal{T}] \rightarrow \mathbb{R}^{n}$ is a continuous function, $\mathcal{T} = \kappa\iota$ is a fixed terminal time ahead, κ is a natural number fixed and $\varpi \in \mathbb{R}^{n \times n}$ is a constant matrix.

The chief aim of this paper is to explore the explicit solution of system (1). To this end, we firstly explore the homogeneous fractional differential oscillatory system

$$\begin{cases} {}^{R}\mathcal{D}_{-\iota^{+}}^{\rho} Y(\varsigma) = \varpi Y(\varsigma - \iota), & \varsigma \in (0, \mathcal{T}], \iota > 0, \\ Y(\varsigma) = \varphi(\varsigma), & -\iota < \varsigma \leq 0, \\ \mathcal{J}_{-\iota^{+}}^{2-\rho} Y(-\iota^{+}) = \mathtt{a}, \quad {}^{R}\mathcal{D}_{-\iota^{+}}^{\rho-1} Y(-\iota^{+}) = \mathtt{b}. \end{cases} \tag{2}$$

The contents and structure of this paper are as follows: The second part describes some definitions and constructs two functions that extended the Mittag-Leffler function. The third part contains the exact solution of system (1). In the fourth part, some sufficient conditions that the system (1) is FTS are given. Ultimately, an example is given to prove the rationality of the conclusions.

2 | PRELIMINARIES

In this section, we describe some definitions and constructs two functions that extended the Mittag-Leffler function. Let $\mathtt{k} \in \Pi = \{0, 1, 2, ..., \kappa\}$, θ express zero matrix, I express identity matrix and $\Gamma(\cdot)$ is the Gamma function.

Definition 1. (See [2]) The Riemann-Liouville integral of order $\rho \in (1,2)$ of an integrable function $f : [-\iota, +\infty) \rightarrow \mathbb{R}^{n}$ is defined as follows

$$\mathcal{J}_{-\iota^{+}}^{\rho} f(\varsigma) = \frac{1}{\Gamma(\rho)} \int_{-\iota}^{\varsigma} (\varsigma - s)^{\rho} f(s) ds, \quad \varsigma > -\iota.$$

Definition 2. (See [2]) The Riemann-Liouville derivative of order $\rho \in (1,2)$ of an integrable function $f : [-\iota, +\infty) \to \mathbb{R}^n$ is defined as follows

$$ {}^{R}\mathcal{D}^{\rho}_{-\iota^+} f(\varsigma) = \frac{1}{\Gamma(2-\rho)} \frac{d^2}{d\varsigma^2} \int_{-\iota}^{\varsigma} (\varsigma - s)^{1-\rho} f(s) ds, \quad \varsigma > -\iota. $$

Remark 1. If $f : [-\iota, +\infty) \to \mathbb{R}^n$ is an integrable function and $\mathcal{J}^{2-\rho}_{-\iota^+} f \in AC^2([-\iota, +\infty), \mathbb{R}^n)$, which the second derivative of $\mathcal{J}^{2-\rho}_{-\iota^+} f$ is an absolutely continuous function. Then the relation between Riemann-Liouville integrals and derivatives of order $\rho \in (1,2)$ is as follows

$$ {}^{R}\mathcal{D}^{\rho}_{-\iota^+} f(\varsigma) = \frac{d^2}{d\varsigma^2} \mathcal{J}^{2-\rho}_{-\iota^+} f(\varsigma). $$

The following are two Mittag-Leffler type matrix functions with time delay that we newly define in this paper.

Definition 3. The delayed Mittag-Leffler type matrix function $P^{\iota}_{\rho}(\varsigma) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined as

$$ P^{\iota}_{\rho}(\varsigma) = \begin{cases} \theta, & -\infty < \varsigma \le -\iota, \\ I \frac{(\varsigma+\iota)^{\rho-1}}{\Gamma(\rho)}, & -\iota < \varsigma \le 0, \\ I \frac{(\varsigma+\iota)^{\rho-1}}{\Gamma(\rho)} + \varpi \frac{\varsigma^{2\rho-1}}{\Gamma(2\rho)}, & 0 < \varsigma \le \iota, \\ \cdots & \cdots \\ I \frac{(\varsigma+\iota)^{\rho-1}}{\Gamma(\rho)} + \varpi \frac{\varsigma^{2\rho-1}}{\Gamma(2\rho)} + \varpi^2 \frac{(\varsigma-\iota)^{3\rho-1}}{\Gamma(3\rho)} + \cdots + \varpi^{\mathtt{k}} \frac{(\varsigma-(\mathtt{k}-1)\iota)^{(\mathtt{k}+1)\rho-1}}{\Gamma((\mathtt{k}+1)\rho)}, & (\mathtt{k}-1)\iota < \varsigma \le \mathtt{k}\iota. \end{cases} $$

Definition 4. The delayed Mittag-Leffler type matrix function $H^{\iota}_{\rho}(\varsigma) : \mathbb{R} \to \mathbb{R}^{n\times n}$ is defined as

$$ H^{\iota}_{\rho}(\varsigma) = \begin{cases} \theta, & -\infty < \varsigma \le -\iota, \\ I \frac{(\varsigma+\iota)^{\rho-2}}{\Gamma(\rho-1)}, & -\iota < \varsigma \le 0, \\ I \frac{(\varsigma+\iota)^{\rho-2}}{\Gamma(\rho-1)} + \varpi \frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)}, & 0 < \varsigma \le \iota, \\ \cdots & \cdots \\ I \frac{(\varsigma+\iota)^{\rho-2}}{\Gamma(\rho-1)} + \varpi \frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)} + \varpi^2 \frac{(\varsigma-\iota)^{3\rho-2}}{\Gamma(3\rho-1)} + \cdots + \varpi^{\mathtt{k}} \frac{(\varsigma-(\mathtt{k}-1)\iota)^{(\mathtt{k}+1)\rho-2}}{\Gamma((\mathtt{k}+1)\rho-1)}, & (\mathtt{k}-1)\iota < \varsigma \le \mathtt{k}\iota. \end{cases} $$

Lemma 1. For $P^{\iota}_{\rho}(\varsigma)$ and $H^{\iota}_{\rho}(\varsigma)$, we acquire
(i) $\frac{d}{d\varsigma} P^{\iota}_{\rho}(\varsigma) = H^{\iota}_{\rho}(\varsigma)$ for all $\varsigma \in \mathbb{R} \setminus \{-\iota\}$.
(ii) $P^{\iota}_{\rho}(\varsigma)$ is a solution of equation (2), which meets starting conditions $P^{\iota}_{\rho}(\varsigma) = I \frac{(\varsigma+\iota)^{\rho-1}}{\Gamma(\rho)}$, $-\iota < \varsigma \le 0$, $\mathcal{J}^{2-\rho}_{-\iota^+} P^{\iota}_{\rho}(-\iota^+) = 0$ and ${}^{R}\mathcal{D}^{\rho-1}_{-\iota^+} P^{\iota}_{\rho}(-\iota^+) = 1$.
(iii) $H^{\iota}_{\rho}(\varsigma)$ is a solution of equation (2), which meets starting conditions $H^{\iota}_{\rho}(\varsigma) = I \frac{(\varsigma+\iota)^{\rho-2}}{\Gamma(\rho-1)}$, $-\iota < \varsigma \le 0$, $\mathcal{J}^{2-\rho}_{-\iota^+} H^{\iota}_{\rho}(-\iota^+) = 1$ and ${}^{R}\mathcal{D}^{\rho-1}_{-\iota^+} H^{\iota}_{\rho}(-\iota^+) = 0$.

