Prescribing transport equation solution's decay via multiplicity manifold and autoregressive boundary control

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Abstract

This paper addresses the boundary control problem of the transport equation. Namely, we propose a control method, which is merely a delayed output feedback relying on a partial pole placement idea, that consists in assigning an appropriate exponential decay rate to the closed-loop system's solution. The proposed control structure appearing in the transport boundary, which has proven its effectiveness in controlling finite dimensional systems, consists of an autoregressive relation linking the transport equation's input and output. The obtained result provides an analytical lower bound for the solution's exponential decay.

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RESEARCH ARTICLE

Prescribing transport equation solution's decay via multiplicity manifold and autoregressive boundary control

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Abstract

This paper addresses the boundary control problem of the transport equation. Namely, we propose a control method, which is merely a delayed output feedback relying on a partial pole placement idea, that consists in assigning an appropriate exponential decay rate to the closed-loop system's solution. The proposed control structure appearing in the transport boundary, which has proven its effectiveness in controlling finite dimensional systems, consists of an autoregressive relation linking the transport equation's input and output. The obtained result provides an analytical lower bound for the solution's exponential decay.

KEYWORDS:
Transport equation, Difference equation, Interfering delays, Spectral abscissa, Stability and stabilization

1 | INTRODUCTION

An intense research activity over the past decades pertaining to the multiple spectral values of functional differential equations[1,2,3] was initially motivated by a better understanding of the mechanism of coalescence of spectral values and the splitting of emerging branches when varying some parameters, as well as the resulting bifurcations. To complete the picture, several works have been devoted to the qualitative properties of such dynamical systems through the effect of the spectral values' multiplicity. Since then, using a standard complex analysis result from[4], a generic bound for allowable multiplicities has been established and related to the degree[1] of the corresponding characteristic function called *quasipolynomial*. Consequently, a remarkable property called *multiplicity-induced-dominancy* (or MID for short) has been emphasized in[5]. The MID property consists of conditions on the parameters of the dynamical system, guaranteeing that a multiple spectral value corresponds to the spectral abscissa[5]. Mysteriously, by introducing an integral representation of quasipolynomials corresponding to retarded equations[6], it appears that a spectral value with the maximal allowable multiplicity corresponds necessarily to the spectral abscissa. More recently, in the single delay case, it has been shown that a quasipolynomial admitting a root with multiplicity equal to its degree, shares its remaining zeros with the well-known *Kummer hypergeometric function*[7,6]. However, a quasipolynomial admitting a root with intermediate multiplicity shares its remaining zeros with an appropriate linear combination of Kummer hypergeometric functions. By exploiting a more than a century-old result on oscillations in the complex domain, in particular, the *Green-Hill transform*[8], it allows to prescribe some regions (in the complex plane) from containing zeros, shedding some light on the mystery. Since, several works have been dedicated not only to studying the extent of the MID[6,9,10,11], but also to the use of the MID property in practical control applications. The MID property inspired a frequency-domain control methodology called *partial poles placement* in assigning the spectral abscissa as a multiple root of the closed-loop characteristic equation. The MID-based control strategy has shown its effectiveness in the design of standard controllers such as the boundary *proportional-integral-derivative* control of the transport equation with a prescribed stabilization[6].

[1]The sum of the degrees of the involved polynomials plus the number of delays.

It is well known that apart from the tuning strategy, a good choice of control structure is crucial for a cheap implementation as well as for a safe and reliable control process, since the design of low-complexity control laws has the notable advantage of easing the implementation in real-time processes. In this paper, we revisit the boundary control of the transport equation, as formulated in [12,6], but under boundary conditions consisting of a four-parameter autoregressive input/output difference relation. The MID property is then put to use in the design procedure of the said controller.

The structure of this paper is as follows. In Section 2, we recall some prerequisites about qualitative properties of delay difference equations as well as the MID-based partial pole placement. Section 3 is dedicated to the problem formulation. In Section 4 we report the main contribution of the paper, where existence and uniqueness of the solution are established, as well as some sufficient conditions guaranteeing a prescribed decay rate.

2 | ON DELAY DIFFERENCE EQUATIONS WITH TWO INTERFERING DELAYS AND THE PARTIAL POLES PLACEMENT

We consider the problem of stabilizing the following delay difference equation in the continuous time function $\theta(t)$ with two interfering delays, $\tau_1 > 0$ and $\tau_2 > 0$, such that

$$\theta(t) + \alpha\,\theta(t-\tau_1) + \beta\,\theta(t-\tau_2) + \gamma\,\theta(t-\tau_1-\tau_2) = 0, \tag{1}$$

where α, β and γ are some real scalars (with the assumption that $\gamma \neq \alpha\,\beta$) to be tuned for ensuring the exponential stability of (1). The corresponding characteristic function, where $s \in \mathbb{C}$ stands for the complex Laplace variable, reads as

$$Q(s;\,\tau_1,\,\tau_2) := 1 + \alpha\,\mathrm{e}^{-\tau_1 s} + \beta\,\mathrm{e}^{-\tau_2 s} + \gamma\,\mathrm{e}^{-(\tau_1+\tau_2)\,s}, \tag{2}$$

which is a quasipolynomial of degree three if $\tau_1 \neq \tau_2$ and $\alpha\,\beta\,\gamma \neq 0$, and is of degree two if either $\tau_1 = \tau_2$ or only one of the parameters $\alpha,\ \beta, \gamma$ vanish.

