STABILITY OF THE 3D BOUSSINESQ EQUATIONS WITH PARTIAL DISSIPATION NEAR THE HYDROSTATIC BALANCE

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Abstract

The Boussinesq equations with partial or fractional dissipation not only naturally generalize the classical Boussinesq equations, but also are physically relevant and mathematically important. Unfortunately, it is not often well understood for many ranges of fractional powers. This paper focuses on a system of the 3D Boussinesq equations with fractional horizontal (- [?] h) α u and (- [?] h) β ϑ dissipation and proves that if an initial data (u 0 , ϑ 0) in the Sobolev space H 3 (R 3) close enough to the hydrostatic balance state, respectively, the equations with α , β [?] (1 2 , 1] then always lead to a steady solution.

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LIYA JIANG[1], YOUHUA WEI[1,*], KAIGE YANG[1]

Abstract. The Boussinesq equations with partial or fractional dissipation not only naturally generalize the classical Boussinesq equations, but also are physically relevant and mathematically important. Unfortunately, it is not often well understood for many ranges of fractional powers. This paper focuses on a system of the 3D Boussinesq equations with fractional horizontal $(-\Delta_h)^{\alpha}u$ and $(-\Delta_h)^{\beta}\theta$ dissipation and proves that if an initial data $(u_0,\ \theta_0)$ in the Sobolev space $H^3(\mathbb{R}^3)$ close enough to the hydrostatic balance state, respectively, the equations with $\alpha, \beta \in (\frac{1}{2}, 1]$ then always lead to a steady solution.

1. Introduction

The system of the 3D Boussinesq equations is one of the most important models for geophysical fluids such as atmospheric fronts and oceanic currents as well as fluids in our daily life such as the Rayleigh-Bénard convection. It arises from the density-dependent fluid equations by using the so-called Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity (see, e.g., [8, 12, 14, 20, 22]).

For the 3D incompressible generalized Boussinesq system with fractional dissipation and diffusion,

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nu\,(-\Delta)^{\alpha}u - \nabla P, \quad x\in\mathbb{R}^3,\ t>0, \\ \partial_t\Theta + u\cdot\nabla\Theta = -\kappa\,(-\Delta)^{\beta}\Theta, \quad x\in\mathbb{R}^3,\ t>0, \\ \nabla\cdot u = 0, \quad x\in\mathbb{R}^3,\ t>0, \end{cases} \tag{1.1}$$

where $\alpha, \beta > 0$ is real parameter, and $u = u(x,t)\in\mathbb{R}^3$ is the velocity, $\theta = \theta(x,t)\in\mathbb{R}^3$ is the temperature and $P = P(x,t)\in\mathbb{R}$ is the scalar pressure, $\nu > 0$ denotes the kinematic viscosity and $\kappa > 0$ denotes the thermal diffusivity. For notational convenience, we write ∂_i for the partial derivatives $\partial_{x_i}(i{=}1{,}2{,}3)$. The fractional Laplacian operator $(-\Delta)^{\alpha}$ is defined via the Fourier transform,

$$\widehat{(-\Delta)^{\alpha}f}(\xi) = |\xi|^{2\alpha}\widehat{f}(\xi)$$

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for

$$\widehat{f}(\xi)=\frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d}e^{-ix\cdot\xi}f(x)dx.$$

The Boussinesq equations given by (1.1) have recently attracted considerable interests, due to their mathematical importance and physical applications. In physics, Hydrostatic balance is an important equilibrium of many geophysical fluids. In fact, our atmosphere is mainly in hydrostatic balance, between the upward-directed pressure gradient force and the downward-directed force of gravity. Understanding the stability of perturbations near the hydrostatic equilibrium may help gain insight into certain severe weather phenomena (see, e.g., [18, 22]). In mathematics, The partial differential equation system concerned here models fractional dissipation and involve only partial dissipation.

When $\alpha=\beta=1$, (1.1) reduces to the standard 3D incompressible Boussinesq equations, the issue of whether (1.1) has a unique global-in-time solution is an outstanding open problem. In two dimensions, there have been a great deal of researches on the global regularity issue concerning the 2D Boussinesq equations with fractional Laplacian dissipation or with partial dissipation (see, e.g. [1, 2, 5–7, 9–11, 15–17]). In three dimensions, if $\alpha\geq\frac{5}{4},\ \beta=1$ in (1.1), the hyperdissipative Boussinesq equations always possess a unique global solution [26]; if $\alpha\geq\frac{5}{4},\ \kappa=0$ in (1.1), the global existence and regularity result actually holds (see, e.g. [19, 23, 25, 28]). Nevertheless, the stability of (1.1) remains unknown.

The hydrostatic balance given by

$$u^{(0)}\equiv(0,0,0),\quad \Theta^{(0)}=x_3,\quad P^{(0)}=\frac{1}{2}x_3^2 \tag{1.2}$$

is a very special steady-state solution of (1.1) with great geophysical and astrophysical importance. To understand the stability of perturbations near the hydrostatic balance in (1.2), we consider the equations governing the perturbation (u,θ,p) with $\theta=\Theta-\Theta^{(0)},\ p=P-P^{(0)}$,

$$\begin{cases}\partial_t u+u\cdot\nabla u=-\nu\,(-\Delta)^{\alpha}u-\nabla P, & x\in\mathbb{R}^3,\ t>0,\\ \partial_t\Theta+u\cdot\nabla\Theta=-\kappa\,(-\Delta)^{\beta}\Theta, & x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot u=0, \quad x\in\mathbb{R}^3,\ t>0.\end{cases} \tag{1.3}$$

In this paper, we concern (1.3) with only horizontal fractional dissipation

$$\begin{cases}\partial_t u+u\cdot\nabla u=-\nu\,(-\Delta_h)^{\alpha}u-\nabla P+\theta e_3, & x\in\mathbb{R}^3,\ t>0,\\ \partial_t b+u\cdot\nabla\theta=-\kappa\,(-\Delta_h)^{\beta}\theta-u_3, & x\in\mathbb{R}^3,\ t>0,\\ \nabla\cdot u=0, \quad x\in\mathbb{R}^3,\ t>0,\\ (u,\theta)|_{t=0}=(u_0,\theta_0).\end{cases} \tag{1.4}$$

with $\alpha,\beta\in(\frac{1}{2},1]$. The concept of horizontal dissipation comes from geophysical fluid dynamics(see [22]), meteorologists modelize the turbulent diffusion with anisotropic viscosity $-\nu_h\Delta_h-\nu_3\partial_3^2$, where the horizontal kinetic viscosity coefficient ν_h and the vertical kinetic viscosity coefficient ν_3 are empiric constant and satisfy $0<\nu_3\ll\nu_h$. In this paper, we take the limit case $\nu_h=\nu$ and $\nu_3=0$.

