Robust Bounds on the Eigenvalues of Linear Systems with Delays and Their Applications

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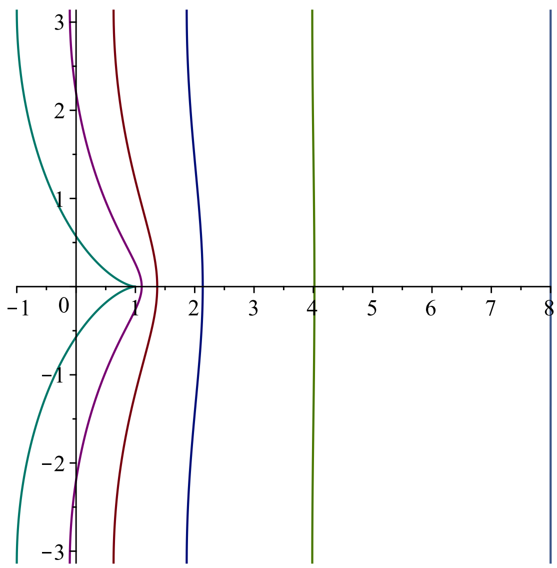
Abstract

We review some known bounds for eigenvalues of matrices and use similar techniques to derive bounds for nonlinear eigen problems and the eigenvalues for LTI systems with delays as a special case. There are two classes of results. The first are based on Hermitian decompositions, the second on Gershgorin's theorem. The bounds are easily computable. We reflect on implications for stability theory, which may be contrasted with bounds that have been obtained via Riccati stability based on Lyapunov-Krasovskii theory.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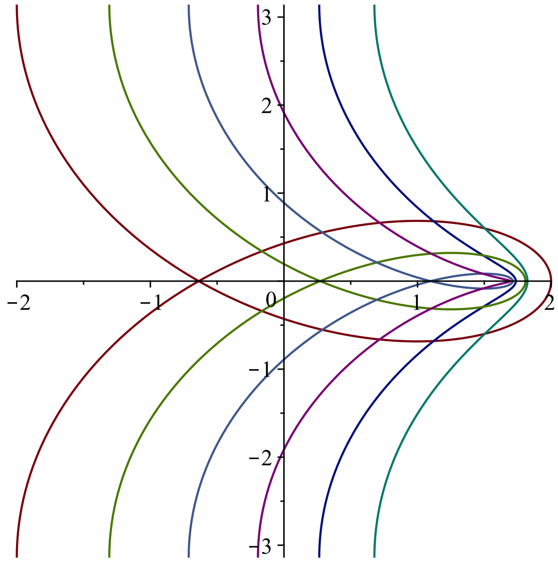
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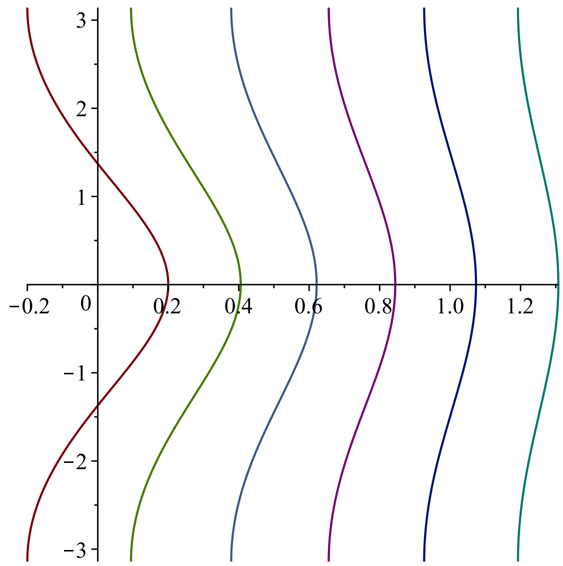
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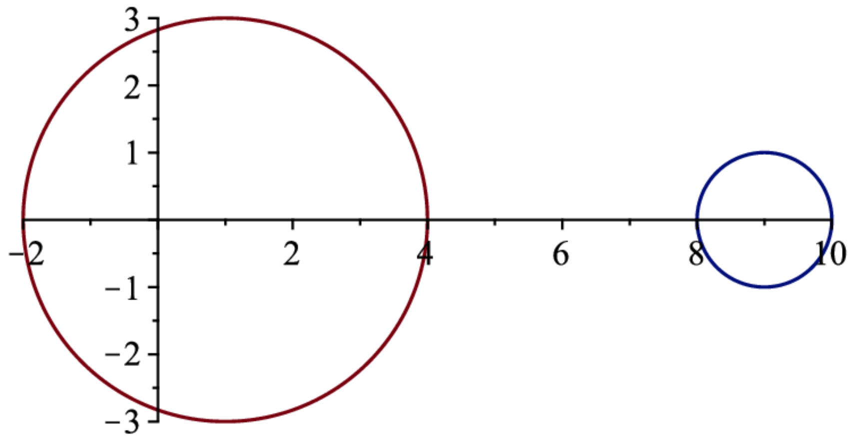
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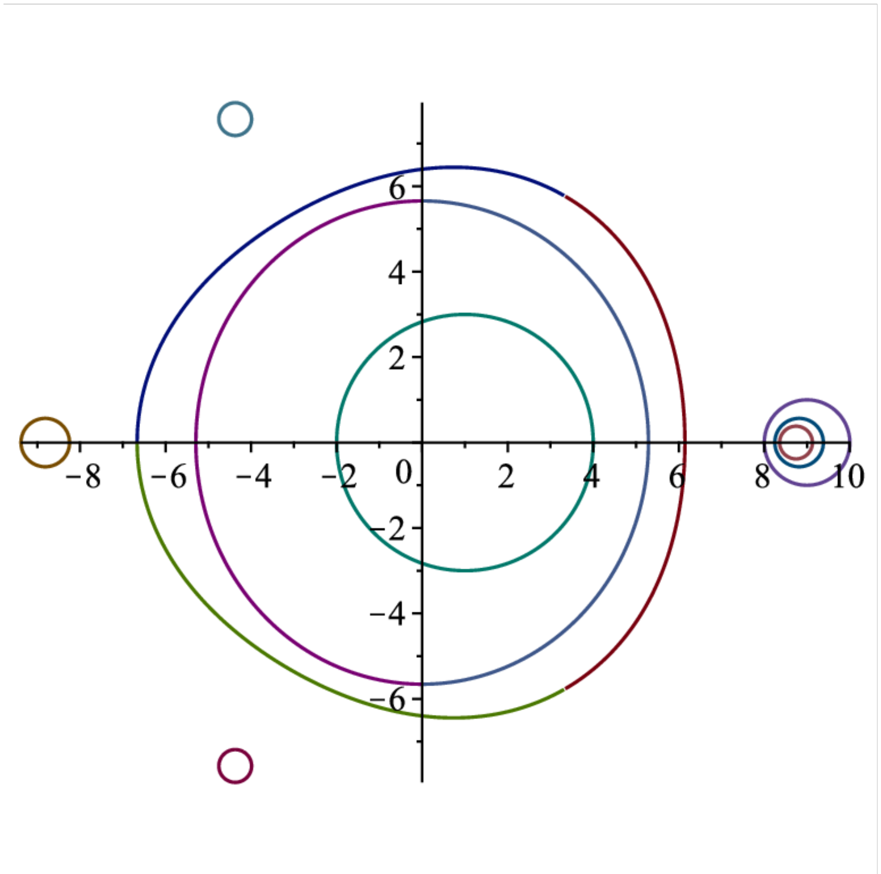

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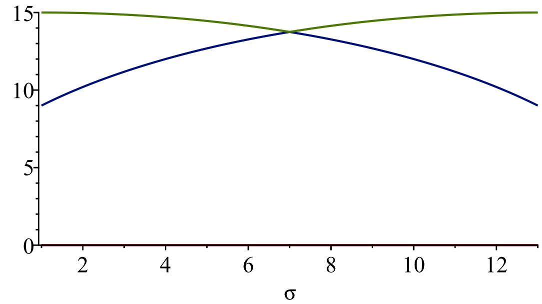
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Robust Bounds on the Eigenvalues of Linear Systems with Delays and Their Applications

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Abstract

We review some known bounds for eigenvalues of matrices and use similar techniques to derive bounds for nonlinear eigen problems and the eigenvalues for LTI systems with delays as a special case. There are two classes of results. The first are based on Hermitian decompositions, the second on Gershgorin's theorem. The bounds are easily computable. We reflect on implications for stability theory, which may be contrasted with bounds that have been obtained via Riccati stability based on Lyapunov-Krasovskii theory.

KEYWORDS:
Time delay systems, stability, eigenvalues, bounds

1 | INTRODUCTION

Determining the eigenvalues of a single n by n matrix has a complexity of order n^3. The basic approach is the QR algorithm. This basic algorithm is made computationally feasible by introducing the Hessenberg decomposition and accelerating its convergence by using shifts. See Golub ad Loan [13] for details. In many applications (e.g. stability) the availability of bounds on the eigenvalues (as domains in he complex plane) suffices. Bounds on the eigenvalues of a matrix can be specified in two simple ways: One way is based on the decomposition of a real matrix in its symmetric and anti-symmetric components, or for a complex matrix in its Hermitian and skew-Hermitian components [17]. The upshot is that the computation of the eigenvalues of a symmetric matrix is considerably simpler. Moreover, the problem of finding eigenvalues for normal matrices (thus also symmetric ones) is always well-conditioned [3]. A second method is a classical result due to Gershgorin [12], also referred to as the circle theorem [13]. Although the proofs are relative simple [16], these results are not widely exploited [4]. For some applications see Brualdi [6] and for networked systems Garren [11] and Huang et al [18]. The goal is to apply these methods to obtains various bounds for more general nonlinear eigen problems of the form

$$F(\lambda; \{A_i\}_{i=1}^N) = 0. \tag{1}$$

These include solutions for polynomial eigenvalue problems of the form

$$\det(\lambda^n I + \lambda^{n-1} A_1 + \lambda^{n-2} A_2 + \ldots + \lambda A_{n-1} + A_n) = 0, \tag{2}$$

a problem that has seen some interest (See [15]). In the scalar case, this is simply a polynomial root problem. It is readily shown that these nonlinear eigenvalues are the (linear) eigenvalues of a block-companion matrix

$$\mathcal{A} = \begin{bmatrix} -A_1 & -A_2 & \cdots & \cdots & -A_n \\ I & 0 & & & 0 \\ 0 & I & \ddots & & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}.$$