Proof. It is obvious that we can directly obtain the property (i) by taking the derivative of $P^{\iota}_{\rho}(\varsigma)$ relative to ς. The approach we adopted to prove the property (ii) and property (iii) is mathematical induction. Firstly, we prove the property (iii).

step1. For $\mathtt{k} = 1$, $0 < \varsigma \le \iota$, one has

$$ Y(\varsigma) = H^{\iota}_{\rho}(\varsigma) = I \frac{(\varsigma+\iota)^{\rho-2}}{\Gamma(\rho-1)} + \varpi \frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)}. $$

Applying the ρ-order Riemann-Liouville derivative on $H^{\imath}_{\rho}(\varsigma)$, then

$$
\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}H^{\imath}_{\rho}(\varsigma) &= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\left[\int_{-\imath}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}I\frac{(\vartheta+\imath)^{\rho-2}}{\Gamma(\rho-1)}d\vartheta+\int_{0}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\varpi\frac{\vartheta^{2\rho-2}}{\Gamma(2\rho-1)}d\vartheta\right]\\
&= \frac{d^2}{d\varsigma^2}\frac{I}{\Gamma(2-\rho)\Gamma(\rho-1)}\int_{-\imath}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}(\vartheta+\imath)^{\rho-2}d\vartheta+\frac{d^2}{d\varsigma^2}\frac{\varpi}{\Gamma(2-\rho)\Gamma(2\rho-1)}\int_{0}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\vartheta^{2\rho-2}d\vartheta\\
&= \frac{d^2}{d\varsigma^2}\frac{I}{\Gamma(2-\rho)\Gamma(\rho-1)}\int_{0}^{1}(\varsigma+\imath)^{1-\rho}(1-y)^{1-\rho}(\varsigma+\imath)^{\rho-2}y^{\rho-2}(\varsigma+\imath)dy\\
&\quad+\frac{d^2}{d\varsigma^2}\frac{\varpi}{\Gamma(2-\rho)\Gamma(2\rho-1)}\int_{0}^{1}\varsigma^{1-\rho}(1-y)^{1-\rho}\varsigma^{2\rho-2}y^{2\rho-2}\varsigma dy\\
&= \frac{d^2}{d\varsigma^2}\frac{I}{\Gamma(2-\rho)\Gamma(\rho-1)}B[2-\rho,\rho-1]+\frac{d^2}{d\varsigma^2}\frac{\varpi\varsigma^{\rho}}{\Gamma(2-\rho)\Gamma(2\rho-1)}B[2-\rho,2\rho-1]\\
&= \varpi\frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)}=\varpi H^{\imath}_{\rho}(\varsigma-\imath).
\end{aligned}
$$

step2. For $\mathtt{k}=2$, $\imath<\varsigma\le 2\imath$, one obtains

$$
\Upsilon(\varsigma)=H^{\imath}_{\rho}(\varsigma)=I\frac{(\varsigma+\imath)^{\rho-2}}{\Gamma(\rho-1)}+\varpi\frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)}+\varpi^2\frac{(\varsigma-\imath)^{3\rho-2}}{\Gamma(3\rho-1)}.
$$

Applying the ρ-order Riemann-Liouville derivative on $H^{\imath}_{\rho}(\varsigma)$, then

$$
\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}H^{\imath}_{\rho}(\varsigma) &= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\left[\int_{-\imath}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}I\frac{(\vartheta+\imath)^{\rho-2}}{\Gamma(\rho-1)}d\vartheta+\int_{0}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\varpi\frac{\vartheta^{2\rho-2}}{\Gamma(2\rho-1)}d\vartheta\right.\\
&\quad\left.+\int_{\imath}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\varpi^2\frac{(\vartheta-\imath)^{3\rho-2}}{\Gamma(3\rho-1)}d\vartheta\right]\\
&= \varpi\frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)}+\frac{d^2}{d\varsigma^2}\frac{\varpi^2}{\Gamma(2-\rho)\Gamma(3\rho-1)}\int_{\imath}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}(\vartheta-\imath)^{3\rho-2}d\vartheta\\
&= \varpi\frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)}+\frac{d^2}{d\varsigma^2}\frac{\varpi^2}{\Gamma(2-\rho)\Gamma(3\rho-1)}(\varsigma-\imath)^{2\rho}B[2-\rho,3\rho-1]\\
&= \varpi\frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)}+\varpi^2\frac{(\varsigma-\imath)^{2\rho-2}}{\Gamma(2\rho-1)}=\varpi H^{\imath}_{\rho}(\varsigma-\imath).
\end{aligned}
$$

step3. Assume the conclusion is true when $\mathtt{k}=U$, $(U-1)\imath<\varsigma\le U\imath$, we acquire

$$
{}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}H^{\imath}_{\rho}(\varsigma)=\varpi\frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)}+\varpi^2\frac{(\varsigma-\imath)^{2\rho-2}}{\Gamma(2\rho-1)}+\cdots+\varpi^{U}\frac{(\varsigma-(U-1)\imath)^{U\rho-2}}{\Gamma(U\rho-1)}=\varpi H^{\imath}_{\rho}(\varsigma-\tau).
$$

For $\mathtt{k}=U+1$, $U\imath<\varsigma\le(U+1)\imath$, and

$$
\Upsilon(\varsigma)=H^{\imath}_{\rho}(\varsigma)=I\frac{(\varsigma+\imath)^{\rho-2}}{\Gamma(\rho-1)}+\varpi\frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)}+\varpi^2\frac{(\varsigma-\imath)^{3\rho-2}}{\Gamma(3\rho-1)}+\cdots+\varpi^{U+1}\frac{(\varsigma-U\imath)^{(U+2)\rho-2}}{\Gamma((U+2)\rho-1)}.
$$

By elementary calculations, the formula has

$$\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\imath^{+}} H^{\imath}_{\alpha}(\varsigma) &= \varpi \frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)} + \varpi^{2}\frac{(\varsigma-\imath)^{2\rho-2}}{\Gamma(2\rho-1)} + \cdots + \varpi^{U}\frac{(\varsigma-(U-1)\imath)^{U\rho-2}}{\Gamma(U\rho-1)} \\
&\quad + \frac{1}{\Gamma(2-\rho)}\frac{d^{2}}{d\varsigma^{2}}\frac{\varpi^{U+1}}{\Gamma((U+2)\rho-1)}\int_{U\imath}^{\varsigma}(\varsigma-\chi)^{1-\rho}(\chi-U\imath)^{(U+2)\rho-2}d\chi \\
&= \varpi \frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)} + \varpi^{2}\frac{(\varsigma-\imath)^{2\rho-2}}{\Gamma(2\rho-1)} + \cdots + \varpi^{U}\frac{(\varsigma-(U-1)\imath)^{U\rho-2}}{\Gamma(U\rho-1)} \\
&\quad + \frac{1}{\Gamma(2-\rho)}\frac{d^{2}}{d\varsigma^{2}}\frac{\varpi^{U+1}}{\Gamma((U+2)\rho-1)}(\varsigma-U\imath)^{(U+1)\rho}B[2-\rho,(U+2)\rho-1] \\
&= \varpi \frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)} + \varpi^{2}\frac{(\varsigma-\imath)^{2\rho-2}}{\Gamma(2\rho-1)} + \cdots + \varpi^{U+1}\frac{(\varsigma-U\imath)^{(U+1)\rho-2}}{\Gamma((U+1)\rho-1)} \\
&= \varpi H^{\imath}_{\rho}(\varsigma-\imath).
\end{aligned}$$

In a word, the property (iii) holds.