2.1 | Standard results on delay difference equations stability

The well-known Hale-Silkowski criterion completely characterizes the exponential stability region (in the parameters space) corresponding to (1), see for instance [13, Chapter 9, Theorem 6.1]. However, the estimation of the solution's decay rate is out of its scope. In order to get some estimate of the decay rate, one has to locate the zeros of the corresponding quasipolynomial (2). To do so, one requires the following settings and results established by Henry in [14]. Consider the quasipolynomial

$$\Gamma(p,\kappa,h) := \sum_{k=0}^{N} \kappa_k\ \exp -p\,\chi_k.h \tag{3}$$

where $\kappa = (\kappa_1,\ldots,\kappa_N)^T \in \mathbb{R}^N$, $h = (h_1,\ldots,h_M)^T \in \mathbb{R}_+^M$, $\chi_j = (\chi_{j,1},\ldots,\chi_{j,M})$, $\chi_{j,k} \in \mathbb{N}^*$ ($j \in [\![1,N]\!],\ k \in [\![1,M]\!]$) and $\chi_j.h = \sum_{k=1}^{M} \chi_{j,k} h_k$. We also adopt the notations $\kappa_0 = 1$ and $\chi_0 = (0,\ldots,0)$. Define $Z_\Gamma(\kappa,h) := \left\{\Re(p) :\ \Gamma(p,\kappa,h) = 0\right\}$ and denote its closure by $\bar{Z}_\Gamma(\kappa,h)$. Let us define $\rho_j = \rho_j(\kappa,\ h)$ ($j \in [\![0,N]\!]$), if they exist, by the relation

$$|\kappa_j|\exp -\rho_j\,\chi_j.h = \sum_{k\neq j} |\kappa_k|\exp -\rho_j\,\chi_k.h \quad \text{for} \quad j \in [\![0,N]\!]. \tag{4}$$

If $\chi_N.h \geq \chi_j.h > 0$ for $j \in [\![1,N-1]\!]$, then ρ_N and ρ_0 are uniquely defined and $\rho_N < \rho_0$ for $N \geq 2$.

Lemma 1 ([14]). If $\chi_N.h \geq \chi_{N-1}.h > \ldots > \chi_1.h > 0$, then $\bar{Z}_\Gamma(\kappa,h) \subseteq [\rho_N,\ \rho_0]$.

2.2 | Recent delay difference stability results using the MID paradigm

In this section we exploit the manifold of spectral values' multiplicities to get some insights on the solutions' decay rates. Roughly speaking, the MID property consists in conditions under which a multiple spectral value is dominant. More precisely, in this section, we shall provide some configurations in which the MID applies; this corresponds to the dominancy of spectral values with a multiplicity which is equal to the degree of the considered quasipolynomial. Notice that such a degree may vary when some coefficients are set to be zero or when some delays are set to be equal.

In particular, the case $\tau_1 = \tau_2$ may be considered separately since it allows to decrease the degree of the quasipolynomial Q. In fact, the quasipolynomial Q reads as

$$Q(s\,;\,\tau_2,\tau_2) = 1 + (\alpha+\beta)\,\mathrm{e}^{-\tau_2\,s} + \gamma\,\mathrm{e}^{-2\,\tau_2\,s}, \tag{5}$$

which admits a degree equal to two for $\gamma \neq 0$ and $\alpha \neq -\beta$. In such a case, one can apply the following result which is strongly inspired from[9] and relies on the MID property.

Lemma 2 ([9]). Consider the quasipolynomial $Q(\cdot\,;\,\tau_1,\,\tau_2)$ given by (2) and let $\tau_1 = \tau_2$.
A given real number s_0 is a double root of (5) if, and only if,

$$\gamma = \mathrm{e}^{2\,\tau_2\,s_0}, \quad \alpha+\beta = -2\,\mathrm{e}^{\tau_2\,s_0}. \tag{6}$$

If (6) is satisfied then the MID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot\,;\,\tau_2,\tau_2)$ given by (5). Furthermore, all zeros of (5) are double and lie on the vertical axis $\mathfrak{R}(s) = s_0$.

