Separation method of semi-fixed variables together with integral bifurcation method for solving generalized time-fractional thin-film equations

Weiguo Rui[1] and weijin he[1]

[1]Chongqing Normal University

September 7, 2023

Abstract

It is well known that investigation on exact solutions of nonlinear fractional partial differential equations (PDEs) is a very difficult work compared with integer-order nonlinear PDEs. In this paper, based on the separation method of semi-fixed variables and integral bifurcation method, a combinational method is proposed. By using this new method, a class of generalized time-fractional thin-film equations are studied. Under two kinds of definitions of fractional derivatives, exact solutions of two generalized time-fractional thin-film equations are investigated respectively. Different kinds of exact solutions are obtained and their dynamic properties are discussed. Compared to the results in the existing references, the types of solutions obtained in this paper are abundant and very different from those in the existing references. Investigation shows that the solutions of the model defined by Riemann-Liouville differential operator converge faster than those defined by Caputo differential operator. It is also found that the profiles of some solutions are very similar to solitons, but they are not true soliton solutions. In order to visually show the dynamic properties of these solutions, the profiles of some representative exact solutions are illustrated by 3D-graphs.

Separation method of semi-fixed variables together with integral bifurcation method for solving generalized time-fractional thin-film equations *

Weiguo Rui, Weijun He

School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China

Abstract

It is well known that investigation on exact solutions of nonlinear fractional partial differential equations (PDEs) is a very difficult work compared with integer-order nonlinear PDEs. In this paper, based on the separation method of semi-fixed variables and integral bifurcation method, a combinational method is proposed. By using this new method, a class of generalized time-fractional thin-film equations are studied. Under two kinds of definitions of fractional derivatives, exact solutions of two generalized time-fractional thin-film equations are investigated respectively. Different kinds of exact solutions are obtained and their dynamic properties are discussed. Compared to the results in the existing references, the types of solutions obtained in this paper are abundant and very different from those in the existing references. Investigation shows that the solutions of the model defined by Riemann-Liouville differential operator converge faster than those defined by Caputo differential operator. It is also found that the profiles of some solutions are very similar to solitons, but they are not true soliton solutions. In order to visually show the dynamic properties of these solutions, the profiles of some representative exact solutions are illustrated by 3D-graphs.

Keywords: Separation method of semi-fixed variables; Integral bifurcation method; Nonlinear time-fractional PDEs; Generalized time-fractional thin-film equation; Exact solutions.

PACS: 02.30.Jr; 02.30.Oz; 02.70.-c; 02.70.Mv; 02.90.+p; 04.20.Jb; 05.10.-a

*Corresponding author. E-mail: wgruihhu@163.com; weiguorhhu@aliyun.com

1 Introduction

It has been more than three hundred and twenty years since the concept of fractional calculus was born in 1695. However, compared with the applications of integer-order calculus, the applications of fractional calculus are not broad enough. The number of fractional differential models is really too small compared to the number of fractional differential models though some mathematical models have been established by using fractional calculus since the 1960s. Fortunately, many complex problems in natural science fields such as mathematical mechanics, control theory, signal processing, aerodynamics, chemistry, biology and so forth can be accurately described by fractional differential models now. However, it becomes difficult to solve the nonlinear fractional differential models. The main reason is that many classical methods in the field of integer-order cannot be directly applied to solve fractional differential equations. So, investigations on exact solutions of nonlinear fractional partial differential equations (PDEs) is a very difficult work. On the other hand, the definitions of fractional derivatives are various, without a unified definition like integer-order derivative at all. This brings great inconvenience to the applications, especially in the modeling, choosing which fractional differential operator to build the model, is also often a more trouble for scientific researchers. In the long process of practice, people still feel that the most classic, the most commonly used and best represent the dominance definitions of fractional derivatives are only two, namely Riemann-Liouville fractional derivative and Caputo fractional derivative. Therefore, it is very meaningful to develop the solution method of fractional partial differential equations (PDEs) under the definitions of these two fractional differential operators.

With the deepening of research, some methods for investigating exact solutions or approximate analytical solutions of fractional differential equations have been proposed successively. Some representative methods include Adomian decomposition method [1,2], homotopy analysis method [3,4], invariant analysis method [5,6], fractional variational iteration method [7-9], invariant subspace method [10-12], method of fractional complex transformation [13-15] and the method of separating variables [16-18], etc. However, compared with nonlinear integer-order models, real nonlinear fractional models for developing solution methods are shortage. In order to make up for this deficiency, many researchers directly changed some classical integer-order PDEs into fractional PDEs to study, so as to develop new solution methods for more complex nonlinear fractional PDEs. From a mathematical point of view, it is meaningful and very necessary to do so in the current shortage of mathematical models.

Very unfortunately, in the above methods, we found that the method of fractional complex transformation appeared in Refs. [13-15] is based on a wrong fractional chain rule which given by Jumarie in Refs. [19-21]. Indeed, Jumarie's fractional chain rule has been verified that it is wrong in Refs. [22-24]. This means peoples need to redesign some new methods to solve those more complex nonlinear fractional PDEs. For this purpose, based on the separation method of variables and combined with other methods such as homogenous balanced principle, idea of invariant subspace, integral bifurcation method, we introduced several new combinational methods [24-27] for investigating exact solutions of nonlinear time-fractional PDEs, recently. The common point of these methods is that exact solutions of some nonlinear time-fractional PDEs are obtained by using the modified separation method of variable together with other methods. Obviously, the mentioned several methods in [24-27] unlike the traditional separation method of variables. Specifically speaking, the function $T(t)$ in the traditional separation method of variables $u(x,t) = v(x)T(t)$ is fixed into some specific special functions such as Mittag-Leffler functions or power function. So, we call this modified separation method of variables as separation method of semi-fixed variables or separation method of variables of semi-fixed form.

In this paper, based on the separation method of semi-fixed variables [28] and integral bifurcation method [29-32], we will improve the combinational method named separation variable method combined with integral bifurcation method [33]. By using this improved method, we will investigate exact solutions and their dynamic properties of a class of time-fractional generalized thin-film equations [34] formed as follows:

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} = &-f(u)u_{xxxx} - g(u)u_x u_{xxx} - h(u)(u_{xx})^2 - l(u)(u_x)^2 u_{xx} \\ &+p(u)u_{xx} + q(u)(u_x)^2 + k(u),\end{aligned} \tag{1.1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ can be Riemann-Liouville or Caputo fractional differential operator and $u = u(x,t),\ x \in R,\ t > 0,\ 0 < \alpha < 1$.

Especially, when $f(u) = \eta u, g(u) = 3\eta, h(u) = 2\eta,\ l(u) = 0,\ p(u) = 2\beta u,\ q(u) = 2\beta, k(u) = \delta u$, Eq. (1.1) becomes the following model

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \delta u - \eta[u u_{xxxx} + 3u_x u_{xxx} + 2(u_{xx})^2] + 2\beta(u u_{xx} + u_x^2). \tag{1.2}$$

When $f(u) = \eta u,\ g(u) = 4\eta,\ h(u) = 3\eta,\ l(u) = 0,\ p(u) = 2\beta u + \delta,\ q(u) = 2\beta,\ k(u) = 0$, Eq. (1.1) becomes the following model

$$\frac{\partial^\alpha u}{\partial t^\alpha} = 2\beta(u_x)^2 - \eta[u u_{xxxx} + 4u_x u_{xxx} + 3(u_{xx})^2] + (2\beta u + \delta)u_{xx}. \tag{1.3}$$

The organization of this paper is as follows: In Sec. 2, we will introduce modified method of separation of semi-fixed variables based on the definitions of fractional derivatives and properties of Mittag-Leffler functions. In Sec. 3, we will investigate exact solutions and dynamic properties of the generalized time-fractional thin-film equation (1.2) under the definition of Riemann-Liouville fractional operator. In Sec. 4, similarly we will investigate exact solutions and dynamic properties of Eq. (1.3) under the definition of Riemann-Liouville fractional operator. In Sec. 4, we will investigate exact solutions of Eqs. (1.2) and (1.3) under the definition of Caputo fractional operator.

2 Brief introduction of preliminary knowledge and method

2.1 Definitions of fractional derivatives and properties of Mittag-Leffler functions

Definition 1 ([35]) *If the function $f(t)$ is a continuous function, then the Riemann-Liouville fractional derivative of $f(t)$ of α-order is defined by*

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (Re(\alpha) > 0), \tag{2.1} $$

where $t > a,\ n-1 \leq \alpha < n,\ n = [\alpha]+1,\ n \in \mathbb{N}^+$.

Definition 2 ([36]) *If the function $f(t)$ is a n-order smooth function, then the Caputo fractional derivative of $f(t)$ of α-order is defined by*

$$ {}_{a}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad (Re(\alpha) > 0), \tag{2.2} $$

where $t > a,\ n-1 < \alpha \leq n,\ n = [\alpha]+1,\ n \in \mathbb{N}^+$.

Property 1 ([35, 36]) *Under definitions of Riemann-Liouville fractional derivative and Caputo fractional derivative, the fractional derivatives of Mittag-Leffler functions and power function have the following properties*

$$ {}_{0}^{RL}D_t^{\alpha} \left[t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right) \right] = \lambda t^{\alpha-1} E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \tag{2.3} $$

$$ {}_{0}^{RL}D_t^{\alpha}\ t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \qquad \gamma > -1, \tag{2.4} $$

$$ {}_{0}^{C}D_t^{\alpha}\ E_{\alpha}\left(\lambda t^{\alpha}\right) = \lambda E_{\alpha}\left(\lambda t^{\alpha}\right), \tag{2.5} $$

$$ {}_{0}^{C}D_t^{\alpha}\ t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \qquad \gamma > 0, \tag{2.6} $$

where $0 < \alpha < 1,\ t > 0.$

Proof of (2.3). According to the definition of the two parameter Mittag-Leffler function and formula of derivative of the power function, it is easy to know that

$$
\begin{aligned}
&{}_{0}^{RL}D_t^{\alpha}\left[t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)\right] = {}_{0}^{RL}D_t^{\alpha}\left[t^{\alpha-1}\sum_{k=0}^{\infty}\frac{\lambda^k t^{k\alpha}}{\Gamma(k\alpha+\alpha)}\right] \\
&= {}_{0}^{RL}D_t^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + t^{\alpha-1}\sum_{k=1}^{\infty}\frac{\lambda^k t^{k\alpha}}{\Gamma(k\alpha+\alpha)}\right] = {}_{0}^{RL}D_t^{\alpha}\left[\frac{t^{\alpha-1}}{\Gamma(\alpha)} + \sum_{k=1}^{\infty}\frac{\lambda^k\ t^{k\alpha+\alpha-1}}{\Gamma(k\alpha+\alpha)}\right] \\
&= \frac{t^{-1}}{\Gamma(0)} + \sum_{k=1}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\alpha)}\frac{\Gamma(k\alpha+\alpha)t^{k\alpha-1}}{\Gamma(k\alpha)} = 0 + \lambda t^{\alpha-1}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}\ t^{(k-1)\alpha}}{\Gamma[(k-1)\alpha+\alpha]} \\
&= \lambda t^{\alpha-1}\sum_{n=0}^{\infty}\frac{\lambda^n\ t^{n\alpha}}{\Gamma(n\alpha+\alpha)} = \lambda t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right),
\end{aligned}
$$

where $\frac{t^{-1}}{\Gamma(0)} = 0$ due to $\Gamma(0) = \infty$.