The same approach can be used to prove the property (ii). When $\mathtt{k} = U+1$, $U\imath < \varsigma \le (U+1)\imath$, through preliminary calculations, we gain

$$\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\imath^{+}} P^{\imath}_{\rho}(\varsigma) &= \varpi \frac{\varsigma^{\rho-1}}{\Gamma(\rho)} + \varpi^{2}\frac{(\varsigma-\imath)^{2\rho-1}}{\Gamma(2\rho)} + \cdots + \varpi^{U}\frac{(\varsigma-(U-1)\imath)^{U\rho-1}}{\Gamma(U\rho)} \\
&\quad + \frac{1}{\Gamma(2-\rho)}\frac{d^{2}}{d\varsigma^{2}}\frac{\varpi^{U+1}}{\Gamma((U+2)\rho+\rho)}\int_{U\imath}^{\varsigma}(\varsigma-\chi)^{1-\rho}(\chi-U\imath)^{(U+2)\rho-1}d\chi \\
&= \varpi \frac{\varsigma^{\rho-1}}{\Gamma(\rho)} + \varpi^{2}\frac{(\varsigma-\imath)^{2\rho-1}}{\Gamma(2\rho)} + \cdots + \varpi^{U}\frac{(\varsigma-(U-1)\imath)^{U\rho-1}}{\Gamma(U\rho)} \\
&\quad + \frac{1}{\Gamma(2-\rho)}\frac{d^{2}}{d\varsigma^{2}}\frac{\varpi^{U+1}}{\Gamma((U+2)\rho-1)}(\varsigma-U\imath)^{(U+1)\rho+1}B[2-\rho,(U+2)\rho] \\
&= \varpi \frac{\varsigma^{\rho-1}}{\Gamma(\rho)} + \varpi^{2}\frac{(\varsigma-\imath)^{2\rho-1}}{\Gamma(2\rho)} + \cdots + \varpi^{U+1}\frac{(\varsigma-U\imath)^{(U+1)\rho-1}}{\Gamma((U+1)\rho)} = \varpi P^{\imath}_{\rho}(\varsigma-\imath).
\end{aligned}$$

The proof is end. □

3 | EXACT SOLUTIONS

In this section, the approach we adopted to acquire the exact solutions of sysytem (1) is constant variation method. We first cast about for the display expression $\tilde{Y}(\varsigma)$ of equation (2), then explore the particular solution $\overline{Y}(\varsigma)$ of equation (2) meeting starting condition $Y(0) = 0$. In line with the superposition principle in ODEs, we take $Y(\varsigma) = \tilde{Y}(\varsigma) + \overline{Y}(\varsigma)$ to represent the explicit solution. Denote $\Phi = \{\varphi \in C((-\imath,0],\mathbb{R}^{n}) : \mathcal{J}^{2-\rho}_{-\imath^{+}}\varphi \in AC^{2}((-\imath,0],\mathbb{R}^{n})\}$ and $\Psi = \big\{f \in C((0,\mathcal{T}],\mathbb{R}^{n}), \int_{0}^{\varsigma}P^{\imath}_{\rho}(\varsigma-\imath-\chi)f(\chi)d\chi|_{\varsigma=0} = 0\big\}$. We introduce a Banach space $AC^{2}((-\imath,0],\mathbb{R}^{n}) = \{y : (-\imath,0] \to \mathbb{R}^{n}, (\frac{d}{dx}y)(x) \in AC((-\imath,0],\mathbb{R}^{n})\}$. Clearly, when $f \in C(\big[0,\mathcal{T}\big],\mathbb{R}^{n})$, we acquire $f \in \Psi$.

Theorem 1. Assume that $\mathtt{k} \in \Pi = \{0,1,2,...,\kappa\}$, $1 < \rho < 2$, $\imath > 0$, and $\varphi \in \Phi$. Then the exact soultion of the fractional differential system (2) is given by

$$\tilde{Y}(\varsigma) = P^{\imath}_{\rho}(\varsigma)\mathtt{b} + H^{\imath}_{\rho}(\varsigma)\mathtt{a} + \int_{-\imath}^{0} P^{\imath}_{\rho}(\varsigma-\imath-\chi)({}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}\varphi)(\chi)d\chi, \quad \varsigma \in (-\imath,\mathcal{T}\big]. \tag{3}$$

Proof. Based on Lemma 1, $P^{\imath}_{\rho}(\varsigma)$ and $H^{\imath}_{\rho}(\varsigma)$ are solutions of equation (2) with their own starting conditions. So the exact soultion of the fractional differential system (2) should search in the form

$$\tilde{Y}(\varsigma) = P^{\imath}_{\rho}(\varsigma)z_{1} + H^{\imath}_{\rho}(\varsigma)z_{2} + \int_{-\imath}^{0} P^{\imath}_{\rho}(\varsigma-\imath-\chi)z(\chi)d\chi, \tag{4}$$

where z_1 and z_2 are unknown constant vectors, $z(\cdot)$ is a continuous unknown Riemann-Liouville differentiable vector function on $(-\iota, 0]$.

Let us assume $\varsigma \in (-\iota, 0]$, the integral term in formula (4) can be written in the following form

$$\int_{-\iota}^{0} P_\rho^\iota(\varsigma-\iota-\chi)z(\chi)d\chi = \int_{-\iota}^{\varsigma} P_\rho^\iota(\varsigma-\iota-\chi)z(\chi)d\chi + \int_{\varsigma}^{0} P_\rho^\iota(\varsigma-\iota-\chi)z(\chi)d\chi.$$

Due to when $\chi \in [\varsigma, 0]$, we can get $\varsigma-\iota-\chi \leq -\iota$ and $P_\rho^\iota(\varsigma-\iota-\chi)=0$. The formula (4) can be rewritten into the following form

$$\varphi(\varsigma) = \tilde{Y}(\varsigma) = P_\rho^\iota(\varsigma)z_1 + H_\rho^\iota(\varsigma)z_2 + \int_{-\iota}^{\varsigma} P_\rho^\iota(\varsigma-\iota-\chi)z(\chi)d\chi.$$

For $-\iota < \varsigma \leq 0$, through calculating, one has

$$\begin{aligned}
\mathcal{J}_{-\iota^+}^{2-\rho}\tilde{Y}(\varsigma) &= \mathcal{J}_{-\iota^+}^{2-\rho}\varphi(\varsigma) \\
&= \frac{1}{\Gamma(2-\rho)}\int_{-\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\left[P_\rho^\iota(\vartheta)z_1 + H_\rho^\iota(\vartheta)z_2 + \int_{-\iota}^{\vartheta} P_\rho^\iota(\vartheta-\iota-\chi)z(\chi)d\chi\right]d\vartheta \\
&= \frac{1}{\Gamma(2-\rho)}\int_{-\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\frac{(\vartheta+\iota)^{\rho-1}}{\Gamma(\rho)}z_1 d\vartheta + \frac{1}{\Gamma(2-\rho)}\int_{-\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\frac{(\vartheta+\iota)^{\rho-2}}{\Gamma(\rho-1)}z_2 d\vartheta \\
&\quad + \frac{1}{\Gamma(2-\rho)}\int_{-\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\left[\int_{-\iota}^{\vartheta}\frac{(\vartheta-\chi)^{1-\rho}}{\Gamma(\rho)}z(\chi)d\chi\right]d\vartheta \\
&= \frac{z_1(\varsigma+\iota)}{\Gamma(2-\rho)\Gamma(\rho)}B[2-\rho,\rho] + \frac{z_2}{\Gamma(2-\rho)\Gamma(\rho-1)}B[2-\rho,\rho-1] \\
&\quad + \frac{1}{\Gamma(2-\rho)\Gamma(\rho)}\int_{-\iota}^{\varsigma} z(\chi)\left[\int_{\chi}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}(\vartheta-\chi)^{\rho-1}d\vartheta\right]d\chi \\
&= z_1(\varsigma+\iota) + z_2 + \int_{-\iota}^{\varsigma}(\varsigma-\chi)z(\chi)d\chi.
\end{aligned}$$

In the light of the initial conditions, the following points hold.

Firstly, by elementary calculations, we get

$$\begin{aligned}
\mathtt{a} = \mathcal{J}_{-\iota^+}^{2-\rho}\tilde{Y}(-\iota^+) = \mathcal{J}_{-\iota^+}^{2-\rho}\varphi(-\iota^+) &= \lim_{\varsigma\to-\iota^+}\mathcal{J}_{-\iota^+}^{2-\rho}\varphi(\varsigma) \\
&= \lim_{\varsigma\to-\iota^+}\left[z_1(\varsigma+\iota) + z_2 + \int_{-\iota}^{\varsigma}(\varsigma-\chi)z(\chi)d\chi\right] = z_2.
\end{aligned}$$

Secondly, since $\varphi \in \Phi$, ${}^R\mathcal{D}_{-\iota^+}^{\rho-1}\varphi$ exists. On the basis of Remark 1, it yields the following equation

$${}^R\mathcal{D}_{-\iota^+}^{\rho-1}\tilde{Y}(\varsigma) = {}^R\mathcal{D}_{-\iota^+}^{\rho-1}\varphi(\varsigma) = \frac{d}{d\varsigma}\mathcal{J}_{-\iota^+}^{2-\rho}\varphi(\varsigma) = z_1 + \int_{-\iota}^{\varsigma} z(\chi)d\chi,$$

then

$$\mathtt{b} = {}^R\mathcal{D}_{-\iota^+}^{\rho-1}\tilde{Y}(-\iota^+) = {}^R\mathcal{D}_{-\iota^+}^{\rho-1}\varphi(-\iota^+) = \lim_{\varsigma\to-\iota^+}{}^R\mathcal{D}_{-\iota^+}^{\rho-1}\varphi(\varsigma) = z_1.$$