The special case of quadratic eigen problems is motivated by problems in the dynamic analysis of structural mechanical, and acoustic systems, in electrical circuit simulation, in fluid mechanics, and, more recently, in modeling microelectronic mechanical systems (MEMS). It is also what has motivated Olga Taussky to become a torchbearer for matrix theory [22]. A thorough discussion of the problem is given in [9], and [23]. Discrete-time linear systems with a fixed delay of the form

$$x_{k+1} = Ax_k + Bx_{k-m}$$

are inherently finite dimensional if $m < \infty$. Hence their stability is related to the poles of an augmented finite dimensional linear system of dimension $\dim \chi = m \dim x$:

$$\chi_{k+1} = \mathcal{A}\chi_k$$

This characteristic equation is equivalent (using block-determinant identities) to

$$\det(\lambda^{m+1} I - \lambda^m A - B) = 0,$$

which is a nonlinear eigen problem of the polynomial form (2). Of more interest in this special issue are the eigen problems related to systems with delay in continuous time. For a single delay system, $\dot{x}(t) = Ax(t) + Bx(t-1)$, the characteristic equation is given by

$$\det(\lambda I - A - B\mathrm{e}^{-\lambda}) = 0,$$

which is of the general form (1). It is well-known that the associated infinitesimal generator is compact, from which we may deduce that the spectrum is a pure point spectrum. Consequently, the stability of the delay system is characterized by all the eigenvalues of the system being in the open left hand plane. Sufficient conditions for stability were easily obtained by Lyapunov-Krasovskii methods, leading to the Riccati conditions for stability, as first established by Verriest and Ivanov [28], and extended in a series of papers [28], [29], [10], [25], [30] These techniques were later rebranded also as Linear Matrix Inequalities (LMI), a term coined by Jan Willems [33], as they are equivalent. An overview of the LMI-technique is exposed in the book [5] and lecture notes [21]. We note that these Riccati-methods can be extended to analyze classes of nonlinear systems [27], unlike the bounds to be described here. To make this paper self-contained, we present the proofs for the basic Hermitian (symmetric) decomposition and Gershgorin's theorem respectively in Sections 2 and 5, as they will be the starting point for the main development in this paper. In Section 3, we look at extensions of the Hermitian decomposition for the quadratic eigen problem. In particular conditions for the absence of real eigenvalues and bounds for the imaginary parts in terms of the real parts are presented. This is repeated for the eigen problem associated with (continuous-time) delay systems in Section 4. The remainder of Section 5 presents bounds on the real part of the eigenvalues for the quadratic problem, and similar results are then obtained for delay systems in Section 6.

2 | EIGENVALUE BOUNDS BASED ON HERMITIAN DECOMPOSITIONS.

To set the stage, we first review some classical bounds on the eigenvalues of a matrix $M \in \mathbb{R}^{n \times n}$. For arbitrary $x, y \in \mathbb{R}^n$, consider the complex vector $z = x + jy$. Let z^* be its Hermitian conjugate, $z^* = x^\top - jy^\top$. Let also $M = M_s + M_a$ be the decomposition of M in its symmetric and anti-symmetric parts

$$M_s = \frac{1}{2}(M + M^\top), \quad M_a = \frac{1}{2}(M - M^\top).$$

Since $p^\top M_a p = 0$ for all $p \in \mathbb{R}^n$, the quadratic form

$$z^* M z = z^* M_s z + z^* M_a z.$$

can be expressed as

$$z^* M z = (x^\top M_s x + y^\top M_s y) + 2jx^\top M_a y. \tag{3}$$

If $z = x + jy$ is an eigenvector of the matrix M, corresponding to the eigenvalue $\lambda = \sigma + j\omega$, with $\sigma, \omega \in \mathbb{R}$,then by definition

$$Mz = M(x + jy) = (\sigma + j\omega)(x + jy) = \lambda z. \tag{4}$$

We shall use these relations to obtain bounds of σ and ω.

2.1 | Bounds on Re $\lambda(M)$

Pre-multiply (4) with z^* to get

$$z^* M z = \lambda z^* z. \tag{5}$$

The right hand side of (5) is $(\sigma + j\omega)(x^\top x + y^\top y)$. It follows from (3) that

$$\sigma(\|x|^2 + \|y\|^2) + 2j\omega(x^\top y) = (x^\top M_s x + y^\top M_s y) + 2jx^\top M_a y,$$

and thus by identifying he real and imaginary parts

$$\sigma(\|x|^2 + \|y\|^2) = x^\top M_s x + y^\top M_s y \tag{6}$$

$$\omega(\|x|^2 + \|y\|^2) = 2x^\top M_a y. \tag{7}$$

Let now $\|z\| = 1$, or equivalently, $\|x\|^2 + \|y\|^2 = 1$, then from (6) we get

$$\min_{\|x\|^2+\|y\|^2=1} \left(x^\top M_s x + y^\top M_s y\right) \leq \sigma \leq \max_{\|x\|^2+\|y\|^2=1} \left(x^\top M_s x + y^\top M_s y\right),$$

from which

$$\min_{\alpha\in[0,1]} \left(\min_{\|x\|^2=\alpha} x^\top M_s x + \min_{\|y\|^2=1-\alpha} y^\top M_s y \right) \leq \sigma \leq \max_{\alpha\in[0,1]} \left(\max_{\|x\|^2=\alpha} x^\top M_s x + \max_{\|y\|^2=1-\alpha} y^\top M_s y \right),$$

or

$$\min_{\alpha\in[0,1]} \lambda_{\min}(M_s)(\alpha + 1 - \alpha) \leq \sigma \leq \max_{\alpha\in[0,1]} \lambda_{\max}(M_s)(\alpha + 1 - \alpha).$$

Since this is in fact independent of α, this relation establishes the following bounds on the real parts of the eigenvalues of M

$$\boxed{\lambda_{\min}(M_s) \leq \sigma \leq \lambda_{\max}(M_s)} \tag{8}$$

2.2 | Bounds on Im $\lambda(M)$

By Hermitian conjugation of (5) we get for the real matrix M

$$z^* M^\top z = \bar{\lambda} z^* z.$$

Subtracting this from (5) gives

$$\omega(\|x\|^2 + \|y\|^2) = 2x^\top M_a y.$$

Bounds for the imaginary part follow then by extremizing $2x^\top M_a y$ with the constraint $\|x^2\| + \|y\|^2 = 1$. Introducing the Lagrange multiplier μ, we get the Lagrangian

$$L = 2x^\top M_a y + \mu(\|x\|^2 + \|y\|^2).$$

Necessary conditions for stationarity follow by taking the gradient of L

$$M_a y + \mu x = 0 \tag{9}$$

$$x^\top M_a + \mu y^\top = 0. \tag{10}$$

These equations imply

$$M_a^\top M_a y = -\mu M_a^\top x = \mu^2 y.$$

Hence, y must be an eigenvector of the positive semi-definite symmetric matrix $M_a^\top M_a$ with eigenvalue $\mu^2 \geq 0$. Likewise

$$M_a M_a^\top x = \mu^2 x.$$

Then (9) and (10) imply

$$x^\top M_a y = -\mu\|x\|^2 = -\mu\|y\|^2,$$

so that $\|x\| = \|y\|$ and $\mu = -2x^\top M_a y = -\omega$. This establishes the following bound on the imaginary part of the eigenvalues of M

$$\boxed{|\omega| \leq \sqrt{\lambda_{\max}(M_a^\top M_a)}.} \tag{11}$$

Note that this differs from the statement (without proof) given by Garren[11].

An alternative proof, based on inequalities, can be given for the bound on $2x^\top M_a y$ with the constraint $\|x^2\| + \|y\|^2 = 1$. Set again $\|x\|^2 = \alpha$ and $\|y\|^2 = 1 - \alpha$ for some $0 \le \alpha \le 1$. Denoting $\hat{x} = \frac{x}{\|x\|}$, it follows that

$$x^\top M_a y = \sqrt{\alpha(1-\alpha)}\, \hat{x}^\top M_a \hat{y}.$$

This is maximal for $\alpha = \frac{1}{2}$, so that the problem reduces to finding the maximum of $\hat{x}^\top M_a \hat{y}$ for unit vectors $\hat{x}$ and $\hat{y}$. By Schwarz's inequality,

$$\hat{x}^\top M_a \hat{y} \le \|\hat{x}\| \|M_a \hat{y}\| = \|M_a \hat{y}\|,$$

with equality if $M_a \hat{y} = k\hat{y}$ for some $k > 0$. But, $M_a^\top M_a$ is symmetric and therefore has a real spectrum,

$$\max_{\|\hat{y}\|=1} \|M_a \hat{y}\| = \max_{\hat{y}\|=1} \hat{y}^\top M_a^\top M_a \hat{y} = \lambda_{\max}(M_a^\top M_a).$$

Thus

$$\max_{\|x\|^2+\|y\|^2=1} x^\top M_a y = \frac{1}{2}\sqrt{\lambda_{\max}(M_a^\top M)}.$$

The two results presented here allow to confine each eigenvalue of the $n \times n$ matrix M to a rectangular domain in $\mathbb{C}$, and the vertices are computable by solving a symmetric eigen problem, which as mentioned in the introduction is a simple problem with a more accurate solution.

2.3 | Bound on $\mathrm{Im}\,\lambda(M)$ in function of $\mathrm{Im}\,\lambda(M)$

We derive now a new bound for the imaginary part of an eigenvalue of a real matrix as function of the real part.

Theorem 1 (Imaginary part bounds). Let $M \in \mathbb{R}^{n\times n}$ have eigenvalue $\lambda = \sigma + j\omega$, with $\sigma, \omega \in \mathbb{R}$. Let M_s and M_a respectively be the symmetric and anti-symmetric parts of M, then
i) the imaginary part of the eigenvalues are upper bounded by

$$|\omega| \le \sqrt{-\lambda_{\min}(M_s^2 + M_a^2) + 2\sigma[\lambda_{\max}(M_s)H(\sigma) + \lambda_{\min}(M_s)H(-\sigma)] - \sigma^2}. \tag{12}$$

where $H(\cdot)$ is the Heaviside function,
ii) if $L(M,\sigma) := -\lambda_{\max}(M_s^2 + M_a^2) + 2\sigma[\lambda_{\min}(M_s)H(\sigma) + \lambda_{\max}(M_s)H(-\sigma)] - \sigma^2$ is real and positive, then $\sqrt{L(M,\sigma)}$ is an effective lower bound for $|\omega|$. Else, this lower bound is the zero function.