A natural consideration is how the parameters α and β are determined, and this is what we choose to do in our tentative estimation work, based primarily on energy estimate method. When we bound the $\dot{H}^3$-norm of (u,θ), we plan to utilize a series of anisotropic inequalities derived from a Sobolev embedding inequality and the Gagliardo-Nirenberg (G-N) interpolation inequality [4, 13, 21]. Based on the relationship between the parameters of these inequalities, we ended up choosing $\alpha,\beta \in (\frac{1}{2},1]$ in (1.4) to study the stability of the Boussinesq equations with fractional horizontal dissipation.

To construct steady solution of (1.4), we make use of the bootstrap argument by anisotropic energy

$$\begin{aligned}E(t) = \sup_{0\le\tau\le t}&\{\|u(\tau)\|_{H^3}^2+\|\theta(\tau)\|_{H^3}^2\}\\ &+2\nu\int_0^t \|\Lambda_h^\alpha u(\tau)\|_{H^3}^2\,d\tau + 2\eta\int_0^t \|\Lambda_h^\beta \theta(\tau)\|_{H^3}^2\,d\tau.\end{aligned} \tag{1.5}$$

Here $\Lambda_h = (-\Delta_h)^{\frac{1}{2}}$ denote the zygmund operator. Our precise result is stated in the following theorem.

Theorem 1.1. *Consider (1.4) with initial data $(u_0,\theta_0) \in H^3(\mathbb{R}^3)$ satisfies $\nabla\cdot u_0$ and $\alpha,\beta \in (\frac{1}{2},1]$. Then there exists a constant $\delta = \delta(\nu,\kappa) > 0$ such that, if*

$$\|(u_0,\theta_0)\|_{H^3} \le \delta, \tag{1.6}$$

then (1.4) has a unique global classical solution satisfying,

$$\sup_{0\le\tau\le t}(\|u(\tau)\|_{H^3}^2+\|\theta(\tau)\|_{H^3}^2) + 2\nu\int_0^t \|\Lambda_h^\alpha u(\tau)\|_{H^3}^2\,d\tau + 2\eta\int_0^t \|\Lambda_h^\beta \theta(\tau)\|_{H^3}^2\,d\tau \le C\delta^2,$$

for any $t>0$ and $C = C(\nu,\kappa)$ is a constant.

A natural starting point is to bound $\|u(t)\|_{H^3}+\|\theta(t)\|_{H^3}$ via energy estimate. We are able to derive the following energy inequality

$$E(t) \le E(0) + CE(t)^{\frac{3}{2}}. \tag{1.7}$$

Combined with the bootstrapping argument (see [24]), we can prove Theorem 1.1. However, the proof of Theorem 1.1 is not superficial, due to the lack of the vertical dissipation and vertical magnetic diffusion, some nonlinear terms are not easy to be controlled in terms of $\|u(t)\|_{H^3}+\|\theta(t)\|_{H^3}$ or the dissipation parts $\|\Lambda_h^\alpha u\|_{H^3}$ and $\|\Lambda_h^\beta \theta\|_{H^3}$. One of the most difficult terms is

$$\begin{aligned}&-\int_{\mathbb{R}^3} \partial_3 u_h\cdot\nabla_h\partial_3^2\theta\cdot\partial_3^3\theta\,dx\\ &\lesssim \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla_h\partial_3^2\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\nabla_h\partial_3^2\Lambda_3^\alpha\theta\|_{L^2}^{\frac{1}{2\beta}}.\end{aligned}$$

Clearly, it does not appear possible to bound the subterms

$$\|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}} \tag{1.8}$$

directly in terms of $\|\Lambda_h^\alpha u\|_{H^3}^{1-\frac{1}{2\alpha}}\|\Lambda_h^\beta\theta\|_{H^3}^{1-\frac{1}{2\beta}}$, but in terms of $\|u(t)\|_{H^3}^{1-\frac{1}{2\alpha}}\|\theta(t)\|_{H^3}^{1-\frac{1}{2\beta}}$. Therefore, we hope the sum of the corresponding exponents of the two subterms to be less than or equal to 1 for all given α and β, which is

$$1-\frac{1}{2\alpha}+1-\frac{1}{2\beta}\leq 1. \tag{1.9}$$

To establish the inequality of (1.7), we choose $\alpha,\beta\in(\frac{1}{2},1]$. In the case of

$$1-\frac{1}{2\alpha}+1-\frac{1}{2\beta}=1,$$

the subterms of (1.8) can be estimated directly by

$$\|u(t)\|_{H^3}+\|\theta(t)\|_{H^3}.$$

For the case

$$1-\frac{1}{2\alpha}+1-\frac{1}{2\beta}<1,$$

with the exponent is not enough to 1. Our strategy is to extract part from the rest subterms

$$\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla_h\partial_3^2\theta\|_{L^2}^{1-\frac{1}{2\beta}}$$

to fill the subterms of (1.8) by G-N interpolation inequality. One reason which can not be ignored is that $\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}$ could bounded by either $\|u\|_{H^3}$ or $\|\Lambda_h^\alpha u\|_{H^3}$ and $\|\nabla_h\partial_3^2\theta\|_{L^2}$ could bounded by either $\|\theta\|_{H^3}$ or $\|\Lambda_h^\alpha\theta\|_{H^3}$. In the last section of our paper, we have successfully used this method to solve all similar difficulties in proving stability and obtain inequality (1.7).