Property 2 *The function $t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)$ must be a fundamental solution of the following linear fractional ordinary differential equation (ODE)*

$$
{}_{0}^{RL}D_t^{\alpha}\ y(t) - \lambda y(t) = 0. \tag{2.7}
$$

Proof. Making Laplace transformation in both sides of Eq. (2.7), it yields

$$
\mathfrak{L}[{}_{0}^{RL}D_t^{\alpha}\ y(t); s] - \lambda\mathfrak{L}[y(t); s] = 0.
$$

According to the property of Laplace transformation of Riemann-Liouville fractional derivative, we get

$$
s^{\alpha}Y(s) - {}_{0}^{RL}D_t^{\alpha-1}\ y(t)|_{t=0} - \lambda Y(s) = 0. \tag{2.8}
$$

Writing the initial constant ${}_{0}^{RL}D_t^{\alpha-1}\ y(t)|_{t=0} = C_0$ and then solving Eq. (2.8), we obtain

$$
Y(s) = \frac{C_0}{s^{\alpha}-\lambda}. \tag{2.9}
$$

Making inverse transformation of Laplace transformation in both sides of Eq. (2.9), we obtain a special solution of the linear fractional ODE (2.7) as follows:

$$
y(t) = C_0 t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right), \tag{2.10}
$$

thus the function $t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)$ is indeed a fundamental solution of the fractional linear ODE (2.7).

On the other hand, directly substituting the function $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$ into Eq. (2.7), it is easy to verify that the linear fractional ODE (2.7) is identity. This illustrates the correctness of the property (2.3) by another way.

2.2 Brief introduction of separation method of semi-fixed variables

It is well known that a linear fractional PDE can always be separated into two independent differential systems (equations) by traditional separation method of variables. For example, we discuss a linear fractional wave equation as follows:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = c^2 \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}}, \tag{2.11}$$

where $\frac{\partial^{\alpha}}{\partial t^{\alpha}}$ and $\frac{\partial^{2\beta}}{\partial x^{2\beta}}$ are fractional differential operators of Riemann-Liouville type or Caputo type, and $t > 0,\ x \in \mathbb{R},\ 0 < \alpha < 1,\ 0.5 < \beta < 1$. Let us assume that Eq. (2.11) has solution formed as

$$u(x,t) = v(x)T(t), \tag{2.12}$$

where $v = v(x),\ T = T(t)$. Substituting (2.12) into (2.11), we get

$$v\frac{d^{\alpha}T}{dt^{\alpha}} = c^2 T \frac{d^{2\beta}v}{dx^{2\beta}}. \tag{2.13}$$

Separating the variables, Eq. (2.13) can be reduced to

$$\frac{\frac{d^{\alpha}T}{dt^{\alpha}}}{T} = c^2 \frac{\frac{d^{2\beta}v}{dx^{2\beta}}}{v} = \lambda, \tag{2.14}$$

where λ is nonzero constant. Obviously, Eq. (2.14) can be reduced to the following two independent fractional equations

$$\frac{d^{\alpha}T}{dt^{\alpha}} - \lambda T = 0, \tag{2.15}$$

$$\frac{d^{2\beta}v}{dx^{2\beta}} - \lambda v = 0. \tag{2.16}$$

However, for a nonlinear fractional PDE, we may not be able to separate it into two independent fractional ODEs by traditional separation method of variables. For example, a nonlinear time-fractional PDE formed as

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \delta u + \eta u^2 + \kappa \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) \tag{2.17}$$

cannot be separated into two independent differential systems by use of (2.12). Substituting (2.12) into (2.17), we get

$$v\,\frac{d^{\alpha}T}{dt^{\alpha}}=\delta vT+\eta T^{2}v^{2}+\kappa T^{2}\left(\frac{dv}{dx}\right)^{2}+\kappa T^{2}v\frac{d^{2}v}{dx^{2}}. \tag{2.18}$$

It is easy to find that the equation (2.18) cannot be separated into two independent differential equations as in (2.13). But, if we modify the separation expression of variables (2.12) as

$$u(x,t)=v(x)t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right) \tag{2.19}$$

or

$$u(x,t)=v(x)E_{\alpha}\left(\lambda t^{\alpha}\right). \tag{2.20}$$

When $\frac{\partial^{\alpha}}{\partial t^{\alpha}}={}_{0}^{RL}D_{t}^{\alpha}$ is Riemann-Liouville fractional differential operator, substituting (2.19) into (2.17), we obtain

$$[(\lambda-\delta)v]t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)=\left[\eta v^{2}+\kappa\left(\frac{dv}{dx}\right)^{2}+\kappa v\frac{d^{2}v}{dx^{2}}\right][t^{\alpha-1}E_{\alpha,\alpha}\left(\lambda t^{\alpha}\right)]^{2}. \tag{2.21}$$

Taking $\lambda=\delta$, Eq. (2.21) can be reduced to a nonlinear ODE as follows:

$$\eta v^{2}+\kappa\left(\frac{dv}{dx}\right)^{2}+\kappa v\frac{d^{2}v}{dx^{2}}=0. \tag{2.22}$$

Similarly, when $\frac{\partial^{\alpha}}{\partial t^{\alpha}}={}_{0}^{C}D_{t}^{\alpha}$ is Caputo fractional differential operator, substituting (2.20) into (2.17), we obtain

$$[(\lambda-\delta)v]E_{\alpha}\left(\lambda t^{\alpha}\right)=\left[\eta v^{2}+\kappa\left(\frac{dv}{dx}\right)^{2}+\kappa v\frac{d^{2}v}{dx^{2}}\right][E_{\alpha}\left(\lambda t^{\alpha}\right)]^{2}. \tag{2.23}$$

Taking $\lambda=\delta$, Eq. (2.23) also can be reduced to Eq. (2.22).

Obviously, this modified separation method of variables is very convenient in sometimes. To test the efficiency of the modified separation method of variables introduced above, we next discuss exact solutions and dynamic properties of Eqs. (1.2) and (1.3) under the definitions of Riemann-Liouville and Caputo differential differential operators in the below three sections.

3 Exact solutions of Eq. (1.2) under Riemann-Liouville fractional differential operator

When $\frac{\partial^\alpha}{\partial t^\alpha} = {}^{RL}_{0}D^\alpha_t$ is Riemann-Liouville fractional differential operator, the equation (1.2) can be rewritten as

$$ {}^{RL}_{0}D^\alpha_t u = \delta u - \eta(uu_{xxxx} + 3u_x u_{xxx} + 2u_{xx}^2) + 2\beta(uu_{xx} + u_x^2). \tag{3.1} $$

According separation method of semi-fixed introduced above, we assume that the Eq. (3.1) has solutions formed as

$$ u = [a_0 + a_1 v(x)]\, t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha), \tag{3.2} $$

Obviously, the separation transformation (3.2) is not the same as those in the Refs. [24-28, 33], because we replace the one-parameter Mittag-Leffler function $E_\alpha(\lambda t^\alpha)$ and the power function t^γ with the two-parameter Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ at here. Although this is only a small improvement, but it works very well, please see the discussion below.

Substituting (3.2) into (3.1), we get

$$ \begin{aligned} (\lambda - \delta)(a_0 + a_1 v)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha) = \big[-\eta(a_0 a_1 + a_1^2 v)v_{xxxx} - 3\eta a_1^2 v_x v_{xxx} - 2\eta a_1^2 v_{xx}^2 \\ +2\beta a_1(a_0 + a_1 v)v_{xx} + 2\beta a_1^2 v_x^2\big][t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)]^2. \end{aligned} \tag{3.3} $$

In above equation, taking

$$ \lambda = \delta \tag{3.4} $$

and letting the coefficient of Mittag-Leffler function $[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)]^2$ equal zero, Eq. (3.3) can be reduced to a nonlinear ordinary differential equation (ODE) as follows:

$$ \eta(a_0 a_1 + a_1^2 v)v_{xxxx} + 3\eta a_1^2 v_x v_{xxx} + 2\eta a_1^2 v_{xx}^2 - 2\beta a_1(a_0 + a_1 v)v_{xx} - 2\beta a_1^2 v_x^2 = 0. \tag{3.5} $$

Integrating (3.5) twice and letting the first integral constant equal zero, it yields

$$ \eta a_1(a_0 + a_1 v)v_{xx} + \frac{1}{2}\eta a_1^2 v_x^2 - \beta a_1^2 v^2 - 2\beta a_0 a_1 v = g, \tag{3.6} $$

where g is the second integral constant. Letting $v_x \equiv \dfrac{dv}{dx} = y$, Eq.(3.6) can be reduced to a planar dynamical system as follows:

$$ \begin{cases} \dfrac{dv}{dx} = y, \\ \dfrac{dy}{dx} = \dfrac{-\frac{1}{2}\eta a_1^2 y^2 + \beta a_1^2 v^2 + 2\beta a_0 a_1 v + g}{\eta a_1(a_0 + a_1 v)}. \end{cases} \tag{3.7} $$

Obviously, the $\frac{dy}{dx}$ cannot be defined at $v = -\frac{a_0}{a_1}$, so the system (3.7) is not equivalent to the equation (3.6) at $v = -\frac{a_0}{a_1}$. However, $v = -\frac{a_0}{a_1}$ is a trivial solution of equation (3.6). In order to obtain a completely equivalent system to the equation (3.6) no mater how the function v vary, we make a scalar transformation as follows:

$$dx = (a_0 + a_1 v)d\tau, \tag{3.8}$$

where τ is a parameter. Under the transformation (3.8), the singular system (3.7) is reduced to a regular system as follows:

$$\begin{cases} \dfrac{dv}{d\tau} = (a_0 + a_1 v)y, \\ \dfrac{dy}{d\tau} = -\dfrac{1}{2}a_1 y^2 + \dfrac{\beta a_1}{\eta}v^2 + \dfrac{2\beta a_0}{\eta}v + \dfrac{g}{\eta a_1}. \end{cases} \tag{3.9}$$

Obviously, both systems (3.7) and (3.9) have a same first integral as follows:

$$y^2 = \frac{\frac{2\beta a_1}{3\eta}v^3 + \frac{2\beta a_0}{\eta}v^2 + \frac{2g}{\eta a_1}v + h}{a_0 + a_1 v}, \tag{3.10}$$

where h is a new integral constant.

Case 1. If $a_0 \neq 0,\ g = 0$ and $h = 0$, then Eq. (3.10) can be reduced to

$$y = \pm\frac{v\sqrt{\dfrac{2\beta a_1^2}{3\eta}v^2 + \dfrac{8\beta a_0 a_1}{3\eta}v + \dfrac{2\beta a_0^2}{\eta}}}{a_0 + a_1 v}. \tag{3.11}$$

Substituting (3.11) into first equation of system (3.9) to integrate, we get

$$\int \frac{dv}{v\sqrt{a + bv + cv^2}} = \pm\int d\tau, \tag{3.12}$$

where $a = \dfrac{2\beta a_0^2}{\eta}, b = \dfrac{8\beta a_0 a_1}{3\eta}, c = \dfrac{2\beta a_1^2}{3\eta}$. We write $\Delta = b^2 - 4ac,\ q = 4ac - b^2$ and $\epsilon = \pm 1$. It is easy to know that $\Delta = \dfrac{16\beta^2 a_0^2 a_1^2}{9\eta^2} > 0$ and $q = -\dfrac{16\beta^2 a_0^2 a_1^2}{9\eta^2} < 0$.

When $a > 0,\ \Delta > 0$, completing the integrals in (3.12) and then reducing it, we get

$$\frac{2a + bv + 2\sqrt{a}\sqrt{a + bv + cv^2}}{v} = Ce^{\epsilon\sqrt{a}\tau}, \tag{3.13}$$

where C is integral constant. Solving (3.13), it yields

$$v = \frac{4aC\exp(\epsilon\sqrt{a}\tau)}{C^2\exp(2\epsilon\sqrt{a}\tau) + (b^2 - 4ac) - 2bC\exp(\sqrt{a}\tau)}. \tag{3.14}$$

In order to facilitate the next discussions, in the following calculation processes, we take the integral constant C into some special values for obtaining the exact solutions of parametric form.

