CMMSE:Estimates of approximation numbers, nuclearity of the resolvent of a third order singular differential operator and the completeness of its root vectors

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Abstract

In this paper, we consider a third order singular differential operator L w + µ w = - w "' + q (x) w + µ w in space L 2 (R) originally defined on the set C 0 [?] (R) , where C 0 [?] (R) is the set of infinitely differentiable compactly supported functions, μ[?]0. Regarding the coefficient q (x) , we assume that it is a continuous function in R (- [?] , [?]) and can be a growing function at infinity. The operator L allows closure in the space L 2 (R) and the closure also be denoted by L. In the paper, under certain restrictions on q (x) , in addition to the above condition, the existence of the resolvent of the operator L and the existence of the estimate —— - w "' —— L 2 (R) + —— q (x) w —— L 2 (R) [?] c (—— L w —— L 2 (R) + —— w —— L 2 (R)) (0.1) have been proved, where c>0 is a constant. **Example.** Let q (x) = e 100 | x | , then the estimate (0.1) holds. The compactness of the resolvent is proved and two-sided estimates for singular numbers (s-numbers) are obtained. Here we note that the estimates of singular numbers (s-numbers) show the rate of approximation of the resolvent of the operator L by linear finite-dimensional operators. In the present paper, apparently for the first time, the nuclearity of the resolvent of the third-order differential operator and completeness of its root vectors are proved in the case of an unbounded domain with a greatly growing coefficient q (x) at infinity.

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CMMSE:Estimates of approximation numbers, nuclearity of the resolvent of a third order singular differential operator and the completeness of its root vectors†

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Summary

In this paper, we consider a third order singular differential operator

$$Ly + \mu y = -y''' + q(x)\, y + \mu y$$

in space $L_2(R)$ originally defined on the set $C_0^\infty(R)$, where $C_0^\infty(R)$ is the set of infinitely differentiable compactly supported functions, $\mu \geq 0$.

Regarding the coefficient $q(x)$, we assume that it is a continuous function in $R(-\infty,\ \infty)$ and can be a growing function at infinity.

The operator L allows closure in the space $L_2(R)$ and the closure also be denoted by L.

In the paper, under certain restrictions on $q(x)$, in addition to the above condition, the existence of the resolvent of the operator L and the existence of the estimate

$$\left\|-y'''\right\|_{L_2(R)} + \|q(x)\, y\|_{L_2(R)} \leq c\left(\|Ly\|_{L_2(R)} + \|y\|_{L_2(R)}\right) \tag{0.1}$$

have been proved, where $c > 0$ is a constant.

Example. Let $q(x) = e^{100\,|x|}$, then the estimate (0.1) holds.

The compactness of the resolvent is proved and two-sided estimates for singular numbers (s-numbers) are obtained. Here we note that the estimates of singular numbers (s-numbers) show the rate of approximation of the resolvent of the operator L by linear finite-dimensional operators. In the present paper, apparently for the first time, the nuclearity of the resolvent of the third-order differential operator and completeness of its root vectors are proved in the case of an unbounded domain with a greatly growing coefficient $q(x)$ at infinity.

KEYWORDS:

third order differential operator, singular operator, approximation numbers, nuclearity, resolvent, separability, root vectors.

AMS: 34L20, 47A10, 47B06

1 | INTRODUCTION. FORMULATION OF RESULTS. EXAMPLE

Consider the third-order differential operator with an unbounded coefficient

$$(L + \mu I)\ y = -y''' + q(x)\,y + \mu y \tag{1.1}$$

originally defined on the set $C_0^\infty(\mathbb{R})$ of infinitely differentiable compactly supported functions, $x \in \mathbb{R}$, $\mu \geq 0$.

Let $q(x)$ be satisfied to the conditions:

i) $q(x) \geq \delta > 0$ is a continuous function in $\mathbb{R}(-\infty,\ \infty)$,

ii) $c_0 = \sup\limits_{|x-t|\leq 1} \frac{q(x)}{q(t)} < \infty$.

Here the function $q(x)$ can be a greatly growing function at infinity.

It is easy to see that the operator L admits a closure in the space $L_2(\mathbb{R})$, and the closure will also be denoted by L.

For several decades, the theory of third-order differential equations has been intensively studied due to their importance for applications.

Note that by applying the Fourier variable separation method, it is possible to reduce some equations of mathematical physics to one-dimensional differential equations of the third order.

A very comprehensive bibliography for third-order differential equations is contained in [1-5]. In these works, in the space of continuous functions, the properties of oscillatory and non-oscillating solutions were studied, as well as the boundedness and asymptotic stability of solutions to third-order differential equations.

In the paper [6], a third-order differential equation with a complex coefficient $Q(x)$ was studied in the space $L_{p,l(x)}(\mathbb{R})$ ($l < p < \infty$, $l(x)$ is a weight function), where $Q(x) = q(x) + i\,r\,(x)$, $q(x)$ and $r\,(x)$ can be growing functions at infinity. The smoothness of solutions, the compactness of the resolvent, and the approximative properties of solutions have been studied in [6], when the Levitan-Titchmarsh condition is satisfied (see condition (3) in [6]).