Lemma 1.2. *Assume that $\alpha,\beta,\gamma\in(\frac{1}{2},1]$, $f,g,h,\Lambda_h^\alpha f,\Lambda_h^\beta g,\Lambda_h^\gamma h$ and $\partial_3 h$ are all in $L^2(\mathbb{R}^3)$. Then,*

$$\int_{\mathbb{R}^3}|fgh|\,dx\lesssim\|f\|_{L^2}^{1-\frac{1}{2\alpha}}\|\Lambda_h^\alpha f\|_{L^2}^{\frac{1}{2\alpha}}\|g\|_{L^2}^{1-\frac{1}{2\beta}}\|\Lambda_h^\beta g\|_{L^2}^{\frac{1}{2\beta}}\|h\|_{L^2}^{\frac{1}{2}}\|\partial_3 h\|_{L^2}^{\frac{1}{2}},$$

$$\int_{\mathbb{R}^3}|fgh|\,dx\lesssim\|f\|_{L^2}^{1-\frac{1}{2\alpha}}\|\Lambda_h^\alpha f\|_{L^2}^{\frac{1}{2\alpha}}\|g\|_{L^2}^{1-\frac{1}{2\beta}}\|\Lambda_h^\beta g\|_{L^2}^{\frac{1}{2\beta}}\|h\|_{L^2}^{1-\frac{1}{2\gamma}}\|\Lambda_3^\gamma h\|_{L^2}^{\frac{1}{2\gamma}}.$$

Here we write $A\lesssim B$ to mean that $A\leq CB$ for some constant C and $\Lambda_3=(-\partial_{33})^{\frac{1}{2}}$.

The rest of this paper is divided into two sections. Section 2 provides the proofs of Theorem 1.1 and Lemma 1.2. Section 3 derives the energy inequality (1.7).

2. Proof of Theorem

This section proves Theorem 1.1 and Lemma 1.2.

2.1. **Proof of Theorem 1.1.**

Roughly speaking, the bootstrap argument starts with an ansatz that $E(t)$ is bounded, say

$$E(t)\leq M$$

and shows that $E(t)$ actually admits a smaller bound, say

$$E(t)\leq\frac{1}{2}M$$

when the initial condition is sufficiently small. A rigorous statement of the abstract bootstrap principle can be found in T. Tao's book [24].

It follows that

$$E(t) \le E(0) + CE(t)^{\frac{3}{2}}, \tag{2.1}$$

for some pure constants C. To initiate the bootstrapping argument, we make the ansatz

$$E(t) \le M := \frac{1}{4C^2}. \tag{2.2}$$

We then show that (2.1) allows us to conclude that $E(t)$ actually admits an even smaller bound by taking the initial H^3-norm $E(0)$ sufficiently small. In fact, when (2.2) holds, (2.1) implies

$$E(t) \le E(0) + \frac{1}{2}E(t)$$

or

$$E(t) \le 2E(0). \tag{2.3}$$

Therefore, if we choose $\delta > 0$ sufficiently small such that

$$\delta^2 \le \frac{1}{4}M, \tag{2.4}$$

then

$$E(t) \le \frac{1}{2}M. \tag{2.5}$$

$E(t)$ actually admits a smaller bound in (2.5) than the one in the ansatz (2.2). The bootstrapping argument then assesses that (2.2) holds for all time when $E(0)$ obeys (2.4). This completes the proof. $\square$

2.2. **Proof of Lemma 1.2.**

The proof makes use of the following version of Minkowski's inequality,

$$\left\| \|f\|_{L^q_y(\mathbb{R}^n)} \right\|_{L^p_x(\mathbb{R}^m)} \le \left\| \|f\|_{L^p_x(\mathbb{R}^m)} \right\|_{L^q_y(\mathbb{R}^n)},$$

for any $1 \le q \le p \le \infty$, where $f = f(x, y)$ with $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ is a measurable function on $\mathbb{R}^m \times \mathbb{R}^n$ and the following basic one-dimensional Sobolev embedding inequality [27], for $f \in H^s(\mathbb{R})$,

$$\|f\|_{L^\infty(\mathbb{R})} \le C\|f\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R})} \|\Lambda^s f\|^{\frac{1}{2s}}_{L^2(\mathbb{R})},$$

where $s > \frac{1}{2}$. By the above inequality and Hölder's inequality,

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|\, dx \le & \|f\|_{L^\infty_{x_1} L^2_{x_2} L^2_{x_3}} \|g\|_{L^2_{x_1} L^\infty_{x_2} L^2_{x_3}} \|h\|_{L^2_{x_1} L^2_{x_2} L^\infty_{x_3}} \\ \le & C \left\| \|f\|_{L^2_{x_1}}^{1-\frac{1}{2\alpha}} \|\Lambda_1^\alpha f\|_{L^2_{x_1}}^{\frac{1}{2\alpha}} \right\|_{L^2_{x_2 x_3}} \left\| \|g\|_{L^2_{x_2}}^{1-\frac{1}{2\beta}} \|\Lambda_2^\beta g\|_{L^2_{x_2}}^{\frac{1}{2\beta}} \right\|_{L^2_{x_1 x_3}} \\ & \times \left\| \|h\|_{L^2_{x_3}}^{1-\frac{1}{2\gamma}} \|\Lambda_3^\gamma h\|_{L^2_{x_3}}^{\frac{1}{2\gamma}} \right\|_{L^2_{x_1 x_2}} \\ \le & C \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_1^\alpha f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_2^\beta g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{1-\frac{1}{2\gamma}} \|\Lambda_3^\gamma h\|_{L^2}^{\frac{1}{2\gamma}}.\end{aligned}$$

Let $\gamma = 1$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} |fgh|\, dx \lesssim & \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_1^\alpha f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_2^\beta g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}} \\ \lesssim & \|f\|_{L^2}^{1-\frac{1}{2\alpha}} \|\Lambda_h^\alpha f\|_{L^2}^{\frac{1}{2\alpha}} \|g\|_{L^2}^{1-\frac{1}{2\beta}} \|\Lambda_h^\beta g\|_{L^2}^{\frac{1}{2\beta}} \|h\|_{L^2}^{\frac{1}{2}} \|\partial_3 h\|_{L^2}^{\frac{1}{2}}.\end{aligned}$$