Proof of Theorem 1. From $M(x + jy) = (\sigma + j\omega)(x + jy)$, where $z = x + jy$ is an eigenvector, with $x, y \in \mathbb{R}^n$, of M associated with eigenvalue $\lambda = \sigma + j\omega$, we get the pair of equations

$$Mx = \sigma x - \omega y \tag{13}$$

$$My = \omega x + \sigma y. \tag{14}$$

These imply

$$M^2 x = \sigma Mx - \omega My - \sigma Mx - \omega^2 x - \sigma\omega y = 2\sigma Mx - (\omega^2 + \sigma^2)x,$$

or

$$\omega^2 x = -M^2 x + 2\sigma Mx - \sigma^2 x.$$

Since both sides scale linearly in x, we may set $\|x\| = 1$. Premultiplication by $x^\top$ yields then

$$\omega^2 = -x^\top M^2 x + 2\sigma x^\top Mx - \sigma^2 = -x^\top (M^2)_s x + 2\sigma x^\top M_s x - \sigma^2.$$

Note that

$$M^2 = (M_s + M_a)^2 = M_s^2 + M_s M_a + M_a M_s + M_a^2 \quad \Rightarrow \quad (M^2)_s = M_s^2 - M_a^\top M_a.$$

Using the above established bounds for a symmetric matrix, this leads to

$$\lambda_{\min}(-M_s^2 + M_a^\top M_a + 2\sigma M_s - \sigma^2) \le \omega^2 \le \lambda_{\max}(-M_s^2 + M_a^\top M_a + 2\sigma M_s - \sigma^2)$$

or

$$-\lambda_{\max}((M_s - \sigma I)^2 - M_a^\top M_a) \le \omega^2 \le -\lambda_{\min}((M_s - \sigma I)^2 - M_a^\top M_a) \tag{15}$$

But there is a problem with the bounds in (15) as they need to be evaluated for each σ. In order to get σ outside the matrix bounds, we consider the more restrictive bounds (by extremizing each term individually). Thus one arrives at

$$\min_{\|x\|=1}[-x^\top(M^2)_s x] + 2\sigma \min_{\|x\|=1}[x^\top M_s x] - \sigma^2 \le \omega^2 \le \max_{\|x\|=1}[-x^\top(M^2)_s x] + 2\sigma \max_{\|x\|=1}[x^\top M_s x] - \sigma^2.$$

For $\sigma > 0$, these bounds are evaluated by the quadratic forms in σ:

$$-\lambda_{\max}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\min}(M_s) - \sigma^2 \le \omega^2 \le -\lambda_{\min}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\max}(M_s) - \sigma^2. \tag{16}$$

For the case $\sigma < 0$ this yields instead the quadratic bounds in σ:

$$-\lambda_{\max}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\max}(M_s) - \sigma^2 \le \omega^2 \le -\lambda_{\min}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\min}(M_s) - \sigma^2. \tag{17}$$

From (8) we already know that σ can be restricted to the interval $[\lambda_{\min}(M_s), \lambda_{\max}(M_s)]$. Furthermore, the bounds in (16) and (17) are only informative if they are real and positive.
Consequently, if $\lambda_{\max}(M_s) > 0$ we get for $\sigma \in [0, \lambda_{max}(M_s)] \cap [-\lambda_{\max}(M_s) - \sqrt{D_+}, -\lambda_{\max}(M_s) + \sqrt{D_+}] = [\max(0, -\lambda_{\max}(M_s) - \sqrt{D_+}), \min(\lambda_{\max}(M_s), -\lambda_{\max}(M_s) + \sqrt{D_+})]$ the bounds

$$\sqrt{-\lambda_{\max}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\min}(M_s) - \sigma^2} \le |\omega| \le \sqrt{-\lambda_{\min}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\max}(M_s) - \sigma^2} \tag{18}$$

while if $\lambda_{min}(M_s) < 0$ we get for $\sigma \in [\lambda_{\min}(M_s), 0] \cap [-\lambda_{\max}(M_s) - \sqrt{(}D'_-), -\lambda_{\max}(M_s) + \sqrt{(}D'_-)]$ where $D'_- = \lambda_{\max}(M_s)^2 - \lambda_{\max}(M_s^2 - M_a^\top M_a)$ the bounds

$$\sqrt{-\lambda_{\max}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\max}(M_s) - \sigma^2} \le |\omega| \le \sqrt{-\lambda_{\min}(M_s^2 - M_a^\top M_a) + 2\sigma\lambda_{\max}(M_s) - \sigma^2} \tag{19}$$

Combining the upper bounds proves (12). The inequalities also tell us that there is an effective (i.e., positive) lower bound for $|\omega|$ if the left hand side quadratic forms in (16) and (17) are positive valued. $\square$

Example 1: The matrix $M = \begin{bmatrix} 1 & 13 \\ -13 & 15 \end{bmatrix}$ has eigenvalues $\lambda_\pm = 7 \pm 13.7477270848675j$. We find the bounds $\mathrm{Re}\,\lambda \in [1, 13]$ and $|\mathrm{Im}\,\lambda| \le 15$. The function upper and lower bounds of theorem 1 are shown in Figure 1. Note that the exact eigenvalue corresponds to $\sigma = 7$. Next we explore extensions to the quadratic eigen problem.

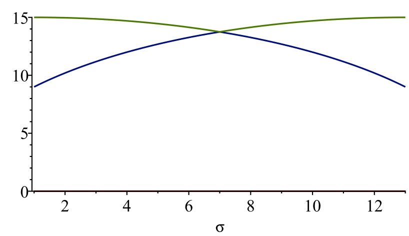

Figure 1 Upper and Lower Bound for $|\omega|$ as function of σ.

3 | QUADRATIC EIGENVALUE PROBLEM

This problem is motivated by Olga Taussky[22], having some importance in the analysis of aerodynamic flutter. Given A and B in $\mathbb{R}^{n\times n}$. Let λ be a complex nonlinear eigenvalue, i.e., a solution of the equation

$$\det(\lambda^2 I + \lambda A + \lambda B) = 0. \tag{20}$$

This implies that a vector, $z \in \mathbb{C}^n$, exists such that

$$(\lambda^2 I + \lambda A + \lambda B)z = 0.$$

As in the previous section, pre-multiply by z^* to get

$$\lambda^2 z^* z + \lambda z^* A z + z^* B z = 0. \tag{21}$$

which yields by conjugation

$$\bar{\lambda}^2 z^* z + \bar{\lambda} z^* A^\top z + z^* B^\top z = 0. \tag{22}$$

Add and subtract (21) and (22), letting again $z = x + jy$ with $x, y \in \mathbb{R}^n$ to get

$$\mathrm{Re}(\lambda^2(\|x\|^2 + \|y\|^2) + \mathrm{Re}(\lambda z^* A z) + \mathrm{Re}(z^* B z) \;=\; 0 \tag{23}$$

$$\mathrm{Im}(\lambda^2(\|x\|^2 + \|y\|^2) + \mathrm{Im}(\lambda z^* A z) + \mathrm{Im}(z^* B z) \;=\; 0. \tag{24}$$

Using (3) with A and B, we get from (23)

$$(\sigma^2 - \omega^2)(\|x\|^2 + \|y\|^2) + \sigma(x^\top A_s x + y^\top A_s y) - 2\omega x^\top A_a y + x^\top B_s x + y^\top B_s y = 0, \tag{25}$$

and in (24)

$$2\sigma\omega(\|x\|^2 + \|y\|^2) + 2\sigma x^\top A_a y + \omega(x^\top A_s x + y^\top A_s y) + 2x^\top B_a y = 0. \tag{26}$$

The equations (25) and (26) obtained this way do not seem very useful. However, if we know that a real eigenvalue exists, or restrict the analysis to a real eigenvalue), $\lambda = \sigma$, then let $\omega = 0$ and $y = 0$ and find

$$\sigma^2\|x\|^2 + \sigma x^\top A_s x + x^\top B_s x = 0 \tag{27}$$

One can deduce from equation (27) the following theorem:

Theorem 2 (Nonexistence of real eigenvalues)**.** No real eigenvalues can exist for the quadratic problem (20) if

$$\max\{\lambda^2_{\max}(A_s), \lambda^2_{\min}(A_s)\} < 4\lambda_{\min}(B_s).$$

Proof of Theorem 2. Solving (27) for σ, no real solutions can exist if

$$(x^\top A_s x)^2 - 4x^\top B_s x\|x\|^2 < 0.$$

Since this condition is scale invariant, it is equivalent to $\max_{\|x\|=1}(x^\top A_s x)^2 < 4\min_{\|x\|=1} x^\top B_s x$, i.e., if

$$\max\{\lambda^2_{\max}(A_s), \lambda^2_{\min}(A_s)\} < 4\lambda_{\min}(B_s).$$

□ □

Example 2: For the quadratic eigenvalue problem $\det(\lambda^2 I + \lambda A + B) = 0$ with

$$A = \begin{bmatrix} 5 & 10 \\ -10 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 10 & -1 \\ 1 & 10 \end{bmatrix}$$

one finds $\lambda_{\max}(A_s) = 5$, and $\lambda_{\min}(A_s) = -3$, and $\max(\lambda^2_{\max}(A_s), \lambda^2_{\min}(A_s)) = 25 < 4\lambda_{\min}(B_s) = 40$. Hence this problem has no real quadratic eigenvalues. In fact the nonlinear eigenvalues are approximately -3.79345 ± 10.83017 and -0.20655 ± 0.85309.

Theorem 3 (Imaginary part bounds)**.** The magnitude of the imaginary part, $|\omega|$, of the eigenvalue of the quadratic eigenvalue problem (20) is upper bounded in terms of its real part, σ, by

$$\lambda^{1/2}_{\max}[(\sigma A_a + B_a)^\top (A_s + 2\sigma I)^{-1}(\sigma A_a + B_a)(A_s + 2\sigma I)^{-1}]. \tag{28}$$

Proof of Theorem 3. : Note that equation (26) is linear in both σ an ω, and scale invariant jointly in x and y, It follows that,

$$\omega = -\frac{2x^\top(\sigma A_a + B_a)y}{x^\top(2\sigma I + A_s)x + y^\top(2\sigma I + A_s)y}.$$

Hence $|\omega|$ is bounded by the maximum of $2|x^\top(\sigma A_a + B_a)y|$ if x and y are constrained by $|x^\top(2\sigma I + A_s)x + y^\top(2\sigma I + A_s)y| = 1$. This maximization problem can be solved as in the previous section, thus giving the bound. □

A more conservative bound is obtained by separately maximizing the numerator and minimizing the absolute value of the denominator. One finds

$$|\omega| \leq \frac{\lambda_{max}^{1/2}[(\sigma A_a + B_a)^\top(\sigma A_a + B_a)]}{\min_{\lambda \in \mathrm{Spec} A_s} |2\sigma + \lambda|}. \tag{29}$$

Observe that (29) is not informative if the real part is $\sigma = -\frac{1}{2}\lambda(A_s)$.