In contrast to [6], in this paper, the Levitan-Titchmarsh type condition is removed and the existence and compactness of the resolvent of the operator $L + \mu I$, separability of the operator (maximum smoothness of solutions to the equation $Ly = f \in L_2(R)$), and estimates of singular numbers (s-numbers) in the space $L_2(\mathbb{R})$ are studied. In addition to the above results, in this paper, apparently for the first time, the nuclearity and completeness of its root vectors are proved for the resolvent of a third-order differential operator in the case of an unbounded domain with the greatly growing coefficient $q(x)$ at infinity.

Let us formulate the main results.

Theorem 1.1. Let the condition i) be satisfied. Then the operator is continuously invertible in the space $L_2(\mathbb{R})$ for $\mu \geq 0$.

Definition 1.1. [7,8]. We will say that a differential operator L is separable if the following estimate

$$\left\|-y'''\right\|_2 + \|q(x)\,y\|_2 \leq c\,\left(\|Ly\|_2 + \|y\|_2\right),$$

holds for all $u \in D(L)$, where $c > 0$ is a constant, $\|\cdot\|_2$ is the norm in $L_2(\mathbb{R})$.

Theorem 1.1. Let the conditions $i) - ii)$ be satisfied. Then the operator L is separable.

Example. Consider the operator

$$(L + \mu I)\ y = -y'''(x) + e^{100|x|}\cdot y(x) + \mu\ \cdot y(x)\ , y(x) \in D(L),$$

$$-\infty < x < \infty.$$

It is easy to check that all conditions of Theorems 1.1 and 1.2 are satisfied. Therefore, the operator $L + \mu I$ is continuously invertible in $L_2(\mathbb{R})$ and separable, i.e. the estimate

$$\left\|-y'''\right\|_2 + \left\|e^{100|x|}y\right\|_2 \leq c\,\left(\|Ly\|_2 + \|y\|_2\right),$$

holds, where $c > 0$ is a constant.

Theorem 1.2. Let the conditions $i) - ii)$ be satisfied. Then the resolvent of the operator L is compact if and only if

$$\lim_{|x|\to\infty} q(x) = \infty.$$

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Definition 1.2. [9]. Let be A a linear completely continuous operator and let $|A| = \sqrt{A^* A}$. The eigenvalues of the operator $|A|$ are called the s-numbers of the operator A.

It is known [9] that estimates of singular numbers (s-numbers) show the rate of approximation of the resolvent of the operator (1.1) by linear finite-dimensional operators.

The nonzero s-numbers of the operator $(L + \mu I)^{-1}$ be numbered according to decreasing magnitude and observing their multiplicities, so that

$$s_k \left((L + \mu I)^{-1}\right) = \lambda_k \left(\left|(L + \mu I)^{-1}\right|\right), k = 1, 2, 3..., \mu \geq 0 .$$

We introduce the counting function $N(\lambda) = \sum_{s_k > \lambda} 1$ of those s_k greater than $\lambda > 0$.

Theorem 1.4. Let the conditions $i) - ii)$ be satisfied. Then the estimate

$$c^{-1} \lambda^{-\frac{1}{3}} mes \left(x \in R : q(x) \leq \lambda^{-1}\right) \leq N(\lambda) \leq c \cdot \lambda^{-\frac{1}{3}} mes \left(x \in R : q(x) \leq \lambda^{-1}\right)$$

holds, where $c > 0$ is a constant which is independent of $q(x)$ and λ.

Definition 1.3. [9] Let A be a linear completely continuous operator. The operator A will be called nuclear if it belongs δ_1, i.e. if

$$\sum_j s_j(A) < \infty.$$

Theorem 1.5. Let the conditions i)-ii) be satisfied. Then the resolvent of the operator L is nuclear if and only if

$$q^{-\frac{2}{3}}(x) \in L_1(R). \tag{1.2}$$

Definition 1.4. [9] A vector $\varphi \neq 0$ is called a root vector for the eigenvalue μ_0 of a linear operator if there exists a natural number n such that

$$\left(A - \mu_0 I\right)^n \varphi = 0.$$

Theorem 1.6. Let the conditions i)-ii) be satisfied. Then the root vectors of the operator L^{-1} are complete in $L_2(\mathbb{R})$, if $q^{-\frac{2}{3}}(x) \in L_1$.

2 | THE EXISTENCE OF THE RESOLVENT. PROOF OF THEOREM 1.1.

Lemma 2.1. Let the condition i) be fulfilled and $\mu \geq 0$. Then the inequality

$$\|(L + \mu I)\, y\|_2 \geq (\delta + \mu)\, \|y\|_2 , \tag{2.1}$$

holds for all $y \in D(L)$, where $\| \cdot \|_2$ is the norm in $L_2(\mathbb{R})$.

Proof. Since it is an operator with a real coefficient, it suffices to prove the estimate for real-valued functions. To do this, we write the following functional $< (L + \mu I)\, y, y >$, $y \in C_0^\infty(\mathbb{R})$, $< \cdot, \cdot >$ is the scalar product. Since $\int_R -y''' y\, dx = 0$ then from functional $< (L + \mu I)\, y, y >$ we find

$$\|(L + \mu I)\, y\|_2 \geq (\delta + \mu)\, \|y\|_2 .$$

Due to the continuity of the norm, the last estimate holds for all $y \in D(L)$. Lemma 2.1 is proved. □

Further, we present a series of statements that reduce the existence of a resolvent and the separability of an operator L with a growing coefficient at infinity to the case with bounded coefficients.