Thirdly, since $\varphi \in \Phi$, ${}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}\varphi$ exists. On the basis of Remark 1, one shows

$$ {}^{R}D^{\rho}_{-\imath^{+}}\varphi(\varsigma) = \frac{d^2}{d\varsigma^2}\mathcal{J}^{2-\rho}_{-\imath^{+}}\varphi(\varsigma) = \frac{d^2}{d\varsigma^2}\left[z_1(\varsigma+\imath) + z_2 + \int_{-\imath}^{\varsigma} (\varsigma-\chi) z(\chi) d\chi \right] = z(\varsigma). $$

After verification, formula (3) does meet equation (2). In addition, it is obvious that the solution of the equation is unique, so formula (3) is the unique solution of equation (2). This proof is finished. □

Theorem 2. Let $\mathtt{k} \in \Pi = \{0, 1, 2, ..., \kappa\}$, $f \in \Psi$. The expression of the inhomogeneous system (1) meeting starting condition $Y(\varsigma) = 0$, $\varsigma \in [-\imath, 0]$ shows the following form

$$ \overline{Y}(\varsigma) = \int_0^{\varsigma} P^{\imath}_{\rho}(\varsigma - \imath - \chi) f(\chi) d\chi, \quad \varsigma \in \left[0, \mathcal{T}\right]. $$

Proof. According to the constant variation method, the solution $\overline{Y}(\varsigma)$ should satisfy the following form

$$ \overline{Y}(\varsigma) = \int_0^{\varsigma} P^{\imath}_{\rho}(\varsigma - \imath - \chi) g(\chi) d\chi, \tag{5} $$

where $g(\cdot)$ is a continuous unknown vector function and $\overline{Y}(0) = 0$.

Applying the ρ-order Riemann-Liouville derivative on $\overline{Y}(\varsigma)$, we acquire the following results.

(i) When $\mathtt{k} = 1$, $0 < \varsigma \leq \imath$, one shows

$$ ({}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}\overline{Y})(\varsigma) = \varpi \overline{Y}(\varsigma - \imath) + f(\varsigma) = f(\varsigma). $$

On the basis of Definition 2, we obtain

$$ \begin{aligned} {}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}\overline{Y}(\varsigma) &= \frac{1}{\Gamma(2-\rho)} \frac{d^2}{d\varsigma^2} \left[\int_{-\imath}^{0} (\varsigma - \vartheta)^{1-\rho} \overline{Y}(\vartheta) d\vartheta + \int_0^{\varsigma} (\varsigma - \vartheta)^{1-\rho} \overline{Y}(\vartheta) d\vartheta \right] \\ &= \frac{1}{\Gamma(2-\rho)} \frac{d^2}{d\varsigma^2} \int_0^{\varsigma} (\varsigma - \vartheta)^{1-\rho} \left[\int_0^{\vartheta} P^{\imath}_{\rho}(\vartheta - \imath - \chi) g(\chi) d\chi \right] d\vartheta \\ &= \frac{1}{\Gamma(2-\rho)} \frac{d^2}{d\varsigma^2} \int_0^{\varsigma} g(\chi) \left[\int_{\chi}^{\varsigma} (\varsigma - \vartheta)^{1-\rho} P^{\imath}_{\rho}(\vartheta - \imath - \chi) d\vartheta \right] d\chi \\ &= \frac{1}{\Gamma(2-\rho)} \frac{d^2}{d\varsigma^2} \int_0^{\varsigma} g(\chi) \left[\int_{\chi}^{\varsigma} (\varsigma - \vartheta)^{1-\rho} \frac{(\vartheta - \chi)^{\rho-1}}{\Gamma(\rho)} d\vartheta \right] d\chi \\ &= \frac{d^2}{d\varsigma^2} \int_0^{\varsigma} g(\chi)(\varsigma - \chi) d\chi \\ &= g(\varsigma). \end{aligned} $$

Thus, we gain $g(\varsigma) = f(\varsigma)$.

(ii)When $\mathtt{k} = 2$, $\imath < \varsigma \leq 2\imath$, one obtains

$$ {}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}\overline{Y}(\varsigma) = \varpi \overline{Y}(\varsigma - \imath) + f(\varsigma) = \varpi \int_0^{\varsigma-\imath} P^{\imath}_{\rho}(\varsigma - 2\imath - \chi) g(\chi) d\chi + f(\varsigma) = \varpi \int_0^{\varsigma-\imath} \frac{(\varsigma - \imath - \chi)^{\rho-1}}{\Gamma(\rho)} g(\chi) d\chi + f(\varsigma). $$

Based on Definition 2, we acquire

$$
\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\iota^+}\overline{Y}(\varsigma) &= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{-\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\overline{Y}(\vartheta)d\vartheta \\
&= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma}g(\chi)\left[\int_{\chi}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}P^{\iota}_{\rho}(\vartheta-\iota-\chi)d\vartheta\right]d\chi \\
&= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma}g(\chi)\left[\int_{\chi}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}I\frac{(\vartheta-\chi)^{\rho-1}}{\Gamma(\rho)}d\vartheta\right]d\chi \\
&\quad+\frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma-\iota}g(\chi)\left[\int_{\chi+\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\varpi\frac{(\vartheta-\iota-\chi)^{2\rho-1}}{\Gamma(2\rho)}d\vartheta\right]d\chi \\
&= \frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma}(\varsigma-\chi)g(\chi)d\chi+\frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma-\iota}g(\chi)\left[\int_{\chi+\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\varpi\frac{(\vartheta-\iota-\chi)^{2\rho-1}}{\Gamma(2\rho)}d\vartheta\right]d\chi \\
&= g(\varsigma)+\varpi\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma-\iota}g(\chi)\frac{(\varsigma-\iota-\chi)^{\rho+1}}{\Gamma(\rho+2)}d\chi \\
&= g(\varsigma)+\varpi\int_{0}^{\varsigma-\iota}\frac{(\varsigma-\iota-\chi)^{\rho-1}}{\Gamma(\rho)}g(\chi)d\chi.
\end{aligned}
$$

Thus, we gain $g(\varsigma)=f(\varsigma)$.

(iii) Suppose that the conclusion is valid when $\mathtt{k}=U$ and $(U-1)\iota<\varsigma\leq U\iota$.

For $\mathtt{k}=U+1$, $U\iota<\varsigma\leq(U+1)\iota$, one shows

$$
\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\iota^+}\overline{Y}(\varsigma) = \varpi\overline{Y}(\varsigma-\iota)+f(\varsigma) &= \varpi\int_{0}^{\varsigma-\iota}P^{\iota}_{\varsigma}(\varsigma-2\iota-\chi)g(\chi)d\chi+f(\varsigma) \\
&= \sum_{q=1}^{U+1}\varpi^{q}\int_{0}^{\varsigma-q\iota}\frac{(\varsigma-q\iota-\chi)^{q\rho-1}}{\Gamma(q\rho)}g(\chi)d\chi+f(\varsigma).
\end{aligned}
$$

On the basis of Definition 2, we get

$$
\begin{aligned}
{}^{R}\mathcal{D}^{\rho}_{-\iota^+}\overline{Y}(\varsigma) &= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{-\iota}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\left[\int_{0}^{\vartheta}P^{\iota}_{\rho}(\vartheta-\iota-\chi)g(\chi)d\chi\right]d\vartheta \\
&= \frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma}g(\chi)\left[\int_{\chi}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}P^{\iota}_{\rho}(\vartheta-\iota-\chi)d\vartheta\right]d\chi \\
&= \frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma}(\varsigma-\chi)g(\chi)d\chi+\sum_{q=1}^{U+1}\frac{1}{\Gamma(2-\rho)}\frac{d^2}{d\varsigma^2}\int_{0}^{\varsigma-q\iota}g(\chi)\left[\int_{q\iota+\chi}^{\varsigma}(\varsigma-\vartheta)^{1-\rho}\varpi^{q}\frac{(\vartheta-q\iota-\chi)^{(q+1)\rho-1}}{\Gamma(q\rho+\rho)}d\vartheta\right]d\chi \\
&= g(\varsigma)+\sum_{q=1}^{U+1}\varpi^{q}\int_{0}^{\varsigma-q\iota}\frac{(\varsigma-q\iota-\chi)^{q\rho-1}}{\Gamma(q\rho)}g(\chi)d\chi.
\end{aligned}
$$

Hence, we get $g(\varsigma)=f(\varsigma)$. The proof is end. □

Remark 2. Compared with previous literatures, we extend the range of f to Ψ in this paper. The original range of f can be changed from left open and right closed interval to closed interval. Moreover, since our proof adopts mathematical induction, the solutions are valid in $[0, \infty)$. Furthermore, our solutions are more universal.