Here $\|f\|_{L^\infty_{x_1} L^2_{x_2} L^2_{x_3}}$ represents the $L^\infty_{x_1}$-norm in the x_1-variable, followed by the $L^2_{x_2}$-norm in x_2 and the $L^2_{x_3}$-norm in x_3. This finishes the proof of Lemma 1.2. □

3. The H^3-stability

Due to the equivalence of $\|(u,\theta)\|_{H^3}$ with $\|(u,\theta)\|_{L^2} + \|(u,\theta)\|_{\dot{H}^3}$, it suffices to bound the L^2-norm and the $\dot{H}^3$-norm of (u,θ). By a simple energy estimate and $\nabla \cdot u = 0$, we find that the L^2-norm of (u,θ) obeys

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2 + 2\nu \int_0^t \|\Lambda_h^\alpha u(\tau)\|_{L^2}^2\, d\tau \\ &+ 2\eta \int_0^t \|\Lambda_h^\beta \theta(\tau)\|_{L^2}^2\, d\tau = \|u_0\|_{L^2}^2 + \|\theta_0\|_{L^2}^2. \end{aligned} \tag{3.1}$$

The rest of the proof focuses on the $\dot{H}^3$-norm. Applying ∂_i^3 to (1.4) and then dotting by $(\partial_i^3 u, \partial_i^3 \theta)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^3 (\|\partial_i^3 u\|_{L^2}^2 + \|\partial_i^3 \theta\|_{L^2}^2) + \nu \|\partial_i^3 \Lambda_h^\alpha u\|_{L^2}^2 + \eta \|\partial_i^3 \Lambda_h^\beta \theta\|_{L^2}^2 = I_1 + I_2 + I_3, \tag{3.2}$$

where

$$\begin{aligned} I_1 &= \sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 \theta e_3 \cdot \partial_3^3 u + \partial_i^3 u_3 \cdot \partial_3^3 \theta\, dx, \\ I_2 &= -\sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u\, dx, \\ I_3 &= -\sum_{i=1}^3 \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla \theta) \cdot \partial_i^3 \theta\, dx, . \end{aligned}$$

Note that, by integration by parts,

$$I_1 = 0.$$

To bound I_2, we decompose it into three pieces,

$$\begin{aligned}
I_2 =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3(u\cdot\nabla u)\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{3}(\int_{\mathbb{R}^3}\partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx+3\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx+3\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx)\\
=& I_{21}+3I_{22}+3I_{23},
\end{aligned}\tag{3.3}$$

where we have used the fact that $\int_{\mathbb{R}^3} u\cdot\nabla\partial_3^3 u\cdot\partial_3^3 u\,dx = 0$. I_{21} is naturally splitted into three parts,

$$\begin{aligned}
I_{21} =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx-\int_{\mathbb{R}^3}\partial_3^3 u_h\cdot\nabla_h u\cdot\partial_3^3 u\,dx-\int_{\mathbb{R}^3}\partial_3^3 u_3\partial_3 u\cdot\partial_3^3 u\\
=& I_{211}+I_{212}+I_{213}.
\end{aligned}$$

By Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{211}| =& |-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^3 u\cdot\nabla u\cdot\partial_i^3 u\,dx|\\
\lesssim& \sum_{i=1}^{2}\|\partial_i^3 u\|_{L^2}^{2-\frac{1}{\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim& \sum_{i=1}^{2}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(2-\frac{1}{\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha} u\|_{L^2}^{(1-\gamma)(2-\frac{1}{\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim& \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned}\tag{3.4}$$

Where we have applied inequality

$$\|\partial_i u\|_{L^2}\le\|\Lambda_h^\alpha u\|_{L^2}^{\gamma}\|\Lambda_h^{1+\alpha} u\|_{L^2}^{1-\gamma}\quad(i=1,2)\tag{3.5}$$

for $\alpha \in (\frac{1}{2}, 1]$ by interpolation inequality. We now turn to I_{212}, by Lemma 1.2,

$$\begin{aligned}
|I_{212}| =& |-\int_{\mathbb{R}^3} \partial_3^3 u_h \cdot \nabla_h u \cdot \partial_3^3 u \, dx| \\
\lesssim& \|\partial_3^3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \|\partial_3^3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
&\times \|\nabla_h u\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\Lambda_h^\alpha u\|_{L^2}^{\gamma(\frac{1}{1-2\alpha})} \|\Lambda_h^{1+\alpha} u\|_{L^2}^{(1-\gamma)(\frac{1}{1-2\alpha})} \\
&\times \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\Lambda_h^\alpha \partial_3 u\|_{L^2}^{\gamma(\frac{1}{1-2\alpha})} \|\Lambda_h^{1+\alpha} \partial_3 u\|_{L^2}^{(1-\gamma)(\frac{1}{1-2\alpha})} \\
\lesssim& \|u\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned} \tag{3.6}$$

In fact, we separate $\|\nabla_h u\|_{L^2}^{\frac{1}{2}} \|\nabla_h \partial_3 u\|_{L^2}^{\frac{1}{2}}$ into two parts and combine with (3.5) to reach our desired bound. Next, we consider the term I_{213},

$$\begin{aligned}
|I_{213}| =& |-\int_{\mathbb{R}^3} \partial_3^3 u_3 \partial_3 u \cdot \partial_3^3 u \, dx| \\
=& |\sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_3^2 \partial_j u_j \partial_3 u \cdot \partial_3^3 u \, dx| \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^2 \partial_j u_j\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_3^\alpha \partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^2 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_3^2 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\gamma\frac{1}{2\alpha}} \|\partial_3^2 \Lambda_h^{\alpha+1} u\|_{L^2}^{(1-\gamma)\frac{1}{2\alpha}} \\
&\times \|\partial_3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}-1} \|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}} \\
\lesssim& \|u\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3}^2,
\end{aligned} \tag{3.7}$$

where we used $\nabla \cdot u = 0$,

$$\|\partial_3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}-1} \leq \|u\|_{H^3}^{\frac{1}{\alpha}-1}$$

and

$$\|\partial_3^2 \Lambda_3^\alpha \partial_j u_j\|_{L^2} \leq \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\gamma} \|\partial_3^2 \Lambda_h^{\alpha+1} u\|_{L^2}^{1-\gamma} \quad (j = 1, 2; \ \gamma = \alpha). \tag{3.8}$$