Now, let us consider the quasipolynomial (2) where $\tau_1 \neq \tau_2$ and $\alpha\,\beta\,\gamma \neq 0$, *i.e.*, the case where the quasipolynomial's degree is equal to three.

Lemma 3 ([9]). Consider the quasipolynomial $Q(\cdot\,;\,\tau_1,\,\tau_2)$ given by (2) and let $\tau_1 \neq \tau_2$.
A given real number s_0 is a triple root of (2) if, and only if,

$$\alpha = \frac{\tau_1+\tau_2}{\tau_1-\tau_2}\exp\tau_1\,s_0, \quad \beta = -\frac{\tau_1+\tau_2}{\tau_1-\tau_2}\exp\tau_2\,s_0, \quad \gamma = -\exp\left(\tau_1+\tau_2\right)\,s_0. \tag{7}$$

If (7) is satisfied and $\tau_1 = k\,\tau_2$ with k an integer greater than one, then the MID holds, that is, s_0 corresponds to the spectral abscissa of the quasipolynomial $Q(\cdot\,;\,k\,\tau_2,\,\tau_2)$ given by (2).

Remark 1. From a control theory viewpoint, the MID property can be exploited by tuning the control parameters as emphasized above after prescribing a negative number s_0 which corresponds to the closed-loop system solution's decay rate.

The proof of Lemma 3 follows the same lines as the proof of Theorem 7.2 from[9]. It relies on properties of self-inversive polynomials, the standard A. Cohn Theorem and an Eneström-Kakeya Theorem. When the MID property fails, one can prescribe a lower bound for the decay rate as will be discussed in the next section.

2.3 | Beyond the MID property when $\tau_1 \neq k\,\tau_2$

By substituting the expressions of and given in system (7) into the expression of Q in expression (2) and by introducing the variable change

$$p := \tau_2\,\left(s - s_0\right)/2 \tag{8}$$

and the new parametrization $\tau = 2\,\tau_1/\tau_2$, it comes that $Q(s;\tau_1,\,\tau_2) := \tilde{Q}(p;\tau)$, where (p;$\tau$) = $1 - \frac{\tau+2}{\tau-2}\,\exp -2\,p + \frac{\tau+2}{\tau-2}\,\exp -\tau\,p -$ $\exp -\tau + 2\,$p, and it remains to examine the roots of $\tilde{Q}(\cdot;\,\tau)$ with respect to τ.

Notice that quasipolynomials with real coefficients admit zeros' distributions which are symmetric with respect to the real axis. In that case, the following lemma emphasizes an additional symmetry structuring the distribution of zeros of $\tilde{Q}(\cdot;\,\tau)$ with respect to the imaginary axis.

Lemma 4 ([9]). Let $p \in \mathbb{C}$ be a zero of $\tilde{Q}(\cdot;\tau)$ defined by expression (2.3). Then, $-p$ is also a zero of $\tilde{Q}(\cdot;\tau)$.

The following lemma provides a vertical strip in the complex plane, which is symmetric with respect to the imaginary axis and contains the set of zeros of $\tilde{Q}(\cdot;\,\tau)$.

Lemma 5 ([9]). The set $\bar{Z}_{\tilde{Q}}(\kappa,h) \subseteq [-\rho^*,\,\rho^*]$ where ρ^* is the unique positive zero of

$$\hat{Q}(\rho,\tau) := 1 - \left|\frac{\tau+2}{\tau-2}\right|\mathrm{e}^{-2\rho} - \left|\frac{\tau+2}{\tau-2}\right|\,\mathrm{e}^{-\tau\rho} - \mathrm{e}^{-(\tau+2)\rho}. \tag{9}$$

Lemma 6 ([9]). Consider the quasipolynomial $\hat{Q}$ given by (9) with $\tau \neq 2$. Then, the spectral abscissa σ of $\hat{Q}$ is upper-bounded by $\hat{\rho}(\tau)$ where $\hat{\rho}$ is given by

$$\hat{\rho}(\tau) := \frac{1}{\min\{\tau,2\}}\ln\left(1 + 2\,\frac{\tau+2}{|\tau-2|}\right). \tag{10}$$

3 | PROBLEM FORMULATION

We study the boundary stabilization of a transport equation in $(0, L) \subset \mathbb{R}$. The system is given by:

$$u_t + \lambda u_x = 0,\ x \in (0, L),\ t > 0, \quad u(0,t) = v(t),\ t > 0, \quad u(x,0) = u_0(x),\ x \in (0, L), \tag{11}$$

where v designates the control law applied at the boundary, L and λ are positive constants, and the initial data u_0 is a given function belonging to a suitable space.