(i) when $\eta\beta > 0$, (i.e. $a > 0,\ c > 0$), taking the integral constant $C = \Delta$, the solution (3.14) can be reduced to

$$v = \frac{2a\ \mathrm{sech}(\sqrt{a}\ \tau)}{\epsilon\sqrt{\Delta} - b\ \mathrm{sech}(\sqrt{a}\ \tau)} = \frac{3a_0\ \mathrm{sech}\left(a_0\sqrt{\frac{2\beta}{\eta}}\ \tau\right)}{\epsilon a_1 - 2a_1\ \mathrm{sech}\left(a_0\sqrt{\frac{2\beta}{\eta}}\ \tau\right)}, \tag{3.15}$$

where τ is a parameter. Substituting (3.15) into the transformation (3.8) and then integrate it, we get

$$x = a_0\tau - \sqrt{\frac{6\eta}{\beta}}\tanh^{-1}\left[\frac{2+\epsilon}{\sqrt{3}}\tanh\left(a_0\sqrt{\frac{\beta}{2\eta}}\ \tau\right)\right]. \tag{3.16}$$

Thus, substituting the Eq.(3.15) and (3.4) into (3.2) and combining with (3.16), we can obtain an exact solution of parametric form of Eq. (3.1) as follows:

$$\begin{cases} u = \left[a_0 + \dfrac{3a_0\ \mathrm{sech}\left(a_0\sqrt{\frac{2\beta}{\eta}}\ \tau\right)}{\epsilon - 2\ \mathrm{sech}\left(a_0\sqrt{\frac{2\beta}{\eta}}\ \tau\right)}\right] t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}), \\ x = a_0\tau - \sqrt{\dfrac{6\eta}{\beta}}\tanh^{-1}\left[\dfrac{2+\epsilon}{\sqrt{3}}\tanh\left(a_0\sqrt{\frac{\beta}{2\eta}}\ \tau\right)\right]. \end{cases} \tag{3.17}$$

(ii) when $\eta\beta < 0$, (i.e. $a < 0,\ c < 0$), as in the case (i), integrating (3.12) and taking the integral constant $C = \frac{\pi}{2}$, we get another exact solution of (3.6) as follows:

$$v = \frac{2a\sec(\sqrt{-a}\ \tau)}{\epsilon\sqrt{\Delta} - b\sec(\sqrt{-a}\ \tau)} = \frac{3a_0\sec\left(a_0\sqrt{-\frac{2\beta}{\eta}}\ \tau\right)}{\epsilon a_1 - 2a_1\sec\left(a_0\sqrt{-\frac{2\beta}{\eta}}\ \tau\right)}. \tag{3.18}$$

where τ is a parameter. Indeed, by using the transformation of $\mathrm{sech}(i\tau) = \sec(\tau)$, $(i = \sqrt{-1})$, the solution (3.18) can also be directly converted by (3.15). Substituting (3.18) into the transformation (3.8), we get

$$x = a_0\tau + \sqrt{-\frac{6\eta}{\beta}}\arctan\left[\frac{2+\epsilon}{\sqrt{3}}\tan\left(a_0\sqrt{-\frac{\beta}{2\eta}}\ \tau\right)\right] \tag{3.19}$$

Substituting the Eq.(3.18) and (3.4) into (3.2) and combining with (3.19), we can obtain an exact solution of Eq.(3.1) as follow:

$$\begin{cases} u = \left[a_0 + \dfrac{3a_0 \sec\left(a_0\sqrt{-\dfrac{2\beta}{\eta}}\ \tau\right)}{\epsilon - 2\sec\left(a_0\sqrt{-\dfrac{2\beta}{\eta}}\ \tau\right)} \right] t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}), \\ x = a_0\tau + \sqrt{-\dfrac{6\eta}{\beta}} \arctan\left[\dfrac{2+\epsilon}{\sqrt{3}} \tan\left(a_0\sqrt{-\dfrac{\beta}{2\eta}}\ \tau\right)\right]. \end{cases} \tag{3.20}$$

(iii) when $\eta\beta > 0$, (i.e. $a > 0,\ c > 0$), taking the integral constant $C = -b$, the solution (3.14) can be reduced to

$$v = \frac{-ab\ \mathrm{sech}^2\left(\frac{\sqrt{a}}{2}\ \tau\right)}{b^2 - ac\left[1 + \epsilon \tanh\left(\frac{\sqrt{a}}{2}\ \tau\right)\right]^2} = \frac{-12a_0\ \mathrm{sech}^2\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)}{16a_1 - 3a_1\left[1 + \epsilon\ \tanh\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)\right]^2}, \tag{3.21}$$

where τ is a parameter. Substituting (3.21) into the transformation (3.8) and then integrate it, we get

$$x = a_0\tau - \frac{1}{\epsilon}\sqrt{\frac{6\eta}{\beta}} \tanh^{-1}\left[\frac{\sqrt{3}}{4}\left(1 + \epsilon\ \tanh\left(a_0\sqrt{\frac{\beta}{2\eta}}\tau\right)\right)\right] \tag{3.22}$$

Substituting the Eq.(3.21) and (3.4) into (3.2) and combining with (3.22), we can obtain an exact solution of Eq.(3.1) as follow:

$$\begin{cases} u = \left[a_0 - \dfrac{12a_0\ \mathrm{sech}^2\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)}{16 - 3\left[1 + \epsilon\ \tanh\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)\right]^2} \right] t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}), \\ x = a_0\tau - \dfrac{1}{\epsilon}\sqrt{\dfrac{6\eta}{\beta}} \tanh^{-1}\left[\dfrac{\sqrt{3}}{4}\left(1 + \epsilon\ \tanh\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)\right)\right]. \end{cases} \tag{3.23}$$

Case 2. If $a_0 \neq 0,\ g \neq 0$ (or $g \neq \frac{2\beta a_0^2}{3}$) and $h = \frac{2a_0(3g - 2\beta a_0^2)}{3\eta a_1^2}$, then Eq. (3.10) can be reduced to

$$y = \pm\sqrt{\frac{2\beta}{3\eta}v^2 + \frac{4\beta a_0}{3\eta a_1}v + \frac{6g - 4\beta a_0^2}{3\eta a_1^2}}. \tag{3.24}$$

(i) when $\eta\beta > 0$ and $\beta(g - \beta a_0^2) > 0$, Eq. (3.24) can be reduced to

$$y = \pm\sqrt{\frac{2\beta}{3\eta}}\sqrt{v^2 + \frac{2a_0}{a_1}v + \frac{3g - 2\beta a_0^2}{\beta a_1^2}} \tag{3.25}$$

Substituting (3.25) into the first equation of (3.7) and then integrating it, we get two solutions of hyperbolic sine function as follows

$$v = \pm\frac{1}{a_1}\sqrt{\frac{3(g-\beta a_0^2)}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right) - \frac{a_0}{a_1}, \tag{3.26}$$

where C is an arbitrary constant and the next cases are the same as this, we will not repeat the reference to this statement. Substituting (3.26) and (3.4) into (3.2), we obtain two exact solutions of Eq. (3.1) as follows:

$$u = \sqrt{\frac{3(g-\beta a_0^2)}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}) \tag{3.27}$$

and

$$u = -\sqrt{\frac{3(g-\beta a_0^2)}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.28}$$

(ii) when $\eta\beta < 0$ and $\beta(g-\beta a_0^2) < 0$, Eq.(3.24) can be written as:

$$y = \pm\sqrt{-\frac{2\beta}{3\eta}}\sqrt{-v^2 - \frac{2a_0}{a_1}v - \frac{3g-2\beta a_0^2}{\beta a_1^2}} \tag{3.29}$$

Similarly, substituting (3.29) into the first equation of (3.7) and integrating it, we get two periodic solutions as follows:

$$v = \pm\sqrt{\frac{3(\beta a_0^2-g)}{\beta a_1^2}}\sin\left(\sqrt{-\frac{2\beta}{3\eta}}\ x + C\right) - \frac{a_0}{a_1}. \tag{3.30}$$

Substituting (3.30) and (3.4) into (3.2), we can obtain two exact solutions of Eq. (3.1) as follows:

$$u = \sqrt{\frac{3(\beta a_0^2-g)}{\beta}}\sin\left(\sqrt{-\frac{2\beta}{3\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}) \tag{3.31}$$

and

$$u = -\sqrt{\frac{3(\beta a_0^2-g)}{\beta}}\sin\left(\sqrt{-\frac{2\beta}{3\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.32}$$

Case 3. If $a_0 \neq 0,\ g = \frac{2\beta a_0^2}{3}$ and $h = 0$, then Eq. (3.10) can be reduced to

$$y = \pm\sqrt{\frac{2\beta}{3\eta}v^2 + \frac{4\beta a_0}{3\eta a_1}v}. \tag{3.33}$$

(i) when $\eta\beta > 0$, substituting (3.33) into the first equation of (3.7) and integrating it, we get a exact solution as follow:

$$v = \frac{2a_0}{a_1}\sinh^2\left(\sqrt{\frac{\beta}{6\eta}}\;x + C\right). \tag{3.34}$$

Substituting (3.34) and (3.4) into (3.2) and setting the integral constant as zero, we obtain an exact solution of Eq.(3.1) as follows:

$$u = a_0\cosh\left(\sqrt{\frac{2\beta}{3\eta}}\;x + C\right)t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.35}$$

(ii) when $\eta\beta < 0$, substituting (3.33) into the first equation of (3.7) and integrating it, we get a exact solution as follow:

$$v = -\frac{2a_0}{a_1}\sin^2\left(\sqrt{-\frac{\beta}{6\eta}}\;x + C\right) \tag{3.36}$$

or

$$v = -\frac{a_0}{a_1}\left[1 - \cos\left(\sqrt{-\frac{2\beta}{3\eta}}\;x + C\right)\right]. \tag{3.37}$$

Respectively substituting (3.36) and (3.37) into (3.2) in the parametric condition (3.4), we can obtain two exact solutions of Eq. (3.1) as follows:

$$u = a_0\left[1 - 2\sin^2\left(\sqrt{-\frac{\beta}{6\eta}}\;x + C\right)\right]t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}) \tag{3.38}$$

or

$$u = a_0\cos\left(\sqrt{-\frac{2\beta}{3\eta}}\;x + C\right)t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.39}$$

Case 4. If $a_0 = 0, g \neq 0$ and $h = 0$, then Eq. (3.10) can be reduced to

$$y = \pm\sqrt{\frac{2\beta}{3\eta}v^2 + \frac{2g}{\eta a_1^2}}. \tag{3.40}$$

(i) when $\eta\beta > 0$ and $g\beta > 0$, Eq.(3.40) can be written as:

$$y = \pm\sqrt{\frac{2\beta}{3\eta}}\sqrt{v^2 + \frac{3g}{\beta a_1^2}}. \tag{3.41}$$

Substituting (3.41) into the first equation of (3.7) and integrating it, we have

$$\int\frac{dv}{\sqrt{v^2 + \frac{3g}{\beta a_1^2}}} = \pm\int\sqrt{\frac{2\beta}{3\eta}}\;dx. \tag{3.42}$$

Solving (3.42), we obtain two solutions of hyperbolic function type of Eq.(3.5) as below:

$$v = \pm\frac{1}{a_1}\sqrt{\frac{3g}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right). \tag{3.43}$$

Substituting (3.43) and (3.4) into (3.2), we obtain two exact solutions of Eq.(3.1) as follows:

$$u = \sqrt{\frac{3g}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right)t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}) \tag{3.44}$$

and

$$u = -\sqrt{\frac{3g}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right)t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.45}$$