As a matter of fact, (3.8) can be verified via G-N interpolation inequality and Plancherel theorem,

$$\begin{aligned}
\|\partial_3^2 \Lambda_3^\alpha \partial_j u_j\|_{L^2} &\leq \|\xi_3^{2+\alpha} |\xi_h| \hat{u}\|_{L^2} \\
&\leq \|\xi_3^3 |\xi_h|^\alpha \hat{u}\|_{L^2}^\alpha \|\xi_3^2 |\xi_h|^{\alpha+1} \hat{u}\|_{L^2}^{1-\alpha} \\
&= \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^\alpha \|\partial_3^2 \Lambda_h^{\alpha+1} u\|_{L^2}^{1-\alpha}.
\end{aligned}$$

To deal with I_{22}, we split it into three parts,

$$
\begin{aligned}
I_{22} =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx\,dx-\int_{\mathbb{R}^3}\partial_3^2 u_h\cdot\nabla_h\partial_3 u\cdot\partial_3^3 u\,dx-\int_{\mathbb{R}^3}\partial_3^2 u_3\partial_3^2 u\cdot\partial_3^3 u\,dx\\
=& I_{221}+I_{222}+I_{223}.
\end{aligned}
$$

Similarly to (3.4),

$$
\begin{aligned}
|I_{221}| =& |-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i u\cdot\partial_i^3 u\,dx|\\
\lesssim& \sum_{i=1}^{2}\|\partial_i^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla\partial_i u\|_{L^2}^{\frac12}\|\nabla\partial_i\partial_3 u\|_{L^2}^{\frac12}\\
\lesssim& \sum_{i=1}^{2}\|\partial_i\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla\partial_i u\|_{L^2}^{\frac12}\|\nabla\partial_i\partial_3 u\|_{L^2}^{\frac12}\\
\lesssim& \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned}\tag{3.9}
$$

Applying Lemma 1.2 and G-N interpolation inequality, we obtain

$$
\begin{aligned}
|I_{222}| =& |-\int_{\mathbb{R}^3}\partial_3^2 u_h\cdot\nabla_h\partial_3 u\cdot\partial_3^3 u\,dx|\\
\lesssim& \|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac12}\\
\lesssim& \|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\nabla_h\partial_3 u\|_{L^2}^{\frac{1}{2\alpha}-\frac12}\|\Lambda_h^\alpha\partial_3 u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\alpha}\partial_3 u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
&\times\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac{1}{2\alpha}-\frac12}\|\Lambda_h^\alpha\partial_3^2 u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\alpha}\partial_3^2 u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
\lesssim& \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned}\tag{3.10}
$$

Note that, we separate out part of $\|\nabla_h\partial_3 u\|_{L^2}^{\frac12}\|\nabla_h\partial_3^2 u\|_{L^2}^{\frac12}$ and make it controlled by $\|\Lambda_h^\alpha u\|_{H^3}$. Similarly,

$$
\begin{aligned}
|I_{223}| =& |-\int_{\mathbb{R}^3}\partial_3^2 u_3\partial_3^2 u\cdot\partial_3^3 u\,dx|\\
=& |\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3\partial_j u_j\partial_3^2 u\cdot\partial_3^3 u\,dx|\\
\lesssim& \sum_{j=1}^{2}\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3\partial_j u_j\|_{L^2}^{\frac12}\|\partial_3^2\partial_j u_j\|_{L^2}^{\frac12}
\end{aligned}
$$

$$
\begin{aligned}
\lesssim & \sum_{j=1}^{2}\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}}\|\partial_3\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_3\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
&\times\|\partial_3^2\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}}\|\partial_3^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_3^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\\
\lesssim & \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2. \qquad (3.11)
\end{aligned}
$$

We deal with I_{23} in the same method, I_{23} is naturally splitted into three parts,

$$
\begin{aligned}
I_{23} =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx\\
=& -\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx-\int_{\mathbb{R}^3}\partial_3 u_h\cdot\nabla_h\partial_3^2 u\cdot\partial_3^3 u\,dx-\int_{\mathbb{R}^3}\partial_3 u_3\partial_3^3 u\cdot\partial_3^3 u\,dx\\
=&I_{231}+I_{232}+I_{233}.
\end{aligned}
$$

By Lemma 1.2 and G-N interpolation inequality, we have

$$
\begin{aligned}
|I_{231}| =& |-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i u\cdot\nabla\partial_i^2 u\cdot\partial_i^3 u\,dx|\\
\lesssim & \sum_{i=1}^{2}\|\nabla\partial_i^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\nabla\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i u\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim & \sum_{i=1}^{2}\|\nabla\partial_i\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\nabla\partial_i\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\nabla\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i u\|_{L^2}^{\frac{1}{2}}\|\partial_i\partial_3 u\|_{L^2}^{\frac{1}{2}}\\
\lesssim & \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2. \qquad (3.12)
\end{aligned}
$$

We estimate I_{232} similarly as I_{213} which yields

$$
\begin{aligned}
|I_{232}| =& |-\int_{\mathbb{R}^3}\partial_3 u_h\cdot\nabla_h\partial_3^2 u\cdot\partial_3^3 u\,dx|\\
\lesssim & \|\nabla_h\partial_3^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\nabla_h\partial_3^2\Lambda_3^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
\lesssim & \|\Lambda_h^\alpha\partial_3^2 u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\alpha}\partial_3^2 u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\gamma\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^{\alpha+1}u\|_{L^2}^{(1-\gamma)\frac{1}{2\alpha}}\\
&\times\|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{\alpha}-1}\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\\
\lesssim & \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2. \qquad (3.13)
\end{aligned}
$$