In view of the equivalence between the quadratic eigen problem and the linear eigen problem for the companion form

$$\mathcal{A} = \begin{bmatrix} -A & -B \\ I & 0 \end{bmatrix}$$

we also know by the fundamental Hermitian method that the real and imaginary components of the quadratic eigenvalue are respectively bounded by

$$\mathrm{Re}\,\lambda \in [\lambda_{\min}(\mathcal{A}_s), \lambda_{\max}(\mathcal{A}_s)]$$

and

$$|\mathrm{Im}\,\lambda| \leq \sqrt{\lambda_{\max}(\mathcal{A}_a^\top \mathcal{A}_a)}.$$

Since

$$\mathcal{A}_s = \frac{1}{2}\begin{bmatrix} -2A_s & I - B \\ I - B^\top & 0 \end{bmatrix},$$

it follows the block determinant identity that Re λ is bounded by the minimum and maximum eigenvalue, μ of the symmetric quadratic problem

$$\det\left[\mu^2 I + \mu A_s - \frac{1}{4}(B - I)(B^\top - I)\right] = 0. \tag{30}$$

The imaginary part of λ is bounded by the square root of the maximal eigenvalue of the symmetric matrix

$$\mathcal{A}_a^\top \mathcal{A}_a = \frac{1}{4}\begin{bmatrix} -2A_a^\top & I + B \\ I - B^\top & 0 \end{bmatrix}\begin{bmatrix} -2A_a & I - B \\ I + B^\top & 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4A_aA_a^\top + (I + B)(I + B^\top) & 2A_a(I - B) \\ 2(I - B^\top)A_a^\top & (I - B^\top)(I - B) \end{bmatrix}. \tag{31}$$

Example 3: Let $A = \alpha I + N$, with $N^\top = -N$ and $\alpha \in \mathbb{R}$. The real part of the quadratic eigenvalues satisfying $\det[\lambda^2 I + \alpha\lambda I + \lambda N + B] = 0$ are bounded by the minimum and maximum quadratic eigenvalues satisfying $\det[\mu^2 I + 2\mu\alpha I - (B - I)(B^\top - I)] = 0$. But this reduces to

$$\det\left[(\mu^2 + \mu\alpha)I - \frac{1}{4}(B - I)(B^\top - I)\right] = 0,$$

and since $(B - I)(B^\top - I)$ is positive semi-definite, its eigenvalues, θ_i, are nonnegative. Identifying $\mu^2 + \alpha mu$ with θ, it follows that the quadratic eigenvalues of the symmetric eigen problem are

$$\mu = -\frac{\alpha}{2} \pm \frac{1}{2}\sqrt{\alpha^2 + \theta},$$

and thus the real parts of the quadratic eigenvalues are constrained by

$$\mathrm{Re}\,\lambda \in \left[-\frac{\alpha}{2} - \frac{1}{2}\sqrt{\alpha^2 + \max\theta}, -\frac{\alpha}{2} + \frac{1}{2}\sqrt{\alpha^2 + \max\theta}\right].$$

In particular, the quadratic eigenvalues for

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

are readily computed as

$$-1,\ 0.392646781702641,\ -0.696323390851321 \pm 1.435949864j,$$

while the bounds give

$$-1.4510 < \mathrm{Re}\,\lambda < 0.4511, \quad \text{and} \quad |\mathrm{Im}\,\lambda| < 3.228.$$

4 | DELAY-SYSTEM EIGENVALUE PROBLEM

$$\dot{x}(t) = ax(t) + bx(t-1). \tag{32}$$

It is well-known that this DDE is asymptotically stable in the domain

$$\Omega = \{\, (a,b) \in \mathbb{R}^2 \mid b > -1,\ a < -b \,\} \cup (a,b) \in \mathbb{R}^2 \mid \exists \omega \in (0,\pi) :\ a + b\cos\omega = 0,\ \omega + b\sin\omega = 0 \,\}.$$

The slope of the boundary for $b \leq 1$ at $b = 1$ ($\omega = 0$)is

$$\frac{\mathrm{d}b/\mathrm{d}\omega}{\mathrm{d}a/\mathrm{d}\omega} = \frac{\cos\omega\sin\omega - \omega}{\omega\cos\omega - \sin\omega} = 1/2.$$

This is shown by setting $x(t) = \mathrm{e}^{(\sigma+j\omega)t}$ in the DDE for $\sigma = 0$. Lines for positive σ correspond to unstable systems. These lines cannot intersect and are to the right of the boundary of Ω in the (a,b)-parameter space. By time scaling, one can always accommodate any fixed delay.

The solution of (32) is expressible in terms of the Lambert W-function, defined as the inverse of $w\,\mathrm{e}^{w} = x$, which has infinitely many (complex) branches. We use the method of characteristics to determine the spectrum and closed form general solution of the homogeneous time delay system. Suppose that $x(t) = C\mathrm{e}^{\lambda t}$, with $C \neq 0$ is a mode of the system where $C, \lambda \in \mathbb{C}$ are constants. Then, substitution in the DDE yields (with $\mathbf{D}$ the differential operator) yields

$$\begin{aligned} \mathbf{D}(C\mathrm{e}^{\lambda t}) = aC\mathrm{e}^{\lambda t} + bC\mathrm{e}^{\lambda(t-1)} \quad &\Leftrightarrow \quad \lambda - a = b\mathrm{e}^{-\lambda} \\ &\Leftrightarrow \quad \mathrm{e}^{\lambda-a}(\lambda - a) = b\mathrm{e}^{-a} \\ &\Leftrightarrow \quad \lambda - a = W(b\mathrm{e}^{a}) \end{aligned}$$

Hence, the k-th eigenvalue ($k \in \mathbb{Z}$) corresponds to the k-th branch of the Lambert W-function, denoted by W_k

$$\lambda_k = a + W_k(b\mathrm{e}^{a}). \tag{33}$$

This proves the following theorem.

Theorem 4 (Spectrum of scalar LTI-DDE)**.** The spectrum of the DDE (32) is given by

$$\text{Spec } = \{\, \lambda_k \mid k \in \mathbb{Z},\ \lambda_k = a + W_k(b\mathrm{e}^{a}) \,\}.$$

No continuous or residual spectrum is present for this infinite dimensional system (functional differential equation). The closed form general solution of the homogeneous system given by (32) will be given as a linear combination of the countably infinite eigen modes (assuming simple poles, i.e., poles of multiplicity one) as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \mathrm{e}^{\lambda_k t}.$$

In what follows we discuss an attempt at generalizing this explicit formula for the eigenvalues to higher dimensions. This is followed by the search for bound of the eigenvalues using an adaptation of the Hermitian approach in sections 2 and 3.

4.1 | Attempts at defining a matrix Lambert function

Consider now the higher-dimensional case, with $A, B \in \mathbb{R}^{n\times n}$

$$\dot{x}(t) = Ax(t) + Bx(t-1). \tag{34}$$

Its eigenvalues correspond to the solutions of the nonlinear eigen problem

$$\det(\lambda I - A - \mathrm{e}^{-\lambda} B) = 0. \tag{35}$$

First, note that in the special case where A and B are simultaneously diagonalizable (which is equivalent to both being diagonalizable and commuting) the equation reduces (letting T be the common diagonalizing transformation) to

$$\det(\lambda I - A - \mathrm{e}^{-\lambda} B) = \det[T(\lambda I - A - \mathrm{e}^{-\lambda} B)T^{-1}] = \det(\lambda I - A_{\text{diag}} - \mathrm{e}^{-\lambda} B_{\text{diag}}) = \prod_{i=1}^{n} (\lambda - a_{ii} - \mathrm{e}^{-\lambda} b_{ii}),$$

where $a_{ii} = (TAT^{-1})_{ii}$ and $b_{ii} = (TBT^{-1})_{ii}$. Stability can be determined by considering the individual factors, but this requires the complex extension of the stability domain for the scalar case (since a_{ii} and/or b_{ii} may be complex. Thus, let us next consider the delay-differential operator $\mathbf{P}_\alpha = \mathbf{D} + \alpha\mathbf{T}_1$ and the subclass of FDD's of the form $p(\mathbf{P}_\alpha)x = 0$ where $\alpha \in \mathbb{R}$ and $p(u) \in \mathbb{R}[u]$ is monic (suggested in Verriest[24]). The characteristic equation is $\prod_{i=1}^n (\mathbf{P} - \lambda_i)$, where the λ_i are the generalized poles of the system (with respect to the operator $\mathbf{P}_\alpha$). The interesting fact is that for these systems, the set of poles is finite. It remains to discover the set $\Omega_\alpha \subset \mathbb{C}$ for which the equation $(\mathbf{P} - \lambda_i)x = 0$ is asymptotically stable. Thus let's consider a potential solution $x(t) = \mathrm{e}^{\mu t}$. Substitution in the 'elementary pole factor' $(\mathbf{P} - \lambda_i)x = 0$ yields $\mu - \alpha \mathrm{e}^{-j\mu} - \lambda = 0$. Letting again $\mu = \sigma + j\omega$ and $\lambda = \mathrm{Re}\,\lambda + j\mathrm{Im}\,\lambda$ and separating gives

$$\mathrm{Re}\,\lambda = \sigma - \alpha \mathrm{e}^{-\sigma}\cos\omega$$
$$\mathrm{Im}\,\lambda = \omega + \alpha \mathrm{e}^{-\sigma}\sin\omega$$

The stability boundary follows by setting $\sigma = 0$, thus giving the parametric form

$$\mathrm{Re}\,\lambda = -\alpha\cos\omega$$
$$\mathrm{Im}\,\lambda = \omega + \alpha\sin\omega.$$

This implies that the stability domain for the elementary operator ¶$_\alpha$ is to the left of the curve described by the parametric equation. In figures 2 and 3 lines of constant $\sigma \geq 0$ are displayed for $\alpha = 0.2$ and $\alpha = 1$ respectively. The leftmost ones (with the cusp in Figure 3) corresponds to $\sigma = 0$ and defines the stability boundary for λ. These curves self intersect (for $\alpha > 1$), in which case the stability domain is the intersection of the left of all branches. Figure 4 shows lines of constant $\sigma \geq 0$ for $\alpha = 2$. Note that these curves extend periodically along the ω axis. In the more general case for $\alpha \in \mathbb{C}$, set $\alpha = |\alpha|\mathrm{e}^{J}\arg(\alpha)$. The parametric

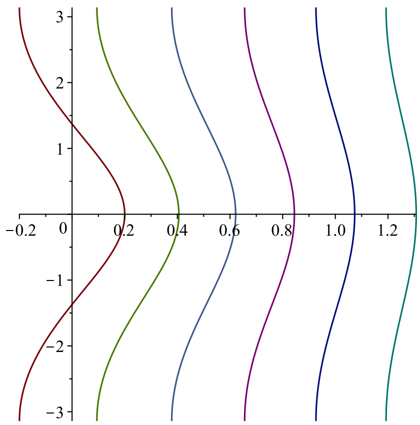

Figure 2 Stability domain for $\mathbf{P}_\alpha$ with $\alpha = 0.2$.

equations are then

$$\mathrm{Re}\,\lambda = -|\alpha|\cos(\omega - \arg\alpha)$$
$$\mathrm{Im}\,\lambda = \omega + |\alpha|\sin(\omega - \arg\alpha).$$

with the effect that the stability region Ω_α is shifted along the ω-axis. This analysis readily extends to FDD's of the form $\prod_{k=1}^{K} p_\kappa(\mathbf{P}_{\alpha_\kappa})x = 0$, with $\alpha_\kappa \in \mathbb{C}$.