Let $\{\varphi_j\}_{j=-\infty}^{\infty} \in C_0^\infty(\mathbb{R})$ be a set of functions such that $\varphi_j(y) \geq 0$, $\operatorname{supp} \varphi_j \subseteq \Delta_j$ $(j \in Z)$, $\sum_{j=-\infty}^{\infty} \varphi_j^2(y) = 1$, where $\Delta_j = (j-1, j+1)$, $j \in Z$, $\bigcup_j \Delta_j = R$ [8, 10].

Let us extend $q(x)$ from Δ_j to the whole R so that its extension $q_j(x)$ is a bounded and periodic function of the same period.

Here we note immediately that any point can belong to at most three segments from the system of segments $\{\Delta_j\}$ [10, 11].

Denote by $L_j + \mu I$ the closure of the operator

$$\left(L_j + \mu I\right) y = -y''' + \left(q_j(x) + \mu\right) y$$

defined on $C_0^\infty(\mathbb{R})$.

Lemma 2.2. Let the conditions i) be fulfilled and $\mu \geq 0$. Then the inequality

$$\left\| \left(L_j + \mu I \right) y \right\|_2 \geq (\delta + \mu) \|y\|_2 . \tag{2.2}$$

holds for all $y \in D(L)$.

Lemma 2.2 is proved in exactly the same way as Lemma 2.1.

Lemma 2.3. Let the conditions i) be fulfilled and $\mu \geq 0$. Then the operator $\left(L_j + \mu I \right)$ has a continuous inverse operator $\left(L_j + \mu I \right)^{-1}$defined in all $L_2(\mathbb{R})$.

Proof. By the estimate (2.2), it suffices to prove that the range of values is dense in $L_2(\mathbb{R})$. Assume that the range is not dense in $L_2(\mathbb{R})$. Then there exists an element $u \in L_2(\mathbb{R})$ such that $< \left(L_j + \mu I \right) y,\ u >= 0$ for all $y \in D\left(L_j \right)$. It means that

$$\left(L_j + \mu I \right)^* u = u''' + q(x) u + \mu u = 0 \tag{2.3}$$

in the sense of the theory of generalized functions. By virtue of boundedness $q_j(x)$, we have that $q_j(x) u \in L_2(\mathbb{R})$. This and (2.3) imply, that $u \in W_2^3(\mathbb{R})$, where $W_2^3(\mathbb{R})$ is the Sobolev space. From the general theory of the embedding theorem we have

$$\lim_{|y| \to \infty} u(y) = 0,\ \lim_{|y| \to \infty} u'(y) = 0,\ \lim_{|y| \to \infty} u''(y) = 0. \tag{2.4}$$

From here and using the calculations used in the proof of estimate (2.2), we obtain

$$\left\| \left(L_j + \mu I \right)^* u \right\|_2 \geq (\delta + \mu) \|u\|_2 .$$

From the last inequality and (2.3), it follows that $u(x) = 0$. Lemma 2.3 is proved. □

Lemma 2.4. Let the condition i) be fulfilled and $\mu \geq 0$. Then the estimate

$$\left\| \left(L_j + \mu I \right)^{-1} \right\|_{2 \to 2} \leq \frac{1}{q\left(x_j \right) + \mu}$$

holds, where $q\left(x_j \right) = \min_{x \in \bar{\Delta}_j} q(x)$, $\| \cdot \|_{2 \to 2}$ is the norm of the operator acting from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$.

Proof. Consider the following functional

$$< \left(L_j + \mu I \right) y,\ y >, \quad y \in C_0^\infty(\mathbb{R}) .$$

Integrating by parts, we have

$$< \left(L_j + \mu I \right) y,\ y >= \int_{-\infty}^{\infty} \left(q_j(x) + \mu \right) |y|^2 \, dx.$$

From the last equality, we obtain

$$\left\| \left(L_j + \mu I \right) y \right\|_2 \cdot \|y\|_2 \geq \min_{x \in \bar{\Delta}_j} \left(q_j(x) + \mu \right) \|y\|_2^2 .$$

Hence, taking $\min_{x \in \bar{\Delta}_j} \left(q_j(x) + \mu \right) = \min_{x \in \bar{\Delta}_j} (q(x) + \mu)$ into account, we find

$$\left\| \left(L_j + \mu I \right) y \right\|_2 \geq \left(q\left(x_j \right) + \mu \right) \|y\|_2 , \tag{2.5}$$

where $q\left(x_j \right) = \min_{x \in \bar{\Delta}} q(x)$.