In [12], a *proportional-integral* controller, i.e., with

$$v(t) = k_p u(L,\, t) + k_i \int_0^t u(L, \mu)\, d\mu,$$

has been used to uniformly stabilize solutions of system (11) using the well-known Walton & Marshall frequency domain approach. More recently, the same problem has been considered in [6] with the aim to provide a uniform stabilization with a prescribed decay. Notice that, in both references, an analysis of a single delay scalar first-order neutral differential equation is performed to obtain the mentioned results.

The aim of this paper is to exploit the effect of an autoregressive input/output difference equation at the boundary as well as a partial poles placement idea as described in Section 2. More precisely,

$$v(t) = -\alpha\, u(0, t-\tau) - \beta\, u(L, t) - \gamma\, u(L, t-\tau)$$

where the delay $\tau \in \mathbb{R}_+^*$ is considered as a control parameter additionally to the gains $\alpha,\ \beta,\ \gamma \in \mathbb{R}$; the aim being to provide a uniform stabilization with a prescribed decay. In this configuration, the analysis reduces to a delay difference equation with two interfering delays, which enables to benefit from the results of Section 2.

4 | AUTOREGRESSIVE BOUNDARY CONTROL OF THE TRANSPORT EQUATION

We study the boundary stabilization of a transport equation in $(0, L) \subset \mathbb{R}$. The system is given by:

$$\begin{cases} u_t + \lambda u_x = 0,\ x \in (0, L),\ t > 0, \\ u(0,t) + \alpha\, u(0, t-\tau_1) + \beta\, u(L,t) + \gamma\, u(L, t-\tau_1) = 0,\ t > 0, \\ u(x,0) = u_0(x),\ x \in (0, L), \\ u(L, t-\tau_1) = 0, u(0, t-\tau_1) = 0,\ t \in (0, \tau_1). \end{cases} \tag{12}$$

The constant $\tau_1 > 0$ is the time delay and $L, \lambda > 0$, α, β, and γ are real numbers and the initial data u_0 is a given function belonging to a suitable space that will be precised later. This section aims to study the problem (12), in which some delay terms are involved in the boundary's expression, where one of them stands for an autoregressive term with respect to the control. We shall show the global existence of a solution for problem (12) by transforming the delay term and by using a semigroup approach. We shall prove that the problem is asymptotically stable with an exponential decay rate.

4.1 | Well-posedness of problem (12)

In order to prove the global existence and the uniqueness of the solution for problem (12), first we transform problem (12) into problem (14) by making the change of variables (13), and second we apply the semigroup approach to prove the existence of the unique solution to problem (14). To overcome the problem of the boundary delay, we introduce the new variables:

$$z(\rho, t) = u\left(0, t - \tau_1 \rho\right), \quad w(\rho, t) = u(L, t - \tau_1 \rho), \quad \rho \in \left(0, 1\right),\ t > 0. \tag{13}$$

Then, for $(\rho,\ t) \in \left(0, 1\right) \times \left(0, +\infty\right)$ one has:

$$\begin{cases} \tau_1\, z_t(\rho, t) + z_\rho(\rho, t) = 0, \\ \tau_1\, w_t(\rho, t) + w_\rho(\rho, t) = 0, \end{cases}$$

Therefore, problem (12) is equivalent to:

$$
\begin{cases}
u_t + \lambda u_x = 0, & x \in (0, L),\ t > 0,\\
u(0,t) + \alpha\, z(1,t) + \beta\, u(L,t) + \gamma\, w(1,t) = 0, & t > 0\\
\tau_1\, z_t(\rho,t) + z_\rho(\rho,t) = 0, & \rho \in (0,1)\,,\ t > 0\\
\tau_1\, w_t(\rho,t) + w_\rho(\rho,t) = 0, & \rho \in (0,1)\,,\ t > 0\\
z(0,t) = u(0,t), \quad w(0,t) = u(L,t), & t > 0\\
u(x,0) = u_0(x), & x \in (0,L),\\
z(\rho,0) = 0, \quad w(\rho,0) = 0, & \rho \in (0,1).
\end{cases}
\tag{14}
$$

In this section we shall provide a sufficient condition that guarantees that the above problem is well-posed in the sense of Hadamard.

For this purpose, we consider a semigroup formulation of the initial-boundary value problem (14). If we denote $V := (u, z, w)^T$ and define the energy space:

$$\mathscr{H} = H^1(0,L) \times L^2(0,1) \times L^2(0,1),$$

clearly $\mathscr{H}$ is a Hilbert space with respect to the inner product

$$\left\langle V_1, V_2 \right\rangle_{\mathscr{H}} = \int_0^L u^1 u^2 dx + \xi_1 \int_0^1 z^1 z^2 d\rho + \xi_2 \int_0^1 w^1 w^2 d\rho$$

for $V_1 = (u^1, z^1, w^1)^T$, $V_2 = (u^2, z^2, w^2)^T$ and $\xi_1, \xi_2 > 0$ a nonnegative real number defined later.
Therefore, if $V_0 \in \mathscr{H}$ and $V \in \mathscr{H}$, problem (14) is formally equivalent to the following abstract evolution equation in the Hilbert space $\mathscr{H}$.

