Stabilizing Model Predictive Control Synthesis using Integral Quadratic Constraints and Full-Block Multipliers*

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Abstract

In this paper, we discuss how to synthesize stabilizing Model Predictive Control (MPC) algorithms based on convexly parameterized Integral Quadratic Constraints (IQCs), with the aid of general multipliers. Specifically, we consider Lur'e systems subject to sector-bounded and slope-restricted nonlinearities. As the main novelty, we introduce point-wise IQCs with storage in order to accordingly generate the MPC terminal ingredients, thus enabling closed-loop stability, strict dissipativity with regard to the nonlinear feedback, and recursive feasibility of the optimization. Specifically, we consider formulations involving both static and dynamic multipliers, and provide corresponding algorithms for the synthesis procedures. The major benefit of the proposed approach resides in the flexibility of the IQC framework, which is capable to deal with many classes of uncertainties and nonlinearities. Moreover, for the considered class of nonlinearities, our method yields larger regions of attraction of the synthesized predictive controllers (with reduced conservatism) if compared to the standard approach to deal with sector constraints from the literature.

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Summary

In this paper, we discuss how to synthesize stabilizing Model Predictive Control (MPC) algorithms based on convexly parameterized Integral Quadratic Constraints (IQCs), with the aid of general multipliers. Specifically, we consider Lur'e systems subject to sector-bounded and slope-restricted nonlinearities. As the main novelty, we introduce point-wise IQCs with storage in order to accordingly generate the MPC terminal ingredients, thus enabling closed-loop stability, strict dissipativity with regard to the nonlinear feedback, and recursive feasibility of the optimization. Specifically, we consider formulations involving both static and dynamic multipliers, and provide corresponding algorithms for the synthesis procedures. The major benefit of the proposed approach resides in the flexibility of the IQC framework, which is capable to deal with many classes of uncertainties and nonlinearities. Moreover, for the considered class of nonlinearities, our method yields larger regions of attraction of the synthesized predictive controllers (with reduced conservatism) if compared to the standard approach to deal with sector constraints from the literature.

KEYWORDS:
Model Predictive Control; Dissipativity; Integral Quadratic Constraints; Lur'e Systems.

1 | INTRODUCTION

Model Predictive Control (MPC) is a standard approach for the regulation of constrained processes[1]. Theoretical means to guarantee closed-loop stability (and recursive feasibility of the MPC optimization) have been established since the seminal results provided by Mayne and colleagues[2]. Specifically, these relate to the use of terminal ingredients (a terminal cost, a terminal constraint, and a terminal feedback law) within the optimization problem that is solved online, during the control implementation. Basically, these elements enable the application of classical Lyapunov arguments with respect to the value of the (to-be-optimized) cost function over time. Regarding the control of Lur'e systems, which are addressed in this work, the usual approach in the MPC literature (e.g.[3,4,5,6,7]) is to replace the nonlinear interconnection equation constraint by sector conditions applied over the interconnection signals, e.g.[8,9].

In this paper, instead of resorting to such an approach, we employ Integral Quadratic Constraints (IQCs) in order to deal with these nonlinearities. The IQC framework is well-known in the robust control community for its flexibility in dealing with various classes of uncertainties and nonlinearities. Moreover, the resulting (robust) stability and performance tests involve often, but not always, the least conservatism if compared to alternative solutions; for Lur'e systems involving a gradient nonlinearity, this has been recently demonstrated in[10]. In particular, we derive novel IQCs on the basis of the so-called dynamic O'Shea-Zames-Falb multipliers for repeated slope-restricted nonlinearities. Furthermore, we show, for the first time, how to synthesize

[0]**Abbreviations:** MPC, Model Predictive Control; IQCs, Integral Quadratic Constraints; OZF, O'Shea-Zames-Falb; LMIs; Linear Matrix Inequalities.

MPC terminal ingredients based on such multipliers. We illustrate how this framework can generate potentially larger regions of attraction for the resulting closed-loop than when using traditional synthesis methods.

Up to our best knowledge, IQCs have not been directly used for synthesizing stabilizing MPC schemes. Only a handful of papers have employed IQCs in the context of MPC algorithms, e.g. [11,12,13,14,15,16]:

(1) In [11,12,13,15], it is exploited that the closed-loop of an LTI system with a synthesized MPC scheme is a particular nonlinear feedback interconnection and this interconnection's stability is analyzed *a posteriori* by means of IQCs;

(2) In [14,16], the IQC approach is used to ensure exponential stability of an error system related to a tube-based MPC scheme.

Moreover, in all of these references merely so-called hard, pointwise or even static IQCs are employed instead of the more general soft IQCs [17] or those involving a terminal cost [18].

Outline. This paper is structured as follows: In Section 2, we present a brief recap of relevant results on robust stability analysis using IQCs, including the link to dissipativity. For the class of Lur'e systems with sector-bounded nonlinearities, we provide a new link from point-wise IQCs with storage to O'Shea-Zames-Falb multipliers. In Section 3, the considered system interconnection and the MPC setup are provided, with a corresponding summary of the theory on stabilizing MPCs. Moreover, we provide the main contribution of this work: a novel approach for synthesizing stabilizing MPC algorithms for Lur'e systems based on IQCs with general multipliers. Accordingly, we provide certificates for closed-loop stability and recursive feasibility of the optimization. In Section 4, we present numerical benchmark examples, comparing the proposed IQC-based MPC (using both static and dynamic multipliers) against the standard Lyapunov-based MPC synthesis with sector arguments. We demonstrate (and numerically assess) how the proposed mechanism offers tighter bounds on the nonlinearity and, thus, larger terminal regions and less conservative control performances. General conclusions are drawn in Section 5.

Notation. $\mathbb{N}$ ($\mathbb{N}_0$) denotes the set of positive (nonnegative) integers and we abbreviate the set $\{i \in \mathbb{N}_0 \mid a \leq i \leq b\}$ by $\mathbb{N}_{[a,b]}$. $\mathbb{S}^n$ stands for the set of symmetric matrices in $\mathbb{R}^{n\times n}$ and ℓ_{2e}^m for the space of sequences with elements in $\mathbb{R}^m$. The $j \times j$ identity matrix is denoted by I_j and $I_{j,\{i\}}$ denotes its i-th column. $\mathrm{col}(v_1, \dots, v_m) := (v_1^\top, \dots, v_n^\top)^\top$ denotes the vectorization operation and $\mathrm{diag}(V_1, \dots, V_n)$ denotes the block diagonal matrix with $V_1, \dots, V_n$ on its diagonal. The predicted value of a given variable $v(k)$ at time instant $k+i$, computed based on the information available at instant k, is denoted as $v(k+i|k)$; in particular, $v(k|k) = v(k)$. Finally, in matrix inequalities, $(\star)$ denotes the corresponding symmetrical transpose.

2 | ROBUST STABILITY VIA INTEGRAL QUADRATIC CONSTRAINTS

In this Section, we recapitulate recent results of interest regarding dissipativity and robust stability analysis using IQCs. In particular, we recall several simple and standard IQCs for dealing with sector-bounded nonlinearities. Next to these repetitions, we construct new IQCs based on dynamic O'Shea-Zames-Falb multipliers for repeated slope-restricted nonlinearities.

2.1 | Dissipativity

Given some real matrices and an initial condition $x(0) \in \mathbb{R}^{n_x}$, we consider a known linear discrete-time system G described in state-space by

$$\begin{pmatrix} x(k+1) \\ z(k) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \end{pmatrix} \tag{1a}$$

for all $k \in \mathbb{N}_0$. This system is interconnected with an uncertain (nonlinear) operator $\Delta : \ell_{2e}^{n_z} \to \ell_{2e}^{n_w}$ via

$$w = \Delta(z). \tag{1b}$$

Here, x stands for the linear system's state, while z and w denote the interconnection signals. Such feedback interconnections are often referred to as Lur'e systems. A corresponding block diagram representation is presented in Fig. 1.

Definition 1 (Dissipativity)**.** Consider $g : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$, $h : \mathcal{X} \times \mathcal{W} \to \mathcal{Z}$ and $s : \mathcal{W} \times \mathcal{Z} \to \mathbb{R}$. We say that the nonlinear system $x^+ = g(x, w)$, $z = h(x, w)$ is strictly dissipative with respect to the supply rate s, if there exists a storage function $J : \mathcal{X} \to \mathbb{R}$ and some positive scalar $\epsilon > 0$ such that the following inequality holds for all admissible trajectories and all

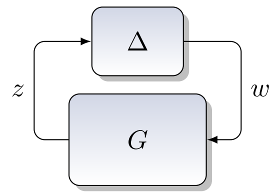


FIGURE 1 The uncertain feedback interconnection from (1).

discrete-time instants $k_1, k_2 \in \mathbb{N}_0$ with $k_1 \leq k_2$:

$$J(x(k_2)) - J(x(k_1)) \leq \sum_{k=k_1}^{k_2-1} s(z(k), w(k)) - \epsilon \sum_{k=k_1}^{k_2-1} \left(\|z(k)\|^2 + \|w(k)\|^2 \right) . \tag{2}$$

The system is said to be dissipative with respect to s if this holds for $\epsilon = 0$.

Note that (2) is satisfied for all discrete-time instants $k_1, k_2 \in \mathbb{N}_0$ with $k_1 \leq k_2$ if and only if the following difference dissipation inequality holds for all $k \in \mathbb{N}_0$:

$$J(x(k+1)) - J(x(k)) \leq s(z(k), w(k)) - \epsilon(\|z(k)\|^2 + \|w(k)\|^2) . \tag{3}$$

Theorem 1 (Characterization of strict dissipativity for LTI systems, from [19]). Consider the linear system G as given in (1a) and the homogeneous quadratic supply rate $s_P : (z, w) \mapsto \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)^\top P \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$ for some $P = P^\top \in \mathbb{R}^{(n_z+n_w)\times(n_z+n_w)}$. Then, G is strictly dissipative with respect to the supply rate s_P if and only if there exits a symmetric matrix X satisfying

$$(\star) \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} < (\star)\, P \begin{pmatrix} C & D \\ 0 & I \end{pmatrix} . \tag{4}$$

The IQC theorem recalled in the sequel is tightly linked to the concept of dissipation. Indeed, it involves an LMI of the form (4) for a filtered version of the known linear part G of the interconnection in (1) and an inequality of the form (3) with $\epsilon = 0$ for a filtered version of the unknown part. It is essential that both of these inequalities are coupled through (almost) the same supply rate.

2.2 | IQCs

Due to the uncertain operator Δ, it is in general not possible to analyze the interconnection (1) directly. Instead, the general idea of the IQC framework [17] is to find numerically suitable quadratic constraints that this operator enforces on the interconnection signals z and w based on the available information on Δ. Afterwards, one analyzes the behavior of the interconnection's linear part (1a) subject to the identified constraints.

Suitable quadratic constraints are classically formulated in terms of so-called multipliers that are factorized as $\Pi = \Psi^* M \Psi$ with Ψ being a fixed stable dynamic outer factor with $n_z + n_w$ inputs and m outputs, as well as a middle matrix $M \in \mathbb{S}^m$, which serves as a variable subject to constraints.