(ii) when $\eta\beta > 0$ and $g\beta < 0$, Eq.(3.40) can be written as:

$$y = \pm\sqrt{\frac{2\beta}{3\eta}}\sqrt{v^2 - \left(-\frac{3g}{\beta a_1^2}\right)}. \tag{3.46}$$

Substituting (3.46) into the first equation of (3.7) and integrating it, we have

$$\int \frac{dv}{\sqrt{v^2 - \left(-\frac{3g}{\beta a_1^2}\right)}} = \pm\int\sqrt{\frac{2\beta}{3\eta}}\ dx. \tag{3.47}$$

Solving (3.47), we obtain a hyperbolic cosine function solution of Eq.(3.5) as below:

$$v = \frac{1}{a_1}\sqrt{-\frac{3g}{\beta}}\cosh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right). \tag{3.48}$$

Substituting (3.48) and (3.4) into (3.2), we can obtain an exact solution of Eq.(3.1) as follows:

$$u = \sqrt{-\frac{3g}{\beta}}\cosh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right)t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.49}$$

(iii) when $\eta\beta < 0$ and $g\beta < 0$, Eq.(3.40) can be written as:

$$y = \pm\sqrt{-\frac{2\beta}{3\eta}}\sqrt{\left(-\frac{3g}{\beta a_1^2}\right) - v^2}. \tag{3.50}$$

Substituting (3.50) into the first equation of (3.7) and integrating it, we have

$$\int \frac{dv}{\sqrt{\left(-\frac{3g}{\beta a_1^2}\right) - v^2}} = \pm\int\sqrt{-\frac{2\beta}{3\eta}}\ dx. \tag{3.51}$$

Solving (3.51), we obtain a smooth periodic solution of Eq.(3.5) as below:

$$v = \frac{1}{a_1}\sqrt{-\frac{3g}{\beta}}\cos\left(\sqrt{-\frac{2\beta}{3\eta}}\ x + C\right). \tag{3.52}$$

Substituting (3.52) and (3.4) into (3.2), we can obtain an exact solution of Eq.(3.1) as follows:

$$u = \sqrt{-\frac{3g}{\beta}}\cos\left(\sqrt{-\frac{2\beta}{3\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.53}$$

Case 5. If $\eta\beta > 0,\ a_0 = 0,\ g = 0$ and $h = 0$, then Eq. (3.10) can be reduced to

$$y = \pm\sqrt{\frac{2\beta}{3\eta}}\ v. \tag{3.54}$$

Substituting Eq.(3.54) into the first equation of (3.7) and integrating it, we can get two general solutions of exponential function type as follows:

$$v = C\exp\left(\sqrt{\frac{2\beta}{3\eta}}\ x\right) \tag{3.55}$$

and

$$v = C\exp\left(-\sqrt{\frac{2\beta}{3\eta}}\ x\right). \tag{3.56}$$

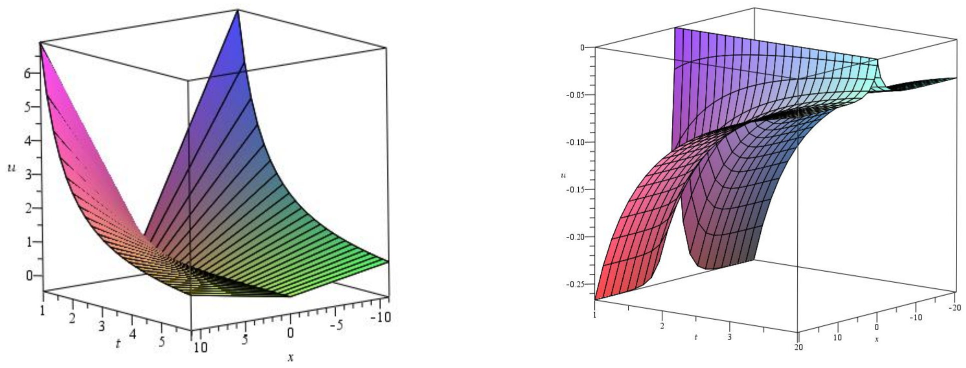
Respectively substituting the (3.55) and (3.56) into Eq.(3.2) in the parametric condition (3.4), we can get two unbounded solutions of Eq. (3.1) as follows:

$$u = a_1 C\exp\left(\sqrt{\frac{2\beta}{3\eta}}\ x\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}) \tag{3.57}$$

and

$$u = a_1 C\exp\left(-\sqrt{\frac{2\beta}{3\eta}}\ x\right) t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha}). \tag{3.58}$$

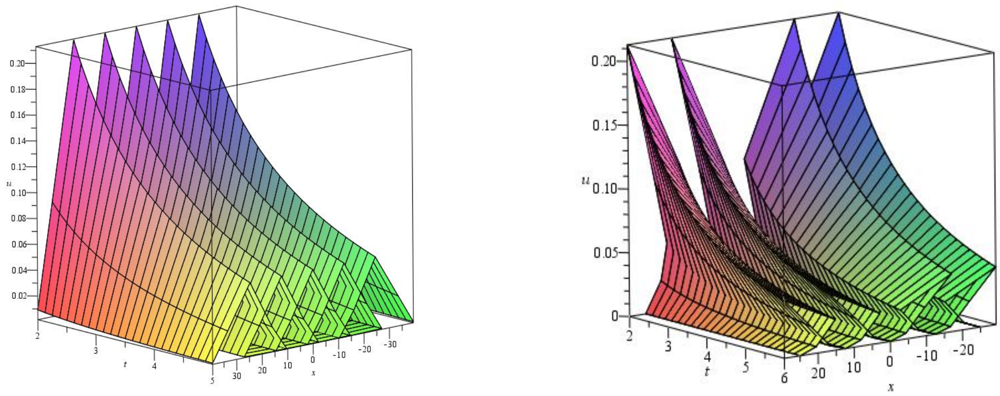
In order to intuitively show the dynamic profiles and properties of above solutions, as examples, the 3D-graphs of the solutions (3.17), (3.20), (3.23), (3.38) and (3.49) are illustrated, which are shown in Figs.1-5, respectively.



(**a**) case of $\epsilon = 1$ (**b**) case of $\epsilon = -1$

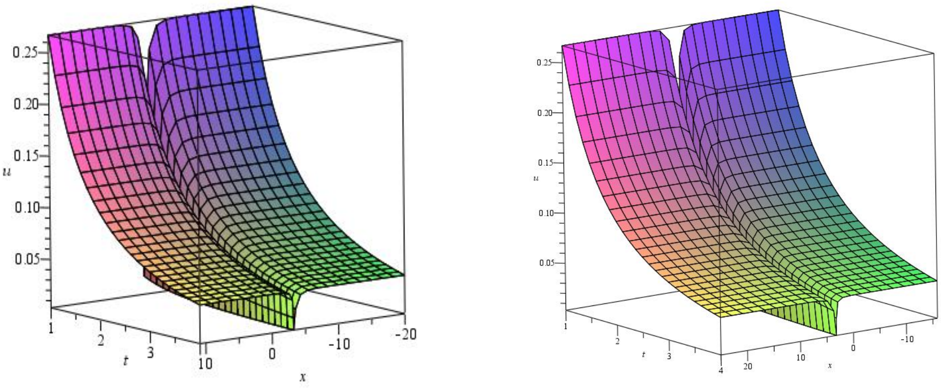
Fig. 1. The 3D-graphs of dynamical profiles of the solution (3.17) under the fixed parameters $a_0 = -5,\ \eta = 4,\ \beta = 0.5,\ \delta = -2,\ \alpha = 0.5.$

As can be seen from Fig.1b, the profile of the solution (3.17) is very similar to a bright soliton when $\epsilon = -1$. However, the solution (3.17) is not a soliton solution after all because it is not a traveling wave solution, which has been explained very clearly in Ref. [37].



(**a**) case of $\epsilon = 1$ (**b**) case of $\epsilon = -1$

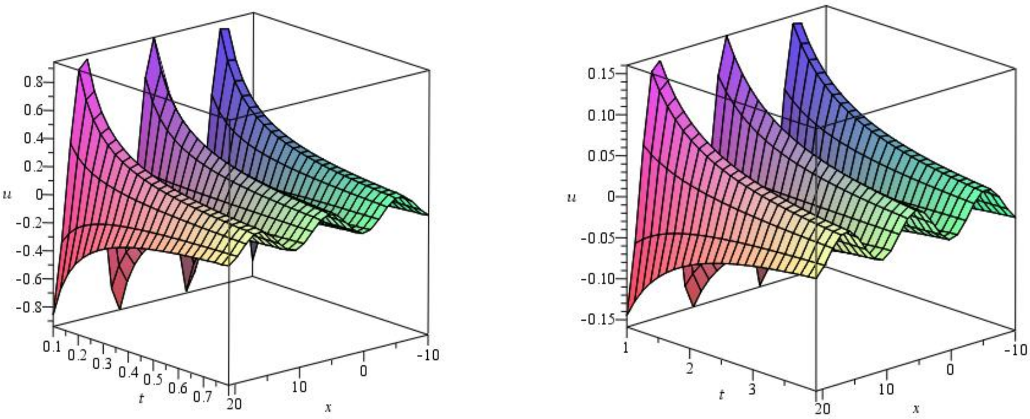
Fig. 2. The 3D-graphs of dynamical profiles of the solution (3.20) under the fixed parameters $a_0 = -5,\ \eta = -4,\ \beta = 0.5,\ \delta = -2,\ \alpha = 0.5.$



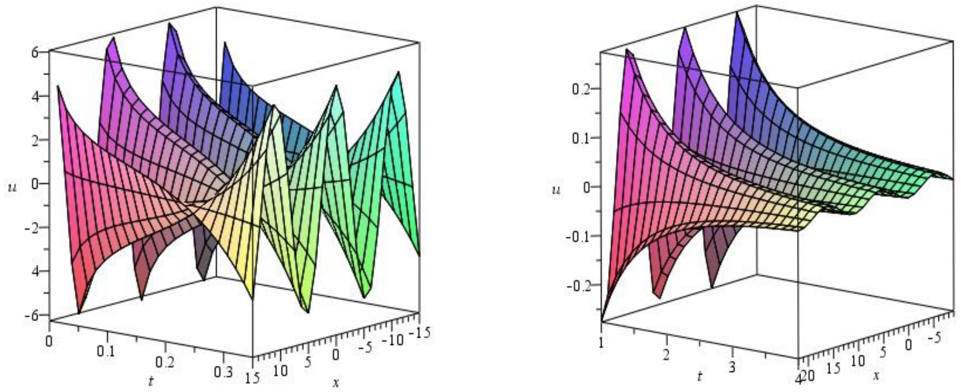
(**a**) case of $\epsilon = 1$ (**b**) case of $\epsilon = -1$

Fig. 3. The 3D-graphs of dynamical profiles of the solution (3.23) under the fixed parameters $a_0 = 5,\ \eta = 4,\ \beta = 0.5,\ \delta = -2,\ \alpha = 0.5.$

As can be seen from Fig.3a and Fig.3b, the profiles of the solution (3.23) are very similar to two dark solitons. But, the solution (3.23) is not soliton solution yet due to it is not travelling wave solution. So, do any other forms of soliton solutions exist in this kind of nonlinear time-fractional PDEs such as Eq. (1.2)? For now, this is also a very challenging issue.