As an alternative method, and by analogy to the scalar case, it was suggested to find a matrix $S \in \mathbb{C}^{n\times n}$ solving

$$S - A - B\exp(-S) = 0. \tag{36}$$

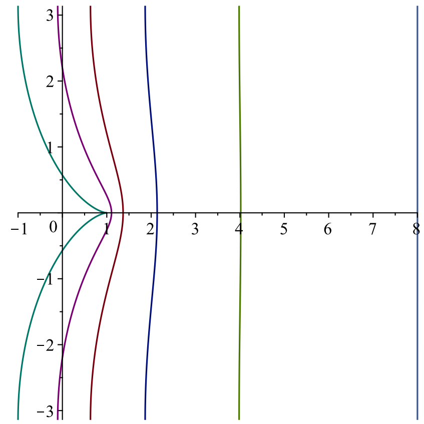

Figure 3 Stability domain for $\mathbf{P}_\alpha$ with $\alpha = 1$.

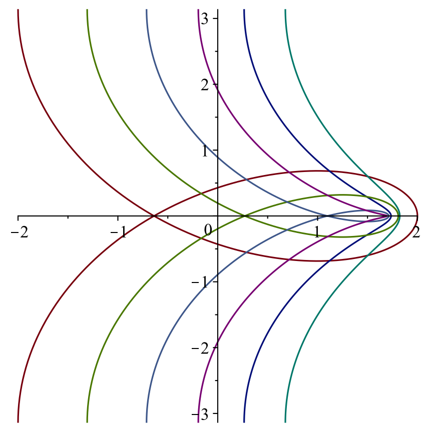

Figure 4 Stability domain for $\mathbf{P}_\alpha$ with $\alpha = 2$.

In order to generalize the theory and circumvent the highly restricted commutativity requirement of the matrices A and B, a matrix Q is introduced in Yi *et al.*[37] which satisfies the following equation:

$$(S - A)\exp(S - A) = BQ. \tag{37}$$

The above equation is in a matrix Lambert-W friendly format and its solution was postulated as

$$S_k = A + W_k(BQ). \tag{38}$$

The set of all the eigenvalues of the matrix $S_k, k \in \mathbb{Z}$ constitutes the spectrum of the higher order time delay system. Substitution of (38) in (36)

$$W_k(BQ)\exp(W_k(BQ) + A) - B = 0. \tag{39}$$

This specified an algorithm:

Repeat for $k = 0, \pm 1, \pm 2, \ldots$:
Step 1: Solve the nonlinear transcendental equation $W_k(M_k)\exp(W_k(M_k) + A) - B = 0$ for M_k.

Step 2: Compute S_k corresponding to $M_k = BQ_k$ by

$$S_k = W_k(M_k) + A$$

Step 3: Compute the eigenvalues of S_k.

However, observe that (36) in Yi *et al.*[36] was derived by assuming a solution of the form $x(t) = \exp(St)x_0$ to the functional equation (34) which, after substitution in (34), yields the relation

$$(S - A - B\exp(-S))x(t) = 0.$$

But it holds that if $P \in \mathbb{R}^{n\times n}$, $x \in \mathbb{R}^n$ and $Px = 0$, then the nontrivial solution for x exists iff ker $(P) = 0$, equivalently iff P is singular i.e., $\det(P) = 0$. Here, ker (P) denotes the kernel (null space) of the matrix P. In the light of the above fact, equation (36) is an incorrect characteristic equation. The correct characteristic equation should be,

$$\det(S - A - B\exp(-S)) = 0.$$

Unfortunately when this equation is employed in the analysis, it does not boil down to matrix Lambert W-functions in general. Corless *et al.*[7], have shown that the matrix Lambert W function evaluated at the matrix A does not represent all possible solutions of $S\exp(S) = A$.
Thus, the idea of using matrix Lambert functions presented in subsequent work (See Asl and Ulsoy[2], Yi and Ulsoy[35], Yi *et al.*[36], Yi *et al.*[37], Yi *et al.*[38]) has some limitations and one needs to be very careful in drawing conclusions about the spectrum of higher order system using this approach. Ahmed[1] generated a set of counterexamples that illustrate that one needs to take care while drawing conclusion about the spectrum and eigenvalues of higher order time delay systems using Yi and Ulsoy's Algorithm as presented in Yi *et al.*[38] and Yi *et al.*[37]. In particular, the algorithm does not produce satisfactory results when the modes have higher multiplicity, i.e. for repeated roots. In some cases, the algorithm produces unnecessary and redundant roots which are not the actual modes of the system under consideration; in other cases it fails to catch all the poles of the system. In addition, the algorithm may give an incorrect judgement of the dominant poles. These examples can be reduced via factorization described in Verriest[24] to the scalar Lambert W friendly format. Ahmed also verified these results using the QPmR algorithm given in[31].

In fact, this caution is not limited to delay systems. We illustrate this with a simple ODE system

$$\ddot{x} = Bx, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

If we follow Yi *et al.*[37], on page 14, we should look for solutions in the form

$$x(t) = \exp(St)c.$$

Substituting this in the ODE gives

$$(S^2 - B)\exp(St)c = 0.$$

Likewise, consider the difference system in continuous time

$$x_{k+2} = Bx_k.$$

Substitute $x_k = S^k x_0$ to obtain $S^{k+2}x_0 = BS^k x_0$, which can at most give solutions at even steps. To get the odd steps, one needs x_1 as well. So $x_{2k+1} = S^{2k}x_1$. The key is that the recursion must be of the form

$$(S^2 - B)\xi = 0.$$

Thus, $(S^2 - B)$ must have a nontrivial null-space, which implies again $\det(S^2 - B) = 0$. If one works with the matrix directly instead of the determinant, thus trying to solve $S^2 = B$, there may not be a solution. This we prove as follows:

Suppose S is a solution, then a similarity transformation exists with unit determinant such that TST^{-1} is in one of following forms (working over $\mathbb{C}$)

$$TST^{-1} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \quad \text{or} \quad TST^{-1} = \begin{bmatrix} \lambda & \\ 1 & \lambda \end{bmatrix}.$$

Let $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. But then, with $T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$ this gives $TBT^{-1} = \begin{bmatrix} t_{12}t_{22} & -t_{12}^2 \\ t_{22}^2 & -t_{12}t_{22} \end{bmatrix}$. So, in the first case $t_{12} = t_{22} = 0$, which is impossible since T must be nonsingular. In the Jordan case, we find $S^2 = \begin{bmatrix} \lambda^2 & 0 \\ 2\lambda & \lambda^2 \end{bmatrix}$. It follows that $t_{12} = 0$, and thus $\lambda = 0$ and $t_{22} = 0$, giving the same contradiction.
What can be concluded from $(S^2 - B)\xi = 0$ if we also introduce the vector ξ? In the diagonalizable case,

$$(\lambda_1^2 - t_{12}t_{22})\xi_1 + t_{12}^2\xi_2 = 0$$
$$-t_{22}^2\xi_1 + (\lambda_2^2 + t_{12}t_{22})\xi_2 = 0.$$

In the Jordan case

$$(\lambda^2 - t_{12}t_{22})\xi_1 + t_{12}^2\xi_2 = 0$$
$$(2\lambda - t_{22}^2)\xi_1 + (\lambda^2 + t_{12}t_{22})\xi_2 = 0.$$

which is only possible for nonsingular T if $\xi_1 = 0$.

We conclude that working with the matrix problem directly (as opposed to its determinant) is not correct. Hence doing the same technique for a delay system is bound to be problematic.

4.2 | Eigenvalue bounds for continuous-time delay systems

We shall now proceed to determine useful bounds on these eigenvalues by mimicking the ideas leading to the Hermitian decomposition based bounds. If z is the corresponding (complex) eigenvector to eigenvalue λ, we get

$$z^*(\lambda I - A - \mathrm{e}^{-\lambda} B)z = 0.$$

From this, we get, setting $\|z\| = 1$, and $\lambda = \sigma + j\omega$

$$(\sigma + j\omega) = z^* A z + \mathrm{e}^{-\sigma}(\cos\omega - j\sin\omega) z^* B z.$$

This yields

$$\sigma + j\omega = x^\top A_s x + y^\top A_s y + 2jx^\top A_a y + \mathrm{e}^{-\sigma}\cos\omega(x^\top B_s x + y^\top B_s y) + 2\mathrm{e}^{-\sigma}\sin\omega x^\top B_a y - j\mathrm{e}^{-\sigma}\sin\omega(x^\top B_s x + y^\top B_s y) + 2j\mathrm{e}^{-\sigma}\cos\omega\, x^\top B_a y.$$

Separating, the real and imaginary parts yields

$$\sigma = x^\top A_s x + y^\top A_s y + \mathrm{e}^{-\sigma}\cos\omega\,(x^\top B_s x + y^\top B_s y) + 2\mathrm{e}^{-\sigma}\sin\omega\, x^\top B_a y \quad (40)$$
$$\omega = 2x^\top A_a y - \mathrm{e}^{-\sigma}\sin\omega\,(x^\top B_s x + y^\top B_s y) + 2\mathrm{e}^{-\sigma}\cos\omega\, x^\top B_a y \quad (41)$$

Theorem 5 (Exponential bound for imaginary part)**.** The imaginary part of the eigenvalues of the system (34) as function of its real part is exponentially bounded by

$$\boxed{|\omega| \le \sqrt{\lambda_{\max}(A_a^\top A_a)} + \mathrm{e}^{-\sigma}\sqrt{\lambda_{\max}(B_a^\top B_a) + \lambda_{\max}(B_s^2)}.} \quad (42)$$

Proof of theorem 5. It follows from (41) that

$$|\omega| \le 2|x^\top A_a y| + \mathrm{e}^{-\sigma}\sqrt{(x^\top B_s x + y^\top B_s y)^2 + 4(x^\top B_a y)^2}.$$

together with the constraint $\|x\|^2 + \|y\|^2 = 1$. Further bounding, similar to the quadratic eigenvalue problem, leads to (42). This gives exponential bounds for the imaginary part of the eigenvalues of the delay system in terms of the real part. ☐

Note that the bound increases as $\sigma \to -\infty$.