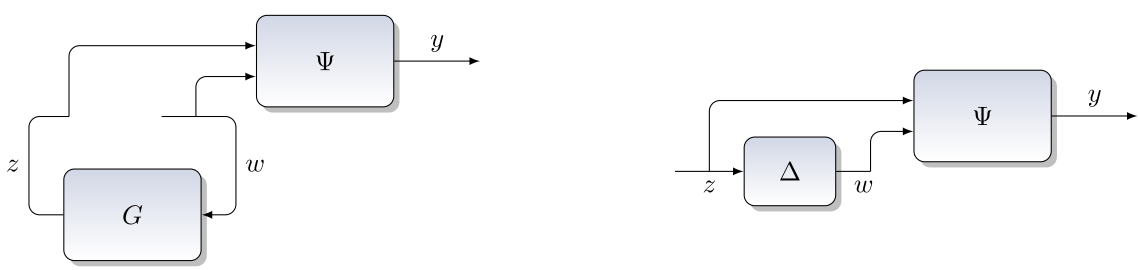


FIGURE 2 (Left) augmented system (6) and (right) filtered uncertainty.

In the sequel, we provide a variant of the IQC theorem proposed in[18]. In order to state this analysis result, we require some state-space description of the filter Ψ and of the corresponding augmented (or filtered) system that is depicted on the left of Fig. 2. To this end, suppose the filter Ψ has an output $y \in \ell_{2e}^{m}$, given in response to an input $u \in \ell_{2e}^{n_z+n_w}$, whose dynamics are represented by

$$\begin{pmatrix} \xi(k+1) \\ y(k) \end{pmatrix} = \begin{pmatrix} A_\Psi & B_\Psi \\ C_\Psi & D_\Psi \end{pmatrix} \begin{pmatrix} \xi(k) \\ u(k) \end{pmatrix}, \tag{5}$$

for all $k \in \mathbb{N}_0$ and with a vanishing initial condition, i.e. $\xi(0) = 0 \in \mathbb{R}^{n_\xi}$. Then, the augmented system is described by

$$\begin{pmatrix} \xi(k+1) \\ x(k+1) \\ \hline y(k) \end{pmatrix} = \left(\begin{array}{cc|c} A_\Psi & B_\Psi \left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) & B_\Psi \left(\begin{smallmatrix} D \\ I_{n_w} \end{smallmatrix}\right) \\ 0 & A & B \\ \hline C_\Psi & D_\Psi \left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) & D_\Psi \left(\begin{smallmatrix} D \\ I_{n_w} \end{smallmatrix}\right) \end{array}\right) \begin{pmatrix} \xi(k) \\ x(k) \\ \hline w(k) \end{pmatrix} \tag{6}$$

for all $k \in \mathbb{N}_0$. Accordingly, our variant of the IQC theorem reads as follows.

Theorem 2 (IQC Theorem for point-wise integral quadratic constraints with storage)**.** Suppose that the interconnection (1) is well-posed and that $\Delta : \ell_{2e}^{n_z} \to \ell_{2e}^{n_w}$ satisfies a *point-wise IQC with storage* with respect to the filter Ψ and the matrices $M \in \mathbb{S}^m$ and $Z \in \mathbb{S}^{n_\xi}$. The latter means that, for all $k \in \mathbb{N}_0$, the inequality

$$\xi(k+1)^\top Z \xi(k+1) - \xi(k)^\top Z \xi(k) + y(k)^\top M y(k) \geq 0 \tag{IQC}$$

holds for any state and output trajectory of the filter in (5) driven by the input $u = \left(\begin{smallmatrix} z \\ \Delta(z) \end{smallmatrix}\right)$ and with any $z \in \ell_{2e}^{n_z}$. Then, the interconnection in (1) is asymptotically stable if there exists a matrix $X \in \mathbb{S}^{n_\xi + n_x}$ satisfying

$$(\star) \left(\begin{array}{cc|c} X & 0 & \\ 0 & -X & \\ \hline & & M \end{array}\right) \left(\begin{array}{cc|c} A_\Psi & B_\Psi \left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) & B_\Psi \left(\begin{smallmatrix} D \\ I_{n_w} \end{smallmatrix}\right) \\ 0 & A & B \\ I_{n_\xi} & 0 & 0 \\ 0 & I_{n_x} & 0 \\ \hline C_\Psi & D_\Psi \left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) & D_\Psi \left(\begin{smallmatrix} D \\ I_{n_w} \end{smallmatrix}\right) \end{array}\right) < 0 \quad \text{and} \quad X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} > 0. \tag{7}$$

Moreover, if the inequalities in (7) are satisfied, it follows that

$$\begin{pmatrix} \xi(k+1) \\ x(k+1) \end{pmatrix}^\top \left(X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \xi(k+1) \\ x(k+1) \end{pmatrix} \leq \begin{pmatrix} \xi(k) \\ x(k) \end{pmatrix}^\top \left(X - \begin{pmatrix} Z & 0 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \xi(k) \\ x(k) \end{pmatrix} \tag{8}$$

holds for all $k \in \mathbb{N}_0$ and any initial condition $x(0) \in \mathbb{R}^{n_x}$.

Note, at first, that (IQC) is exactly a difference dissipation inequality of the form (3) with $\epsilon = 0$ and the quadratic supply rate $s : (y, z) \mapsto y^\top M y$ for the uncertain nonlinear interconnection depicted on the right of Fig. 2; this inequality is where all the information about the behavior of the uncertainty is encoded. Secondly, one observes that, by Theorem 1, the first inequality in (7) translates into another difference dissipation inequality, which actually involves the quadratic supply rate $\tilde{s} : (y, w) \mapsto -y^\top M y$. The key argument of this IQC theorem's proof is that the particular structure of the supply rates allows us to combine both dissipation inequalities in order to obtain (8), which is an inequality only involving the corresponding storage functions. By strictness of the inequalities in (7) and a minor modification of the stated arguments, this guarantees asymptotic stability of the uncertain interconnection (1).

Remark 1. Relation to other notions of IQCs. The discrete-time version of the main result from[18] involves a so-called *finite-horizon IQC with terminal cost*, which is characterized by an inequality of the form

$$\xi(k+1)^\top Z \xi(k+1) + \sum_{t=0}^{k} y(t)^\top M y(t) \geq 0 \quad \text{for all} \quad k \in \mathbb{N}_0 . \tag{9}$$

In[18], the authors provide tight links to classical *soft IQCs* as introduced, for example, in[17] and as represented by

$$\sum_{t=0}^{\infty} y(t)^\top M y(t) \geq 0 .$$

So-called *point-wise IQCs* have been very recently considered, for example, in[20] and are inequalities of the form (IQC), but with the restriction $Z = 0$ which can be severe. Even for $Z \neq 0$ and due to the vanishing initial condition of the filter in (5), it is immediate that (9) holds if (IQC) is satisfied; the converse does not hold, in general. Hence, demanding that Δ satisfies a point-wise IQC with storage is more restrictive than requiring it to satisfy a finite-horizon IQC with terminal cost and less than requiring it to satisfy a point-wise IQC[1]. Nevertheless, one can show that many of the uncertainties and nonlinearities in the catalogs found in[17,21] satisfy point-wise IQCs (with storage). In particular, we will show at the end of this section that such an IQC is valid for repeated slope-restricted nonlinearities based on dynamic O'Shea-Zames-Falb multipliers and with a nontrivial matrix $Z \neq 0$.

2.3 | Specific multipliers

In order to apply Theorem 2 in practice, it is necessary to find a filter and (classes of) matrices Z and M such that (IQC) is satisfied for the operator Δ of interest by considering its input-output behavior.

Next, we repeat a few IQCs for the reader's convenience and corresponding to nonlinear operators Δ that are given by

$$\Delta(z)(k) := f(z(k)) \quad \text{with} \quad f : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z},\ z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n_z} \end{pmatrix} \mapsto \begin{pmatrix} \phi(z_1) \\ \vdots \\ \phi(z_{n_z}) \end{pmatrix}, \tag{10}$$

for some nonlinear function $\phi : \mathbb{R} \to \mathbb{R}$. In other words, the interconnection (1) involves a memory-less, repeated nonlinearity. In the sequel, we assume that the function ϕ is only known to be

- *sector-bounded*[8] with bounds $a \leq b$, i.e. it satisfies

$$(\phi(z) - az)(bz - \phi(z)) \geq 0, \quad \text{for all } z \in \mathbb{R},$$

- and *slope-restricted*[22] with bounds $\mu \leq \varrho$, i.e., it satisfies

$$\phi(0) = 0 \text{ and } \mu \leq \frac{\phi(x) - \phi(y)}{x - y} \leq \varrho \text{ for all } x \neq y.$$

Note that, for example, the standard saturation sat $: \mathbb{R} \to \mathbb{R}$ is sector-bounded with bounds $(a, b) = (0, 1)$ and slope-restricted with bounds $(\mu, \varrho) = (0, 1)$. We recall that, for such an operator, one of the simplest ways to guarantee (IQC) is to use a static filter Ψ, i.e., a filter without any dynamics, and a variant of the so-called DG-scalings; these scalings also appear in the context of structured singular value theory[23].

Lemma 1 (Static DG-scalings)**.** The operator Δ in (10) satisfies (IQC) for the static filter $\Psi = I$ with $n_\xi = 0$ and for any matrix M contained in the set

$$\mathcal{M}_{\mathrm{DG}} := \left\{ (\star) \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \begin{pmatrix} bI & -I \\ -aI & I \end{pmatrix} \;\middle|\; S \in \mathbb{S}^{n_z} \text{ is diagonal and positive definite} \right\}$$

Proof. Let $z \in \ell_{2e}^{n_z}$ and $k \in \mathbb{N}_0$ be arbitrary. Then one observes that

$$w(k) = f(z(k)) = \begin{pmatrix} \phi(z_1(k)) \\ \vdots \\ \phi(z_{n_z}(k)) \end{pmatrix} = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_{n_z} \end{pmatrix} z(k) \quad \text{holds for} \quad \delta_j := \begin{cases} \frac{\phi(z_j(k))}{z_j(k)}, & \text{if } z_j(k) \neq 0, \\ b, & \text{otherwise.} \end{cases}$$

Here, it is crucial to note that sector-boundedness of ϕ implies $\delta_j \in [a, b]$ for all j. Then, it is a matter of direct calculations to infer that

$$\begin{pmatrix} z(k) \\ w(k) \end{pmatrix}^\top M \begin{pmatrix} z(k) \\ w(k) \end{pmatrix} = \sum_{j=1}^{n_z} (\delta_j - a)(b - \delta_j)\, |z_j(k)|^2\, S_{jj} \geq 0.$$

Since z and k were arbitrary, we obtain (IQC) with an empty matrix Z. □

Another one out of several possibilities, which usually leads to less conservative stability tests, is to employ the so-called full-block multipliers. The proof can be carried out in a similar fashion as for DG-scalings, but relies on demonstrating and exploiting concavity of the map $\mathbb{R}^{n_z \times n_z} \to \mathbb{R}^{n_z \times n_z}$, $\Delta \mapsto \left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right)^\top M \left(\begin{smallmatrix} I \\ \Delta \end{smallmatrix}\right)$.

[1] As brief illustration of this discussion, notice that demanding $y(k)^T M y(k) \geq 0$ for all $k \in \mathbb{N}_0$ is a stronger requirement than demanding $\sum_{t=0}^{k} y(t)^T M y(t) \geq 0$ for all $k \in \mathbb{N}_0$. For example, suppose $y \in \ell_{2e}$ satisfies $y(0)^T M y(0) = 1$, $y(1)^T M y(1) = -1$, and $y(k)^T M y(k) = 0$ for all $k \geq 2$. Then the latter inequality is satisfied, but not the former.