From inequality (2.5), according to the definition of the operator norm, we obtain

$$\left\| \left(L_j + \mu I \right)^{-1} \right\|_{2 \to 2} \leq \frac{1}{q\left(x_j \right) + \mu}.$$

Lemma 2.4 is proved. □

Lemma 2.5. Let the conditions i)-ii) be fulfilled and $\mu \geq 0$. Then the estimate

$$\|-y'''\|_2 + \left\| q_j(x) y \right\|_2 + \mu \|y\| \leq c_0 \left(\left\| \left(L_j + \mu I \right) y \right\|_2 \right), \tag{2.6}$$

holds, where $c_0 > 0$ is a constant independent of $q_j(x)$, R and μ.

Proof. Lemma 2.3 implies that there exists a bounded inverse operator $\left(L_j + \mu I\right)^{-1}$ defined in all $L_2(\mathbb{R})$. Hence we have $y = \left(L_j + \mu I\right)^{-1} f$.

Using the last equality, we estimate the norm $\left\|q_j(x)\, y\right\|$:

$$\left\|q_j(x)\, y\right\|_2 = \left\|q_j(x)\left(L_j + \mu I\right)^{-1} f\right\|_2 \leq \max_{x\in R} q_j(x) \left\|\left(L_j + \mu I\right)^{-1}\right\|_{2\to 2} \cdot \|f\|_2 . \tag{2.7}$$

Since $\max_{x\in R} q_j(x) = \max_{x\in \bar{\Delta}_j} q(x)$ then, by virtue of condition ii) and Lemma 2.4, we find from (2.7)

$$\left\|q_j(x)\, y\right\|_2 \leq \frac{\max_{x\in R} q_j(x)}{q\left(x_j\right)} \cdot \|f\|_2 \leq \frac{\max_{x\in\bar{\Delta}_j} q(x)}{\min_{x\in\bar{\Delta}_j} q(x)} \cdot \|f\|_2 \leq \sup_{|x-t|\leq 2} \frac{q(x)}{q(t)} \cdot \|f\|_2 \leq c_0 \|f\|_2^2 .$$

Hence

$$\left\|q_j(x)\, y\right\|_2 \leq c_0 \left\|\left(L_j + \mu I\right)^{-1} y\right\|_2 , \tag{2.8}$$

where $\left(L_j + \mu I\right) y = f$.

Now, using estimates (2.2) and (2.8), we obtain

$$\begin{aligned} \|-y'''\|_2 = \left\|\left(L_j + \mu I\right) y - \left(q_j(x)\, y + \mu y\right)\right\|_2 \leq \left\|\left(L_j + \mu I\right) y\right\|_2 + \\ + c_0 \left\|\left(L_j + \mu I\right) y\right\|_2 + \mu \left\|\left(L_j + \mu I\right) y\right\|_2 \leq c_1 \left\|\left(L_j + \mu I\right) y\right\|_2 , \end{aligned} \tag{2.9}$$

where $c_1 = 2 + c_0$.

Estimates (2.7), (2.8), and (2.9) prove Lemma 2.5. □

Lemma 2.6. Let the conditions i)-ii) be fulfilled and $\mu \geq 0$. Then the estimate

$$\left\|D^\alpha \left(L_j + \mu I\right)^{-1}\right\|_{2\to 2} \leq \frac{c}{\mu^{1-\frac{\alpha}{3}}},$$

holds, where$\alpha = 0, 1, 2, 3$, c is a constant independent of $y(x)$.

Proof. According to Lemma 2.5 we have

$$\left\|D^\alpha \left(L_j + \mu I\right)^{-1}\right\|_{2\to 2} \leq \sup_{y\in D(L_j)} \frac{\|D^\alpha y(x)\|_2}{\left\|\left(L_j + \mu I\right) y\right\|_2} \leq c \sup_{y\in D(L_j)} \frac{\|D^\alpha y(x)\|_2}{\|-y'''\|_2 + \mu \|y\|_2}.$$

From this and applying similarity transformations $x = \frac{t}{\mu^{\frac{1}{3}}}$ we obtain that

$$\left\|D^\alpha \left(L_j + \mu I\right)^{-1}\right\|_{2\to 2} \leq c \sup_{y\in D(L_j)} \frac{\mu^{\frac{\alpha}{3}} \left\|D_t^\alpha y\right\|_2}{\mu \left(\left\|-y_t'''\right\|_2 + \|y\|_2\right)} \leq \frac{c}{\mu^{1-\frac{\alpha}{3}}}.$$

Here we have used the following inequality

$$\left\|D_t^\alpha y\right\| \leq \left\|y_t'''\right\|_2 + \|y\|_2 \ \text{ for } \alpha = 0,\ 1,\ 2,\ 3.$$

Lemma 2.6 is proved. □

Assume

$$K_\mu f = \sum_{\{j\}} \varphi_j(x) \left(L_j + \mu I\right)^{-1} \varphi_j f,\ f \in C_0^\infty(\mathbb{R}),$$

$$B_\mu f = \sum_{\{j\}} \varphi_j'''(x) \left(L_j + \mu I\right)^{-1} \varphi_j f + 3\left(\sum_{\{j\}} \varphi_j''(x)\, D_x \left(L_j + \mu I\right)^{-1} \varphi_j f + \right.$$

$\left. + \sum_{\{j\}} \varphi_j'(x)\, D_x^2 \left(L_j + \mu I\right)^{-1} \varphi_j f\right)$, where $D_x = \frac{\partial}{\partial x}$.
It is easy to verify that

$$(L + \mu I)\, K_\mu f = f - B_\mu f. \tag{2.10}$$

Lemma 2.7. Let the condition i) be fulfilled. Then there exists a number $\mu_0 > 0$ such that $\left\|B_\mu\right\|_{2\to 2} < 1$ for all $\mu \geq \mu_0 > 0$.