We next consider the term I_{233}, utilizing the incompressible condition again, we have

$$\begin{aligned}
|I_{233}| =&| - \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3^3 u \cdot \partial_3^3 u \, dx| \\
=&| \sum_{j=1}^{2} \int_{\mathbb{R}^3} \partial_j u_j \partial_3^3 u \cdot \partial_3^3 u \, dx| \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^3 u\|_{L^2}^{2-\frac{1}{\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}} \|\partial_j u_j\|_{L^2}^{\frac{1}{2}} \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \sum_{j=1}^{2} \|\partial_3^3 u\|_{L^2}^{2-\frac{1}{\alpha}} \|\partial_3^3 \Lambda_h^\alpha u\|_{L^2}^{\frac{1}{\alpha}} \|\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \\
&\times \|\partial_3 \partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}} \|\partial_3 \Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})} \|\partial_3 \Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})} \\
\lesssim& \|u\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3}^2.
\end{aligned} \tag{3.14}$$

Combined with (3.3)-(3.14), we obtain

$$I_2(\tau) \lesssim \|u\|_{H^3} \|\Lambda_h^\alpha u\|_{H^3}^2.$$

Now, we try to bound I_3, we split it into three parts,

$$\begin{aligned}
I_3 =& - \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 (u \cdot \nabla \theta) \cdot \partial_i^3 \theta \, dx \\
=& - \sum_{i=1}^{3} (\int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla \theta \cdot \partial_i^3 \theta \, dx + 3 \int_{\mathbb{R}^3} \partial_i^2 u \cdot \nabla \partial_i \theta \cdot \partial_i^3 \theta \, dx + 3 \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2 \theta \cdot \partial_i^3 \theta \, dx) \\
=& I_{31} + 3 I_{32} + 3 I_{33}.
\end{aligned} \tag{3.15}$$

I_{31} can be divided directly into three parts,

$$\begin{aligned}
I_{31} =& - \sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla \theta \cdot \partial_i^3 \theta \, dx \\
=& - \sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i^3 u \cdot \nabla \theta \cdot \partial_i^3 \theta \, dx - \int_{\mathbb{R}^3} \partial_3^3 u_h \cdot \nabla_h \theta \cdot \partial_3^3 \theta \, dx - \int_{\mathbb{R}^3} \partial_3^3 u_3 \partial_3 \theta \cdot \partial_3^3 \theta \, dx \\
=& I_{311} + I_{312} + I_{313}.
\end{aligned}$$

By Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{311}| =& |-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^3 u\cdot\nabla\theta\cdot\partial_i^3\theta\,dx| \\
\lesssim& \sum_{i=1}^{2}\|\partial_i^3 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_i^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\partial_3\theta\|_{L^2}^{\frac12} \\
\lesssim& \sum_{i=1}^{2}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^\beta\theta\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\partial_i^2\Lambda_h^{1+\beta}\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \\
&\times\|\partial_i^3\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\partial_3\theta\|_{L^2}^{\frac12} \\
\lesssim&\|\theta\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta\theta\|_{H^3}.
\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}
|I_{312}| =& |-\int_{\mathbb{R}^3}\partial_3^3 u_h\cdot\nabla_h\theta\cdot\partial_3^3\theta\,dx| \\
\lesssim&\|\partial_3^3 u_h|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla_h\theta\|_{L^2}^{\frac12}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac12} \\
\lesssim&\|\partial_3^3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \\
&\times\|\Lambda_h^\beta\theta\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\Lambda_h^{1+\beta}\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2\beta}-\frac12} \\
&\times\|\Lambda_h^\beta\partial_3\theta\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\beta}\partial_3\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\nabla_h\theta\|_{L^2}^{\frac{1}{2\alpha}-\frac12} \\
\lesssim&\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|\theta\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta\theta\|_{H^3}^{2-\frac{1}{2\alpha}}.
\end{aligned}\tag{3.17}$$

By divergence-free condition $\nabla\cdot u=0$ and

$$\|\partial_3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{\beta}-1}\le\|\theta\|_{H^3}^{\frac{1}{\beta}-1},$$

we have

$$\begin{aligned}
|I_{313}| =& |-\int_{\mathbb{R}^3}\partial_3^3 u_3\partial_3\theta\cdot\partial_3^3\theta\,dx| \\
=& |\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3^2\partial_j u_j\partial_3\theta\cdot\partial_3^3\theta\,dx| \\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2\partial_j u_j\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_3^\alpha\partial_j u_j\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_3^2\Lambda_h^{1+\alpha}u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_3^3\Lambda_h^\alpha u\|_{L^2}^{\gamma\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^{\alpha+1}u\|_{L^2}^{(1-\gamma)\frac{1}{2\alpha}} \\
&\times\|\partial_3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{\beta}-1}\|\partial_3\Lambda_h^\beta\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\alpha\theta\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim&\|\theta\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta\theta\|_{H^3}.
\end{aligned}\tag{3.18}$$

Similarity, I_{32} can also be divided directly into three terms,

$$\begin{aligned}
I_{32} =& -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i\theta\cdot\partial_i^3\theta\,dx\\
=& -\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i\theta\cdot\partial_i^3\theta\,dx-\int_{\mathbb{R}^3}\partial_3^2 u_h\cdot\nabla_h\partial_3\theta\cdot\partial_3^3\theta\,dx-\int_{\mathbb{R}^3}\partial_3^2 u_3\partial_3^2\theta\cdot\partial_3^3\theta\,dx\\
=&I_{321}+I_{322}+I_{323}.
\end{aligned}$$