Lemma 2 (Static full-block multipliers)**.** Let us denote by $\Delta_1, \ldots, \Delta_N$ the generators of the convex set $\left\{\mathrm{diag}(\delta_1, \ldots, \delta_{n_z}) \,\middle|\, \delta_j \in [a, b] \text{ for all } j\right\}$. Then, the operator Δ in (10) satisfies (IQC) for the static filter $\Psi = I$ with $n_\xi = 0$ and for any matrix M contained in the set

$$\mathcal{M}_{\mathrm{FB}} := \left\{ M \,\middle|\, \begin{pmatrix} I \\ \Delta_j \end{pmatrix}^\top M \begin{pmatrix} I \\ \Delta_j \end{pmatrix} \geq 0 \text{ for all } j \text{ and } \begin{pmatrix} 0 \\ I \end{pmatrix}^\top M \begin{pmatrix} 0 \\ I \end{pmatrix} \leq 0 \right\}.$$

In many situations, it has been demonstrated that considering IQCs described by dynamic multipliers with dynamic filters (instead of static ones) can lead to superior analysis tests. Hence, we provide a result involving such dynamic multipliers next. Concretely, we consider a variant of the well-known O'Shea-Zames-Falb multipliers which are described in detail, for example in[24], in the context of soft IQCs. Here, we show for the first time how to employ them for generating a point-wise IQC with nontrivial storage, i.e., with $Z \neq 0$, for repeated slope-restricted nonlinearities (10).

Theorem 3 (Dynamic O'Shea-Zames-Falb multipliers)**.** Let $\psi(z) := I_{n_w} \otimes (\frac{1}{z^l}, \ldots, \frac{1}{z}, 1)^\top$ for some length l, which admits the realization $\left[I_{n_w} \otimes J_l, I_{n_w} \otimes e_l, I_{n_w} \otimes C_l, I_{n_w} \otimes e_{l+1}\right]$ where $e_l := \left(\begin{smallmatrix} 0_{l-1} \\ 1 \end{smallmatrix}\right)$, $C_l := \left(\begin{smallmatrix} I_l \\ 0_{1\times l} \end{smallmatrix}\right)$, $e_{l+1} := \left(\begin{smallmatrix} 0_l \\ 1 \end{smallmatrix}\right)$ and where $J_l \in \mathbb{R}^{l\times l}$ is an upper Jordan block with eigenvalue zero. Then, the operator Δ in (10) satisfies (IQC) for the filter $\Psi := \left(\begin{smallmatrix} \psi & 0 \\ 0 & \psi \end{smallmatrix}\right)$ and any pair of matrices

$$Z = (\star)\begin{pmatrix} 0 & E^\top \\ E & 0 \end{pmatrix}\left(\begin{pmatrix} \varrho & -1 \\ -\mu & 1 \end{pmatrix} \otimes I_{ln_w}\right) \quad \text{and} \quad M = (\star)\begin{pmatrix} 0 & L^\top \\ L & 0 \end{pmatrix}\left(\begin{pmatrix} \varrho & -1 \\ -\mu & 1 \end{pmatrix} \otimes I_{(l+1)n_w}\right)$$

defined by $E \in \mathbb{R}^{ln_w \times ln_w}$ and $L \in \mathbb{R}^{(l+1)n_w \times (l+1)n_w}$ with the property that the matrix

$$\tilde{L} := (\tilde{L}_{ij}) = L - \begin{pmatrix} E & 0 \\ 0 & 0_{n_w\times n_w} \end{pmatrix} + \begin{pmatrix} 0_{n_w\times n_w} & 0 \\ 0 & E \end{pmatrix} \in \mathbb{R}^{(l+1)n_w \times (l+1)n_w}$$

satisfies

$$\tilde{L}_{ij} \leq 0 \text{ for all } i \neq j \quad \text{as well as} \quad \tilde{L}\mathbf{1} \geq 0 \quad \text{and} \quad \mathbf{1}^\top \tilde{L} \geq 0\,;$$

here, $\mathbf{1} \in \mathbb{R}^{(l+1)n_w}$ denotes the all-ones vector. In the sequel, we abbreviate the set of such matrix pairs by $\mathcal{M}_{\mathrm{OZF}}$.

Proof. Let $k \in \mathbb{N}_0$ and $z \in \ell_{2e}^{n_w}$ be arbitrary and let $w := \Delta(z)$. Note at first that the choice of ψ is motivated as follows:

- Its output at time k in response to z and for zero initial conditions is given by

$$\psi(z)(k) := \mathrm{col}(z(k-l), \ldots, z(k-1), z(k)) \in \mathbb{R}^{(l+1)n_w} \quad \text{where we set} \quad z(t) := 0 \text{ for } t < 0.$$

- Its corresponding state at time k and $k+1$ can be expressed, respectively, as

$$\begin{pmatrix} I_{ln_w} & 0 \end{pmatrix} \psi(z)(k) \quad \text{and} \quad \begin{pmatrix} 0 & I_{ln_w} \end{pmatrix} \psi(z)(k).$$

This holds analogously for the response to w. Consequently, the state ξ of the filter $\Psi = \left(\begin{smallmatrix} \psi & 0 \\ 0 & \psi \end{smallmatrix}\right)$ in response to $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$ satisfies

$$\xi(k) = \begin{pmatrix} (\,I\ 0\,)\,\psi(z)(k) \\ (\,I\ 0\,)\,\psi(w)(k) \end{pmatrix} = \begin{pmatrix} (\,I\ 0\,) & 0 \\ 0 & (\,I\ 0\,) \end{pmatrix} y(k) \quad \text{and} \quad \xi(k+1) = \begin{pmatrix} (\,0\ I\,)\,\psi(z)(k) \\ (\,0\ I\,)\,\psi(w)(k) \end{pmatrix} = \begin{pmatrix} (\,0\ I\,) & 0 \\ 0 & (\,0\ I\,) \end{pmatrix} y(k),$$

where y is the filter's output. Accordingly, we obtain

$$\xi(k+1)^\top Z \xi(k+1) - \xi(k)^\top Z \xi(k) + y(k)^\top M y(k) = y(k)^\top \Omega y(k)$$

with the matrix Ω given by

$$(\star)\begin{pmatrix} 0 & E^\top \\ E & 0 \end{pmatrix}\begin{pmatrix} \varrho I & -I \\ -\mu I & I \end{pmatrix}\begin{pmatrix} (\,0\ I\,) & 0 \\ 0 & (\,0\ I\,) \end{pmatrix} - (\star)\begin{pmatrix} 0 & E^\top \\ E & 0 \end{pmatrix}\begin{pmatrix} \varrho I & -I \\ -\mu I & I \end{pmatrix}\begin{pmatrix} (\,I\ 0\,) & 0 \\ 0 & (\,I\ 0\,) \end{pmatrix} + (\star)\begin{pmatrix} 0 & L^\top \\ L & 0 \end{pmatrix}\begin{pmatrix} \varrho I & -I \\ -\mu I & I \end{pmatrix}$$

$$= (\star)\begin{pmatrix} 0 & \star \\ \left(\begin{smallmatrix} 0 & 0 \\ 0 & E \end{smallmatrix}\right) - \left(\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right) + L & 0 \end{pmatrix}\begin{pmatrix} \varrho I & -I \\ -\mu I & I \end{pmatrix} = (\star)\begin{pmatrix} 0 & \tilde{L}^\top \\ \tilde{L} & 0 \end{pmatrix}\begin{pmatrix} \varrho I & -I \\ -\mu I & I \end{pmatrix}.$$

Then, it remains to observe that $\tilde{L}$ and $y(k)$ are such that we can apply Corollary 6.10 of[24] in order to infer that $y(k)^\top \Omega y(k)$ is non-negative. This yields the claim since $k \in \mathbb{N}_0$ and $z \in \ell_{2e}^{n_z}$ were arbitrary. □

Finally, it is well-known within the robust control community how to combine several IQCs for a given uncertain operator Δ. However, for the novel type of IQCs, it is briefly illustrated next how to combine static full-block with dynamic O'Shea-Zames-Falb multipliers.

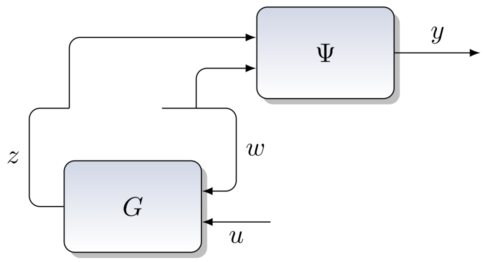


FIGURE 3 Augmented system (12) involving the filter Ψ in (5) and G, the known linear part of (11).

Lemma 3 (Combined static full-block and dynamic O'Shea-Zames-Falb multipliers)**.** With the notation from the previous lemmas, the operator Δ in (10) satisfies (IQC) for the filter

$$\Psi := \begin{pmatrix} \psi & 0 \\ 0 & \psi \\ I & 0 \\ 0 & I \end{pmatrix}$$

and any pair of matrices (Z, M) contained in

$$\mathcal{M}_{\mathrm{OZF+FB}} := \left\{ \left(Z, \begin{pmatrix} M_{\mathrm{OZF}} & 0 \\ 0 & M_{\mathrm{FB}} \end{pmatrix} \right) \,\middle|\, (Z, M_{\mathrm{OZF}}) \in \mathcal{M}_{\mathrm{OZF}} \text{ and } M_{\mathrm{FB}} \in \mathcal{M}_{\mathrm{FB}} \right\}.$$

3 | MPC SETUP

In the remainder of this paper, we consider the discrete-time feedback interconnection

$$\begin{pmatrix} x(k+1) \\ z(k) \end{pmatrix} = \begin{pmatrix} A & B & B_u \\ C & D & D_u \end{pmatrix} \begin{pmatrix} x(k) \\ w(k) \\ u(k) \end{pmatrix}, \quad w(k) = \Delta(z)(k) \tag{11}$$

for all $k \in \mathbb{N}_0$. Apart from the additional control input signal u, this interconnection is of the same structure as in (1) and involves a nonlinear operator $\Delta : \ell_{2e}^{n_z} \to \ell_{2e}^{n_z}$. In the sequel, we assume that this operator is known and memory-less, i.e., it satisfies $\Delta(z)(k) = f(z(k))$, for all $k \in \mathbb{N}_0$ and for some function $f : \mathbb{R}^{n_z} \to \mathbb{R}^{n_z}$. Moreover, we suppose that the map $z \mapsto z - Df(z)$ is bijective which means that the interconnection (11) is well-posed. Finally, we assume that the feedback operator satisfies (IQC) for some given stable LTI filter Ψ with realization as given in (5).

In order to benefit from our IQC result, we will again need an augmented version of the known linear part in (11) involving the filter Ψ. This augmented system is given by

$$\begin{pmatrix} \nu(k+1) \\ y(k) \end{pmatrix} = \begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B}_u \\ \mathcal{C} & \mathcal{D} & \mathcal{D}_u \end{pmatrix} \begin{pmatrix} \nu(k) \\ w(k) \\ u(k) \end{pmatrix} \tag{12}$$

for all $k \in \mathbb{N}_0$ with stacked states $\nu : k \mapsto \mathrm{col}(\xi(k), x(k))$ and matrices

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} & \mathcal{B}_u \\ \mathcal{C} & \mathcal{D} & \mathcal{D}_u \end{pmatrix} := \left(\begin{array}{cc|c|c} A_\Psi & B_\Psi \left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) & B_\Psi \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) & B_\Psi \left(\begin{smallmatrix} D_u \\ 0 \end{smallmatrix}\right) \\ 0 & A & B & B_u \\ \hline C_\Psi & D_\Psi \left(\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right) & D_\Psi \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) & D_\Psi \left(\begin{smallmatrix} D_u \\ 0 \end{smallmatrix}\right) \end{array}\right).$$

A block diagram of this system is depicted in Fig. 3.