Proof. Using Lemma 2.6 and repeating the computations and arguments used in the proof of Lemma 9 in [11] and 3.2 in [14], we obtain the proof of Lemma 2.7. □

Lemma 2.8. Let the condition i) be fulfilled. Then the operator $L + \mu I$ is continuously invertible in the space $L_2(\mathbb{R})$ when $\mu \geq \mu_0 > 0$ and the equality

$$(L + \mu I)^{-1} = K_\mu \left(I - B_\mu\right)^{-1}$$

holds.

Proof. Using the representation (2.10) and Lemmas 2.1 and 2.7 we obtain the proof of Lemma 2.8. □

Proof of Theorem 1.1.. Lemma 2.8 implies that Theorem 1.1 holds for all $\mu \geq \mu_0 > 0$.

Now, it remains to prove that Theorem 1.1 is also valid for $\mu = 0$. To do this, consider the equation

$$Ly = -y''' + q(x)\, y = f \in L_2(\mathbb{R}). \tag{2.11}$$

Here we note that the question of the existence of the bounded operator L^{-1} the closed operator L in space $L_2(\mathbb{R})$ is equivalent to the following problem: to find a unique solution $y(x)$ of the equation $Ly = f \in L_2(\mathbb{R})$ belonging to the space $L_2(\mathbb{R})$.

Further, we rewrite equation (2.11) as follows:

$$-y''' + (q(x) + \mu)\, y - \mu y = f. \tag{2.12}$$

Let $\mu \geq \mu_0 > 0$, then by Lemma 2.8 there exists $(L + \mu I)^{-1}$ and equation (2.12) takes the form

$$\vartheta - A_\mu \vartheta = f, \tag{2.13}$$

here

$$\vartheta = (L + \mu I)\, y, \tag{2.14}$$

where $A_\mu = \mu \cdot (L + \mu I)^{-1}$.

Taking Lemma 2.1 into account, it is easy to obtain the estimate

$$\left\|A_\mu \vartheta\right\|_2 \leq \mu \cdot \left\|(L + \mu I)^{-1}\right\|_{2\to2} \cdot \|\vartheta\|_2 \leq \frac{\mu}{\delta + \mu} \|\vartheta\|_2 .$$

Hence

$$\left\|A_\mu\right\|_{2\to2} < 1.$$

It follows from the last inequality that the equation (2.13) has a unique solution, i.e.

$$\vartheta = \left(I - A_\mu\right)^{-1} f. \tag{2.15}$$

Now, taking into account that $y = (L + \mu I)^{-1} \vartheta$, from (2.14) and (2.15) we find

$$y = (L + \mu I)^{-1} \left(I - A_\mu\right)^{-1} f. \tag{2.16}$$

Hence we obtain that (2.16) is a unique solution of equation (2.11). Therefore, we have proved that Theorem 1.1 also holds for $\mu = 0$. Theorem 1.1 is completely proved. □

3 | OPERATOR SEPARABILITY. COMPACTNESS OF THE RESOLVENT

To prove the separability of the operator (1.1), we first prove the following lemma.

Lemma 3.1. Let the conditions i)-ii) be fulfilled and $\mu_0 > 0$ such that $\left\|B_\mu\right\|_{2\to2} < 1$ for all $\mu \geq \mu_0 > 0$. Then the following inequality

$$\left\| q(x)\, (L + \mu I)^{-1} \right\|_{2\to2}^2 \leq c \cdot \sup_{\{j\}} \left\|q(x)\, \varphi_j \left(L_j + \mu I\right)^{-1}\right\|_{2\to2}^2, \tag{3.1}$$

holds, where $c > 0$ is a constant.

Proof. Let $f(x) \in L_2(\mathbb{R})$. Taking the properties of the functions φ_j $(j \in Z)$ into account, we have from Lemma 2.8

$$\left\| q(x)\, (L + \mu I)^{-1} f \right\|_2^2 = \left\|q(x)\, K_\mu \left(I - B_\mu\right)^{-1} f\right\|_{2\to2}^2 =$$

$$= \int_{-\infty}^{\infty} \left| q(x) \sum_{\{j\}} \varphi_j \left(L_j + \mu I\right)^{-1} \varphi_j \left(I - B_\mu\right)^{-1} f \right|^2 dx.$$