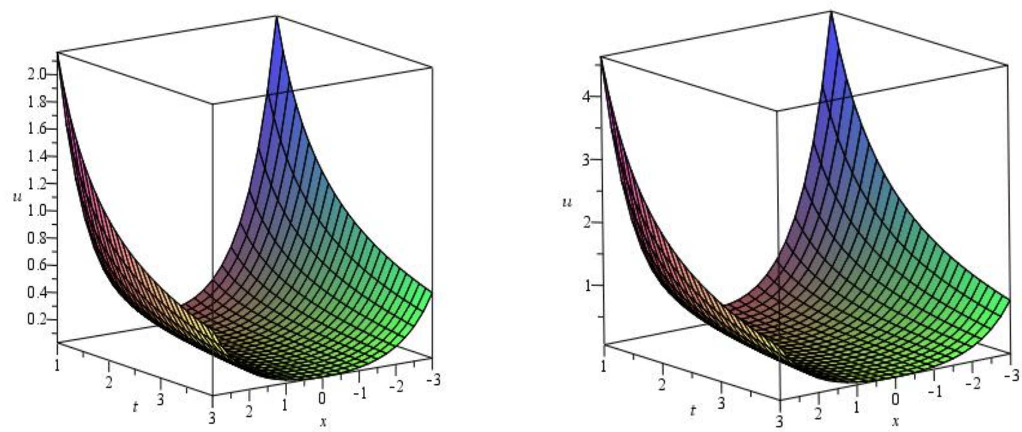
(**a**) case of $\alpha = 0.25$ (**b**) case of $\alpha = 0.5$



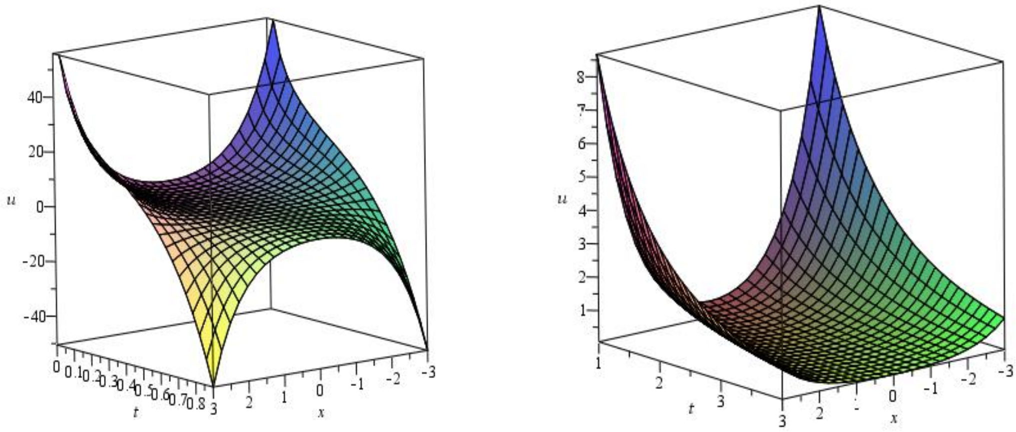
($\mathbf{c}$) case of $\alpha = 0.75$ ($\mathbf{d}$) case of $\alpha = 0.8$

Fig. 4. The 3D-graphs of dynamical profiles of the solution (3.38) under the fixed parameters $a_0 = -3,\ \eta = -1,\ \beta = 0.5,\ \delta = -2,\ C = 0.$

As can be seen from Fig. 4, the profile of the solution (3.38) occurred a mutation phenomenon at $\alpha = 0.75$. This is obviously a very anomalous phenomenon, which is unlikely to occur in cases of integer-order nonlinear PDEs. Currently we do not know the cause of the mutation (anomalous phenomenon), perhaps caused by the singularity of the two-parameter Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha})$.



($\mathbf{a}$) case of $\alpha = 0.25$ ($\mathbf{b}$) case of $\alpha = 0.5$



(**c**) case of $\alpha = 0.75$ (**d**) case of $\alpha = 0.8$

Fig. 5. The 3D-graphs of dynamical profiles of the solution (3.49) under the fixed parameters $a_1 = 1,\ g = -3,\ \eta = 1,\ \beta = 2,\ \delta = -1,\ C = 0$.

Similarly, as can be seen from Fig. 5, the profile of the solution (3.49) appeared a mutation phenomenon at $\alpha = 0.75$, perhaps caused by the singularity of the two-parameter Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha})$, too.

4 Exact solutions of Eq. (1.3) under Riemann-Liouville differential operator

When $\frac{\partial^{\alpha}}{\partial t^{\alpha}} = {}_0^{RL}D_t^{\alpha}$ is Riemann-Liouville fractional differential operator, the equation (1.3) can be rewritten as

$$ {}_0^{RL}D_t^{\alpha}u = 2\beta(u_x)^2 - \eta[uu_{xxxx} + 4u_xu_{xxx} + 3(u_{xx})^2] + (2\beta u + \delta)u_{xx}. \tag{4.1}$$

According separation method of semi-fixed introduced above, we assume that Eq. (4.1) has solutions formed as

$$u = v(x)t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}). \tag{4.2}$$

Substituting (4.2) into (4.1), we have

$$\begin{aligned}(\lambda v - \delta v_{xx})t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha}) = [&-\eta v v_{xxxx} - 4\eta v_x v_{xxx} - 3\eta v_{xx}^2 \\ &+2\beta v v_{xx} + 2\beta v_x^2][t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})]^2.\end{aligned} \tag{4.3}$$

In (4.3), letting each coefficient of the functions $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})$ and $[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})]^2$ equal zero, we obtain

$$\begin{cases} \lambda v - \delta v_{xx} = 0, \\ \eta v v_{xxxx} + 4\eta v_x v_{xxx} + 3\eta v_{xx}^2 - 2\beta v v_{xx} - 2\beta v_x^2 = 0. \end{cases} \tag{4.4}$$

The first equation in (4.4) is linear ODE, but the second equation in (4.4) is nonlinear ODE. From the theory of ODEs, we know that a solution of the first equation in (4.4) is not necessarily a solution of the second equation in (4.4), but a solution of the second equation may be a solution of the first equation. Therefore, we plan to solve the second nonlinear ODE in (4.4) firstly, and then substitute the obtained results into the first linear ODE in (4.4) to obtain corresponding parametric condition.

Integrating the second equation in (4.4) twice and setting the first integral constant as zero, it yields

$$\eta v_x^2 + \eta v v_{xx} - \beta v^2 = g, \tag{4.5}$$

where g is the second integral constant. Letting $\frac{dv}{dx} = y$, Eq. (4.5) can be reduced to the following planar dynamic system

$$\begin{cases} \dfrac{dv}{dx} = y, \\ \dfrac{dy}{dx} = \dfrac{g + \beta v^2 - \eta y^2}{\eta v}. \end{cases} \tag{4.6}$$

Eq. (4.6) has a first integral as follows:

$$y^2 = \frac{\frac{g}{\eta} v^2 + \frac{\beta}{2\eta} v^4 + h}{v^2}, \tag{4.7}$$

where h is new integral constant. Taking the integral constant $h = 0$, Eq. (4.7) can be reduced to

$$y = \pm\sqrt{\frac{g}{\eta} + \frac{\beta}{2\eta} v^2}. \tag{4.8}$$

Substituting (4.8) into the first equation $\frac{dv}{dx} = y$ in (4.6) to integrate, we get

$$\int \frac{dv}{\sqrt{\frac{g}{\eta} + \frac{\beta}{2\eta} v^2}} = \pm \int dx. \tag{4.9}$$

When $\eta\beta > 0$ and $g\beta > 0$, solving (4.9), we obtain two general solutions of the second equation in (4.4) as follows:

$$v = \sqrt{\frac{2g}{\beta}} \sinh\left(\sqrt{\frac{\beta}{2\eta}}\, x + C\right) \tag{4.10}$$

and

$$v = -\sqrt{\frac{2g}{\beta}}\sinh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right), \tag{4.11}$$

where C are two arbitrary constants and we will not repeat the reference to this statement in the below discussions. Respectively plugging (4.10) and (4.11) into the first equation (i.e. $\lambda v - \delta v_{xx} = 0$) in (4.4), it yields

$$\left[\lambda\sqrt{\frac{2g}{\beta}} - \frac{\delta\beta}{2\eta}\sqrt{\frac{2g}{\beta}}\right]\sinh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right) = 0, \tag{4.12}$$

$$-\left[\lambda\sqrt{\frac{2g}{\beta}} - \frac{\delta\beta}{2\eta}\sqrt{\frac{2g}{\beta}}\right]\sinh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right) = 0. \tag{4.13}$$

In Eqs. (4.12) and (4.13), letting the coefficient of the function $\sinh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right)$ equal zero, it yields

$$\lambda\sqrt{\frac{2g}{\beta}} - \frac{\delta\beta}{2\eta}\sqrt{\frac{2g}{\beta}} = 0. \tag{4.14}$$

Solving (4.14), we obtain a parametric condition as follows:

$$\lambda = \frac{\delta\beta}{2\eta}. \tag{4.15}$$

Thus, respectively plugging (4.10), (4.11) and the parametric condition (4.15) into (4.2), we obtain two exact solutions of Eq. (4.1) as follows:

$$u = \sqrt{\frac{2g}{\beta}}\sinh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right) \tag{4.16}$$

and

$$u = -\sqrt{\frac{2g}{\beta}}\sinh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right) t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right). \tag{4.17}$$

When $\eta\beta < 0$ and $g\beta < 0$, solving (4.9), we obtain two general solutions of the second equation in (4.4) as follows:

$$v = \sqrt{-\frac{2g}{\beta}}\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right) \tag{4.18}$$

and

$$v = -\sqrt{-\frac{2g}{\beta}}\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right). \tag{4.19}$$

Respectively plugging (4.18) and (4.19) into the first equation (i.e. $\lambda v - \delta v_{xx} = 0$) in (4.4), it yields

$$\left[\lambda\sqrt{-\frac{2g}{\beta}} - \frac{\delta\beta}{2\eta}\sqrt{-\frac{2g}{\beta}}\right]\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right) = 0, \tag{4.20}$$

$$-\left[\lambda\sqrt{-\frac{2g}{\beta}} - \frac{\delta\beta}{2\eta}\sqrt{-\frac{2g}{\beta}}\right]\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right) = 0. \tag{4.21}$$

In Eqs. (4.20) and (4.21), letting the coefficient of the function $\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right)$ equal zero, it yields

$$\lambda\sqrt{-\frac{2g}{\beta}} - \frac{\delta\beta}{2\eta}\sqrt{-\frac{2g}{\beta}} = 0. \tag{4.22}$$

Solving (4.22), we obtain a parametric condition as follows:

$$\lambda = \frac{\delta\beta}{2\eta}. \tag{4.23}$$

Thus, respectively plugging (4.18), (4.19) and the parametric condition (4.23) into (4.2), we obtain two exact solutions of Eq. (4.1) as follows:

$$u = \sqrt{-\frac{2g}{\beta}}\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right)t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right) \tag{4.24}$$

and

$$u = -\sqrt{-\frac{2g}{\beta}}\sin\left(\sqrt{-\frac{\beta}{2\eta}}\ x + C\right)t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right). \tag{4.25}$$

By using the same method, under $\eta\beta > 0$ and $g\beta < 0$, we obtain an exact solutions of Eq. (4.1) as follows:

$$u = \sqrt{-\frac{2g}{\beta}}\cosh\left(\sqrt{\frac{\beta}{2\eta}}\ x + C\right)t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right). \tag{4.26}$$