In the sequel, we assume that the full state x of (11) can be measured at any time $k \in \mathbb{N}$. Since the input u is generated by the controller and as Δ is assumed to be known as well, this actually implies that we also have access to the state ξ of the filter Ψ and, hence, to the augmented state ν at each time instant. This motivates our main goal to find a stabilizing feedback law of the form

$$u(k) = \kappa(\nu(k)) \tag{13}$$

that is generated by an MPC algorithm outlined in the next section. Clearly, if the filter Ψ is static, this reduces to a classical nonlinear state-feedback law $u(k) = \kappa(x(k))$. In order to achieve this goal, our approach relies on designing suitable terminal ingredients involving a state-feedback gain κ_t and on the analysis criteria in the IQC theorem 2. Our numerical examples will illustrate the strong benefit of this novel construction of terminal ingredients.

3.1 | Model Predictive Control

The implementation of the feedback law (13) relies on solving the optimization problem

$$\begin{aligned}
\min_{U_k} \quad & \overbrace{\left(\sum_{j=0}^{N_p-1} \ell(v(k+j|k), u(k+j|k))\right) + V(v(k+N_p|k))}^{J(v(k),U_k)} \,, \qquad (14)\\
\text{s.t.}: v(k+j+1|k) &= \mathcal{A}v(k+j|k) + \mathcal{B}w(k+j|k) + \mathcal{B}_u u(k+j|k)\,, \quad \text{for all} \quad j \in \mathbb{N}_{[0,N_p-1]},\\
z(k+j|k) &= Cx(k+j|k) + Dw(k+j|k) + D_u u(k+j|k)\,, \quad \text{for all} \quad j \in \mathbb{N}_{[0,N_p-1]},\\
w(k+j|k) &= f\left(z(k+j|k)\right)\,, \quad \text{for all} \quad j \in \mathbb{N}_{[0,N_p-1]},\\
x(k+j|k) &\in \mathcal{X}\,, \quad \text{for all} \quad j \in \mathbb{N}_{[1,N_p]},\\
u(k+j-1|k) &\in \mathcal{U}\,, \quad \text{for all} \quad j \in \mathbb{N}_{[1,N_p]},\\
u(k+N_p-1|k) &= \kappa_t x(k+N_p-1|k),\\
v(k+N_p|k) &\in \mathbf{X}_t,
\end{aligned}$$

at each time instance k. Here, the decision variable $U_k := \mathrm{col}\big(u(k|k), \dots, u(k+N_p-1|k)\big)$ represents possible control input candidates along a horizon of length N_p. Once the problem (14) is (approximately) solved at time k and for some state $v(k)$, the first element of its solution is returned as the control signal and, hence, defines the law (13).

The constraints in (14) involve, on a horizon of length N_p, the dynamics of the augmented system (12), the function f characterizing the nonlinearity Δ, and the hard state and control input constraints defined by the sets $\mathcal{X}$ and $\mathcal{U}$, which are motivated, e.g., by physical limitations. Moreover, they include a static terminal feedback law defined by the gain $\kappa_t \in \mathbb{R}^{n_u \times n_x}$ and another constraint on the state characterized by the terminal set $\mathbf{X}_t$ at the end of the considered horizon.

The cost function J in (14) comprises a so-called stage cost ℓ summed over the prediction horizon N_p weighting the states and control inputs and a terminal cost V, which weights the state at the end of this horizon.

Remark 2. We stress the following points regarding the considered MPC formulation:

- The augmented state $v = \left(\begin{smallmatrix}\xi\\ x\end{smallmatrix}\right)$ at time k enters the MPC problem (14) as a parameter. If the nonlinearity Δ is not known a priori and, hence, the filter's state ξ is not available, one would have to approximate it by including an observer or estimation scheme.

- As we do assume that Δ is known, it is actually possible to replace the terminal feedback law by a full-information law that returns $\kappa_t x(k+N_p-1|k)+\kappa_w w(k+N_p-1|k)$, or by an augmented-state-feedback law that returns $\kappa_v v(k+N_p-1|k)$. In the sequel, we maintain the standard choice of using $u(k+N_p-1|k) = \kappa_t x(k+N_p-1|k)$ and emphasize that the nonlinear feedback law (13) is still a function of the augmented state.

- The prediction model used in (14) turns the MPC optimization problem into a nonlinear one. In order to relax it, the interconnection signal w could be viewed as an unknown input disturbance and dealt with by considering corresponding maximization problem $\max_{\|w(k+j|k)\| \leq \overline{w}} J(v(k), U_k)$ subject to the same constraints (14). This is the so called min.-max. approach[6]. Other possible solutions are to use a nominal model without the term $\mathcal{B}_w w(k+j|k)$ and consider constraints tightening, e.g.[25], or to consider the alternative view of IQC relaxations on dynamic programming, see e.g.[26]. Since the focus of this work are not robust alternatives to (14), we do not proceed in this direction.

The most important properties for our considerations are *recursive feasibility* of the MPC scheme (14) and *stability* of the corresponding closed-loop interconnection with (11). The former means that if (14) is feasible at time $k = 0$, then so it is for all times $k \in \mathbb{N}$, and the latter is defined as follows.

Definition 2 (Stability)**.** The closed-loop interconnection of the system (11) and the feedback law (13) is said to be stable if there exists a constant $c > 0$ such that

$$\|x(k)\| \leq c \text{ for all } k \in \mathbb{N}_0 \quad \text{and} \quad \lim_{k\to\infty} \|x(k)\| = 0$$

for any initial condition $x(0) \in \mathbb{R}^{n_x}$ with the property that the MPC scheme (14) is feasible at time $k = 0$ and for $v(0) = \left(\begin{smallmatrix} 0 \\ x(0) \end{smallmatrix}\right)$.

Recursive feasibility and closed-loop stability can be guaranteed based on the following criteria from[2].

Theorem 4 (Stability and recursive feasibility in MPC)**.** Assume that the stage cost $\ell : \mathbb{R}^{n_v} \times \mathbb{R}^{n_u} \to \mathbb{R}$ and the terminal cost $V : \mathbb{R}^{n_v} \to \mathbb{R}$ are quadratic, continuous and positive definite for all $x, u \in \mathcal{X} \times \mathcal{U}$. Moreover, suppose that the following conditions are satisfied:

(C1) The origin lies in the interior of the terminal set $\mathbf{X}_t$.

(C2) The terminal set $\mathbf{X}_t$ is positively invariant for the system in (12) in closed-loop with the terminal control law $u(k) = \kappa_t x(k)$. That is, for any time instance k, if $v(k)$ is contained in $\mathbf{X}_t$, then, so is $v(k+1)$.

(C3) The stage cost and terminal cost satisfy the following inequality for any closed-loop trajectories, for all $k \in \mathbb{N}_0$, with $v(k) \in \mathbf{X}_t$: $V(v(k+1)) - V(v(k)) \leq -\ell(v(k), \kappa_t x(k))$.

(C4) The terminal control law is admissible, i.e., for any $v = \left(\begin{smallmatrix} \xi \\ x \end{smallmatrix}\right) \in \mathbf{X}_t$ we have $\kappa_t x \in \mathcal{U}$.

(C5) The terminal set $\mathbf{X}_t$ only contains admissible states, i.e., for any $v = \left(\begin{smallmatrix} \xi \\ x \end{smallmatrix}\right) \in \mathbf{X}_t$ we have $x \in \mathcal{X}$.

Then, the MPC scheme (14) is recursively feasible and the resulting closed-loop interconnection is stable.

Proof. Despite the incorporation of the filter dynamics in the MPC optimization problem, the proof remains standard; we refer to[2] for complete details. We only sketch the arguments for showing recursive feasibility, as an illustration. By assumption, the MPC optimization problem (14) is feasible initially. Let $U_0^\star$ be the corresponding optimal predicted sequence of control inputs and note that the corresponding state sequence is given by

$$N_0^\star = \begin{bmatrix} v^\star(0|0) \\ v^\star(1|0) \\ \vdots \\ v^\star(N_p-1|0) \\ v^\star(N_p|0) \end{bmatrix} = \begin{bmatrix} v(0) \\ \mathcal{A}v^\star(0|0) + \mathcal{B}w^\star(0|0) + \mathcal{B}_u u^\star(0|0) \\ \vdots \\ \mathcal{A}v^\star(N_p-2|0) + \mathcal{B}w^\star(N_p-2|0) + \mathcal{B}_u u^\star(N_p-2|0) \\ \left(\mathcal{A} + \mathcal{B}_u\kappa_v\right) v^\star(N_p-1|0) + \mathcal{B}w^\star(N_p-1|0) \end{bmatrix};$$

here, we abbreviate $\kappa_v := (0, \kappa_t)$ and denote by $w^\star(j|0)$ the corresponding output of the nonlinearity. Note that, from the terminal state constraint in (14), it is implied that $v^\star(N_p|0) \in \mathbf{X}_t$.

Regarding the MPC scheme at the following time instant $k = 1$ for $v^\star(1|0)$, we consider the control sequence candidate

$$\hat{U}_1 = \mathrm{col}\left\{ u^\star(1|0) \quad u^\star(2|0) \quad \dots \quad \kappa_v v^\star(N_p-1|0) \quad \kappa_v v^\star(N_p|0) \right\}.$$

By construction, the corresponding state sequence is given by

$$\hat{N}_1 = \mathrm{col}\left\{ v^\star(1|0) \quad v^\star(2|0) \quad \dots \quad v^\star(N_p|0) \quad \hat{v}(N_p+1|1) \right\},$$

where

$$\hat{v}(N_p+1|1) = \left(\mathcal{A} + \mathcal{B}_u\kappa_v\right) v^\star(N_p|0) + \mathcal{B}\hat{w}(N_p|1) \quad \text{and} \quad \hat{w}(N_p|1) = \left(f \circ (\mathrm{id} - Df)^{-1}\right)\left(\left(0 \;\; C + D_u\kappa_t\right) v^\star(N_p|0)\right).$$

By (C2) and $v^\star(N_p|0) \in \mathbf{X}_t$, we then obtain $\hat{v}(N_p+1|1) \in \mathbf{X}_t$. Hence, we can conclude from (C5) that the corresponding state is contained in $\mathcal{X}$, i.e., satisfies the state constraints. Similarly, we infer from $v^\star(N_p|0) \in \mathbf{X}_t$ and (C4) that $\kappa_v v^\star(N_p|0) \in \mathcal{U}$ holds, i.e., the input constraints are satisfied. In total, we have shown that $\hat{U}_1$ is an admissible candidate for the MPC optimization in (14) at time instant $k = 1$, which is thus feasible. The same procedure applies to all following time instances, which means that the MPC optimization is recursively feasible. □

Next and for an a priori given terminal feedback gain κ_t, we provide an LMI test for analyzing whether we can find a suitable terminal cost V and a terminal set $\mathbf{X}_t$. In particular, this test amounts to solving a convex semi-definite program which can nowadays easily be done by off-the-shelf solvers.