It is known by construction that only the functions $\varphi_{j-1},\ \varphi_j,\ \varphi_{j+1}$ are nonzero on the interval Δ_j ($j \in Z$), therefore

$$\left\| q(x)\,(L + \mu I)^{-1} f \right\|_2^2 \le \sum_{j=-\infty}^{\infty} \int_{\Delta_j} \left| q(x) \sum_{k=j-1}^{j+1} \varphi_k \left(L_k + \mu I\right)^{-1} \varphi_k \left(I - B_\mu\right)^{-1} f \right|^2 dx \le$$

$$\le 9 \cdot \sum_{j=-\infty}^{\infty} \left\| q(x) \varphi_j \left(L_j + \mu I\right)^{-1} \varphi_j \left(I - B_\mu\right)^{-1} f \right\|_{L_2(\Delta_j)}^2 \le$$

$$\le 9 \cdot \sup_{j\in Z} \left\| q(x) \varphi_j \left(L_j + \mu I\right)^{-1} \right\|_{L_2(\Delta_j)\to L_2(\Delta_j)}^2 \cdot \int_{-\infty}^{\infty} \left(\sum_j \varphi_j^2 \right) \left| \left(I - B_\mu\right)^{-1} f \right|^2 dx.$$

Since $\sum_{j=-\infty}^{\infty} \varphi_j^2 = 1$, then from the last inequality we have

$$\left\| q(x)\,(L + \mu I)^{-1} f \right\|_2^2 \le 9 \cdot \sup_{j\in Z} \left\| q(x)\,\varphi_j \left(L_j + \mu I\right)^{-1} \right\|_{L_2(\Delta_j)\to L_2(\Delta_j)}^2 \cdot$$

$$\cdot \left\| \left(I - B_\mu\right)^{-1} \right\|_{2\to 2}^2 \cdot \|f\|_2^2. \tag{3.2}$$

Lemmas 2.7 and 2.8 imply that

$$\left\| \left(I - B_\mu\right)^{-1} \right\|_{2\to 2} \le c_1, \tag{3.3}$$

where $c_1 > 0$ is a constant.

Using inequality (3.3), from inequality (3.2) we obtain

$$\left\| q(x)\,(L + \mu I)^{-1} \right\|_{L_2(\mathbb{R})\to L_2(\mathbb{R})} \le c \cdot \sup_{j\in Z} \left\| q(x) \varphi_j \left(L_j + \mu I\right)^{-1} \right\|_{L_2(\Delta_j)\to L_2(\Delta_j)},$$

where $c = 9 \cdot c_1$.

Lemma 3.1 is proved. □

Lemma 3.2. Let the conditions of Lemma 3.1 be satisfied. Then the estimate

$$\left\| q(x)(L + \mu I)^{-1} \right\|_{2\to 2} \le c_2 < \infty, \tag{3.4}$$

holds, where $c_2 > 0$ is a constant.

Proof. Using Lemma 2.4, we obtain

$$\left\| q(x)(L + \mu I)^{-1} \right\|_{2\to 2} \le c \cdot \sup_{j\in Z} \left\| q(x) \varphi_j \left(L_j + \mu I\right)^{-1} \right\|_{L_2(\Delta_j)\to L_2(\Delta_j)} \le$$

$$\le c \cdot \sup_{j\in Z} \max_{x\in\overline{\Delta}_j} q(x) \varphi_j \cdot \left\| \left(L_j + \mu I\right)^{-1} \right\|_{2\to 2} \le c \cdot \frac{\max\limits_{x\in\Delta_j} q(x)\varphi_j}{q\left(x_j\right)} \le$$

$$\le c \cdot \sup_{|x-t|\le 2} \frac{q(x)}{q(t)} \le c \cdot c_0 \le c_2 < \infty.$$

from Lemma 3.1, where $c_2 = c \cdot c_0$.

Lemma 3.2 is proved. □

Proof of Theorem 1.2.. It is easy to see that from estimate (2.1) we have

$$\|(L + \mu I)\, y\|_2 \ge \mu\ \|y\|_2\,. \tag{3.5}$$

Now, using inequalities (3.4) and (3.5), we obtain

$$\|-y'''\|_2 = \|(L + \mu I)\, y - (q(x)\, y + \mu\, y)\|_2 \le \|(L + \mu I)\, y\|_2 + \|q(x)\, y\|_2 + \mu\, \|y\|_2 \le$$

$$\le \|(L + \mu I)\, y\|_2 + \left\| q(x)(L + \mu I)^{-1} (L + \mu I)\, y \right\|_2 + \|(L + \mu I)\, y\|_2 \le$$

$$\le \|(L + \mu I)\, y\|_2 + \left\| q(x)(L + \mu I)^{-1} \right\|_{2\to 2} \cdot \|(L + \mu I)\, y\|_2 + \|(L + \mu I)\, y\|_2 \le$$

$$\leq \left(2+c_2\right)\|(L+\mu I)\,y\|_2 \leq c_3\cdot\|(L+\mu I)\,y\|_2\,,$$

where $c_3 = 2 + c_2$.

From the last inequality we find

$$\left\|-y'''\right\|_2 \leq c_3\cdot\|(L+\mu I)\,y\|_2\,. \tag{3.6}$$

The estimate (3.4) gives that

$$\|q(x)\,y\|_2 \leq c_2\cdot\|(L+\mu I)\,y\|_2\,, \tag{3.7}$$

where $c_2 > 0$ is a constant from Lemma 3.2.

From inequality (3.6) and (3.7) we have

$$\left\|-y'''\right\|_2 + \|q(x)\,y\|_2 \leq c\cdot\|(L+\mu I)\,y\|_2\,,$$

where $c = \max\left\{c_1,\, c_3\right\}$. Theorem 1.2 is proved for $\mu \geq \mu_0 > 0$.