When $g = 0,\ h = 0$ and $\eta\beta > 0$, Eq. (4.7) can be reduced to

$$y = \pm\sqrt{\frac{\beta}{2\eta}}\ v. \tag{4.27}$$

Substituting (4.27) into the first equation $\frac{dv}{dx} = y$ in (4.6) to integrate, we get two solutions of Eq. (4.5) as follows:

$$v = C\exp\left(\sqrt{\frac{\beta}{2\eta}}\ x\right) \tag{4.28}$$

and

$$v = C\exp\left(-\sqrt{\frac{\beta}{2\eta}}\ x\right). \tag{4.29}$$

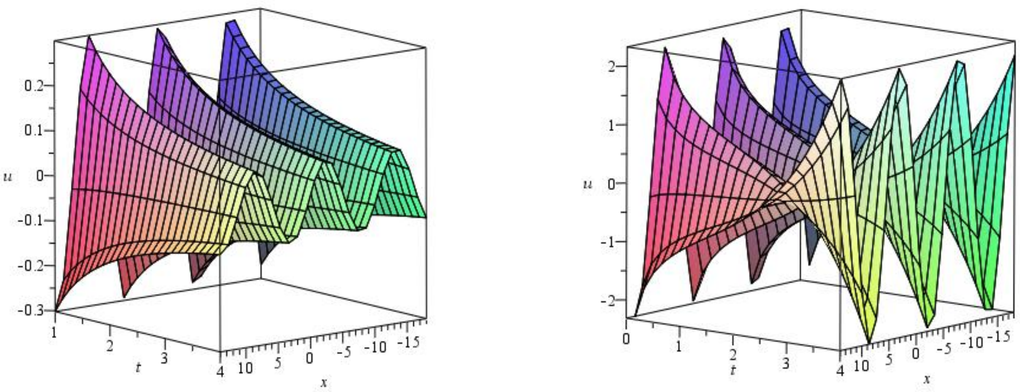
As in determining parametric condition of (4.16) and (4.17), we also obtain $\lambda = \frac{\delta\beta}{2\eta}$. Respectively plugging (4.28), (4.29) and above parametric condition into (4.2), we obtain two exact solutions of Eq. (4.1) as follows:

$$u = C\exp\left(\sqrt{\frac{\beta}{2\eta}}\ x\right) t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right) \tag{4.30}$$

and

$$u = C\exp\left(-\sqrt{\frac{\beta}{2\eta}}\ x\right) t^{\alpha-1}E_{\alpha,\alpha}\left(\frac{\delta\beta}{2\eta}t^{\alpha}\right). \tag{4.31}$$

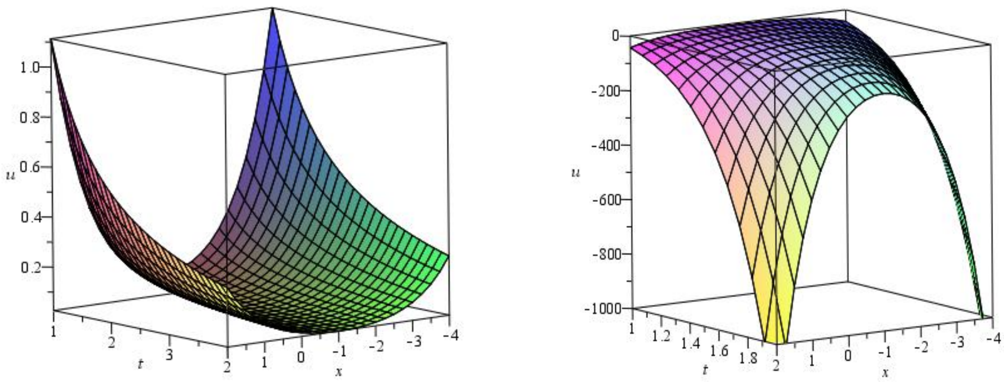
In order to intuitively show the dynamic profiles and properties of above solutions, as examples, the 3D-graphs of the solutions (4.24) and (4.26) are illustrated, which are shown in Fig. 6 and Fig.7.



(**a**) case of $\alpha = 0.25$ (**b**) case of $\alpha = 0.75$

Fig. 6. The 3D-graphs of dynamical profiles of the solution (4.24) under the fixed parameters $C = 3,\ g = -4,\ \eta = -3,\ \beta = 2,\ \delta = 1$.

As can be seen from Fig. 6, the profile of the solution (4.24) occurred mutation phenomenon at $\alpha = 0.75$ caused by the singularity of the two-parameter Mittag-Leffler function $t^{\alpha-1}E_{\alpha,\alpha}(\delta t^{\alpha})$.



(**a**) case of $\alpha = 0.25$ (**b**) case of $\alpha = 0.75$

Fig. 7. The 3D-graphs of dynamical profiles of the solution (4.26) under the fixed parameters $C = 1,\ g = -3,\ \eta = 1,\ \beta = 2,\ \delta = -1.$

Similarly, as can be seen from Fig. 7 that the profile of the solution (4.26) appeared a mutation phenomenon at $\alpha = 0.75$, too.

5 Exact solutions of Eqs. (1.2) and (1.3) under Caputo fractional differential operator

5.1 Exact solutions of Eq. (1.2) under Caputo operator

When $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^C D_t^\alpha$ is Caputo differential operator, Eq. (1.2) can be rewritten as

$$ {}_0^C D_t^\alpha u = \delta u - \eta(u u_{xxxx} + 3u_x u_{xxx} + 2u_{xx}^2) + 2\beta(u u_{xx} + u_x^2). \tag{5.1}$$

According separation method of semi-fixed introduced above, we assume that Eq. (5.1) has solutions formed as

$$u = [a_0 + a_1 v(x)]\, E_\alpha(\lambda t^\alpha). \tag{5.2}$$

Substituting (5.2) into (5.1), we get

$$\begin{aligned}(\lambda - \delta)(a_0 + a_1 v) E_\alpha(\lambda t^\alpha) = \big[-\eta(a_0 a_1 + a_1^2 v) v_{xxxx} - 3\eta a_1^2 v_x v_{xxx} - 2\eta a_1^2 v_{xx}^2 \\ +2\beta a_1(a_0 + a_1 v) v_{xx} + 2\beta a_1^2 v_x^2\big] [E_\alpha(\lambda t^\alpha)]^2.\end{aligned} \tag{5.3}$$

In Eq. (5.3), we directly take

$$\lambda = \delta, \tag{5.4}$$

so that Eq. (5.3) can be reduced to

$$\eta(a_0a_1 + a_1^2v)v_{xxxx} + 3\eta a_1^2 v_x v_{xxx} + 2\eta a_1^2 v_{xx}^2 - 2\beta a_1(a_0 + a_1 v)v_{xx} - 2\beta a_1^2 v_x^2 = 0. \tag{5.5}$$

Obviously, Eq.(5.5) is same to Eq. (3.5) completely, so they have the same exact solutions. Therefore, directly plugging those exact solutions of Eq. (3.5) given in Sec. 3 into Eq. (5.2), we can easily obtain different kinds of exact solutions of Eq. (5.1) as follows:

Case 1. When $a_0 \neq 0,\ g = 0$ and $h = 0$, Eq. (5.1) has exact solutions as follows:

$$\begin{cases} u = \left[a_0 + \dfrac{3a_0 \operatorname{sech}\left(a_0\sqrt{\dfrac{2\beta}{\eta}}\ \tau\right)}{\epsilon - 2\operatorname{sech}\left(a_0\sqrt{\dfrac{2\beta}{\eta}}\ \tau\right)}\right] E_\alpha(\delta t^\alpha), \\ x = a_0\tau - \sqrt{\dfrac{6\eta}{\beta}}\tanh^{-1}\left[\dfrac{2+\epsilon}{\sqrt{3}}\tanh\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\ \tau\right)\right], \end{cases} \tag{5.6}$$

and

$$\begin{cases} u = \left[a_0 - \dfrac{12a_0 \operatorname{sech}^2\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)}{16 - 3\left[1 + \epsilon\ \tanh\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)\right]^2}\right] E_\alpha(\delta t^\alpha), \\ x = a_0\tau - \dfrac{1}{\epsilon}\sqrt{\dfrac{6\eta}{\beta}}\tanh^{-1}\left[\dfrac{\sqrt{3}}{4}\left(1 + \epsilon\ \tanh\left(a_0\sqrt{\dfrac{\beta}{2\eta}}\tau\right)\right)\right], \end{cases} \tag{5.7}$$

where $\eta\beta > 0$.

$$\begin{cases} u = \left[a_0 + \dfrac{3a_0 \sec\left(a_0\sqrt{-\dfrac{2\beta}{\eta}}\ \tau\right)}{\epsilon - 2\sec\left(a_0\sqrt{-\dfrac{2\beta}{\eta}}\ \tau\right)}\right] E_\alpha(\delta t^\alpha), \\ x = a_0\tau + \sqrt{-\dfrac{6\eta}{\beta}}\arctan\left[\dfrac{2+\epsilon}{\sqrt{3}}\tan\left(a_0\sqrt{-\dfrac{\beta}{2\eta}}\ \tau\right)\right], \end{cases} \tag{5.8}$$

where $\eta\beta < 0$.

Case 2. When $a_0 \neq 0,\ g \neq 0$ (or $g \neq \frac{2\beta a_0^2}{3}$) and $h = \frac{2a_0(3g-2\beta a_0^2)}{3\eta a_1^2}$, Eq. (5.1) has exact solutions as follows:

$$u = \sqrt{\frac{3(g-\beta a_0^2)}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x + C\right) E_\alpha(\delta t^\alpha) \tag{5.9}$$

and

$$u=-\sqrt{\frac{3(g-\beta a_0^2)}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha), \tag{5.10}$$

where $\eta\beta>0$ and $\beta(g-\beta a_0^2)>0$.

$$u=\sqrt{\frac{3(\beta a_0^2-g)}{\beta}}\sin\left(\sqrt{-\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha) \tag{5.11}$$

and

$$u=-\sqrt{\frac{3(\beta a_0^2-g)}{\beta}}\sin\left(\sqrt{-\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha) \tag{5.12}$$

where $\eta\beta<0$ and $\beta(g-\beta a_0^2)<0$.

Case 3. When $a_0\neq 0,\ g=\frac{2\beta a_0^2}{3}$ and $h=0$, Eq. (5.1) has exact solutions as follows:

$$u=a_0\cosh\left(\sqrt{\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha), \tag{5.13}$$

where $\eta\beta>0$.

$$u=a_0\left[1-2\sin^2\left(\sqrt{-\frac{\beta}{6\eta}}\ x+C\right)\right]E_\alpha(\delta t^\alpha) \tag{5.14}$$

or

$$u=a_0\cos\left(\sqrt{-\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha), \tag{5.15}$$

where $\eta\beta<0$.

Case 4. When $a_0=0, g\neq 0$ and $h=0$, Eq. (5.1) has exact solutions as follows:

$$u=\sqrt{\frac{3g}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha) \tag{5.16}$$

and

$$u=-\sqrt{\frac{3g}{\beta}}\sinh\left(\sqrt{\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha), \tag{5.17}$$

where $\eta\beta>0$ and $g\beta>0$.

$$u=\sqrt{-\frac{3g}{\beta}}\cosh\left(\sqrt{\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha), \tag{5.18}$$

where $\eta\beta>0$ and $g\beta<0$.

$$u=\sqrt{-\frac{3g}{\beta}}\cos\left(\sqrt{-\frac{2\beta}{3\eta}}\ x+C\right)E_\alpha(\delta t^\alpha), \tag{5.19}$$

where $\eta\beta<0$ and $g\beta<0$.