Corollary 1 (Bound on the real part)**.** The eigenvalues of the system (34) lie to the left of $\lambda_{\max}(A_s) + W_0(\mathrm{e}^{-\lambda_{\max}(A_s)}\beta(B))$ where where

$$\beta(B) = \sqrt{\lambda_{\max}(B_a^\top B_a) + \lambda_{\max}(B_s^2)}. \quad (43)$$

Proof of corollary 1. It follows from (40) that

$$\lambda_{\min}(A_s) - \mathrm{e}^{-\sigma}\sqrt{\lambda_{\max}(B_a^\top B_a) + \lambda_{\max}(B_s^2)} \leq \sigma \leq \lambda_{\max}(A_s) + \mathrm{e}^{-\sigma}\sqrt{\lambda_{\max}(B_a^\top B_a) + \lambda_{\max}(B_s^2)}.$$

These bounds are of the form $\sigma \leq a + \mathrm{e}^{-\sigma} b$ and $\sigma \geq a' - \mathrm{e}^{-\sigma} b$, with $b > 0$. Equivalently,

$$(\sigma - a)\mathrm{e}^{\sigma - a} \leq \mathrm{e}^{-a} b \quad \text{and} \quad (\sigma - a')\mathrm{e}^{\sigma - a'} \geq -\mathrm{e}^{-a'} b.$$

From the graph of $w\mathrm{e}^w = x$ we can see that $w\mathrm{e}^w$ is monotonically increasing for $w > -1$ and decreasing for $w < -1$. Moreover $(w, x) = (-1, \mathrm{e}^{-1})$ corresponds to a minimum. Thus, since $\mathrm{e}^{-a} b > 0$, and $\sigma \in \mathbb{R}$, we can express this in terms of the real branches of the Lambert W-function

$$\sigma \leq a + W(\mathrm{e}^{-a} b),$$

and for $\mathrm{e}^{-a'} b < \mathrm{e}^{-1}$

$$\{\, \sigma \geq a' + W_0(-\mathrm{e}^{-a'} b) \,\} \bigvee \{\, \sigma \leq a' + W_{-1}(-\mathrm{e}^{-a'} b) \,\}$$

where W_0 and W_{-1} are the real branches of the Lambert W function. In terms of the given matrices only the first yields useful information, which in terms of the original matrices is

$$\sigma \leq \lambda_{\max}(A_s) + W_0(\mathrm{e}^{-\lambda_{\max}(A_s)} \beta(B)), \tag{44}$$

with $\beta(B)$ as defined in (43). □

We infer form this the following sufficient condition for stability

Theorem 6 (Asymptotic Stability)**.** Let β be as in (43). The delay system is asymptotically stable if

$$\boxed{\lambda_{\max}(A_s) \leq -\beta(B).} \tag{45}$$

Proof of theorem 8. Obviously, if the right hand side in (44) is negative, the upper bound

$$\lambda_{\max}(A_s) + W(\mathrm{e}^{-\lambda_{\max}(A_s)} \beta(B))) < 0$$

ensures that $\sigma < 0$ and all eigenvalues must lie in the open left half plane. By the definition of the Lambert W function, this simplifies to the condition (45). □

We close this section by providing bounds for any real eigenvalue of the system (34).

Theorem 7 (Bounds on real eigenvalues)**.** If $\lambda \in \mathbb{R}$ is an eigenvalue of the delay system (34) with B_s positive semi-definite, then

$$\lambda_{\min}(A_s) + W_0(\mathrm{e}^{-\lambda_{\max}(A_s)} \lambda_{\min}(B_s)) \leq \lambda \leq \lambda_{\max}(A_s) + W_0(\mathrm{e}^{-\lambda_{\min}(A_s)} \lambda_{\max}(B_s)). \tag{46}$$

Proof of theorem 7. With a real eigenvalue λ corresponds a real eigenvector, x such that $Ax + \mathrm{e}^{-\lambda} Bx = \lambda x$. Premultiplication with $x^\top$ yields

$$x^\top A_s x + \mathrm{e}^{-\lambda} x^\top B_s x = \lambda.$$

which is reorganized in the form

$$(\lambda - x^\top A_s x)\mathrm{e}^{\lambda - x^\top A_s x} = \mathrm{e}^{-x^\top A_s x} x^\top B_s x.$$

Now invoke the real branch of the Lambert W function to get

$$\lambda = x^\top A_s x + W_0(\mathrm{e}^{-x^\top A_s x} x^\top B_s x).$$

Noting that $W_0(u)$ is monotone for $u > 0$, it follows from the positive semi-definiteness of B_s that for all x, $x^\top B_s x \geq 0$. Then observe that

$$\mathrm{e}^{-\lambda_{\max}(A_s)} \lambda_{\min}(B_s) \leq \mathrm{e}^{-x^\top A_s x} x^\top B_s x \leq \mathrm{e}^{-\lambda_{\min}(A_s)} \lambda_{\max}(B_s).$$

The bounds (46) follow by monotonicity of W_0 for positive argument and the bounds on the real part of an eigenvalue of a matrix. □

5 | GERSHGORIN'S THEOREM

In this section we first review the classical theorem by Gershgorin. This is a very simple and easily shown result, which remarkably is rather seldomly exploited. It provides bounds in terms of the entries of a matrix. This theorem has found some use in networks by Lee and Spong[20] and Wang and Elia[32]. Using the semi-discretization technique of Insperger and Stépán[19], an LTI delay system with constant delay may be represented by an infinite-dimensional ODE, and approximated by a finite-dimensional one, which when solved leads to a large dimensional LTI discrete dynamics, representing the state transition in successive intervals with length equal to the fixed delay time. Gershgorin can then be applied to the resulting dynamical matrix, which actually is an approximation of the infinitesimal operator. What's more, if the delay is periodically varying, but with the causality constraint $\dot{\tau} < 1$ imposed, then a semi-discretization approximation with fixed dimension is still applicable. The necessity of this rate-constraint for well-posedness of the problem is argued in Verriest[26]. The MIMO state-augmentation technique of Helmke and Verriest[14] (which differs from the usual monodromy system) yields then a linear time-invariant a representation for which Gershgorin is directly applicable to constrain the eigenvalues of this alternate monodromy system. We also adapt Gershgorin's technique for the nonlinear eigenvalue problem directly, including LTI functional differential equations.

5.1 | Gershgorin's theorem

Let $A \in \mathbb{C}^{n\times n}$, and let λ be an eigenvalue of A. Then there exists $x \in \mathbb{C}^n$ such that $\lambda x = Ax$. Componentwise this means

$$\forall i, \quad \lambda x_i = (Ax)_i = \sum_{j=1}^{n} A_{ij} x_j$$

and thus

$$\forall i: \quad (\lambda - A_{ii}) x_i = \sum_{j\neq i} A_{ij} x_j .$$

Taking the modulus leads to the inequalities

$$\forall i: \quad |\lambda - A_{ii}||x_i| \leq \sum_{j\neq i} |A_{ij}||x_j|$$

One of these satisfies $|x_i| \geq |x_j|$, for all $j \neq i$. Hence for this maximizing index i

$$|\lambda - A_{ii}| \leq \sum_{j\neq i} |A_{ij}| \frac{|x_j|}{|x_i|} \leq \sum_{j\neq i} |A_{ij}|. \tag{47}$$

This means that λ belongs to a disk, $D(A_{ii}, \rho_i)$, centered at A_{ii} with radius $\rho_i = \sum_{j\neq i} |A_{ij}|$. But as we do not have knowledge of which i is maximizing, all we can say is that λ must belong to one of the disks $D(A_{ii}, \rho_i)$, and therefore to the union of all such disks in (47)

$$\lambda \in \bigcup_{i=1}^{n} D(A_{ii}, \rho_i).$$

Moreover, the spectrum of A is the same as the spectrum of $A^\top$, so that we may directly conclude that also

$$\lambda \in \bigcup_{i=1}^{n} D(A_{ii}, \rho_i'), \quad \rho_i' = \sum_{j\neq i} |A_{ji}|.$$

Consequently, all eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ lie inside

$$\boxed{\lambda \in \bigcup_{i=1}^{n} D(A_{ii}, \rho_i) \;\cap\; \bigcup_{i=1}^{n} D(A_{ii}, \rho_i').} \tag{48}$$

5.2 | Gershgorin for matrix polynomials

Since $\lambda \in$ SpecA implies $p(\lambda) \in \mathrm{Spec}\, p(A)$ for any polynomial, p, we get also

Theorem 8 (polynomial form)**.** If $p \in \mathbb{R}[x]$, and $\lambda \in \operatorname{Spec} A$, then

$$p(\lambda) \in \bigcup_{i=1}^{n} D\left((p(A))_{ii}, \sum_{j\neq i}(|p(A)|_{ij}\right). \tag{49}$$

Note that although the Cayley-Hamilton theorem makes considering polynomials of degree larger than $n-1$ redundant, it does lead to different forms in Gershgorin's theorem as $[p(A)]_{ii} \neq p(A_{ii})$.

In particular, the powers of λ are constrained to:

$$\lambda^k \in \bigcup_{i=1}^{n} D\left((A^k)_{ii}, \sum_{j\neq i}(|(A^k)|_{ij}\right). \tag{50}$$

Equation (50) means that λ belongs to the inverse map of the disk under the power map. This yields k possibly overlapping domains in $\mathbb{C}$. If λ^m is in a disk region $D(c,\rho)$, then the parameterized form of the boundary of λ^m is

$$[x(t) = c + \rho\cos(t),\ y(t) = \rho\sin(t),\ t = 0..2\pi].$$

It follows that the boundary of the domains in which λ lies is given by the union

$$\bigcup_{k=0}^{m-1}[x(t,k), y(t,k), t = 0..2\pi]$$

where

$$x(t,k) = \left((c+\rho\sin t)^2 + \rho^2\cos^2 t\right)^{1/(2m)} \cdot \cos\left(\frac{2k\pi}{m} + \frac{1}{m}\arctan\left(\frac{\rho\sin t}{c+\rho\cos t}\right)\right),$$

$$y(t,k) = \left((c+\rho\sin t)^2 + \rho^2\cos^2 t\right)^{1/(2m)} \cdot \sin\left(\frac{2k\pi}{m} + \frac{1}{m}\arctan\left(\frac{\sin t}{c+\cos t}\right)\right),$$

The following are then evident:

Corollary 2 (Specification of Spec (A) from A^k)**.** If $\lambda \in \operatorname{Spec} A$, where A is real, then λ^m is in the regions parameterized by $m \in \mathbb{Z}_+$ bounded by

$$\bigcup_{k=0}^{m-1}\left[\ x(t) = (-(A^m)_{ii} + \sum_{j\neq i}|(A^m)_{ij}|\cos t,\ y(t) = (-(A^m)_{ii} + \sum_{j\neq i}|(A^m)_{ij}|\sin t,\ t = 0..2\pi\ \right]$$

Corollary 3 (Companion form)**.** If A is in companion form, with characteristic polynomial $a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$, the disks are $D(-a_1, |a_2| + |a_3| + \cdots + |a_n|)$, and $D(0,1)$ (the latter $n-1$ times). Hence all roots of the polynomial $a(s)$ lie in the union of the disks $D(0,1)$ and $D(-a_1, \sum_{j\neq 1}|a_j|)$.

Example 4: The figures below show the domains constraining the spectrum of $A = \begin{bmatrix} 9 & 1 \\ -3 & 1 \end{bmatrix}$ given A^k for $k = 1,2,3$. The exact eigenvalues are $\lambda_1 = 1.3945$, $\lambda_2 = 8.6056$. One can see that the domains given by successive powers (Figure 6) shrinks to a small neighborhood of λ_2, while the domains for λ_1 enlarge with increasing power k.