Then Lemma 1.2 and G-N interpolation inequality implies

$$\begin{aligned}
|I_{321}| =&|-\sum_{i=1}^{2}\int_{\mathbb{R}^3}\partial_i^2 u\cdot\nabla\partial_i\theta\cdot\partial_i^3\theta\,dx|\\
\lesssim&\sum_{i=1}^{2}\|\partial_i^2 u\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_i^2\Lambda_h^\alpha u\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_i^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_i^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla\partial_i\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_i\partial_3\theta\|_{L^2}^{\frac{1}{2}}\\
\lesssim&\sum_{i=1}^{2}\|\partial_i\Lambda_h^\alpha u\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\partial_i\Lambda_h^{1+\alpha}u\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\partial_i^2\Lambda_h^\alpha u|_{L^2}^{\frac{1}{2\alpha}}\\
&\times\|\partial_i^2\Lambda_h^\beta\theta\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\partial_i^2\Lambda_h^{1+\beta}\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\partial_i^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla\partial_i\theta\|_{L^2}^{\frac{1}{2}}\|\nabla\partial_i\partial_3\theta\|_{L^2}^{\frac{1}{2}}\\
\lesssim&\|\theta\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta\theta\|_{H^3}
\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}
|I_{322}| =&|-\int_{\mathbb{R}^3}\partial_3^2 u_h\cdot\nabla_h\partial_3\theta\cdot\partial_3^3\theta\,dx|\\
\lesssim&\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2}}\|\nabla_h\partial_3^2\theta\|_{L^2}^{\frac{1}{2}}\\
\lesssim&\|\partial_3^2 u_h\|_{L^2}^{1-\frac{1}{2\alpha}}\|\partial_3^2\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\\
&\times\|\Lambda_h^\beta\partial_3\theta\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\Lambda_h^{1+\beta}\partial_3\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\nabla_h\partial_3\theta\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}}\\
&\times\|\Lambda_h^\beta\partial_3^2\theta\|_{L^2}^{\gamma(1-\frac{1}{2\alpha})}\|\Lambda_h^{1+\beta}\partial_3^2\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\alpha})}\|\nabla_h\partial_3^2\theta\|_{L^2}^{\frac{1}{2\alpha}-\frac{1}{2}}\\
\lesssim&\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|\theta\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta\theta\|_{H^3}^{2-\frac{1}{2\alpha}}.
\end{aligned}\tag{3.20}$$

I_{323} can also be bounded via $\nabla\cdot u=0$, Lemma 1.2 and G-N interpolation inequality,

$$\begin{aligned}
|I_{323}| =&|-\int_{\mathbb{R}^3}\partial_3^2 u_3\partial_3^2\theta\cdot\partial_3^3\theta\,dx|\\
=&|\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_3\partial_j u_j\partial_3^2\theta\cdot\partial_3^3\theta\,dx|\\
\lesssim&\sum_{j=1}^{2}\|\partial_3^2\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^2\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}}\|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2}}\|\partial_3^2\partial_j u_j\|_{L^2}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
\lesssim & \sum_{j=1}^{2} \|\partial_3^2\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^2\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \\
& \times \|\partial_3\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_3\Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
& \times \|\partial_3^2\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_3^2\Lambda_h^{1+\alpha} u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_3^2\partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
\lesssim & \|u\|_{H^3}^{\frac{1}{\beta}-1} \|\theta\|_{H^3}^{2-\frac{1}{\beta}} \|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}} \|\Lambda_h^\beta\theta\|_{H^3}^{\frac{1}{\beta}}.
\end{aligned} \tag{3.21}
$$

To deal with I_{33}, we rewrite it as

$$
\begin{aligned}
I_{33} =& -\sum_{i=1}^{3} \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2\theta \cdot \partial_i^3\theta \, dx \\
=& -\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2\theta \cdot \partial_i^3\theta \, dx - \int_{\mathbb{R}^3} \partial_3 u_h \cdot \nabla_h \partial_3^2\theta \cdot \partial_3^3\theta \, dx - \int_{\mathbb{R}^3} \partial_3 u_3 \partial_3^3\theta \cdot \partial_3^3\theta \, dx \, dx \\
=& I_{331} + I_{332} + I_{333}.
\end{aligned}
$$

Again by Lemma 1.2 and G-N interpolation inequality,

$$
\begin{aligned}
|I_{331}| =& |-\sum_{i=1}^{2} \int_{\mathbb{R}^3} \partial_i u \cdot \nabla \partial_i^2\theta \cdot \partial_i^3\theta \, dx| \\
\lesssim & \sum_{i=1}^{2} \|\nabla\partial_i^2\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\nabla\partial_i^2\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i^3\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_i^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i u\|_{L^2}^{\frac{1}{2}} \|\partial_i\partial_3 u\|_{L^2}^{\frac{1}{2}} \\
\lesssim & \sum_{i=1}^{2} \|\nabla\partial_i\Lambda_h^\beta\theta\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\nabla\partial_i\Lambda_h^{1+\beta}\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_i^2\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \\
& \times \|\partial_i^2\Lambda_h^\beta\theta\|_{L^2}^{\gamma(1-\frac{1}{2\beta})} \|\partial_i^2\Lambda_h^{1+\beta}\theta\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})} \|\partial_i^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\partial_i u\|_{L^2}^{\frac{1}{2}} \|\partial_i\partial_3 u\|_{L^2}^{\frac{1}{2}} \\
\lesssim & \|u\|_{H^3} \|\Lambda_h^\beta\theta\|_{H^3}^2.
\end{aligned} \tag{3.22}
$$

The estimate for I_{332} is more complex, utilizing Lemma 1.2 and G-N interpolation inequality, we have

$$
\begin{aligned}
|I_{332}| =& |-\int_{\mathbb{R}^3} \partial_3 u_h \cdot \nabla_h \partial_3^2\theta \cdot \partial_3^3\theta \, dx| \\
\lesssim & \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}} \|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\nabla_h\partial_3^2\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\nabla_h\partial_3^2\Lambda_3^\alpha\theta\|_{L^2}^{\frac{1}{2\beta}} \\
\lesssim & \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\nabla_h\partial_3^2\Lambda_3^\alpha\theta\|_{L^2}^{\frac{1}{2\beta}} \\
& \times \|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\lambda\frac{1}{2\alpha}} \|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{(1-\lambda)\frac{1}{2\alpha}} \|\nabla_h\partial_3^2\theta\|_{L^2}^{\lambda(1-\frac{1}{2\beta})} \|\nabla_h\partial_3^2\theta\|_{L^2}^{(1-\lambda)(1-\frac{1}{2\beta})} \\
\lesssim & \|\partial_3 u_h\|_{L^2}^{1-\frac{1}{2\alpha}} \|\partial_3^3\theta\|_{L^2}^{1-\frac{1}{2\beta}} \|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{2\beta}} \|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\gamma\frac{1}{2\beta}} \|\partial_3^2\Lambda_h^{1+\beta}\theta\|_{L^2}^{(1-\gamma)\frac{1}{2\beta}} \\
& \times \|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\lambda\frac{1}{2\alpha}} \|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{(1-\lambda)\frac{1}{2\alpha}} \\
& \times \|\nabla_h\partial_3^2\theta\|_{L^2}^{\lambda(1-\frac{1}{2\beta})} \|\Lambda_h^\beta\partial_3^2\theta\|_{L^2}^{\gamma(1-\lambda)(1-\frac{1}{2\beta})} \|\Lambda_h^{1+\beta}\partial_3^2\theta\|_{L^2}^{(1-\gamma)(1-\lambda)(1-\frac{1}{2\beta})}
\end{aligned}
$$