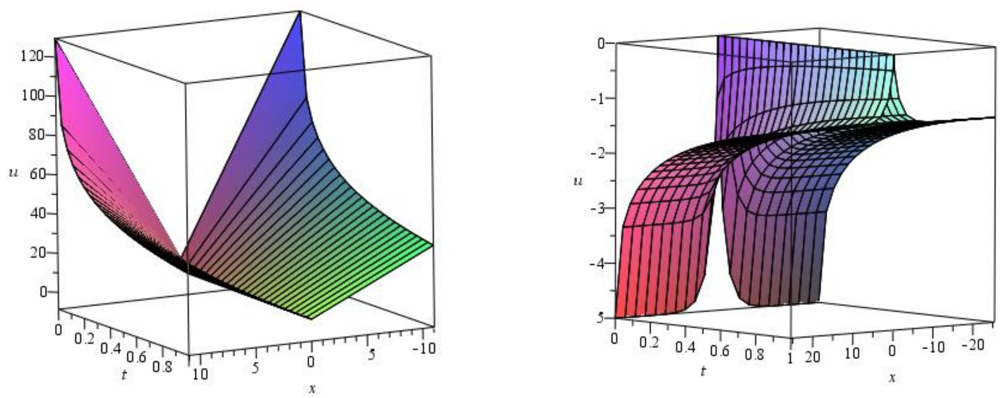
Case 5. When $\eta\beta>0,\ a_0=0,\ g=0$ and $h=0$, Eq. (5.1) has exact solutions as follows:

$$u=a_1C\exp\left(\sqrt{\frac{2\beta}{3\eta}}\ x\right)E_\alpha(\delta t^\alpha) \tag{5.20}$$

and

$$u=a_1C\exp\left(-\sqrt{\frac{2\beta}{3\eta}}\ x\right)E_\alpha(\delta t^\alpha). \tag{5.21}$$

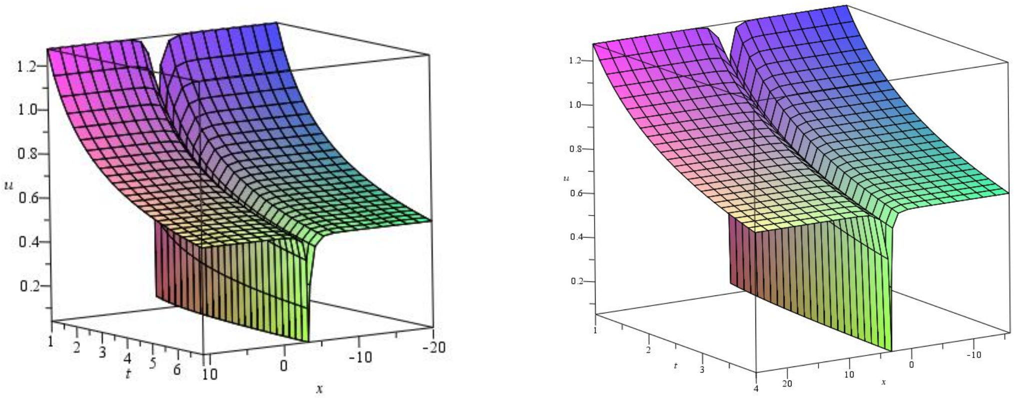
In order to intuitively show the dynamic profiles and properties of above solutions, as examples, the 3D-graphs of the solutions (5.6) and (5.7) are illustrated, which are shown in Fig. 8 and Fig.9.



(**a**) case of $\epsilon=1$ (**b**) case of $\epsilon=-1$

Fig. 8. The 3D-graphs of dynamical profiles of the solution (5.6) under the fixed parameters $a_0=-5,\ \eta=4,\ \beta=0.5,\ \delta=-2,\ \alpha=0.5$.

Comparing Fig. 8 and Fig. 1, it is easy to find that their shapes are very similar, only the amplitude decay degree is different. This is due to the difference of the Mittag-Leffler functions $t^{\alpha-1}E_{\alpha,\alpha}(\delta t^\alpha)$ and $E_\alpha(\delta t^\alpha)$ in the solutions (5.6) and (3.17). It can be seen that the solution (3.17) is only one more factor $t^{\alpha-1}$ than the solution (5.6). So that the solution (3.17) converges faster than the solution (5.6) when $t \to +\infty$.



(**a**) case of $\epsilon = 1$ (**b**) case of $\epsilon = -1$

Fig. 9. The 3D-graphs of dynamical profiles of the solution (5.7) under the fixed parameters $a_0 = 5,\ \eta = 4,\ \beta = 0.5,\ \delta = -2,\ \alpha = 0.5$.

Similarly, comparing Fig. 9 and Fig. 3, it can be found that their shapes are also similar and only their amplitude decay degree is different. This is due to the difference of the Mittag-Leffler functions $t^{\alpha-1}E_{\alpha,\alpha}(\delta t^\alpha)$ and $E_\alpha(\delta t^\alpha)$ in the solutions (5.7) and (3.20). Also, it is easy to find that the solution (3.20) is only one more factor $t^{\alpha-1}$ than the solution (5.7). So that the solution (3.20) converges faster than the solution (5.7) when $t \to +\infty$.

5.2 Exact solutions of Eqs.(1.3) under Caputo operator

When $\frac{\partial^\alpha}{\partial t^\alpha} = {}_0^C D_t^\alpha$ is Caputo differential operator, Eq. (1.3) can be rewritten as

$$ {}_0^C D_t^\alpha u = 2\beta(u_x)^2 - \eta[u u_{xxxx} + 4u_x u_{xxx} + 3(u_{xx})^2] + (2\beta u + \delta)u_{xx}. \tag{5.22}$$

According separation method of semi-fixed introduced above, we assume that Eq. (5.22) has solutions formed as

$$u = v(x)E_\alpha(\lambda t^\alpha). \tag{5.23}$$

Substituting (5.23) into (5.22), we get

$$
\begin{aligned}
(\lambda v - \delta v_{xx}) E_\alpha(\lambda t^\alpha) = [&-\eta v v_{xxxx} - 4\eta v_x v_{xxx} - 3\eta v_{xx}^2 \\
&+2\beta v v_{xx} + 2\beta v_x^2][E_\alpha(\lambda t^\alpha)]^2.
\end{aligned} \tag{5.24}
$$

In (5.24), letting each coefficient of the functions $t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)$ and $[t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^\alpha)]^2$ equal zero, we obtain

$$
\begin{cases}
\lambda v - \delta v_{xx} = 0, \\
\eta v v_{xxxx} + 4\eta v_x v_{xxx} + 3\eta v_{xx}^2 - 2\beta v v_{xx} - 2\beta v_x^2 = 0.
\end{cases} \tag{5.25}
$$

Obviously, Eq. (5.25) is same to Eq. (4.4) completely, so they have the same exact solutions. Therefore, directly plugging those exact solutions of Eq. (4.4) given in Sec. 4 into Eq. (5.23), we can easily obtain different kinds of exact solutions of Eq. (5.22) as follows:

When $\eta\beta > 0$ and $g\beta > 0$, we obtain two general solutions of Eq. (5.22) as follows:

$$
u = \sqrt{\frac{2g}{\beta}} \sinh\left(\sqrt{\frac{\beta}{2\eta}}\, x + C\right) E_\alpha\left(\frac{\delta\beta}{2\eta} t^\alpha\right) \tag{5.26}
$$

and

$$
u = -\sqrt{\frac{2g}{\beta}} \sinh\left(\sqrt{\frac{\beta}{2\eta}}\, x + C\right) E_\alpha\left(\frac{\delta\beta}{2\eta} t^\alpha\right). \tag{5.27}
$$

When $\eta\beta < 0$ and $g\beta < 0$, we obtain two general solutions of Eq. (5.22) as follows:

$$
u = \sqrt{-\frac{2g}{\beta}} \sin\left(\sqrt{-\frac{\beta}{2\eta}}\, x + C\right) E_\alpha\left(\frac{\delta\beta}{2\eta} t^\alpha\right) \tag{5.28}
$$

and

$$
u = -\sqrt{-\frac{2g}{\beta}} \sin\left(\sqrt{-\frac{\beta}{2\eta}}\, x + C\right) E_\alpha\left(\frac{\delta\beta}{2\eta} t^\alpha\right). \tag{5.29}
$$

When $\eta\beta > 0$ and $g\beta < 0$, we obtain an exact solutions of Eq. (5.22) as follows:

$$
u = \sqrt{-\frac{2g}{\beta}} \cosh\left(\sqrt{\frac{\beta}{2\eta}}\, x + C\right) E_\alpha\left(\frac{\delta\beta}{2\eta} t^\alpha\right). \tag{5.30}
$$

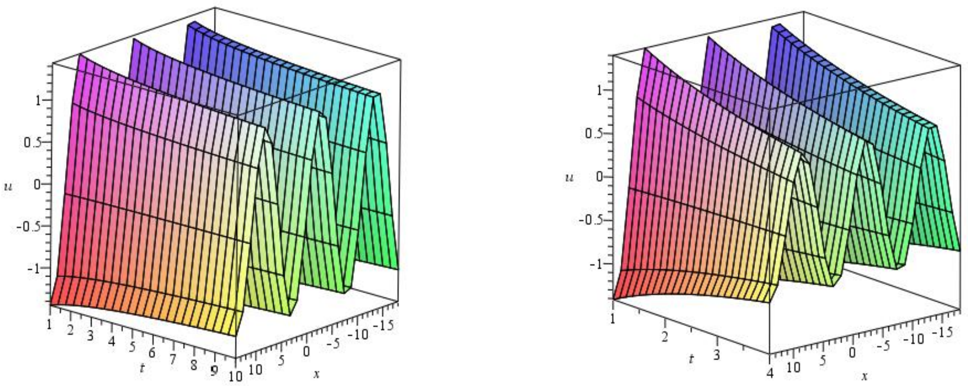
When $g = 0,\ h = 0$ and $\eta\beta > 0$, we obtain two general solutions of Eq. (5.22) as follows:

$$
u = C \exp\left(\sqrt{\frac{\beta}{2\eta}}\, x\right) E_\alpha\left(\frac{\delta\beta}{2\eta} t^\alpha\right) \tag{5.31}
$$

and

$$u = C \exp\left(-\sqrt{\frac{\beta}{2\eta}}\, x\right) E_{\alpha}\left(\frac{\delta\beta}{2\eta} t^{\alpha}\right). \tag{5.32}$$

In order to intuitively show the dynamic profiles and properties of above solutions, as examples, the 3D-graphs of the solutions (5.28) and (5.30) are illustrated, which are shown in Fig. 10 and Fig. (11).

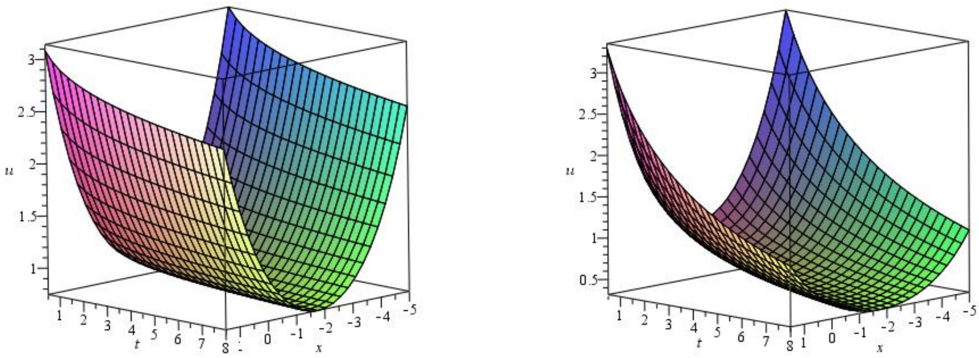


(**a**) case of $\alpha = 0.25$ (**b**) case of $\alpha = 0.75$

Fig. 10. The 3D-graphs of dynamical profiles of the solution (5.28) under the fixed parameters $C = 3,\ g = -4,\ \eta = -3,\ \beta = 2,\ \delta = 1$.

As can be seen from Fig. 10 that the profile of the solution (5.28) has not mutation phenomenon at $\alpha = 0.75$, this is because the one-parameter Mittag-Lefler function $E_{\alpha}\left(-\frac{\delta\beta}{2\eta} t^{\alpha}\right)$ in the solution (5.28) has no singularity.