5.3 | Gershgorin for Quadratic Polynomials

Consider the autonomous systems in discrete and continuous time, respectively described by

$$x_{k+2} + Ax_{k+1} + Bx_k = 0$$
$$\ddot{x} + A\dot{x} + Bx = 0$$

Exponential solutions, $e^{\lambda t}x$ (for continuous time) or λ^k (for discrete time) are characterizable by a common quadratic equation

$$(\lambda^2 I + \lambda A + B)x = 0. \tag{51}$$

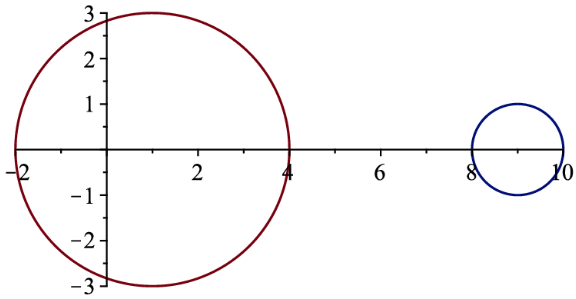

Figure 5 Gershgorin circles for A (by row)

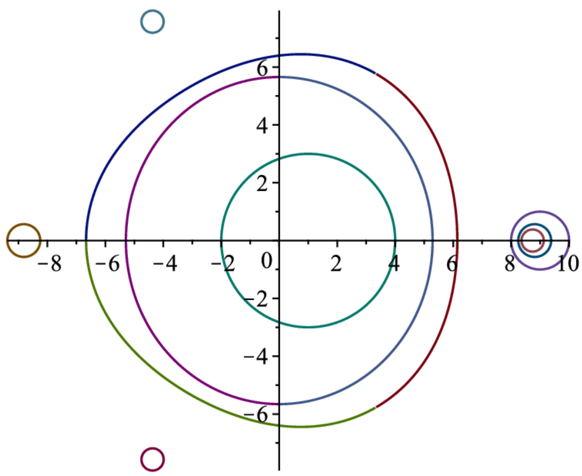

Figure 6 Gershgorin domains using A^k for $k = 1, 2, 3$ (by row)

This is a nonlinear (here quadratic) eigen-problem already considered in section 3. Applying the ideas in the proof of Gershgorin's theorem, The following "no-go" theorem is deduced:

Theorem 9 (No solution)**.** Let γ be a positive number. The equation $\det(\lambda^2 I + \lambda A + B) = 0$ has no solutions with $|\lambda| < \gamma$ if for $i = 1, \ldots, n$

$$\gamma^2 + \alpha_i \gamma + \beta_i < |B_{ii}|.] \tag{52}$$

where

$$\alpha_i = \sum_{j=1}^{n} |A_{ij}|, \quad \beta_i = \sum_{j \neq i} |B_{ij}|. \tag{53}$$

Proof of theorem 5.3. The relation (51) implies that componentwise

$$(\lambda^2 - B_{ii})x_i = -\lambda \sum_{j=1}^{n} A_{ij}x_j - \sum_{j\neq i} B_{ij}x_j.$$

It follows that for some i in $\{1,\dots,n\}$

$$|\lambda^2 - B_{ii}| \le |\lambda| \sum_{j=1}^{n} |A_{ij}| + \sum_{j\neq i} |B_{ij}|.$$

For any pole of the system satisfying $|\lambda| < \gamma$ it must hold that

$$|\lambda^2 - B_{ii}| \le \gamma \sum_{j=1}^{n} |A_{ij}| + \sum_{j\neq i} |B_{ij}|.$$

and all poles with norm less than γ must lie in the union of disks $\bigcup_{i=1}^{n} D(B_{ii}, \gamma\alpha_i + \beta_i)$. Hence poles with $|\lambda| < \gamma$ cannot exist if for all i

$$D(0,\gamma^2) \cap D(B_{ii}, \gamma\alpha_i + \beta_i) = \emptyset. \tag{54}$$

Since B_{ii} is real, (54) is equivalent to $B_{ii} - \gamma\alpha_i - \beta_i > \gamma^2$ if $B_{ii} > 0$ or $B_{ii} + \gamma\alpha_i + \beta_i < -\gamma^2$ if $B_{ii} < 0$. In either case, the condition is that for all i

$$\gamma^2 + \alpha_i\gamma + \beta_i < |B_{ii}|.$$

$\square$

Example 5: Consider $\det\left(\lambda^2 I + \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 0$. The poles are solutions to $\det\begin{bmatrix} \lambda^2+1 & \lambda \\ -\lambda & \lambda^2+2 \end{bmatrix} = (\lambda^2+1)(\lambda^2+2)+\lambda^2 = 0$ which are $\lambda \in \{\sqrt{2+\sqrt{2}}j, -j\sqrt{2+\sqrt{2}}, \sqrt{2-\sqrt{2}}j, -j\sqrt{2-\sqrt{2}}\} = \{\pm 1.847759065j, \pm 0.7653668650j\}$. The theorem states the nonexistence of eigenvalues with modulus less than γ if

$$\begin{aligned} \gamma^2 + \gamma &< 1 \\ \gamma^2 + \gamma &< 2 \end{aligned}$$

Both inequalities are satisfied for $\gamma < 0.618034$, so that one would conclude that there are no poles with modulus less than 0.618034.

For the same system we can alternatively work as follows: If $\lambda \neq 0$, (51) is equivalent to

$$\left(\lambda I + A + \frac{1}{\lambda}B\right)x = 0.$$

Again componentwise,

$$(\lambda + A_{ii})x_i = -\sum_{j\neq i} A_{ij}x_j - \frac{1}{\lambda}\sum_{j=1}^{n} B_{ij}x_j$$

Bounding the magnitude

$$|\lambda + A_{ii}||x_i| \le \sum_{j\neq i} |A_{ij}||x_j| + \frac{1}{|\lambda|}\sum_{j=1}^{n} |B_{ij}||x_j|$$

Thus for some $i \in 1,\dots,n$ it holds that

$$|\lambda + A_{ii}| = \sum_{j\neq i} |A_{ij}| + \frac{1}{|\lambda|}\sum_{j} |B_{ij}|$$

Consequently, if λ is such a nonlinear eigenvalue, then

$$\lambda \in \bigcup_{i=1}^{n} D\left(-A_{ii}, \sum_{j\neq i} |A_{ij}| + \frac{1}{|\lambda|}\sum_{j} |B_{ij}|\right).$$

Consider the subset (possibly empty) of nonlinear eigenvalues for which $|\lambda| \ge \gamma$ then $\frac{1}{|\lambda|} \le \frac{1}{\gamma}$ and

$$\lambda \in \bigcup_{i=1}^{n} D\left(-A_{ii}, \sum_{j\neq i} |A_{ij}| + \frac{1}{\gamma}\sum_{j} |B_{ij}|\right).$$

Clearly, this set is empty if for all $i \in \{1, \dots, n\}$ it holds that

$$-A_{ii} + \sum_{j \neq i} |A_{ij}| + \frac{1}{\gamma} \sum_{j} |B_{ij}| < \gamma. \tag{55}$$

We conclude then with the statement

Theorem 10 (No solution II)**.** All nonlinear eigenvalues, i.e., solutions of $\det(\lambda^2 I + \lambda A + B) = 0$, have norm less than γ if for all i

$$\gamma - \alpha_i' - \frac{1}{\gamma}\beta_i' > |A_{ii}|. \tag{56}$$

$$\alpha_i' = \sum_{j \neq i}^{n} |A_{ij}|, \quad \beta_i' = \sum_{j=1}^{n} |B_{ij}|. \tag{57}$$

Proof of theorem 10. The condition (56) is readily seen to be equivalent to (55) for all $i = 1, \dots, n$. □

For the above example 5, the inequalities read

$$\begin{aligned} \gamma^2 - \gamma - 1 &> 0 \\ \gamma^2 - \gamma - 2 &> 0 \end{aligned}$$

Both inequalities are satisfied for $\gamma > 2$, so that one would conclude that there are no poles with modulus larger than 2.

6 | GERSHGORIN FOR SYSTEM WITH DELAY

In this section we apply ideas similar to the derivation of Gershgorin's theorem to obtain bound on the eigenvalues of a simple delay systems. Then we extend this to systems with multiple delays and distributed delays.

6.1 | Systems with a single fixed delay

Consider first the LTI delay system

$$\dot{x}(t) = Ax(t) + Bx(t-1). \tag{58}$$

Any other fixed delay τ can be accommodated by time scaling the result. The characteristic equation is

$$\det(\lambda I - A - \mathrm{e}^{-\lambda} B) = 0. \tag{59}$$

Theorem 11 (Bounds for single delay)**.** Let γ be a positive number. The poles with real part exceeding $\gamma \in \mathbb{R}$ of the delay system $\dot{x}(t) = Ax(t) + Bx(t-1)$ lie in the union of the disks $D(A_{ii}, \sum_{j \neq i} |A_{ij}| + \mathrm{e}^{-\gamma} \sum_{j=1}^{n} |B_{ij}|)$.

Proof of Theorem 11. Equation (59) implies that an $x \in \mathbb{C}^n$ exists such that

$$(\lambda I - A - \mathrm{e}^{-\lambda} B)x = 0$$

Hence, componentwise

$$\lambda x_i - (Ax)_i - \mathrm{e}^{-\lambda}(Bx)_i = 0.$$

Assume that the real part of λ exceeds α, then for $\lambda = \sigma + j\omega$ it holds that

$$|\mathrm{e}^{-\lambda}| = \mathrm{e}^{-\sigma} |\mathrm{e}^{-j\omega}| \leq \mathrm{e}^{-\alpha}.$$

Thus,

$$\begin{aligned} |\lambda - A_{ii}||x_i| &\leq \sum_{j \neq i} |A_{ij}||x_j| + \mathrm{e}^{-\lambda} \sum_{j} |B_{ij}||x_j| \\ &\leq \sum_{j \neq i} |A_{ij}||x_j| + \mathrm{e}^{-\alpha} \sum_{j} |B_{ij}||x_j|. \end{aligned}$$

Letting again x_i be the maximum among $\{|x_j|\}_{j=1}^n$, then

$$|\lambda - A_{ii}| \le \sum_{j\neq i} |A_{ij}| + \mathrm{e}^{-\alpha} \sum_j |B_{ij}|.$$

Thus any pole with real part larger than α lies in the disk $D(A_{ii}, \sum_{j\neq i} |A_{ij}| + \mathrm{e}^{-\alpha} \sum_j |B_{ij}|)$. Consequently, λ such that $\operatorname{Re}\lambda > \alpha$, lies in the union ($i = 1 \dots n$) of these disks. Now, if the rightmost point of this union is less than α, it means that no such α can exists. Thus there are no poles with real part larger than α if for all $i = 1, \dots, n$

$$\forall i: \quad A_{ii} + \sum_{j\neq i} |A_{ij}| + \mathrm{e}^{-\alpha} \sum_j |B_{ij}| < \alpha.$$

□

In particular, we obtain a sufficient condition for stability

Corollary 4 (Asymptotic stability via Gershgorin)**.** The delay system (58) is asymptotically stable if

$$\forall i: \quad A_{ii} + \sum_{j\neq i} |A_{ij}| + \sum_{j=1}^{n} |B_{ij}| < 0. \tag{60}$$

Theorem 12 (Wedge theorem)**.** The poles of the delay system $\dot{x}(t) = Ax(t) + Bx(t-1)$ lie within the wedge bounded by $y = \pm\zeta(x)$, where $\zeta(x) = \max_i(\alpha_i + \mathrm{e}^{-x}\beta_i)$.