Now, let us prove separability does not depend on μ. To do this, we prove the following lemma.

Lemma 3.3. The operator $Ly = -y''' + q(x)\,y$ is separable if only if $L + \mu I$ is separable for any μ.

Proof. Necessity. Let L be separable. Since the operator L is separable then from $y \in D(L) \in L_2(\mathbb{R})$ and $Ly \in L_2(\mathbb{R})$ it follows $y''' \in L_2(\mathbb{R})$, i.e. the operator $L + \mu I$ is separable.

Sufficiency. Let the operator $L+\mu I$ be separable. This implies $y \in D(L)$, $(L+\mu I)\,y \in L_2(\mathbb{R})$ and $y''' \in L_2(\mathbb{R})$. Therefore, $y \in D(L)$, $Ly \in L_2(\mathbb{R})$ and $y''' \in L_2(\mathbb{R})$, i.e. the operator L is separable. □

Using Lemma 3.3, we obtain a complete proof of Theorem 1.2. □

Lemma 3.4. Let the conditions i)-ii) be fulfilled. Then the inequality

$$c^{-1}q^{\frac{1}{3}}(x) \leq q^*(x) \leq c\cdot q^{\frac{1}{3}}(x)\,,$$

holds, where $c > 0$ is a constant, the function $q^*(x)$ is a special averaging of the function [12], i.e.

$$q^*(x) = \inf\left\{d^{-1} \,:\, d^{-5} \geq \int\limits_{x-\frac{d}{2}}^{x+\frac{d}{2}} q^2(t)\,dt\right\}.$$

Lemma 3.4 is proved in exactly the same way as Lemma 12 in [13].

Proof of Theorem 1.3.. Theorem 2.1 implies $R\left(L^{-1}\right) = L^3_{2,q}(\mathbb{R})$, where $R\left(L^{-1}\right)$ is the range of the operator L^{-1}, $L^3_{2,q}(\mathbb{R})$ is the space obtained by completion $C_0^\infty(\mathbb{R})$ with respect to the norm

$$\left\|y \,:\, L^3_{2,q}(\mathbb{R})\right\| = \left(\int\limits_{-\infty}^{\infty}\left(\left|y'''\right|^2 + q^2(x)\,|y|^2\right)dx\right)^{\frac{1}{2}}.$$

To complete the proof, it remains to show that the embedding operator of the space $L^3_{2,q}(\mathbb{R})$ into $L_2(\mathbb{R})$ is compact. The answer for this question follows from the results of [12]. In this paper, it is proved that any bounded set of space $L^3_{2,q}(\mathbb{R})$ is compact in $L_2(\mathbb{R})$ if and only if

$$\lim_{|x|\to\infty} q^*(x) = \infty. \tag{3.8}$$

Now, the proof of Theorem 1.3 follows from Lemma 3.4. Theorem 1.3 is proved. □

4 | ESTIMATES FOR SINGULAR NUMBERS. NUCLEARITY OF THE RESOLVENT. COMPLETENESS OF ROOT VECTORS

To study the singular values of the operator L^{-1}, we need the following lemmas.

We introduce the following sets

$$M = \left\{u \in L_2(\mathbb{R}) \,:\, \|Ly\|_2^2 + \|y\|_2^2 \leq 1\right\},$$

$$\tilde{M}_C = \left\{u \in L_2(\mathbb{R}) \,:\, \left\|-y'''\right\|_2^2 + \|q(x)\,y\|_2^2 \leq c\right\};$$

$$\tilde{M}_{C^{-1}} = \left\{ u \in L_2(\mathbb{R}) : \left\|-y'''\right\|_2^2 + \|q(x)y\|_2^2 \le c^{-1} \right\},$$

where $\|\cdot\|_2$is the norm in $L_2(\mathbb{R})$ and $A > 0$ is a constant independent of $y(x)$.

Lemma 4.1. Let the conditions i)-ii) be fulfilled. Then the inclusions

$$\tilde{M}_{c^{-1}} \subseteq M \subseteq \tilde{M}_c$$

hold.

Proof. Let $y \in \tilde{M}_{c^{-1}}$. Then we have

$$\|Ly\|_2^2 + \|y\|_2^2 \le \left\|-y'''\right\|_2^2 + \|q(x)y\|_2^2 + \|y\|_2^2 \le c \cdot \left(\left\|-y'''\right\|_2^2 + \|q(x)y\|_2^2 \right), \tag{4.1}$$

where $c > 0$ is a constant independent of $y(x)$.

Here we took into account the following estimate:

$$\|y\|_2^2 \le \frac{1}{\delta^2} \|q(x)y\|_2^2.$$

Taking into account that $y \in \tilde{M}_{c^{-1}}$, we find from (4.1)

$$\|Ly\|_2^2 + \|y\|_2^2 \le c \cdot \left(\left\|-y'''\right\|_2^2 + \|q(x)y\|_2^2 \right) \le c \cdot c^{-1} \le 1.$$

Hence it follows that $\tilde{M}_{c^{-1}} \subseteq M$.