Theorem 5 (IQC-based stability and recursive feasibility test for a given terminal gain)**.** Suppose that the nonlinear operator Δ is memory-less and satisfies (IQC) for some filter Ψ and matrices $Z \in \mathbb{S}^{n_\xi}$, as well as $M \in \mathbb{S}^{m}$. Moreover, let $\kappa_t \in \mathbb{R}^{n_u \times n_v}$ be some static feedback gain and let $Q \in \mathbb{S}^{n_x}$ and $R \in \mathbb{S}^{n_u}$ be known positive definite weighting matrices. Finally, let the admissibility sets be given by $\mathcal{X} := \{x \in \mathbb{R}^{n_x} : -\bar{x} \le x \le \bar{x}\}$ and $\mathcal{U} := \{u \in \mathbb{R}^{n_u} : -\bar{u} \le u \le \bar{u}\}$, for some $\bar{x} \in \mathbb{R}^{n_x}$ and some $\bar{u} \in \mathbb{R}^{n_u}$. If there exist a matrix $\mathfrak{X} = \left(\begin{smallmatrix} X_\Psi & W \\ \star & X \end{smallmatrix}\right) \in \mathbb{S}^{n_\xi + n_x}$ satisfying

$$
(\star)\left(\begin{array}{cc|c|cc}
\mathfrak{X} & 0 & & & \\
0 & -\mathfrak{X} & & & \\
\hline
 & & M & & \\
\hline
 & & & Q & 0 \\
 & & & 0 & R
\end{array}\right)
\left(\begin{array}{ccc}
A_\Psi & B_\Psi \left(\begin{smallmatrix} C+D_u\kappa_t \\ 0 \end{smallmatrix}\right) & B_\Psi \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \\
0 & A + B_u\kappa_t & B \\
I & 0 & 0 \\
0 & I & 0 \\
\hline
C_\Psi & D_\Psi \left(\begin{smallmatrix} C+D_u\kappa_t \\ 0 \end{smallmatrix}\right) & D_\Psi \left(\begin{smallmatrix} D \\ I \end{smallmatrix}\right) \\
\hline
0 & I & 0 \\
0 & \kappa_t & 0
\end{array}\right) < 0, \qquad
\begin{pmatrix} X_\Psi - Z & W \\ \star & X \end{pmatrix} > 0\,, \tag{15a}
$$

as well as

$$
\left(\begin{array}{cc|c}
X_\Psi - Z & W & 0 \\
\star & X & I_{n_x,j} \\
\hline
0 & \star & \bar{x}_j^2
\end{array}\right) > 0 \text{ and }
\left(\begin{array}{cc|c}
X_\Psi - Z & W & 0 \\
\star & X & \kappa_t^\top I_{n_u,i} \\
\hline
0 & \star & \bar{u}_i^2
\end{array}\right) > 0 \tag{15b}
$$

for all $j \in \mathbb{N}_{[1,n_x]}$ and $i \in \mathbb{N}_{[1,n_u]}$, respectively, then all of the assumptions in Theorem 4 are satisfied for:

- the stage cost $\ell : \mathbb{R}^{n_v} \times \mathbb{R}^{n_u} \to \mathbb{R}$, $(x, u) \mapsto x^\top Q x + u^\top R u$,
- the terminal cost $V : \mathbb{R}^{n_v} \to \mathbb{R}$, $v \mapsto v^\top \left(\begin{smallmatrix} X_\Psi - Z & W \\ \star & X \end{smallmatrix}\right) v$ and
- the terminal set $\mathbf{X}_t := \{v \in \mathbb{R}^{v} : v^\top \left(\begin{smallmatrix} X_\Psi - Z & W \\ \star & X \end{smallmatrix}\right) v \le 1\}$.

In particular, then the MPC scheme (14) is recursively feasible and the resulting closed-loop interconnection is stable.

Theorem 5 involves several assumptions that are quite standard in the MPC literature. Here, they are supplemented by the hypothesis that the nonlinearity Δ satisfies (IQC) and by non-standard inequalities with a similar structure as (7) for the interconnection the augmented system (12) in feedback with the terminal gain κ_t.

In particular, the inequality on the left hand side of (15a) together with the assumption that Δ satisfies (IQC) permit us to assure (C3) from Theorem 4. Indeed, this is essentially a consequence of the IQC Theorem 2. Moreover, the inequality on the right hand side of (15a) allows the natural construction of a quadratic and positive definite terminal cost. Finally, since we use a terminal set defined as a sublevel set of the terminal cost, the inequalities in (15b) guarantee the satisfaction of the state and input constraints once the state is in the terminal set. This addresses (C4) and (C5) of Theorem 4.

Proof. We verify each condition in Theorem 4 individually. Note, at first, that by the assumption on Q and R, as well as by the right hand side of (15a), it is implied that that $\ell(\cdot,\cdot)$ and $V(\cdot)$ are quadratic, continuous and positive definite. Consequentially, (C1) is satisfied.
Proof of (C2): By the definiteness of $\ell(\cdot,\cdot)$ and assuming (C3) holds, we directly obtain

$$
V(v(k+1)) \le V(v(k))\,.
$$

Together with the definition of $\mathbf{X}_t$, this implies that $v(k+1) = \left(\begin{smallmatrix} \xi(k+1) \\ x(k+1) \end{smallmatrix}\right) \in \mathbf{X}_t$ holds whenever $v(k) = \left(\begin{smallmatrix} \xi(k) \\ x(k) \end{smallmatrix}\right) \in \mathbf{X}_t$.
Proof of (C3): Let $v(k) = \left(\begin{smallmatrix} \xi(k) \\ x(k) \end{smallmatrix}\right) \in \mathbf{X}_t$ for some $k \in \mathbb{N}_0$. Then, the left hand side of (15a) implies

$$
\begin{aligned}
v(k+1)^\top \mathfrak{X} v(k+1) - v(k)^\top \mathfrak{X} v(k) &= (\star) \begin{pmatrix} \mathfrak{X} & 0 \\ 0 & -\mathfrak{X} \end{pmatrix} \begin{pmatrix} \mathcal{A} + \mathcal{B}_u\kappa_t & \mathcal{B} \\ I & 0 \end{pmatrix} \begin{pmatrix} v(k) \\ w(k) \end{pmatrix} \\
&\le -(\star) M \begin{pmatrix} \mathcal{C} + \mathcal{D}_u\kappa_t & \mathcal{D} \end{pmatrix} \begin{pmatrix} v(k) \\ w(k) \end{pmatrix} - (\star) \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \left(\begin{array}{cc|c} 0 & I & 0 \\ 0 & \kappa_t & 0 \end{array}\right) \begin{pmatrix} v(k) \\ w(k) \end{pmatrix} \\
&= -y(k)^\top M y(k) - \ell(x(k), \kappa_t x(k))\,.
\end{aligned}
$$

By combining this inequality with (IQC), we are able to obtain the desired inequality

$$
V(v(k+1)) - V(v(k)) = v(k+1)^\top \mathfrak{X} v(k+1) - v(k)^\top \mathfrak{X} v(k) - \xi(k+1)^\top Z \xi(k+1) + \xi(k)^\top Z \xi(k) \le -\ell(x(k), \kappa_t x(k))\,.
$$

Proof of (C4) and (C5): These conditions follow from a direct application of the Schur complement to (15b), considering the ellipsoidal form and definition of $\mathbf{X}_t$. Indeed, for any $j \in \mathbb{N}_{[1,n_x]}$, Schur implies $\begin{pmatrix} X_\Psi - Z & W \\ \star & X \end{pmatrix} - \begin{pmatrix} 0 \\ I_{n_x,j} \end{pmatrix} \frac{1}{\bar{x}_j^2}(\star) > 0$ and, hence,

$$x_j^2 = x^\top I_{n_x,j} I_{n_x,j}^\top x = \frac{\bar{x}_j^2}{\bar{x}_j^2} \begin{pmatrix} \xi \\ x \end{pmatrix}^\top \begin{pmatrix} 0 \\ I_{n_x,j} \end{pmatrix} \begin{pmatrix} 0 \\ I_{n_x,j} \end{pmatrix}^\top \begin{pmatrix} \xi \\ x \end{pmatrix} \le \bar{x}_j^2 \nu^\top \begin{pmatrix} X_\Psi - Z & W \\ \star & X \end{pmatrix} \nu \le \bar{x}_j^2 \quad \text{for any} \quad \nu = \begin{pmatrix} \xi \\ x \end{pmatrix} \in \mathbf{X}_t.$$

This shows (C5) and (C4) follows analogously. □

3.2 | Concrete algorithms for designing all terminal ingredients

Recall that the inequalities in Theorem 5 are convex in the involved variables only if the terminal gain κ_t is fixed. If we instead search for a suitable terminal cost, terminal set *and* a terminal gain κ_t satisfying these inequalities, then we are facing a challenging non-convex problem; in principle, this problem can be tackled, for example, via a coordinate descent approach, but this is rarely efficient. The LMI literature contains numerous approaches to render such and other challenging design problem convex by means of suitable algebraic manipulations such as variants of the well-known convexifying parameter transformation[27,28].

Let us now assume that the nonlinear operator Δ satisfies (IQC) for a static filter Ψ. Then we can apply a special case of the latter transformation which leads to the following result.

Proposition 1 (Static multipliers)**.** Let $M \in \mathbb{S}^{n_z+n_w}$ be non-singular and set $N := M^{-1}$. Then, there exists a stabilizing state-feedback gain κ_t such that the inequalities in (15) are feasible for a static filter $\Psi = I$ if and only if there exist a symmetric matrix $Y \in \mathbb{S}^{n_x}$ and a rectangular matrix $F \in \mathbb{R}^{n_u \times n_x}$ satisfying

$$(\star) \left(\begin{array}{cc:c:cc} Y & 0 & & & \\ 0 & -Y^{-1} & & & \\ \hdashline & & N & & \\ \hdashline & & & Q^{-1} & 0 \\ & & & 0 & R^{-1} \end{array}\right) \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ -(AY + B_u F)^\top & -(CY + D_u F)^\top & -Y & -F^\top \\ \hdashline 0 & I & 0 & 0 \\ -B^\top & -D^\top & 0 & 0 \\ \hdashline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) > 0\,, \tag{16a}$$

as well as

$$\left(\begin{array}{c|c} Y & Y I_{n_x,j} \\ \hline \star & \bar{x}_j^2 \end{array}\right) > 0 \;\text{ and }\; \left(\begin{array}{c|c} Y & F^\top I_{n_u,i} \\ \hline \star & \bar{u}_i^2 \end{array}\right) > 0 \tag{16b}$$

for all $j \in \mathbb{N}_{[1,n_x]}$ and $i \in \mathbb{N}_{[1,n_u]}$, respectively. Moreover, if the above inequalities are feasible, then a suitable feedback gain is given by $\kappa_t := FY^{-1}$.

Proof. We suppose that the inequalities (16) are satisfied. Then, we demonstrate that the inequalities in (15) are satisfied using $X := Y^{-1}$, $M = N^{-1}$ and $\kappa_t := FY^{-1}$; the converse follows from reversing the arguments.

Note, at first, that the left upper blocks of LMIs (16b) imply $Y > 0$ and, hence, the right inequality in (15a). Then, we can perform a congruence transformations using $\begin{pmatrix} Y^{-1} & 0 \\ 0 & 1 \end{pmatrix}$ for the inequalities in (16b) in order to infer that the inequalities in (15b) are satisfied for $X = Y^{-1}$ and $\kappa_t := FY^{-1}$.