$$V'(t) = \mathscr{A} V(t), \quad t > 0,\ V\left(0\right) = V_0, \tag{15}$$

where $'$ denotes the derivative with respect to time t, $V_0 := \left(u_0, 0, 0\right)^T$ and the operator $\mathscr{A}$ is defined by

$$\mathscr{A}\,(u,\ z,\ w)^T = \left(-\lambda\, u_x,\ -{\tau_1}^{-1} z_\rho,\ -{\tau_1}^{-1} w_\rho\right)^T.$$

The domain of $\mathscr{A}$ is the set of $V = (u, z, w)^T$ such that $(u, z, w)^T \in H^1(0,L) \times H^1(0,1) \times H^1(0,1)$, $u(0) = z(0)$, $u(L) = w(0)$, $u(0) + \alpha z(1) + \beta u(L) + \gamma w(1) = 0$. Assume $(\alpha, \beta, \gamma, \xi_1, \xi_2)$ satisfies the following conditions

$$\alpha^2 < 1/3, \quad \beta^2 < 1/3, \quad \gamma^2 < 1/3, \quad \frac{1-3\beta^2}{6} > \max\left(\alpha^2, \gamma^2\right), \quad \xi_1 = \xi_2, \quad \max\left(\frac{3\alpha^2}{1-3\alpha^2}, \frac{3\gamma^2}{1-3\gamma^2}\right) < \frac{\xi_1}{\lambda\tau_1} < \frac{1-3\beta^2}{1+3\beta^2}, \tag{16}$$

then the well-posedness of problem (14) is ensured by the following theorem.

Theorem 1. Let $V_0 \in \mathscr{H}$, then there exists a unique solution $V \in C\left(\mathbb{R}_+; \mathscr{H}\right)$ of problem (15). Moreover, if $V_0 \in \mathscr{D}\left(\mathscr{A}\right)$, then $V \in C\left(\mathbb{R}_+; \mathscr{D}\left(\mathscr{A}\right)\right) \cap C^1\left(\mathbb{R}_+; \mathscr{H}\right)$.

Proof. In order to prove the existence and uniqueness of the solution of problem (15), we use the semigroup approach and the Lumer-Phillips' theorem. Indeed, let $V = (u, z, w)^T \in \mathscr{D}\left(\mathscr{A}\right)$. By definition of the operator $\mathscr{A}$ and the scalar product of $\mathscr{H}$, we have:

$$\langle \mathscr{A}V, V \rangle_{\mathscr{H}} = -\lambda \int_0^L u_x u\, dx + -\frac{\xi_1}{\tau_1} \int_0^1 z z_\rho\, d\rho - \frac{\xi_2}{\tau_1} \int_0^1 w w_\rho\, d\rho.$$

Hence, we obtain

$$\langle \mathscr{A}V, V \rangle_{\mathscr{H}} = -\frac{\lambda}{2}\left(u^2(L) - u^2(0)\right) - \frac{\xi_1}{2\tau_1}\left(z^2(1) - z^2(0)\right) - \frac{\xi_2}{2\tau_1}\left(w^2(1) - w^2(0)\right).$$

As a consequence, the last equation becomes:

$$
\begin{aligned}
\langle \mathscr{A}V, V \rangle_{\mathscr{H}} =& (\frac{\lambda}{2} + \frac{\xi_1}{2\tau_1})(\alpha z(1) + \beta w(0) + \gamma w(1))^2 + \left(-\frac{\lambda}{2} + \frac{\xi_2}{2\tau_1}\right) u^2(L)\\
&- \frac{\xi_1}{2\tau_1} z^2(1) - \frac{\xi_2}{2\tau_1} w^2(1).
\end{aligned}
\tag{17}
$$

To treat the first term in the preceding equation, Young's inequality yields

$$\begin{aligned}\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \leq & \left(-\frac{\xi_1}{2\tau_1}+3\alpha^2\left(\frac{\lambda}{2}+\frac{\xi_1}{2\tau_1}\right)\right)z^2(1)+\left(-\frac{\lambda}{2}+\frac{\xi_2}{2\tau_1}+3\beta^2\left(\frac{\lambda}{2}+\frac{\xi_1}{2\tau_1}\right)\right)u^2(L)\\ & +\left(-\frac{\xi_2}{2\tau_1}+3\gamma^2\left(\frac{\lambda}{2}+\frac{\xi_1}{2\tau_1}\right)\right)w^2(1).\end{aligned} \tag{18}$$

According to condition (16), we have $\langle \mathscr{A}V, V\rangle_{\mathscr{H}} \leq 0$. Thus the operator $\mathscr{A}$ is well dissipative.