Theorem 3. Let $\mathtt{k} \in \Pi = \{0, 1, 2, ..., \kappa\}$, $1 < \rho < 2$, $\imath > 0$, $\varphi \in \Phi$ and $f \in \Psi$. Then the expression of the nonhomogeneous system (1) shows the following form

$$Y(\varsigma) = P^{\imath}_{\rho}(\varsigma)\mathrm{b} + H^{\imath}_{\rho}(\varsigma)\mathrm{a} + \int_{-\imath}^{0} P^{\imath}_{\rho}(\varsigma - \imath - \chi)({}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}\varphi)(\chi)d\chi + \int_{0}^{\varsigma} P^{\imath}_{\rho}(\varsigma - \imath - \chi)f(\chi)d\chi, \quad \varsigma \in [0, \mathcal{T}].$$

Remark 3. To begin with, we wanted to find the explicit solution by means of Laplace transformation. Through calculation, we found that the results are complicated. By virtue of the lower limit of integral in the definition of fractional derivative starts at $-\imath$,

$$\begin{aligned}\mathscr{L}\left[({}^{R}\mathcal{D}^{\rho}_{-\imath^{+}}Y)(\varsigma)\right](\lambda) &= \mathscr{L}\left[(g_{2-\rho} * Y)''_{-\imath}(\varsigma)\right](\lambda)\\ &= \lambda^{2}\mathscr{L}\left[(g_{2-\rho} * Y)_{-\imath}(\varsigma)\right](\lambda) - \lambda(g_{2-\rho} * Y)_{-\imath}(0) - (g_{2-\rho} * Y)'_{-\imath}(0)\end{aligned}$$

and because of $(g_{2-\rho} * Y)_{-\imath}(\varsigma)$, one has

$$(g_{2-\rho} * Y)_{-\imath}(\varsigma) = \int_{-\imath}^{\varsigma} \frac{(\varsigma-\sigma)^{1-\rho}}{\Gamma(2-\rho)}Y(\sigma)d\sigma = \int_{-\imath}^{0} \frac{(\varsigma-\sigma)^{1-\rho}}{\Gamma(2-\rho)}Y(\sigma)d\sigma + \int_{0}^{\varsigma} \frac{(\varsigma-\sigma)^{1-\rho}}{\Gamma(2-\rho)}Y(\sigma)d\sigma.$$

When $Y(\varsigma) = P^{\imath}_{\rho}(\varsigma)$, one has the following form of integral $\int_{-\imath}^{0} \frac{(\varsigma-\sigma)^{1-\rho}}{\Gamma(2-\rho)} P^{\imath}_{\rho}(\sigma)d\sigma$. This integral cannot be applied to Beta function, so it is difficult to calculate this integral directly. The approach we adopted to present the accurate solution is constant variation method here.

Remark 4. Compared with Mahmudov's, if $A = \theta$, $d = 1$, $l = 2$ and $\rho \in (1, 2)$, the equation has the following form ${}^{R}\mathcal{D}^{\rho}_{0^{+}}y(\varsigma) = A_1 y(\varsigma - h_1) + f(\varsigma)$ and the solution should be satisfied with the form

$$y(\varsigma) = X_{\rho,\rho}(\varsigma)q_1 + X_{\rho,\rho-1}(\varsigma)q_2 + \int_{-h_1}^{\varsigma-h_1} X_{\rho,\rho}(\varsigma - r - h_1)A_1\varphi(r)dr + \int_{0}^{\varsigma} X_{\rho,\rho}(\varsigma - r)f(r)dr,$$

where $X_{\rho,\rho}(\varsigma) = \sum_{i=0}^{\infty}\sum_{j=0}^{k-1} Q_{i+1}(jh_1)\frac{(\varsigma-jh_1)^{(i+1)\rho-1}}{\Gamma(i\rho+\rho)}$. Due to the construction of $X_{\rho,\rho}(\varsigma)$ via the multivariate function Q_{i+1} which is a delayed simulation of the polynomial formula of the non-commutative matrixes, estimating the stability is still a difficult problem. Therefore, in the next section, we will use the explicit solution obtained here.

4 | FINITE TIME STABILITY RESULTS

Let $\mathtt{k} \in \Pi \setminus \{0\} = \{1, 2, ...\kappa\}$. For $\mu \geq 0$, we denote $Z_{\mu} = \left\{Y(\cdot) \in C((a, b], \mathbb{R}^{n}) : (\cdot - a)^{\mu}Y(\cdot) \in C([a, b], \mathbb{R}^{n})\right\}$. Then $\|Y\|_{Z_{\mu}} = \sup_{a\leq\varsigma\leq b} \|(\varsigma - a)^{\mu}Y(\varsigma)\|$.

Definition 5. (See [8]) The inhomogeneous system (1) is finite time stable concerning $\{0, [(\mathtt{k}-1)\imath, \mathtt{k}\imath], \imath, \delta, \eta\}$, $\delta < \eta$, iff $\|\varphi\| < \delta$, $\|a\| < \delta$ and $\|b\| < \delta$, imply the solution of system (1) satisfying $\|Y\|_{Z_{\mu}} < \eta$.

Lemma 2. For $\varsigma \in ((\mathtt{k}-1)\imath, \mathtt{k}\imath]$, $\mathtt{k} \in \Pi \setminus \{0\}$, $1 < \rho < 2$ and $\mu \geq 0$, we obtain
(i) $\|(\varsigma - (\mathtt{k}-1)\imath)^{\mu}P^{\imath}_{\rho}(\varsigma)\| \leq (\varsigma + \imath)^{\rho+\mu-1}E_{\rho,\rho}(\|\varpi\|(\varsigma + \imath)^{\rho})$.
(ii) $\|(\varsigma - (\mathtt{k}-1)\imath)^{\mu}H^{\imath}_{\rho}(\varsigma)\| \leq \varsigma^{\rho+\mu-2}E_{\rho,\rho-1}(\|\varpi\|\varsigma^{\rho})$.