It remains to show that the inequality from the left of (15a) is satisfied. To this end, we express (16a) as

$$(\star) \left(\begin{array}{cc|c:cc} X^{-1} & 0 & & & \\ 0 & -X^{-1} & & & \\ \hline & & M^{-1} & & \\ \hdashline & & & Q^{-1} & 0 \\ & & & 0 & R^{-1} \end{array}\right) \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ -(A + B_u \kappa_t)^\top & -(C + D_u \kappa_t)^\top & -I & -\kappa_t^\top \\ \hline 0 & I & 0 & 0 \\ -B^\top & -D^\top & 0 & 0 \\ \hdashline 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right) > 0\,.$$

It remains to apply the dualization lemma, as stated for example in[29], which results in the desired inequality in (15a) for the static filter $\Psi = I$. This concludes the proof. □

Let us now more concretely assume that the nonlinear operator Δ is memory-less and repeated; this means that it has the form (10). Moreover, we suppose that its characterizing function $\phi : \mathbb{R} \to \mathbb{R}$ is sector-bounded and slope-restricted. Hence, this operator satisfies (IQC) for any of the multiplier sets recapitulated in Section 2. Since we are, at the moment, still considering

IQCs based on a static filter, this leaves us with the DG-scalings in Lemma 1 and the full-block multipliers in Lemma 2. However, since $N = M^{-1}$ is involved in the inequalities (16) (instead of M), we require the corresponding so-called dual multiplier sets. These are given by

$$\mathcal{N}_{\mathrm{DG}} := \left\{ (\star) \begin{pmatrix} 0 & S \\ S & 0 \end{pmatrix} \begin{pmatrix} I & aI \\ I & bI \end{pmatrix} \;\middle|\; S \in \mathbb{S}^{n_z} \text{ is diagonal and positive definite} \right\}$$

for $\mathcal{M}_{\mathrm{DG}}$ in Lemma 1 and by

$$\mathcal{N}_{\mathrm{FB}} := \left\{ M \;\middle|\; \begin{pmatrix} -\Delta_j^\top \\ I \end{pmatrix}^\top M \begin{pmatrix} -\Delta_j^\top \\ I \end{pmatrix} \leq 0 \text{ for all } j \text{ and } \begin{pmatrix} I \\ 0 \end{pmatrix}^\top M \begin{pmatrix} I \\ 0 \end{pmatrix} \geq 0 \right\}$$

for $\mathcal{M}_{\mathrm{FB}}$ in Lemma 2. Indeed, one easily checks that $M \in \mathcal{M}_{\mathrm{DG}}$ is equivalent to $M^{-1} \in \mathcal{N}_{DG}$, which holds in the same fashion for full-block multipliers.

With this we have collected all pieces of the puzzle and can combine them to obtain the following result that provides efficient criteria for designing all terminal ingredients of the MPC scheme (14) and which is based on IQCs with a static filter Ψ.

Corollary 1 (Static multipliers)**.** Let $\mathcal{N}$ be either $\mathcal{N}_{\mathrm{DG}}$ or $\mathcal{N}_{\mathrm{FB}}$. Moreover, suppose that there exist matrices $Y \in \mathbb{S}^{n_x}$, $N \in \mathcal{N}$ and $F \in \mathbb{R}^{n_u \times n_x}$ satisfying the inequalities (16) and that the MPC scheme (14) is operated using the terminal cost $V : x \mapsto x^\top Y^{-1} x$, the terminal set $\mathbf{X}_t = \{x : V(x) \leq 1\}$ and the terminal law $\kappa_t := FY^{-1}$. Then, the MPC is recursively feasible and the resulting closed-loop is stable.

Note that the inequalities (16) can be easily rendered convex in Y, N and F by applying the Schur complement. Hence, they can indeed be solved efficiently. Moreover, we can search for suitable terminal ingredients and simultaneously maximize the size of the corresponding terminal region $\mathbf{X}_t$. This can be achieved by minimizing an upper bound on the trace of Y^{-1}, which can be represented as an LMI constraint, subject to the Schur'ed version of the inequality (16). Likewise, minimizing $-\log(\det(Y))$ also imposes the maximization of the resulting ellipsoid, and is supported as a cost function in standard solvers.

Remark 3. If we suppose that $D = 0$ holds and that the involved nonlinearity ϕ is sector-bounded with bounds $(a, b) = (0, 1)$, we note that the choice $\mathcal{N} = \mathcal{N}_{DG}$ leads to the following Schur'ed version of (16):

$$\begin{pmatrix} Y & -(AY + B_u F)^\top & -(CY + D_u F)^\top & -Y & -F^\top \\ \star & Y & -SB & 0 & 0 \\ \star & \star & S + S & 0 & 0 \\ \star & \star & \star & Q^{-1} & 0 \\ \star & \star & \star & \star & R^{-1} \end{pmatrix} > 0 . \tag{17}$$

We emphasize that exactly this LMI is often seen in synthesis condition used in the MPC literature for Lur'e systems subject to sector-bounded nonlinearities. From an IQC point of view, however, the use static DG-scalings is often too restrictive and there are various, typically much better, alternatives. In particular, this reveals that our results seamlessly encompass existing ones and extends them.

Now, let us still assume that Δ is given as concretely as before, but, this time, we employ IQCs involving *dynamic* filters for its description. A major difference if compared to the previous design based on static filters is that the matter of designing all terminal ingredients cannot be turned into a convex problem. Nevertheless, based on introducing slack variables by applying the projection (or elimination) lemma[30], we arrive at the following result. Let us emphasize that the projection lemma is a very powerful result for convexifying controller design problems and for reducing the conservatism in convex approximations of non-convex design problems[31].

Proposition 2 (Dynamic multipliers)**.** The inequalities (15) are satisfied for some given matrices $\mathfrak{X}$, κ_t, Z and M if and only if there exists some slack variable $F \in \mathbb{R}^{n_u \times (n_v + n_w + n_u)}$ satisfying

$$(\star) \left(\begin{array}{cc|c:cc} \mathfrak{X} & 0 & & & \\ 0 & -\mathfrak{X} & & & \\ \hline & & M & & \\ \hdashline & & & Q & 0 \\ & & & 0 & R \end{array}\right) \left(\begin{array}{ccc} \mathcal{A} & \mathcal{B} & \mathcal{B}_u \\ I & 0 & 0 \\ \hline \mathcal{C} & \mathcal{D} & \mathcal{D}_u \\ \hdashline \begin{pmatrix} 0 & I_{n_x} \end{pmatrix} & 0 & 0 \\ 0 & 0 & I \end{array}\right) + (\star) + F^\top \begin{pmatrix} 0_{n_u \times n_\xi} & \kappa_t & 0_{n_u \times n_w} & -I \end{pmatrix} < 0 , \tag{18a}$$

$$\begin{pmatrix} X_\Psi - Z & W \\ \star & X \end{pmatrix} \succ 0 \text{ as well as } \left(\begin{array}{cc|c} X_\Psi - Z & W & 0 \\ \star & X & I_{n_x,j} \\ \hline 0 & \star & \bar{x}_j^2 \end{array}\right) \succ 0\,, \text{ and } \left(\begin{array}{cc|c} X_\Psi - Z & W & 0 \\ \star & X & \kappa_t^\top I_{n_u,i} \\ \hline 0 & \star & \bar{u}_i^2 \end{array}\right) \succ 0 \tag{18b}$$

for all $j \in \mathbb{N}_{[1,n_x]}$ and $i \in \mathbb{N}_{[1,n_u]}$, respectively.

The proof of Proposition 2 follows by noticing that

$$\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & \kappa_t & 0 \end{pmatrix} \text{ is a basis matrix of the kernel of } \begin{pmatrix} 0_{n_u \times n_\xi} & \kappa_t & 0_{n_u \times n_w} & -I \end{pmatrix},$$

and from a direct application of the projection lemma[30].

This brings us to the following design criteria for all terminal ingredients of the MPC scheme (14) based on IQCs with dynamic filters. In particular, we can employ the combination of static full-block multipliers and dynamic O'Shea-Zames-Falb multipliers as given in Lemma 3. We emphasize that these multipliers yield (IQC) for the considered operator Δ with a nontrivial storage $Z \neq 0$.

Corollary 2 (Dynamic multipliers)**.** Let Ψ be as in Lemma 3. Moreover, suppose that there exist $\mathfrak{X} \in \mathbb{S}^{n_\xi + n_x}$, $\kappa_t \in \mathbb{R}^{n_u \times n_x}$, $(Z, M) \in \mathcal{M}_{\text{OZF+FB}}$ and $F \in \mathbb{R}^{n_u \times (n_v + n_w + n_u)}$ satisfying (18) and that the MPC scheme (14) is operated using the terminal cost $V : \nu \mapsto \nu^\top \left(\mathfrak{X} - \left(\begin{smallmatrix} Z & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)\nu$, the terminal set $\mathbf{X}_t = \{\nu : V(\nu) \leq 1\}$ and the terminal law κ_t. Then, the MPC is recursively feasible and the resulting closed-loop is stable.

We emphasize that minimizing $\mathrm{trace}\left(\mathfrak{X} - \left(\begin{smallmatrix} Z & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$ or $\log\left(\det\left(\mathfrak{X} - \left(\begin{smallmatrix} Z & 0 \\ 0 & Z \end{smallmatrix}\right)\right)\right)$ subject to feasibility of the inequalities (18) is a difficult non-convex problem due to the involved product of the slack variable F and the feedback gain κ_t. However, this issue can be tackled based on the following coordinate descent scheme, which is a variant of the one proposed in[31] in the context of static output-feedback synthesis for LTI systems: Suppose that we have an initial controller κ_t satisfying (15) for Ψ as in Lemma 3 and $M \in \mathcal{M}_{\text{OZF+FB}}$. Such a controller can, for instance, be obtained by solving the static design problem with full-block multipliers. Then, we can alternately apply the following steps until the cost function stops decreasing:

- Minimize the cost subject to feasibility of the inequalities (18), with κ_t being fixed to the given feedback gain. Save the resulting optimal slack variable F;
- Minimize the cost subject to feasibility of the inequalities (18), with F being fixed to the given slack variable. Save the resulting optimal feedback gain κ_t.

For a discussion and summary of alternative approaches for dealing with related non-convex design problems, the reader can refer to[31] and[32]. Here, we emphasize that the optimization problems considered in this procedure are guaranteed to be recursively feasible and the corresponding optimal values form a monotonically decreasing sequence that is bounded from below, but this sequence is not assured to converge to the optimal value of the original problem. Still, in practice, one often obtains good approximations of the latter value. If compared to alternative approaches such as applying a coordinate descent scheme directly to the inequalities (15), the procedure outlined above is rather efficient since the variables $\mathfrak{X}$ and M are free at each of the steps.

Remark 4. Since we are applying an iterative procedure, and as we are mainly interested in the behavior of the closed-loop within the terminal set $\mathbf{X}_t$, we have the opportunity to determine upper and lower bounds on the the values of the interconnection signal z after each of the iterations. Depending on the available information on Δ, these upper bounds can permit us to employ refined local IQCs that are more accurate than global ones as stated in (IQC). See, for example,[33] for a more comprehensive debate regarding such local IQCs.