Now we want to show that the operator $\mathscr{A}$ is invertible. To do so, let us introduce the following. For $F = (f_1, f_2, f_3)^T \in \mathscr{H}$, let $V = (u, z, w)^T \in \mathscr{D}(\mathscr{A})$ be a solution of $\mathscr{A}V = F$, which gives:

$$-\lambda u_x = f_1, -\frac{1}{\tau_1}z_\rho = f_2, -\frac{1}{\tau_1}w_\rho = f_3. \tag{19}$$

To find $V = (u, z, w)^T \in \mathscr{D}(\mathscr{A})$ solution of system (19), we suppose that u is determined with the appropriate regularity.

Thus, from the last equalities in (19), z and w are given, respectively, by

$$\begin{cases} z(\rho) = z(0) + \int\limits_0^\rho f_2(s)\,ds, \\ w(\rho) = u(L) + \int\limits_0^\rho f_3(s)\,ds, \quad \rho \in (0,1). \end{cases} \tag{20}$$

As a result, knowing u, we may deduce z and w by (20). Therefore, using the preceding expression and assumption (16), we get th expression of u

$$\begin{aligned} u(x) = & -\frac{1}{\lambda}\int\limits_0^x f_1(y)\,dy \\ & + \frac{1}{1+\alpha+\beta+\gamma}\left(\alpha\tau_1\int\limits_0^1 f_2(s)\,ds + \gamma\tau_1\int\limits_0^1 f_3(s)\,ds + \frac{\gamma-\beta}{\lambda}\int\limits_0^L f_1(y)\,dy\right), \quad x\in(0,L), \end{aligned} \tag{21}$$

such that $u \in H^1(0, L)$ verifies (19). Thereby, we have found $V = (u, z, w)^T \in \mathscr{D}(\mathscr{A})$ the unique solution of $\mathscr{A}V = F$.

The operator $\mathscr{A}$ generates a C_0 semigroup of contractions $e^{t\mathscr{A}}$ on $\mathscr{H}$. Hence, according to the Lumer-Phillips' theorem[15], there exists a unique solution $V \in C\left(\mathbb{R}_+; \mathscr{H}\right)$ of problem (15). This completes the proof of the theorem. ☐

4.2 | Asymptotic behavior

In this section, we show that under condition (16), the semigroup $e^{t\mathscr{A}}$ decays exponentially to the null steady state. To obtain this, our technique is based on a frequency domain method and combines a contradiction argument with the multiplier technique to carry out a special analysis for the resolvent.

Theorem 2. Suppose that condition (16) holds. Then, there exist constants $C, \omega > 0$ such that, for all $V_0 \in \mathcal{H}$, the semigroup $e^{t\mathscr{A}}$ satisfies the following estimate

$$\left\|e^{t\mathscr{A}}V_0\right\|_{\mathscr{H}} \leq C\,e^{-\omega t}\,\left\|V_0\right\|_{\mathscr{H}}, \forall\, t > 0. \tag{22}$$

Proof of Theorem 2. We use the frequency domain theorem for uniform stability from Huang-Prüss[16,17] of a C_0 semigroup of contractions on a Hilbert space. ☐

Lemma 7. A C_0 semigroup $e^{t\mathcal{L}}$ of contractions on a Hilbert space $\mathcal{H}$ satisfies

$$||e^{t\mathcal{L}}U_0||_{\mathcal{H}} \leq C\,e^{-\theta t}||U_0||_{\mathcal{H}}$$

for some constant $C > 0$ and for $\theta > 0$ if and only if

$$\rho(\mathcal{L}) \supset \left\{ i\delta \;\middle|\; \delta \in \mathbb{R} \right\} \equiv i\mathbb{R}, \tag{23}$$

and

$$\limsup_{\delta\in\mathbb{R},|\delta|\to\infty} \|(i\delta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \tag{24}$$

where $\rho(\mathcal{L})$ denotes the resolvent set of the operator $\mathcal{L}$.

Then we look at the point spectrum of $\mathscr{A}$.

Lemma 8. The spectrum of $\mathscr{A}$ contains no point on the imaginary axis.