Comparing Fig. 10 and Fig. 6, it can be found that their shapes are also similar and only their amplitude decay degree is different. This is due to the difference of the Mittag-Leffler functions $t^{\alpha-1} E_{\alpha,\alpha}(\delta t^{\alpha})$ and $E_{\alpha}(\delta t^{\alpha})$ in the solutions (5.28) and (4.24). Also, it is easy to find that the solution (4.24) is only one more factor $t^{\alpha-1}$ than the solution (5.28). Therefore, the solution (4.24) converges faster than the solution (5.28) as $t \to +\infty$.



(**a**) case of $\alpha = 0.25$ (**b**) case of $\alpha = 0.75$

Fig. 11. The 3D-graphs of dynamical profiles of the solution (5.30) under the fixed parameters $C = 1,\ g = -3,\ \eta = 1,\ \beta = 2,\ \delta = -1$.

As can be seen from Fig. 11 that the profile of the solution (5.30) has not mutation phenomenon at $\alpha = 0.75$ yet due to the one-parameter Mittag-Lefler function $E_\alpha\left(-\frac{\delta\beta}{2\eta}t^\alpha\right)$ in the solution (5.30) has no singularity.

Similarly, comparing Fig. 11 and Fig. 7, it can be found that their shapes are also similar and only their amplitude decay degree is different. This is due to the difference of the Mittag-Leffler functions $t^{\alpha-1}E_{\alpha,\alpha}(\delta t^\alpha)$ and $E_\alpha(\delta t^\alpha)$ in the solutions (5.30) and (4.26). Also, it is easy to find that the solution (4.26) is only one more factor $t^{\alpha-1}$ than the solution (5.30). Thus, the solution (4.26) converges faster than the solution (5.30) as $t \to +\infty$.

6 Conclusions

In this work, based on a modified separation method of variables and the integral bifurcation method, a combinational method is proposed. By using this new method, two generalized time-fractional thin-film equations are studied. Under two definitions of Riemann-Liouville and Caputo fractional derivatives, exact solutions of the two generalized time-fractional thin-film equations are investigated respectively.

Under the definition of Riemann-Liouville fractional differential operator, when $g = h = 0,\ a_0 \neq 0$, we obtained three kinds of exact solutions of parametric form such as (3.17), (3.20) and (3.23) of the generalized time-fractional thin-film equation (1.2). In the other parametric conditions, we obtained twelve explicit solutions such as (3.27), (3.28), (3.31), (3.32), (3.38), (3.39), (3.44), (3.45), (3.49), (3.53), (3.57) and (3.58) of the generalized time-fractional thin-film equation (1.2). Under the definition of Riemann-Liouville fractional differential operator, when $h = 0$, we obtained six explicit solutions such as (4.16), (4.17), (4.24), (4.25), (4.30) and (4.31) of the generalized time-fractional thin-film equation (1.3). Under the definition of Caputo fractional differential operator, by using similar method, we obtained different kinds of exact solutions of two generalized time-fractional thin-film equations (1.2) and (1.3). Obviously, the types of exact solution of these two time-fractional generalized time-fractional thin-film equations are very richer more than those in the existing references.

The investigations shew that the solutions of Eqs. (1.2) and (1.3) defined by Caputo fractional differential operator are very similar to those of Eqs. (1.2) and (1.3) defined by Riemann-Liouville fractional differential operator, only their Mittag-Leffler functions are different. It is found that the solutions of Eqs. (1.2) and (1.3) defined by Riemann-Liouville differential operator converge faster than those defined by Caputo differential operator. It is also found that all solutions of Eqs. (1.2) and (1.3) defined by Riemann-Liouville differential operator occurred mutation phenomenon at $\alpha = 0.75$. This is obviously a very anomalous phenomenon, which is unlikely to occur in cases of integer-order nonlinear PDEs.

Among these solutions of Eqs. (1.2) and (1.3) mentioned above, we found that all of them are not soliton solutions. However, when $\alpha = 1$, the integer-order generalized thin-film equations exist soliton solutions. This implies that its soliton solutions will disappear when an integer-order nonlinear PDE is changed into a nonlinear time-fractional PDE. So, must there necessarily be any forms of soliton solutions for nonlinear fractional PDEs? This is a very interesting question, but in the present way, we cannot answer it with certainty, maybe someone (readers) can answer it in the future.

Acknowledgement

This work was supported by the National Natural Science Foundation of China (Grant No. 11361023). This work was also supported by the Natural Science Project of Chongqing Education Commission (Grant No. CXQT21014).

CRediT author ship contribution statement

Weiguo Rui: Improvements of method (Lead), Investigation (Equal), Writing (Lead).

Weijun He: Investigation (Equal), Graph drawing (Supporting).

Data Availability Statement

The authors assure that all data present within the text of the manuscript are available and reliable.

Conflict of Interest:

The authors declare that they have no conflict of interest.

References

[1] Daftardar-Gejji V., Jafari H.: Adomian decomposition: a tool for solving a system of fractional differential equations. Journal of Mathematical Analysis and Applications **301**(2), 508-518 (2005)

[2] T. Bakkyaraj, R. Sahadevan, An approximate solution to some classes of fractional nonlinear partial differential difference equation using Adomian decomposition method. Journal of Fractional Calculus and Applications **5**(1), 37-52 (2014)

[3] Bakkyaraj T., Sahadevan R.: Approximate analytical solution of two coupled time fractional nonlinear Schrodinger equations. International Journal of Applied and Computational Mathematics **2**(1), 113-135 (2016)

[4] Bakkyaraj T., Sahadevan R.: On solutions of two coupled fractional time derivative Hirota equations. Nonlinear Dynamics **77**(4), 1309-1322 (2014)

[5] Sahadevan R., Bakkyaraj T.: Invariant analysis of time fractional generalized Burgers and Korteweg-de Vries equations. Journal of Mathematical Analysis and Applications **393**(2), 341-347 (2012)

[6] Bakkyaraj T., Sahadevan R.: Invariant analysis of nonlinear fractional ordinary differential equations with Riemann-Liouville derivative. Nonlinear Dynamics **80**(1-2), 447-455 (2015)

[7] Odibat Z.M., Shaher M.: The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics. Computers and Mathematics with Applications **58**, 2199-2208 (2009)

[8] Wu G., Lee E.W.M.: Fractional variational iteration method and its application. Physics Letters A **374**(25), 2506-2509 (2010)

[9] Momani S., Zaid O.: Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations. Computers and Mathematics with Applications **54**(7-8), 910-919 (2007)

[10] Sahadevan R., Bakkyaraj T.: Invariant subspace method and exact solutions of certain nonlinear time fractional partial differential equations. Fractional Calculus and Applied Analysis 18(1), 146-162 (2015)

[11] Harris P.A., Garra R.: Analytic solution of nonlinear fractional Burgers-type equation by invariant subspace method. Nonlinear Studies **20**(4), 471-481 (2013)

[12] Sahadevan R., Prakash P.: Exact solution of certain time fractional nonlinear partial differential equations. Nonlinear Dynamics **85**(1), 659-673 (2016)

[13] Elsayed M.E.Z., Yasser A.A., Reham M.A.S.: The fractional complex transformation for nonlinear fractional partial differential equations in the mathematical physics. Journal of Association of Arab Universities for Basic and Applied Sciences **19**, 59-69 (2016)

[14] Li Z.B., Zhu W.H., He J.H.: Exact solutions of time-fractional heat conduction equation by the fractional complex transform. Thermal Science 16(2), 335-338 (2012)

[15] Li Z.B., He J.H., Fractional complex transform for fractional differential equations. Mathematical Computational Applications **15**(5), 970-973 (2010)

[16] Chen J., Liu F., Anh V.: Analytical solution for the time-fractional telegraph equation by the method of separating variables. Journal of Mathematical Analysis and Applications **338**, 1364-1377 (2008)

[17] Jiang H., Liu F., Turner I., Burrage K.: Analytical solutions for the multi-term time-fractional diffusion-wave/diffusion equations in a finite domain. Computers and Mathematics with Applications **64**, 3377-3388 (2012)

[18] Y. Luchko, Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation. Computers and Mathematics with Applications 2010; 59: 1766-1772.

[19] Jumarie G.: Fractional partial differential equations and modified Riemann-Liouville derivative new methods for solution. Journal of Applied Mathematics and Computing **24**(1-2), 31-48 (2007)

[20] Jumarie G.: Modified Riemann-Liouville derivative and fractional Talor series of non-differentiable functions further results. Computers and Mathematics with Applications **51**(9-10), 1367-1376 (2006)

[21] Jumarie G.: Cauchy's integral formula via the modified Riemann-Liouville derivative for analytic functions of fractional order. Applied Mathematics Letters **23**(12), 1444-1450 (2010)

[22] He J.H.: Geometrical explanation of the fractional complex trnsform and derivative chain rule for fractional calculus. Physica Letters A **376**, 257-259 (2012)

[23] Tarasov V.E.: On chain rule for fractional derivatives. Communications in Nonlinear Science and Numerical Simulation **30**(1), 1-4 (2016)

[24] Rui W.: Applications of homogenous balanced principle on investigating exact solutions to a series of time fractional nonlinear PDEs. Communications in Nonlinear Science and Numerical Simulation **47**, 253-266 (2017)

[25] Rui W.: Applications of integral bifurcation method together with homogeneous balanced principle on investigating exact solutions of time fractional nonlinear PDEs. Nonlinear Dynamics **91**(1), 697-712 (2018)

[26] Wu C., Rui W.: Method of separation variables combined with homogenous balanced principle for searching exact solutions of nonlinear time-fractional biological population model. Communications in Nonlinear Science and Numerical Simulation **63**, 88-100 (2018)

[27] Rui W.: Idea of invariant subspace combined with elementary integral method for investigating exact solutions of time-fractional NPDEs. Applied Mathematics and Computation **339**, 158-171 (2018)

[28] Rui W.: Separation method of semi-fixed variables together with dynamical system method for solving nonlinear time-fractional PDEs with higher-order terms. Nonlinear Dynamics **109**, 943-961 (2022)

[29] Rui W., He B., Long Y., Chen C.: The integral bifurcation method and its application for solving a family of third-order dispersive PDEs. Nonlinear Analysis **69**, 1256-1267 (2008)

[30] He B., Rui W., Chen C., Li S.: Exact travelling wave solutions of a generalized Camassa-Holm equation using the integral bifurcation method. Applied Mathematics and Computation **206**, 141-149 (2008)

[31] Rui W., He B., Xie S., Long Y.: Application of the integral bifurcation method for solving modified Camassa-Holm and Degasoeris-Procesi equations. Nonlinear Analysis **71**, 3456-3470 (2009)

[32] Rui W., Long Y., He B., Li Z.: Integral bifurcation method combined with computer for solving a higher order wave equation of KdV type. International Journal of Computer Mathematics **87**(1), 119-128 (2010)

[33] Rui W., Zhang H.: Separation variable method combined with integral bifurcation method for solving nonlinear time-fractional reaction-diffusion models. Computational and Applied Mathematics **39**, 299 (2020)

[34] Invariant subspaces and exact solutions: (1+1) and (2+1)-dimensional generalized time-fractional thin-film equations. Computational and Applied Mathematics **42**, 97 (2023)

[35] Podlubny I.: Fractional Differential Equations. Academic Press, New York, 1999.

[36] Caputo M.: Elasticitàe Dissipazione. Zanichelli, Bologna, 1969.

[37] Rui W.: Comments on whether nonlinear fractional partial differential equations have soliton solutions. Partial Differential Equations in Applied Mathematics **5**, 100396 (2022)