4 | NUMERICAL EXAMPLE

In this Section, we illustrate that employing IQCs in the context of stabilizing MPC schemes can be very beneficial. To this end, we consider the following discrete-time Lur'e system, with three states, two outputs, and one input, which operates at a sampling rate of 10 Hz:

$$
\begin{aligned}
x(k+1) &= \overbrace{\begin{bmatrix} 0.8 & 0.6 & 1 \\ -0.4 & 1.2 & 1 \\ 0 & 0.3 & 0.9 \end{bmatrix}}^{A} x(k) + \overbrace{\begin{bmatrix} -0.18 & 0.04 \\ 0.5 & 0.5 \\ 0 & 0.01 \end{bmatrix}}^{B} w(k) + \overbrace{\begin{bmatrix} 0.3 \\ 0 \\ 0.3 \end{bmatrix}}^{B_u} u(k)\,, \\
z(k) &= \overbrace{\begin{bmatrix} -0.1 & -1.8 & 0 \\ 1 & 1 & 0 \end{bmatrix}}^{C} x(k) + \overbrace{\begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}}^{D_u} u(k)\,, \\
w(k) &= \Delta(z)(k)\,.
\end{aligned}
$$

Here, we suppose that the nonlinearity is given as in (10) with the standard saturation function $\phi(z) = \text{sat}(z) = \max(-1, \min(1, z))$. Note that this saturation is sector-bounded and slope-restricted, both with bounds $(0, 1)$.

This system is said to be *admissibly operated* as long as $x(k) \in \mathcal{X}$ and $u(k) \in \mathcal{U}$ hold for all $k \geq 0$, where the respective sets are defined as

$$
\mathcal{X} := \left\{ x \in \mathbb{R}^{n_x} \,:\, |x_j| \leq 1, \forall j \in \mathbb{N}_{[1,n_x]} \right\} \quad \text{and} \quad \mathcal{U} := \{ u \in \mathbb{R} \,:\, |u| \leq 1 \}\,.
$$

Accordingly, we use both static and dynamic IQC multipliers in order to compute the terminal ingredients of the corresponding MPC optimization, as proposed in the main contribution of this work, Theorem 5. The static multiplier solution is enabled using Proposition 1, while the dynamic approach is based on Proposition 2. Taking into account the specificity of the considered system class, we use the corresponding Corollaries (1 and 2, respectively). We stress that the results presented in the sequel were obtained in an M1 Pro Macintosh environment using Matlab and Yalmip[34], with Gurobi[35] and SDPT3[36] solvers (for the MPC optimization and the LMIs, respectively).

In order to thoroughly debate the proposed MPC formulation synthesized with the IQC arguments, we compare it to the standard approach in the literature involving Lyapunov arguments to handle the nonlinearity using the sector condition, as done in[5,6,7]. Henceforth, we refer to this approach as *traditional* MPC and recall that the corresponding terminal ingredients are synthesized using Corollary 1 with $\mathcal{N} = \mathcal{N}_{\text{DG}}$ (see Remark 3).

Next, we compare the obtained regulation results of the considered system, when applying the *traditional* MPC, with the ones with static and dynamic multipliers. To this end, we use of prediction horizon of $N_p = 4$ (samples) and the weights $Q = \text{diag}(0.5\,, 0.5\,, 0.6)$ and $R = 0.8$.

In the sequel, by *static multiplier* MPC, we mean the MPC scheme with terminal ingredients synthesized based on Corollary 1, using $\mathcal{N} = \mathcal{N}_{\text{FB}}$. By *dynamic multiplier* MPC, we refer to the ingredients synthesized using dynamic full-block O'Shea-Zames-Falb multipliers with the non-trivial storage, as given by Corollary 2. We note that, in the dynamic approach, we use an LTI filter Ψ with $n_\xi = 4$, as done in[21], and we employ the local refinements as mentioned in Remark 4.

Accordingly, we consider three simulation scenarios: first, we compare the nominal performances of the three controllers, i.e., without the presence of any disturbances; then, we test the multiplier-based MPCs when the system is subject to disturbances; finally, we assess the issue of feasibility and its relationship to the size of the MPC prediction horizon.

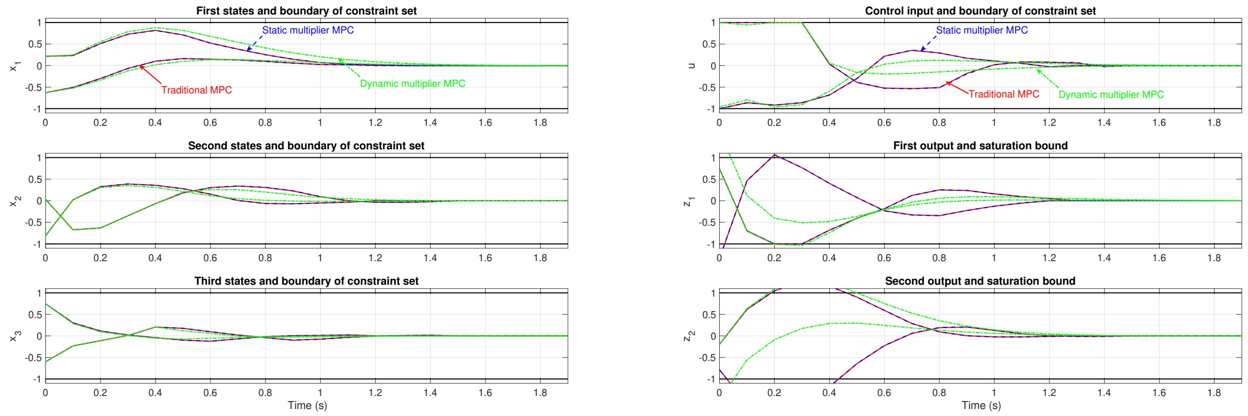
When applying Corollary 1 by choosing $\mathcal{N}$ to be $\mathcal{N}_{\text{DG}}$, we actually recover the same solution as with traditional design. Since the results are very close to what is obtained with the static full-block multipliers, this traditional design is only included in the first simulation scenario.

First simulation: We consider the closed-loop system's responses to two different initial conditions contained in the admissibility set:

- In Figure 4a, we present the state trajectories obtained with each controller. The results obtained with the static multiplier synthesis are roughly the same as those obtained with the traditional synthesis, as already mentioned before. Overall, the dynamic multiplier synthesis provides smoother regulation.

- In Figure 4b, we present the corresponding control inputs, considering the three controllers (traditional MPC, static multiplier MPC, and dynamic multiplier MPC), as well the two components of the interconnection signal z.

- In order to numerically assess the obtained results, we summarize some key performance indices in Table 1. Therein, we present the average computational time required to evaluate the MPC optimization (t_c), the average RMS index for each system state (over all initial conditions) with each controller; we also give the TV index, which measures the total



(a) System state trajectories with the three controllers.

(b) Control inputs and system outputs.

FIGURE 4 First simulation.

variance of the control signal over time (i.e. $\sum_{k=0}^{T} \|u(k+1)-u(k)\|$). These indices indicate quantitatively how the dynamic synthesis offer less conservative results (better state regulation with a smoother control signal).

- Finally, we present the synthesized terminal set (projection onto the x plane), and the corresponding state trajectories, in Figure 5. As one can see, the dynamic multiplier approach provides a much larger terminal region, due to flexibility of these multipliers. Numerically, we obtain the following traces related to these ellipsoids: 338105 (Traditional MPC), 2101511 (Static multiplier MPC with full-block multipliers), and 29.44 (Dynamic multiplier MPC with local full-block multipliers). We note that a wider terminal region allows for less conservative control. Moreover, the size of the terminal region affects the feasibility of the initial conditions, which we assess in the third simulation results.

TABLE 1 First simulation: Performance comparison.

Method	RMS: x_1	x_2	x_3	TV	t_c (ms)
Traditional MPC	0.2820	0.2524	0.1769	1.1393	1.70
Static multiplier MPC	**0.2813**	0.2523	0.1770	1.1494	1.67
Dynamic multiplier MPC	0.2859	**0.2449**	**0.1715**	**0.7009**	1.80

Second simulation: Next, we consider the system dynamics over a simulation run of nine seconds. In this scenario, we consider the presence of disturbances; these are emulated by three sudden time-decaying signals, which are added to each state dynamics. These disturbances are chosen as randomly, yet given within $\mathcal{X}$; they occur at the middle of the simulation run, as shown in Figure 6.

Accordingly, the synthesized MPC algorithms regulate the state dynamics towards the origin, departing from twenty different initial conditions, and subject to a sudden time-decaying load disturbance added to each state dynamics, occurring at the middle of the simulation run, as shows Figure. 6. Accordingly, the obtained result are presented in Figure 7, where state, output and input trajectories are presented. In Table 2, we give some performances indices to assess these results. Again, we can conclude that the dynamic IQC synthesis offers overall a less conservative performance, being able to regulate the system states faster and with a smoother control law (as shown by the TV performance index).

Third simulation: Finally, we assess how the size of the terminal region affects the initial feasibility of the MPC algorithm. As discussed in Theorems 4 and 5, the control algorithm is able to ensure recursive feasibility as long the first MPC optimization problem is feasible for the initial condition $v(0) = \left(\begin{smallmatrix} 0 \\ x(0) \end{smallmatrix}\right)$. The initial feasibility is linked to the length of the prediction horizon N_p, i.e. the MPC should be able to steer the states to the interior of the terminal set $\mathbf{X}_t$ within N_p steps. This means that when a given initial condition x_0 is not feasible for a given horizon length, this horizon can be enlarged to ensure feasibility. A thorough

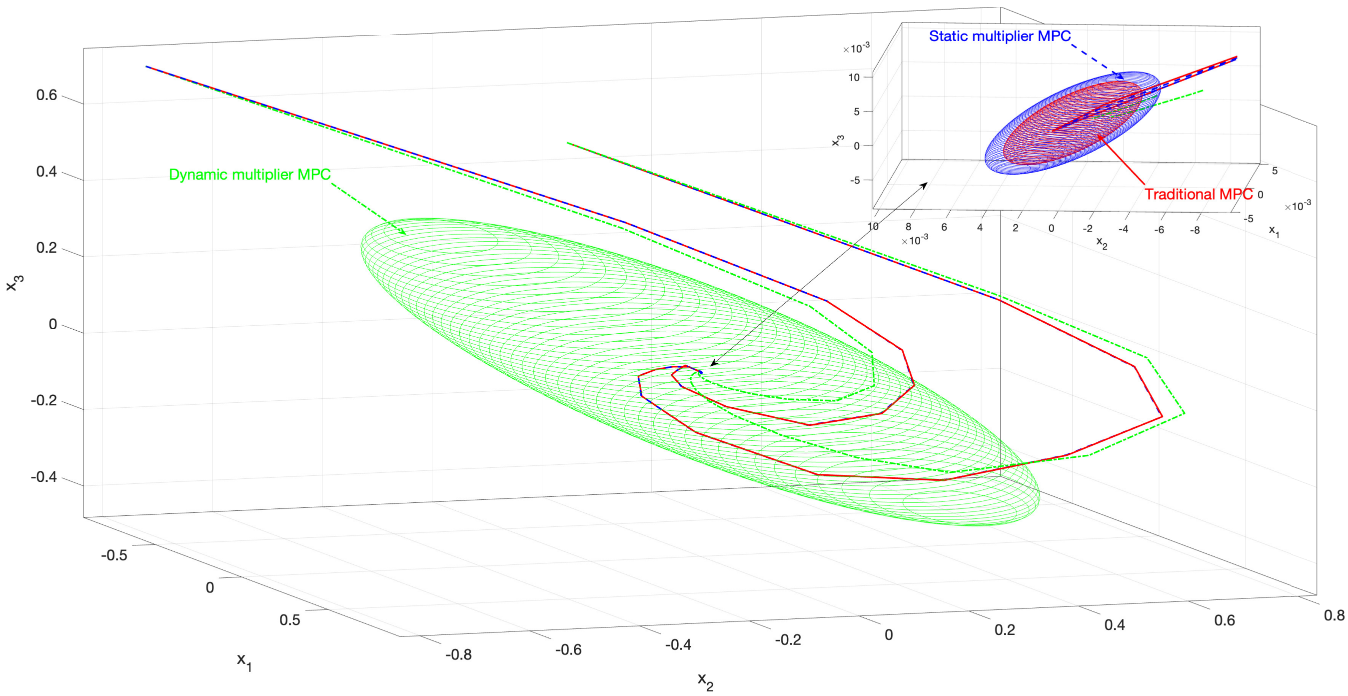


FIGURE 5 First simulation: System state trajectories and terminal ellipsoid.