Proof. Since $\mathscr{A}$ has a compact resolvent, its spectrum $\sigma(\mathscr{A})$ only consists of eigenvalues of $\mathscr{A}$. We shall show that the equation

$$\mathscr{A} Z = i\delta Z \tag{25}$$

with $Z = (u, z, w)^T \in \mathscr{D}(\mathscr{A})$ and $\delta \in \mathbb{R}$ admits only the trivial solution.
Equation (25) reads as

$$i\delta u + \lambda u_x = 0, \quad i\delta z + {\tau_1}^{-1} z_\rho = 0, \quad i\delta w + {\tau_1}^{-1} w_\rho = 0.$$

By taking the inner product of (25) with Z and using the bound (18), we get $w(0) = 0, w(1) = 0$ and $z(1) = 0$. Thus, we have $z = 0, w = 0$. Since $u(0) = z(0)$, we also obtain $u = 0$. So, the only solution of (25) is the trivial one. □

Lemma 9. The resolvent operator of $\mathscr{A}$ satisfies the condition $\limsup_{\delta\in\mathbb{R},|\delta|\to\infty} \|(i\delta I - \mathcal{L})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Proof. Suppose that the condition is false. By the Banach-Steinhaus Theorem, there exists a sequence of real numbers $\delta_n \to +\infty$ and a sequence of vectors $Z_n = (u_n, z_n, w_n)^t \in \mathscr{D}(\mathscr{A})$ with $\|Z_n\|_{\mathscr{H}} = 1$ such that

$$||(i\delta_n I - \mathscr{A}) Z_n||_{\mathscr{H}} \to 0 \quad \text{as} \quad n \to \infty, \tag{26}$$

i.e.,

$$\begin{cases} i\delta_n u_n + \lambda u_n' \equiv f_n \to 0 \text{ in } L^2(0, L), \\ i\delta_n z_n + \dfrac{1}{\tau_1}\partial_\rho z_n \equiv g_n \to 0 \text{ in } L^2(0, 1), \\ i\delta_n w_n + \dfrac{1}{\tau_1}\partial_\rho w_n \equiv h_n \to 0 \text{ in } L^2(0, 1). \end{cases}$$

Our goal is to derive from limit (26) that $||Z_n||_{\mathscr{H}}$ converges to zero, to obtain a contradiction. First, notice that we have

$$||(i\delta_n I - \mathscr{A}) Z_n||_{\mathscr{H}} \geq |\Re \left(\langle (i\delta_n I - \mathscr{A}) Z_n, Z_n \rangle_{\mathscr{H}} \right)|. \tag{27}$$

Then, by (18) and (26),

$$z_n(1) \to 0, \quad w_n(0) \to 0, \quad w_n(1) \to 0. \tag{28}$$

Moreover, since $Z_n \in \mathscr{D}(\mathscr{A})$, we have, by (28), $z_n(0) = u_n(0) \to 0$. That entails

$$u_n(x) = u_n(0)\, e^{-i\frac{\delta_n}{\lambda} x} + \frac{1}{\lambda} \int_0^x e^{-i\frac{\delta_n}{\lambda}\tau_1(x-y)}\, f_n(y)\, dy, \tag{29}$$

$$z_n(\rho) = z_n(0)\, e^{-i\delta_n\tau_1\rho} + \tau_1 \int_0^\rho e^{-i\delta_n\tau_1(\rho-s)}\, g_n(s)\, ds, \tag{30}$$

$$w_n(\rho) = w_n(1)\, e^{-i\delta_n\tau_1(\rho-1)} - \tau_1 \int_\rho^1 e^{-i\delta_n\tau_1(\rho-s)}\, h_n(s)\, ds. \tag{31}$$

According to (28), we get the implication that

$$u_n \to 0 \text{ in } L^2(0, L), \quad z_n \to 0 \text{ in } L^2(0, 1), \quad w_n \to 0 \text{ in } L^2(0, 1)$$

which contradicts $\|Z_n\|_{\mathscr{H}} = 1$, thereby terminating the proof. □

4.3 | Prescribed stabilization of the transport equation

Let (α, β, γ) satisfy (16) and $u_0 \in L^2(0, L)$. The Laplace transform applied to problem (12) yields

$$sv + \lambda\, v_x = u_0,\ x \in (0, L), \Re s > 0,\ \ v(0) + \alpha v(0) e^{-s\tau_1} + \beta v(L) + \gamma v(L) e^{-s\tau_1} = 0.$$

So that $v(x) = -\frac{(\beta+\gamma e^{-s\tau_1})\, e^{-\frac{sx}{\lambda}}}{\lambda\, Q(s;\tau_1,\frac{L}{\lambda})} \int_0^L e^{-\frac{s}{\lambda}(L-y)} u_0(y)\, dy + \frac{1}{\lambda} \int_0^x e^{-\frac{s}{\lambda}(x-y)} u_0(y)\, dy$, $\forall\, x \in (0, L)$,

where $Q(s; \tau_1, \frac{L}{\lambda}) = 1 + \alpha e^{-\tau_1 s} + \beta e^{-\frac{L}{\lambda} s} + \gamma e^{-\left(\tau_1 + \frac{L}{\lambda}\right) s}$ as defined in (2).

Thanks to the above results and to the lemmas presented in Section 2, the proof of the following theorem, which gives a certified decay rate's lower bound for the closed-loop system's solution, is immediate.