Proof. For $\varsigma \in ((\mathtt{k}-1)\imath, \mathtt{k}\imath]$, in the light of Definition 3 and Definition 4, we obtain
(i)

$$\begin{aligned}\|(\varsigma - (\mathtt{k}-1)\imath)^{\mu}P^{\imath}_{\rho}(\varsigma)\| &\leq (\varsigma + \imath)^{\mu}\left[\frac{(\varsigma+\imath)^{\rho-1}}{\Gamma(\rho)} + \|\varpi\|\frac{\varsigma^{2\rho-1}}{\Gamma(\rho+\rho)} + \|\varpi\|^{2}\frac{(\varsigma-\imath)^{3\rho-1}}{\Gamma(2\rho+\rho)} + \cdots + \|\varpi\|^{\mathtt{k}}\frac{(\varsigma-(\mathtt{k}-1)\imath)^{(\mathtt{k}+1)\rho-1}}{\Gamma(\mathtt{k}\rho+\rho)}\right]\\ &\leq (\varsigma + \imath)^{\mu}\left[\frac{(\varsigma+\imath)^{\rho-1}}{\Gamma(\rho)} + \|\varpi\|\frac{(\varsigma+\imath)^{2\rho-1}}{\Gamma(\rho+\rho)} + \|\varpi\|^{2}\frac{(\varsigma+\imath)^{3\rho-1}}{\Gamma(2\rho+\rho)} + \cdots + \|\varpi\|^{\mathtt{k}}\frac{(\varsigma+\imath)^{(\mathtt{k}+1)\rho-1}}{\Gamma(\mathtt{k}\rho+\rho)}\right]\\ &\leq (\varsigma + \imath)^{\rho+\mu-1}E_{\rho,\rho}(\|\varpi\|(\varsigma + \imath)^{\rho}).\end{aligned}$$

(ii)

$$\begin{aligned}
\|(\varsigma-(\mathrm{k}-1)\imath)^{\mu}H^{\imath}_{\rho}(\varsigma)\| &\leq (\varsigma-(\mathrm{k}-1)\imath)^{\mu}\left[\frac{(\varsigma+\imath)^{\rho-2}}{\Gamma(\rho-1)}+\|\varpi\|\frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)}+\|\varpi\|^{2}\frac{(\varsigma-\imath)^{3\rho-2}}{\Gamma(3\rho-1)}+\cdots+\|\varpi\|^{\mathrm{k}}\frac{(\varsigma-(\mathrm{k}-1)\imath)^{(\mathrm{k}+1)\rho-2}}{\Gamma((\mathrm{k}+1)\rho-1)}\right]\\
&\leq \varsigma^{\mu}\left[\frac{\varsigma^{\rho-2}}{\Gamma(\rho-1)}+\|\varpi\|\frac{\varsigma^{2\rho-2}}{\Gamma(2\rho-1)}+\|\varpi\|^{2}\frac{\varsigma^{3\rho-2}}{\Gamma(3\rho-1)}+\cdots+\|\varpi\|^{\mathrm{k}}\frac{\varsigma^{(\mathrm{k}+1)\rho-2}}{\Gamma((\mathrm{k}+1)\rho-1)}\right]\\
&\leq \varsigma^{\rho+\mu-2}E_{\rho,\rho-1}(\|\varpi\|\varsigma^{\rho}).
\end{aligned}$$

This proof is end. □

Lemma 3. For $\varsigma\in((\mathrm{k}-1)\imath,\mathrm{k}\imath]$, $\mathrm{k}\in\Pi\setminus\{0\}$ and $1<\rho<2$, one obtains

$$\int_{-\imath}^{0}\|P^{\imath}_{\rho}(\varsigma-\imath-\chi)\|d\chi\leq\sum_{q=1}^{\mathrm{k}}\frac{\|\varpi\|^{q-1}}{\Gamma(q\rho+1)}\left[(\varsigma-(q-2)\imath)^{q\rho}-(\varsigma-(q-1)\imath)^{q\rho}\right]+\|\varpi\|^{\mathrm{k}}\frac{(\varsigma-(\mathrm{k}-1)\imath)^{(\mathrm{k}+1)\rho}}{\Gamma((\mathrm{k}+1)\rho+1)}.$$

Proof. For $\varsigma\in((\mathrm{k}-1)\tau,\mathrm{k}\tau]$, in the light of Definition 3, one acquires

$$\begin{aligned}
\int_{-\imath}^{0}\|P^{\imath}_{\rho}(\varsigma-\imath-\chi)\|d\chi &= \int_{-\imath}^{\varsigma-\mathrm{k}\imath}\|P^{\imath}_{\rho}(\varsigma-\imath-\chi)\|d\chi+\int_{\varsigma-k\imath}^{0}\|P^{\imath}_{\rho}(\varsigma-\imath-\chi)\|d\chi\\
&\leq \int_{-\imath}^{\varsigma-\mathrm{k}\imath}\left[\frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)}+\|\varpi\|\frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(\rho+\rho)}+\cdots+\|\varpi\|^{\mathrm{k}-1}\frac{(\varsigma-(\mathrm{k}-1)\imath-\chi)^{\mathrm{k}\rho-1}}{\Gamma((\mathrm{k}-1)\rho+\rho)}\right]d\chi\\
&\quad+\int_{\varsigma-\mathrm{k}\imath}^{0}\left[\frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)}+\|\varpi\|\frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(\rho+\rho)}+\cdots+\|\varpi\|^{\mathrm{k}-1}\frac{(\varsigma-(\mathrm{k}-1)\imath-\chi)^{\mathrm{k}\rho-1}}{\Gamma((\mathrm{k}-1)\rho+\rho)}\right]d\chi\\
&\quad+\int_{-\imath}^{\varsigma-\mathrm{k}\imath}\|\varpi\|^{\mathrm{k}}\frac{(\varsigma-\mathrm{k}\imath-\chi)^{(\mathrm{k}+1)\rho-1}}{\Gamma(\mathrm{k}\rho+\rho)}d\chi\\
&\leq \int_{-\imath}^{0}\left[\frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)}+\|\varpi\|\frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(\rho+\rho)}+\cdots+\|\varpi\|^{\mathrm{k}-1}\frac{(\varsigma-(\mathrm{k}-1)\imath-\chi)^{\mathrm{k}\rho-1}}{\Gamma((\mathrm{k}-1)\rho+\rho)}\right]d\chi\\
&\quad+\int_{-\imath}^{\varsigma-\mathrm{k}\imath}\|\varpi\|^{\mathrm{k}}\frac{(\varsigma-\mathrm{k}\imath-\chi)^{(\mathrm{k}+1)\rho-1}}{\Gamma(\mathrm{k}\rho+\rho)}d\chi\\
&\leq \int_{-\imath}^{\varsigma-\mathrm{k}\imath}\|\varpi\|^{\mathrm{k}}\frac{(\varsigma-\mathrm{k}\imath-\chi)^{(\mathrm{k}+1)\rho-1}}{\Gamma(\mathrm{k}\rho+\rho)}d\chi+\int_{-\imath}^{0}\sum_{q=1}^{\mathrm{k}}\frac{\|\varpi\|^{q-1}}{\Gamma(q\rho)}(\varsigma-(q-1)\imath-\chi)^{q\rho-1}d\chi\\
&\leq \|\varpi\|^{\mathrm{k}}\frac{(\varsigma-(\mathrm{k}-1)\imath)^{(\mathrm{k}+1)\rho}}{\Gamma((\mathrm{k}+1)\rho+1)}+\sum_{q=1}^{\mathrm{k}}\frac{\|\varpi\|^{q-1}}{\Gamma(q\rho+1)}\left[(\varsigma-(q-2)\imath)^{q\rho}-(\varsigma-(q-1)\imath)^{q\rho}\right].
\end{aligned}$$

This proof is end. □

Lemma 4. For $\varsigma\in((\mathrm{k}-1)\imath,\mathrm{k}\imath]$, $\mathrm{k}\in\Pi\setminus\{0\}$ and $1<\rho<2$, one acquires

$$\int_{0}^{\varsigma}\|P^{\imath}_{\rho}(\varsigma-\imath-\chi)\|d\chi\leq\sum_{q=0}^{\mathrm{k}}\frac{\|\varpi\|^{q}}{\Gamma((q+1)\rho+1)}(\varsigma-q\imath)^{(q+1)\rho}.$$