Proof of theorem 12. If λ is a pole with real part σ, we know by Theorem 1 that λ lies within the union of the Gershgorin disks $D(A_{ii}, \alpha_i + \mathrm{e}^{-\sigma}\beta_i)$, where $\alpha_i = \sum_{j\neq i} |A_{ij}|$ and $\beta = \sum_{j=1}^n |B_{ij}|$. Consequently, its imaginary component $j\omega$ lies in the union of the disks $D(A_{ii} - \sigma, \alpha_i + \mathrm{e}^{-\sigma}\beta_i)$ It follows that $\omega \in [-\zeta(\sigma), \zeta(\sigma)]$, where $\zeta(\sigma) = \max_i(\alpha_i + \mathrm{e}^{-\sigma}\beta_i)$.

□

Example 6:

$$A = \begin{bmatrix} a_1 & \\ & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} & b_1 \\ b_2 & \end{bmatrix}, \quad \tau = 1.$$

If $\operatorname{Re}\lambda > \alpha$

$$\lambda \in D(a_1, \mathrm{e}^{-\alpha}|b_1|) \ \cup \ D(a_2, \mathrm{e}^{-\alpha}|b_2|)$$

Sufficient condition for stability: $a_1 + |b_1| < 0$ and $a_2 + |b_2| < 0$ (as then there cannot exits a pole with positive real part).

6.2 | Systems with multiple fixed delays

Consider now the functional differential equation

$$\dot{x}(t) = Ax(t) + \sum_{\ell=1}^{m} B^{(\ell)} x(t - \tau_\ell). \tag{61}$$

The characteristic equation is

$$\det\left(\lambda I - A - \sum_{\ell=1}^{m} B^{(\ell)} \mathrm{e}^{-\lambda\tau_\ell}\right) = 0. \tag{62}$$

Theorem 13 (Eigenvalue bounds for multiple delays)**.**] Let γ be positive number. The functional differential equation $\dot{x}(t) = Ax(t) + \sum_{\ell=1}^m B^{(\ell)} x(t - \tau_\ell)$ has a pole with real part larger than γ if for all i:

$$A_{ii} + \sum_{j\neq i}^{n} A_{ij} + \sum_{\ell=1}^{m} \sum_{j=1}^{n} \mathrm{e}^{-\tau_\ell \gamma} |B_{ij}^{(\ell)}| < \gamma. \tag{63}$$

Proof of theorem 13. Again, any λ solving the (62) is an eigenvalue, i.e., a pole of the delay differential equation (61). Thus, for every eigenvalue λ_k there exists a vector $x^{(k)} \in \mathbb{C}^n$ such that

$$\left(\lambda_k I - A - \sum_{\ell=1}^{m} B^{(\ell)} \mathrm{e}^{-\lambda\tau_\ell}\right) x^{(k)} = 0$$

and for all $i = 1, \dots, n$

$$(\lambda_k - A_{ii})x_{ii}^{(k)} = \sum_{j\neq i} A_{ij}x_j + \sum_{\ell=0}^{m}\sum_j \mathrm{e}^{-\tau_\ell\lambda}B_{ij}^{\ell}x_j.$$

Let now λ_k be an eigenvalue with real part exceeding γ, then

$$|\lambda_k - A_{ii}||x_{ii}^{(k)}| \le \sum_{j\neq i} |A_{ij}||x_j| + \sum_{\ell=0}^{m}\sum_j \mathrm{e}^{-\tau_\ell\gamma}|B_{ij}^{\ell}||x_j|.$$

It follows that from the component, i, with largest absolute value

$$\lambda_k \in D\left(A_{ii}, \sum_{j\neq i}|A_{ij}| + \sum_{\ell=0}^{m}\sum_j \mathrm{e}^{-\tau_\ell\gamma}|B_{ij}^{\ell}|\right).$$

and thus

$$\lambda_k \in \bigcup_{i=1}^{n} D\left(A_{ii}, \sum_{j\neq i}|A_{ij}| + \sum_{\ell=0}^{m}\sum_j \mathrm{e}^{-\tau_\ell\gamma}|B_{ij}^{\ell}|\right).$$

□

Theorem 13 implies a sufficient condition for stability.

Corollary 5 (Stability with multiple delays)**.** If for all i,

$$A_{ii} + \sum_{j\neq i}^{n}|A_{ij}| + \sum_{\ell=1}^{m}\sum_{j=1}^{n}|B_{ij}^{(\ell)}| < 0, \tag{64}$$

then an eigenvalue with positive real part cannot exist, and the delay system is asymptotically stable.

6.3 | Systems with a Distributed Delay

As a last class, consider the LTI systems with distributed delay

$$\dot{x}(t) = Ax(t) + \int_0^\infty B(\tau)x(t-\tau), \mathrm{d}\tau. \tag{65}$$

where the support of B may be finite. The characteristic equation for this system is

$$\det\left(\lambda I - A - \int_0^\infty B(\tau)\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau\right) = 0 \tag{66}$$

Observe that the integral $\int_0^\infty B(\tau)\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau$ in (66) is the (unilateral) Laplace transform, $\widehat{B}(\lambda)$, of the matrix function $B(t)$. We get then

$$\det\left(\lambda I - A - \widehat{B}(\lambda)\right) = 0$$

Proceeding as n the previous subsections, this means that for some i

$$|\lambda - A_{ii}| \le \sum_{j\neq i}|A_{ij}| + \sum_j |\widehat{B}_{ij}(\lambda_i)|.$$

The problem is that this implicit inequality in λ is not easy to handle. A more accessible but also more restrictive form is

$$|\lambda - A_{ii}| \le \sum_{j\neq i}|A_{ij}| + \sum_j \int_0^\infty |B_{ij}(\tau)|\,\mathrm{e}^{-\tau\mathrm{Re}\,\lambda}\,\mathrm{d}\tau. \tag{67}$$

Since $\mathrm{Re}\lambda > \gamma$ implies $\mathrm{e}^{-\tau\mathrm{Re}\,\lambda} < \mathrm{e}^{-\gamma}$, the right hand side side of (67) is further bounded by $\sum_{j\neq i}|A_{ij}| + \sum_j \int_0^\infty |B_{ij}(\tau)|\,\mathrm{e}^{-\tau\gamma}\,\mathrm{d}\tau$. Poles with real part exceeding γ cannot exist if the intersection of $\mathbb{C}_+(\gamma)$ with the union of the Gershgorin disks

$$\bigcup_{i=1}^{n} D\left(A_{ii}, \sum_{j\neq i}|A_{ij}| + \sum +j = 1^n \int_0^\infty |B_{ij}(\tau)|\mathrm{e}^{-\tau\gamma}\,\mathrm{d}\tau\right)$$

is empty.

$$\lambda \in \bigcup_{i=1}^{n} D\left(A_{ii}, \sum_{j\neq i} |A_{ij}| + \sum_{j} \int_0^\infty |B_{ij}(\tau)|\, \mathrm{e}^{-\tau\gamma}\, \mathrm{d}\tau\right) \leq \gamma.$$

Moreover, if the $B_{ij} \in L_1$, then the Laplace transforms, $|\widehat{|B|}_{ij}(\gamma)|$ of the moduli $|B_{ij}|$ exists and converge on the imaginary axis. The disks can then be described as

$$\bigcup_{i=1}^{n} D\left(A_{ii}, \sum_{j\neq i} |A_{ij}| + \sum_{j} |\widehat{|B|}_{ij}(\gamma)|\right) \leq \gamma.$$

Example 7: Consider for $\gamma > 0$ the Volterra system

$$\dot{x}(t) = Ax(t) + B\int_0^\infty \mathrm{e}^{-\gamma\tau} x(t-\tau)\,\mathrm{d}\tau.$$

Its characteristic function is

$$\det(sI - A - B\int_0^\infty e^{-\gamma\tau}\mathrm{e}^{-s\tau}\,\mathrm{d}\tau = \det\left(sI - A - \frac{1}{s+\gamma}B\right).$$

provided the integral converges, i.e., Res $+\,\gamma > 0$. We note that this is a quadratic system in disguise. By Gershgorin,

$$|s - A_{ii}| \leq \underbrace{\sum_{j\neq i} |A_{ij}|}_{=\alpha_i} + \underbrace{\sum_{j} |B_{ij}|}_{=\beta_i} \left|\frac{1}{s+\gamma}\right|.$$

The eigenvalues lie in the union of the disks $D(A_{ii}, \alpha_i + \frac{\beta_i}{\sigma_1+\gamma})$. Let $\mathbb{C}_{\sigma_1,\sigma_2}$ denote the strip $\sigma_1 < \mathrm{Re}\, s < \sigma_2$, then $\max_{s\in\mathbb{C}_{\sigma_1,\sigma_2}} \left|\frac{1}{s+\gamma}\right| = \frac{1}{\sigma_1+\gamma}$ for $\sigma > -\gamma$. Based on the geometry of the Gershgorin disks, there cannot exist eigenvalues in $\mathbb{C}_{\sigma_1,\sigma_2}$ if for all i either

$$A_{ii} + \alpha_i + \frac{\beta_i}{\sigma_1+\gamma} < \sigma_1 < \sigma_2$$

or

$$A_{ii} - \alpha_i - \frac{\beta_i}{\sigma_1+\gamma} > \sigma_2 < \sigma_1.$$

Combining,

$$(\sigma_1+\gamma)^2 - (\sigma_1+\gamma)(A_{ii} + \alpha_i + \gamma) - \beta_i > 0.$$

In particular, the system is asymptotically stable if for all i

$$\begin{aligned}(A_{ii} + \alpha_i)\gamma + \beta_i \;&>\; 0\\ A_{ii} + \alpha_i) \;&>\; \gamma.\end{aligned}$$

7 | CONCLUSIONS

We reviewed classical results on Hermitian decompositions and Gershgorin's theorem to obtain bounds on the location of eigenvalues of a matrix. It was shown that these results can be applied to obtain bounds for nonlinear eigenvalue problems. These conditions lead also to sufficient conditions for asymptotic stability for some classes of systems.
In particular we analyzed the quadratic eigen problem as a warm-up for extensions to the problem of finding some interesting bounds for eigenvalues of delay systems. Further results on bounding the eigenvalues of a matrix may potentially be explored, based on works of Wolkowicz and Styan [34], and Higham and Tisseur [15] among other.

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