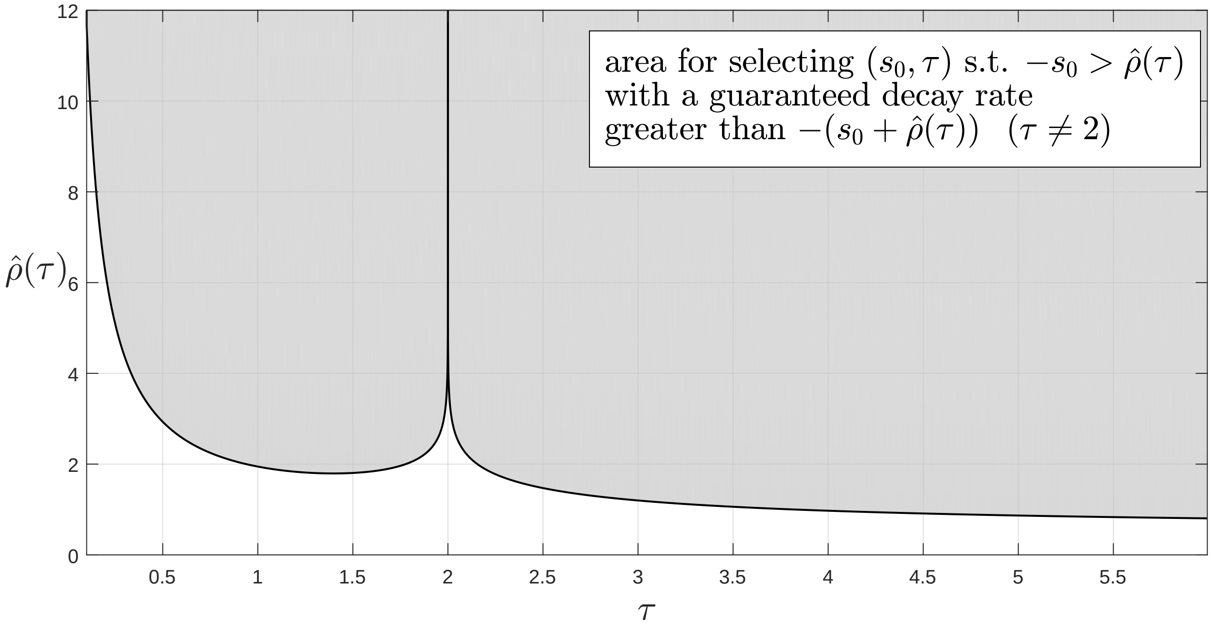

Figure 1 $\hat{\rho}$ given by expression (10) as a function of $\tau = 2\,\tau_1/\tau_2$.

Theorem 3. Consider the output feedback stabilization of the wave equation in (12) with an arbitrary positive feedback-delay $\tau_1 > 0$ and transport delay $\tau_2 = L/\lambda$ then the following assertions hold:
• If $\tau_1 = k\,\tau_2$ where k is an integer equal to (respectively greater than) one, then the control parameter tuning prescribed in system (6) (respectively (7)) enables the assignment of the solution's exponential decay rate to an arbitrary $-s_0$.
• If $\tau_1 \neq k\,\tau_2$, then the control parameter tuning prescribed in system (7) enables a closed-loop solution decaying exponentially faster than $-s_0 - 2\,\hat{\rho}(2\,\tau_1/\tau_2)/\tau_2$, where $\hat{\rho}$ is defined by expression (10).

Proof. By applying the Laplace transform in the frequency domain to the aforementioned output feedback stabilization, we obtain the characteristic quasipolynomial function (2). Next, using the normalization (8), we end up with expression (2.3). The first assertion is a direct consequence of Lemmas 2 and 3. The last assertion follows from Lemma 6 and is illustrated by Figure 1. □

Figure 1 shows the locus of $\hat{\rho}$ given by (10) (the proposed upper-bound on the real parts of the zeros of the quasipolynomial $\hat{Q}$) as a function of the delay $\tau = 2\,\tau_1\,\lambda/L$.

Thanks to the linear change of variables (8), an appropriate pair (s_0, τ) in the filled gray region, providing an upper-bound on the spectral abscissa of the quasipolynomial Q given in (2) may be selected. As asserted in Theorem 3, the desired decay rate towards the steady state equilibrium is greater than $-s_0 - 2\,\hat{\rho}(2\,\tau_1/\tau_2)/\tau_2$.

5 | CONCLUSION

This paper discusses the spectral abscissa of linear time-invariant dynamical systems represented by continuous-time delay-difference equations with interfering delays by exploiting the valuable benefits of the multiplicity manifolds and in particular the well established MID property for quasipolynomials. It proposes an autoregressive control design enabling the closed-loop system's solution to obey a prescribed decay rate, opening perspectives in the control of partial differential equations such as in[9,18]. In particular, the proposed methodology is illustrated through the the boundary control of the transport equation.

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