Proof. For $\varsigma \in ((\mathrm{k}-1)\imath, \mathrm{k}\imath]$, in the light of Definition 3, one has

$$\begin{aligned}
\int_0^{\varsigma} \|P_\rho^\imath(\varsigma-\imath-\chi)\| d\chi \le & \int_0^{\varsigma-\mathrm{k}\imath} \left[\frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)} + \|\varpi\| \frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(\rho+\rho)} + \cdots \|\varpi\|^{\mathrm{k}-1} \frac{(\varsigma-(\mathrm{k}-1)\imath-\chi)^{\mathrm{k}\rho-1}}{\Gamma((\mathrm{k}-1)\rho+\rho)}\right. \\
& \left. + \|\varpi\|^{\mathrm{k}} \frac{(\varsigma-\mathrm{k}\imath-\chi)^{(\mathrm{k}+1)\rho-1}}{\Gamma(\mathrm{k}\rho+\rho)}\right] d\chi + \int_{\varsigma-\mathrm{k}\imath}^{\varsigma-(\mathrm{k}-1)\imath} \left[\frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)} + \|\varpi\| \frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(\rho+\rho)} + \cdots\right. \\
& \left. + \|\varpi\|^{\mathrm{k}-1} \frac{(\varsigma-(\mathrm{k}-1)\imath-\chi)^{\mathrm{k}\rho-1}}{\Gamma((\mathrm{k}-1)\rho+\rho)}\right] d\chi + \cdots + \int_{\varsigma-2\imath}^{\varsigma-\imath} \left[\frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)} + \|\varpi\| \frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(\rho+\rho)}\right] d\chi \\
& + \int_{\varsigma-\imath}^{\varsigma} \frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)} d\chi \\
& \le \int_0^{\varsigma} \frac{(\varsigma-\chi)^{\rho-1}}{\Gamma(\rho)} d\chi + \int_0^{\varsigma-\imath} \|\varpi\| \frac{(\varsigma-\imath-\chi)^{2\rho-1}}{\Gamma(2\rho)} d\chi + \cdots + \int_0^{\varsigma-\mathrm{k}\imath} \|\varpi\|^{\mathrm{k}} \frac{(\varsigma-\mathrm{k}\imath-\chi)^{(\mathrm{k}+1)\rho-1}}{\Gamma((\mathrm{k}+1)\rho)} d\chi \\
& \le \sum_{q=0}^{\mathrm{k}} \frac{\|\varpi\|^q}{\Gamma((q+1)\rho)} \int_0^{\varsigma-q\imath} (\varsigma-q\imath-\chi)^{(q+1)\rho-1} d\chi \le \sum_{q=0}^{\mathrm{k}} \frac{\|\varpi\|^q}{\Gamma((q+1)\rho+1)} (\varsigma-q\imath)^{(q+1)\rho}.
\end{aligned}$$

This proof is end. □

Before presenting the finite time stability results, we put forward the following assumptions:
$[\Omega_1]$ ${}^R\mathcal{D}^\rho_{-\imath^+}\varphi \in C((-\imath,0], \mathbb{R}^n)$ and $M = \sup_{-\imath<\chi\le 0} \|({}^R\mathcal{D}^\rho_{-\imath^+}\varphi)(\chi)\| < \infty$.
$[\Omega_2]$ Assume that $f \in C([0,\mathcal{T}], \mathbb{R}^n)$ and $\|f\|_C = \max_{0\le\varsigma\le\mathcal{T}} \|f(\varsigma)\| < \infty$.

For convenience of representation, we define
$\psi_1(\varsigma) = (\varsigma+\imath)^{\rho+\mu-1} E_{\rho,\rho}(\|\varpi\|(\varsigma+\imath)^\rho)$,
$\psi_2(\varsigma) = \varsigma^{\rho+\mu-2} E_{\rho,\rho-1}(\|\varpi\|\varsigma^\rho)$,
$\psi_3(\varsigma) = \|\varpi\|^{\mathrm{k}} \frac{(\varsigma-(\mathrm{k}-1)\imath)^{(\mathrm{k}+1)\rho}}{\Gamma((\mathrm{k}+1)\rho+1)} + \sum_{q=1}^{\mathrm{k}} \frac{\|\varpi\|^{q-1}}{\Gamma(q\rho+1)} \left[(\varsigma-(q-2)\imath)^{q\rho} - (\varsigma-(q-1)\imath)^{q\rho}\right]$,
$\psi_4(\varsigma) = \sum_{q=0}^{k} \frac{\|B\|^q}{\Gamma((q+1)\rho+1)} (\varsigma-q\imath)^{(q+1)\rho}$.

Theorem 4. Assume that $\mathrm{k} \in \Pi \setminus \{0\}$, $1<\rho<2$, $\imath>0$, $\mu\ge 0$, $[\Omega_1]$ and $[\Omega_2]$ hold. Equation (1) is finite time stable with regard to $\{0, [(\mathrm{k}-1)\imath, \mathrm{k}\imath], \imath, \delta, \eta\}$ as long as

$$\sup_{(\mathrm{k}-1)\imath\le\varsigma\le\mathrm{k}\imath} \left\{\delta\left[\psi_1(\varsigma)+\psi_2(\varsigma)\right] + (\varsigma-(\mathrm{k}-1)\imath)^\mu \left[M\psi_3(\varsigma) + \|f\|_C \psi_4(\varsigma)\right]\right\} < \eta. \tag{6}$$

Proof. By using the Lemmas 2, 3 and 4, one obtains

$$\begin{aligned}
\|Y\|_{Z_\mu} &= \sup_{(\mathtt{k}-1)\iota\le\varsigma\le\mathtt{k}\iota} \|(\varsigma-(\mathtt{k}-1)\iota)^\mu Y(\varsigma)\| \\
&\le \sup_{(\mathtt{k}-1)\iota\le\varsigma\le\mathtt{k}\iota} \Big\{ \|(\varsigma-(\mathtt{k}-1)\iota)^\mu P_\rho^\iota(\varsigma)\mathtt{b}\| + \|(\varsigma-(\mathtt{k}-1)\iota)^\mu H_\rho^\iota(\varsigma)\mathtt{a}\| \\
&\quad + \int_{-\iota}^{0} \|(\varsigma-(\mathtt{k}-1)\iota)^\mu P_\rho^\iota(\varsigma-\iota-\chi)\|\|({}^R\mathcal{D}^\rho_{-\iota^+}\varphi)(\chi)\|d\chi \\
&\quad + \int_{0}^{\varsigma} \|(\varsigma-(\mathtt{k}-1)\iota)^\mu P_\rho^\iota(\varsigma-\iota-\chi)\|\|f(\chi)\|d\chi \Big\} \\
&\le \sup_{(\mathtt{k}-1)\iota\le\varsigma\le\mathtt{k}\iota} \Big\{ \delta\|(\varsigma-(\mathtt{k}-1)\iota)^\mu P_\rho^\iota(\varsigma)\| + \delta\|(\varsigma-(\mathtt{k}-1)\iota)^\mu H_\rho^\iota(\varsigma)\| \\
&\quad + (\varsigma-(\mathtt{k}-1)\iota)^\mu M \int_{-\iota}^{0} \|P_\rho^\iota(\varsigma-\iota-\chi)\|d\chi \\
&\quad + (\varsigma-(\mathtt{k}-1)\iota)^\mu \|f\|_C \int_{0}^{\varsigma} \|P_\rho^\iota(\varsigma-\iota-\chi)\|d\chi \Big\} \\
&\le \sup_{(\mathtt{k}-1)\iota\le\varsigma\le\mathtt{k}\iota} \Big\{ \delta\left[\psi_1(\varsigma)+\psi_2(\varsigma)\right] + (\varsigma-(\mathtt{k}-1)\iota)^\mu \left[M\psi_3(\varsigma)+\|f\|_C\psi_4(\varsigma)\right] \Big\} < \eta.
\end{aligned}$$

This proof is end. □

5 | INSTANCE

In this section, an instance is given to verify the rationality of the theoretical results.

Example 5.1. Assume that $\rho = 1.8$, $\iota = 0.15$, $\mathtt{k} = 4$, $\mu = 0.6$ and $\mathcal{T} = 0.6$. Consider

$$\begin{cases} {}^R\mathcal{D}^{1.8}_{-0.15^+}Y(\varsigma) = \varpi Y(\varsigma-0.15) + f(\varsigma), & 0<\varsigma\le 0.6, \\ \varphi(\varsigma) = ((\varsigma+0.15)^2, (\frac{(\varsigma+0.15)^3}{2}))^\top, & -0.15<\varsigma\le 0, \\ \mathcal{J}^{0.2}_{-0.15^+}Y(-0.15^+) = \mathtt{a} = 0, \\ {}^R\mathcal{D}^{0.8}_{-0.15^+}Y(-0.15^+) = \mathtt{b} = 0, \end{cases} \tag{7}$$

where $Y(\varsigma) = (Y_1(\varsigma), Y_2(\varsigma))^\top$, $\varpi = \begin{pmatrix} 0.45 & 0 \\ 0 & 0.5 \end{pmatrix}$, $f(\varsigma) = (\frac{\varsigma^2}{2}, \varsigma^3)^\top$.