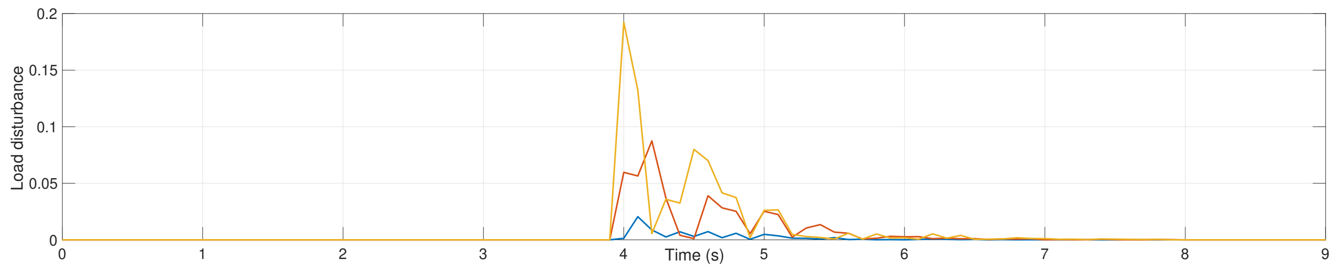


FIGURE 6 Second simulation: Load disturbances.

TABLE 2 Second simulation: Performance comparison.

Method	RMS: x_1	x_2	x_3	TV	t_c (ms)
Static multiplier MPC	**0.1653**	0.1261	0.0665	2.8033	1.53
Dynamic multiplier MPC	0.1718	**0.1220**	**0.0616**	**1.7548**	2.06

discussion on the role of the prediction horizon is presented in [37]. Accordingly, using the initial condition[2] $x_0 = [1\,,\, 0.9\,,\, 0.26]^T$, we obtain **infeasibility** using the static multiplier synthesis with $N_p = 4$, which means that the controller is unable to steer the state trajectories to the terminal set (the one shown in Figure 5, in blue) with this horizon length. Feasibility is enabled only with $N_p = 8$, for this initial condition. Since the terminal region is much larger with the dynamic multipliers, we are able to obtain feasibility already with $N_p = 4$, as shown in Figure 8, which is a significant advantage.

With the result shown in the prequel, we emphasize some key messages:

[2]In this simulation, the bounds on the first state are actually $|x_1| \leq 1$ so that a boundary initial point could be used.

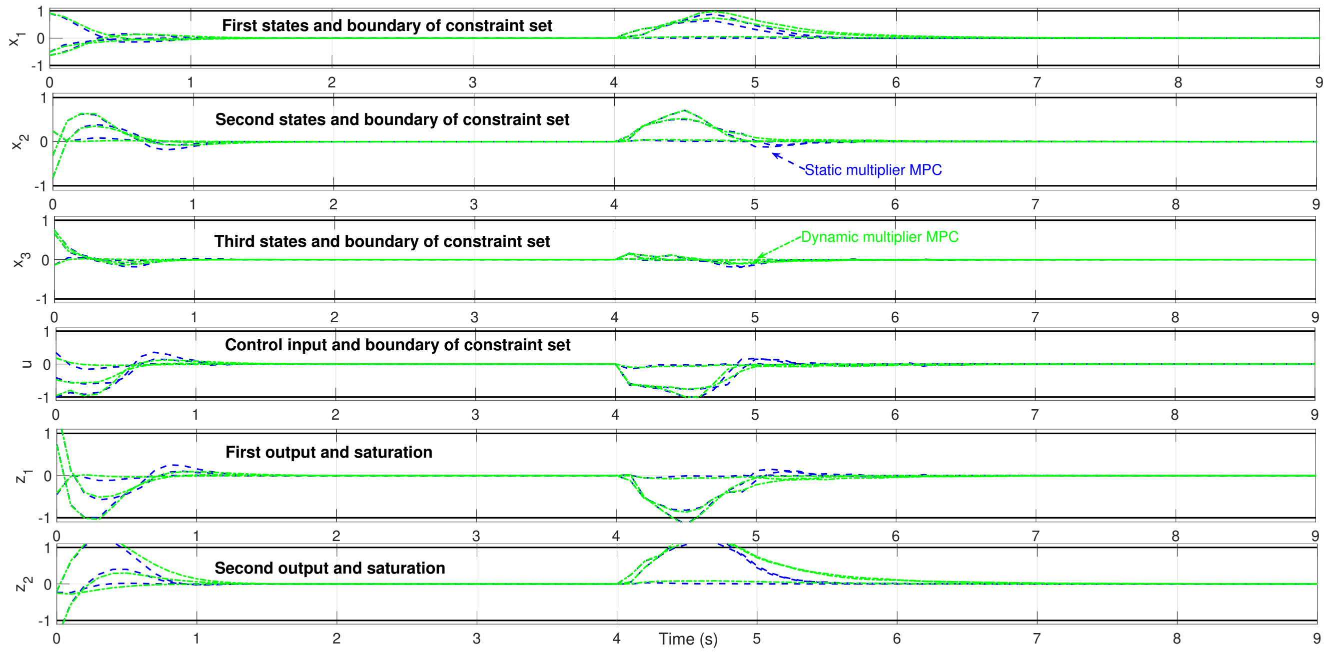


FIGURE 7 Second simulation: States, outputs and control input.

- The proposed stability framework, as summarized in Theorem 5, allows one to choose an specific IQC that matches the considered feedback operator Δ. In practice, this is very generic and can be exploited in the context of many system classes.
- Accordingly, the theorem allows to render the MPC optimization recursively feasible, the closed-loop stable, and the dynamics dissipative with regard to the uncertainty. Moreover, the IQC formulations mentioned so far allow to benefit from local information and enables better control performances, in general.
- Furthermore, the dynamic IQC formulation usually offers the most freedom in capturing the input-output behavior of the uncertainty operator Δ. In this way, we obtain less conservatism and a larger terminal region. This fact implies that the related MPC controller has a wider region of attraction, i.e. initial feasibility can be satisfied for a larger set of initial conditions and for the same fixed horizon length.

5 | CONCLUSIONS

In this work, we provide a novel synthesis procedure for MPC algorithms for a class of Lur'e systems. Our procedure relies on capturing the input-output behavior of the involved nonlinear operator by means of IQCs with (static and dynamic) multipliers. In particular, we introduce point-wise IQCs with storage that are more general than point-wise IQCs and related to finite-horizon IQCs with terminal cost [18]. We show that this notion of IQCs can be employed for efficiently designing all terminal ingredients involved in the MPC scheme with less conservatism if compared to traditional approaches. We restrict our attention to systems affected by sector-bounded and slope-restricted operators, but our approach is capable of dealing with other (uncertain) operators due to the flexibility of the IQC framework.

By the means of a numerical benchmark with a repeated saturation, we test and compare the proposed MPC synthesis with a standard approach from the literature (relying on Lyapunov arguments and sector conditions for the nonlinearity). We illustrate that the terminal ingredients synthesized by our method yields larger terminal regions and lead to smoother regulation of the closed-loop system.

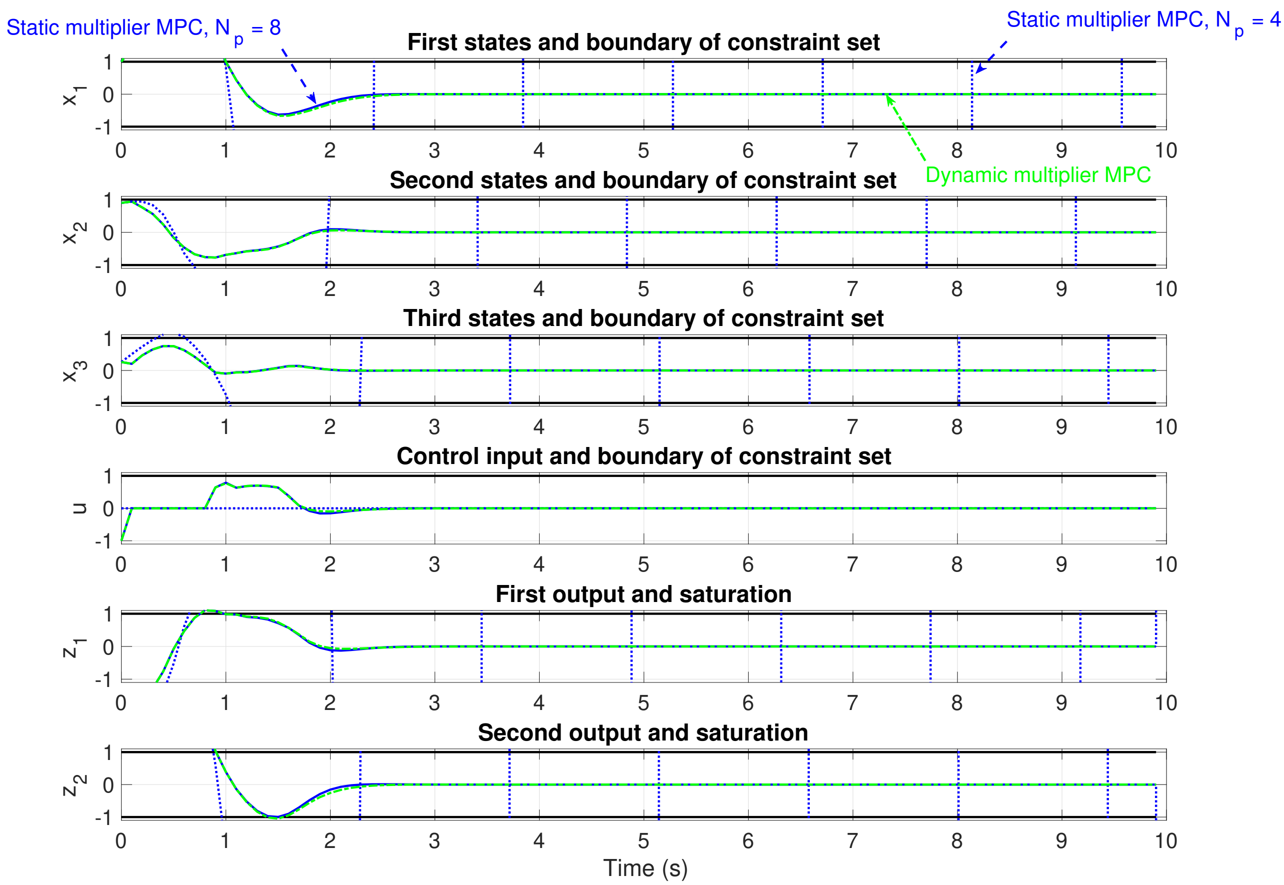


FIGURE 8 Third simulation: States, outputs and control input.

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Author contributions

All authors have contributed equally for this paper.

Financial disclosure

None reported.

Conflict of interest

The authors declare no potential conflict of interests.

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