$$\lesssim \|u\|_{H^3}^{(1-\frac{1}{2\alpha})+\lambda\frac{1}{2\alpha}}\|\theta\|_{H^3}^{(1-\frac{1}{2\beta})+\lambda(1-\frac{1}{2\beta})}\|\Lambda_h^\alpha u\|_{H^3}^{(1-\lambda)\frac{1}{2\alpha}}\|\Lambda_h^\beta \theta\|_{H^3}^{(1-\lambda)(1-\frac{1}{2\beta})+\frac{1}{\beta}}, \tag{3.23}$$

where $\lambda = \frac{\frac{1}{2\alpha}+\frac{1}{2\beta}-1}{\frac{1}{2\alpha}-\frac{1}{2\beta}+1}, 1-\lambda = \frac{2-\frac{1}{\beta}}{\frac{1}{2\alpha}-\frac{1}{2\beta}+1}$. It is worth noting that

$$\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\frac{1}{2\alpha}}\|\nabla_h\partial_3^2\theta\|_{L^2}^{1-\frac{1}{2\beta}}$$

allows us to extract

$$\|\partial_3\Lambda_h^\alpha u_h\|_{L^2}^{\lambda\frac{1}{2\alpha}}\|\nabla_h\partial_3^2\theta\|_{L^2}^{\lambda(1-\frac{1}{2\beta})}$$

which can be bounded by $\|u\|_{H^3}^{\lambda\frac{1}{2\alpha}}\|\theta\|_{H^3}^{\lambda(1-\frac{1}{2\beta})}$, and brings us the hope of controlling I_{332} suitably. We estimate I_{333} by the same way as I_{323}, which is

$$\begin{aligned}
|I_{333}| =& |-\int_{\mathbb{R}^3}\partial_3 u_3\partial_3^3\theta\cdot\partial_3^3\theta\,dx\,dx| \\
=& |\sum_{j=1}^{2}\int_{\mathbb{R}^3}\partial_j u_j\partial_3^3\theta\cdot\partial_3^3\theta\,dx| \\
\lesssim& \sum_{j=1}^{2}\|\partial_3^3\theta\|_{L^2}^{2-\frac{1}{\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{\beta}}\|\partial_j u_j\|_{L^2}^{\frac{1}{2}}\|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2}} \\
\lesssim& \sum_{j=1}^{2}\|\partial_3^3\theta\|_{L^2}^{2-\frac{1}{\beta}}\|\partial_3^3\Lambda_h^\beta\theta\|_{L^2}^{\frac{1}{\beta}}\|\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\Lambda_h^{1+\alpha}u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
&\times\|\partial_3\Lambda_h^\alpha u_j\|_{L^2}^{\gamma(1-\frac{1}{2\beta})}\|\partial_3\Lambda_h^{1+\alpha}u_j\|_{L^2}^{(1-\gamma)(1-\frac{1}{2\beta})}\|\partial_3\partial_j u_j\|_{L^2}^{\frac{1}{2\beta}-\frac{1}{2}} \\
\lesssim& \|u\|_{H^3}^{\frac{1}{\beta}-1}\|\theta\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\beta\theta\|_{H^3}^{\frac{1}{\beta}}.
\end{aligned} \tag{3.24}$$

Combining with (3.15)-(3.24), we obtain

$$\begin{aligned}
I_3(\tau) \lesssim& \|u\|_{H^3}\|\Lambda_h^\beta\theta\|_{H^3}^2 \\
&+\|\theta\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}\|\Lambda_h^\beta\theta\|_{H^3} \\
&+\|u\|_{H^3}^{1-\frac{1}{2\alpha}}\|\theta\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\alpha u\|_{H^3}^{\frac{1}{2\alpha}}\|\Lambda_h^\beta\theta\|_{H^3}^{2-\frac{1}{2\alpha}} \\
&+\|u\|_{H^3}^{\frac{1}{\beta}-1}\|\theta\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\alpha u\|_{H^3}^{2-\frac{1}{\beta}}\|\Lambda_h^\beta\theta\|_{H^3}^{\frac{1}{\beta}} \\
&+\|u\|_{H^3}^{(1-\frac{1}{2\alpha})+\theta\frac{1}{2\alpha}}\|\theta\|_{H^3}^{(1-\frac{1}{2\beta})+\theta(1-\frac{1}{2\beta})}\|\Lambda_h^\alpha u\|_{H^3}^{(1-\theta)\frac{1}{2\alpha}}\|\Lambda_h^\beta\theta\|_{H^3}^{(1-\theta)(1-\frac{1}{2\beta})+\frac{1}{\beta}} \\
\lesssim& (\|u\|_{H^3}+\|\theta\|_{H^3})(\|\Lambda_h^\alpha u\|_{H^3}^2+\|\Lambda_h^\beta\theta\|_{H^3}^2).
\end{aligned}$$

Adding (3.1), (3.2) and integrating in time, we have

$$E(t) \lesssim E(0) + \int_0^t I_2(\tau) + I_3(\tau)\,d\tau$$

and inserting all the bounds obtained above for I_2 and I_3. We obtain (1.7). For example, the bounds for I_2 yield

$$\begin{aligned}\int_0^t |I_2(\tau)|d\tau \lesssim& \int_0^t \|u\|_{H^3}\|\Lambda_h^\alpha u\|_{H^3}^2\, d\tau\\ \lesssim& \sup_{\tau\in[0,t]}\|u(\tau)\|_{H^3}\int_0^t \|\Lambda_h^\alpha u\|_{H^3}^2\, d\tau\\ \lesssim& E(t)^{\frac{3}{2}}.\end{aligned}$$

The time integral of I_3 is similarly bounded, which completes the proof of (1.7). $\square$

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