Based on the Theorem 3 and $\varsigma \in (0, 0.6]$, the solution of system (7) shows the following form

$$Y(\varsigma) = P^{0.15}_{1.8}(\varsigma)\mathtt{b} + H^{0.15}_{1.8}(\varsigma)\mathtt{a} + \int_{-0.15}^{0} P^{0.15}_{1.8}(\varsigma-0.15-\chi)({}^R\mathcal{D}^{1.8}_{-0.15^+}\varphi)(\chi)d\chi + \int_{0}^{\varsigma} P^{0.15}_{1.8}(\varsigma-0.15-\chi)f(\chi)d\chi,$$

where

$$\int_{-0.15}^{0} P^{0.15}_{1.8}(\varsigma-0.15-\chi)({}^R\mathcal{D}^{1.8}_{-0.15^+}\varphi)(\chi)d\chi = \int_{-0.15}^{0} P^{0.15}_{1.8}(\varsigma-0.15-\chi)\begin{pmatrix} \frac{2.64}{\Gamma(0.2)}(\varsigma+0.15)^{0.2}B[3,0.2] \\ \frac{3.52}{\Gamma(0.2)}(\varsigma+0.15)^{1.2}B[4,0.2] \end{pmatrix} d\chi.$$

By elementary calculations, one shows $M = \sup_{-0.15<\chi\le 0}\|({}^R\mathcal{D}^{1.8}_{-0.15^+}\varphi)(\chi)\| = 5.5891$, $\|\varphi\| = 0.2267$, $\|\varpi\| = 0.5$ and $\|f\|_C = 0.396$. Furthermore, $\sup_{0.45\le\varsigma\le 0.6}\psi_1(\varsigma) \le \psi_1(0.6) = 0.7727$, $\sup_{0.45\le\varsigma\le 0.6}\psi_2(\varsigma) \le \psi_2(0.6) = 0.8171$, $\sup_{0.45\le\varsigma\le 0.6}\psi_3(\varsigma) \le \psi_3(0.6) = 0.1214$ and $\sup_{0.45\le\varsigma\le 0.6}\psi_4(\varsigma) \le \psi_4(0.6) = 0.2399$.

Assume that $\delta = 0.23$ and $\eta = 0.62$, the system (7) is finite time stable on $[0, 0.6]$. In the light of Figure 1, we can make a decision that Theorem 4 is valid and reasonable, thanks to $\|Y(0.6)\|_{Z_\mu} = 0.17794 < \eta = 0.62$. That is to say, in a fixed time interval $[0, 0.6]$, the state function $Y(\varsigma)$ will not exceed this threshold $\eta = 0.62$.

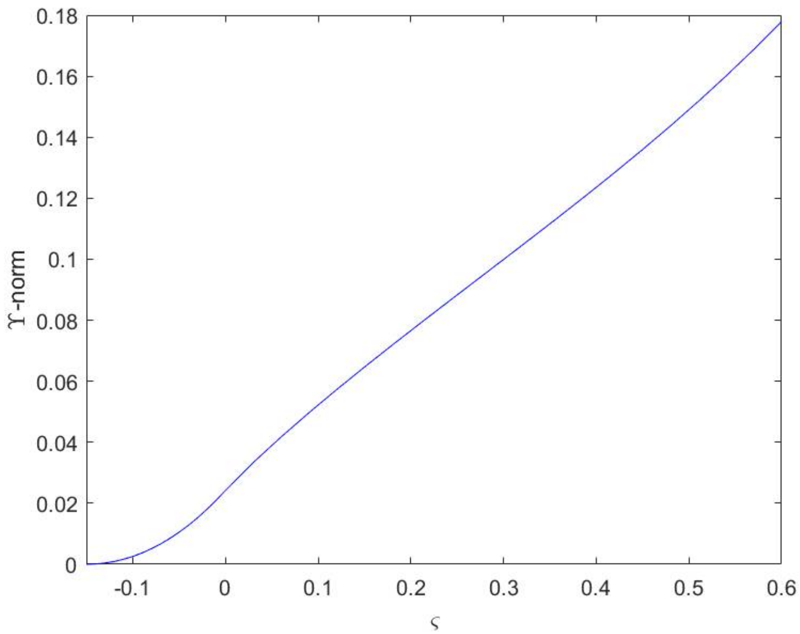


Figure 1 The norm of the state vector of system (7) with $\rho = 1.8$, $\imath = 0.15$ and $\mathcal{T} = 0.6$.

6 | CONCLUSION

In this paper, we derived the exact solutions of Riemann-Liouville type linear nonhomogengous fractional differential oscillating syetems with order $\rho \in (1, 2)$ through the two newly defined delayed perturbations of Mittag-Leffler matrix functions and constant variation method. Ultimately, in the light of the exact solutions, we further study the finite time stability. Due to the good properties of our solution, we can continue to study other stability and controllability problems in the subsequent research. Through our method, the public problems left by Mahmudov in 2022 were partially solved.

ACKNOWLEDGEMENTS

This work was supported by the National Natural Science Foundation of China [grant numbers 11871064, 11571300].

References

[1] Diethelm K. The Analysis of Fractional Differential Equations. Springer. 2010.

[2] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies. 2006;204.

[3] Fan Z, Li G. Existence results for semilinear differential equations with nonlocal and impulsive conditions. J. Funct. Anal. 2010;258:1709-1727.

[4] Ntouyas SK, Alsaedi A, Ahmad B. Existence theorems for mixed Riemann-Liouville and Caputo fractional differential equations and inclusions with nonlocal fractional integro-differential boundary conditions. Fractal Fract. 2019;3(2):21.

[5] Li M, Wang J. Existence results and Ulam type stability for conformable fractional oscillating system with pure delay. Chaos Solitons Fractals. 2022;161:112317.

[6] Bhat SP, Bernstein DS. Finite-time stability of continuous autonomous systems. SIAM J. Control Optim. 2000;38(3):751-766.

[7] An TV, Vu H, Hoa NV. Finite-time stability of fractional delay differential equations involving the generalized Caputo fractional derivative with non-instantaneous impulses. Math. Meth. Appl. Sci. 2022;45:4938-4955.

[8] Lazarević MP, Spasić AM. Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach. Math. Comput. Model. 2009;49:475-481.

[9] Khusainov DY, Shuklin GV. Linear autonomous time-delay system with permutation matrices solving. Stud. Univ. Žilina Math. Ser. 2003;17:101-108.

[10] Li M, Wang J. Finite time stability of fractional delay differential equations. Appl. Math. Lett. 2017;64:170-176.

[11] Li M, Wang J. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. Appl. Math. Comput. 2018;324:254-265.

[12] Liu L, Dong Q, Li G. Exact solutions and Hyers-Ulam stability for fractional oscillation equations with pure delay. Appl. Math. Lett. 2021;112:106666.

[13] Elshenhab AM, Wang X. Representation of solutions for linear fractional systems with pure delay and multiple delays. Math. Meth. Appl. Sci. 2021;44(17):12835-12850.

[14] Elshenhab AM, Wang X, Cesarano C, Almarri B, Moaaz O. Finite-time stability analysis of fractional delay systems. Mathematics. 2022;10:1883.

[15] Li M, Wang J. Representation of solution of a Riemann-Liouville fractional differential equation with pure delay. Appl. Math. Lett. 2018;85:118-124.

[16] Li M, Wang J. Finite time stability and relative controllability of Riemann-Liouville fractional delay differential equations. Math. Meth. Appl. Sci. 2019;42:6607-6623.

[17] Mahmudov NI. Multi-delayed perturbation of Mittag-Leffler type matrix functions. J. Math. Anal. Appl. 2022;505:125589.

[18] Mahmudov NI. Delayed perturbation of Mittag-Leffler functions and their applications to fractional linear delay differential equations. Math. Meth. Appl. Sci. 2018;42(16):5489-5497.

[19] Aydin M, Mahmudov NI. On a study for the neutral Caputo fractional multi-delayed differential equations with noncommutative coefficient matrices. Chaos Solitons Fractals. 2022;161:112372.