Now, let's prove the right inclusion. Let $y(x) \in M$. It means that $y(x) \in D(L)$. Therefore, by virtue of Theorem 1.2, we find

$$\left\|-y'''\right\|_2^2 + \|q(x)y\|_2^2 \le c \left(\|Ly\|_2^2 + \|y\|_2^2 \right),$$

where A is a constant independent of $y(x)$.

Since $y(x) \in M$, then from the last inequality we obtain

$$\left\|-y'''\right\|_2^2 + \|q(x)y\|_2^2 \le c.$$

We find from the last inequality that $y(x) \in \tilde{M}_c$, i.e. $M \subseteq \tilde{M}_c$. Lemma 4.1 is proved. □

Definition 4.1. [9] The Kolmogorov k-width of a set M in the space $L_2(\mathbb{R})$ is called the quantity

$$d_k = \inf_{\{y_k\}} \sup_{y \in M} \inf_{v \in y_k} \|y - v\|_2,$$

where $\{y_k\}$ is the set of all subspaces in the space $L_2(\mathbb{R})$, whose dimensions do not exceed k.

The following lemmas hold.

Lemma 4.2. Let the conditions i)-ii) be fulfilled. Then the estimate

$$c^{-1}\tilde{d}_k \le s_{k+1} \le c\,\tilde{d}_k, k = 1, 2, ...,$$

holds, where $c > 0$ is a constant, s_k −−s-numbers (singular numbers) of the operator L^{-1}, d_k, $\tilde{d}_k$ is the Kolmogorov widths of the corresponding sets M, $\tilde{M}$.

Lemma 4.3. Let the conditions i)-ii) be fulfilled. Then the estimate

$$\tilde{N}(c\lambda) \le N(\lambda) \le \tilde{N}\left(c^{-1}\lambda\right)$$

holds, where $N(\lambda) = \sum_{S_{k+1} > \lambda} 1$ is the counting function of s_{k+1} of the operator L^{-1} greater $\lambda > 0$, $\tilde{N}(\lambda) = \sum_{d_k > \lambda} 1$ is the counting function of $\tilde{d}_k$ greater $\lambda > 0$.

Lemma 4.2 and 4.3 is proved in exactly the same way as Lemmas 4.3 and 4.4 in [14].

Proof of Theorem 1.4.. Lemma 4.1 implies that $M \subset L^3_{2,q}(\mathbb{R})$. Now, using Lemma 3.4 and repeating the calculations and arguments that were studied in the proof of Theorem 1.4 in [14], we obtain the proof of Theorem 1.4 of this paper. □

Proof of Theorem 1.5.. Using Lemma 3.4 and Theorem 1.4 from Theorem 8.3 in [12], we obtain the proof of Theorem 1.5. □

To prove Theorem 1.6, we first prove the following lemmas.

Definition 4.2. [15]. A linear operator A acting on a Hilbert space H is called accretive if

$$Re < Au, \ u > \geq 0 \text{ for all } u \in D(A),$$

where $< \cdot, \ \cdot >$ is the scalar product in H.

Remark. The operator $-A$ in this case is called dissipative. In some papers, the authors define a dissipative operator by the condition $Im < Au, u > \geq 0$.

Lemma 4.4. Let A^{-1} exist. Then if A is an accretive operator, then A^{-1} is also an accretive operator.

Proof. Indeed, $Re < Au, \ u > \geq 0$ for all $u \in D(A)$, therefore

$$Re < A^{-1}y, \ y > = Re < u, \ Au > = Re < Au, u > = Re < \overline{Au, u} > \geq 0,$$

where $u = A^{-1}y$, $y \in R(A)$, $R(A)$ is the range of the operator A. Lemma 4.4 is proved. □

Lemma 4.5. Let the conditions i)-ii) be fulfilled. Then the operator L^{-1} is accretive.

Here L^{-1} is the inverse operator to the operator $Ly = -y''' + q(x)y$, $y \in D(L)$.

Proof. Theorem 1.2 implies that any element $y \in D(L)$ has generalized derivatives up to the third order, inclusive, belonging to the space $L_2(\mathbb{R})$. Taking into account the above and the properties of the Fourier transform in space $L_2(\mathbb{R})$, we obtain

$$< Ly, \ y > = \int_{-\infty}^{\infty} \left(-y''' + q(x)y\right) \bar{y}\, dx = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (i\xi)^3 \, \hat{y} \, \bar{\hat{y}} \, d\xi + \int_{-\infty}^{\infty} q(x) \, |y|^2 \, dx,$$

for any $y \in D(L)$, where $\hat{y}$ is the Fourier transform of the function $y(x)$.

Hence $Re < Ly, \ y > \geq 0$. Therefore, according to Definition 4.2, the operator L is an accretive operator. By Lemma 4.4, L^{-1} is also an accretive operator. Lemma 4.5 is proved. □

Proof of Theorem 1.6.. Theorem 1.5 implies that the operator L^{-1} is a nuclear operator when $q^{-\frac{2}{3}}(x) \in L$. According to Lemmas 4.4 and 4.5, the operator L^{-1} is accretive. Therefore, according to Lidsky's theorem [9], we obtain that the system of root vectors of the operator L^{-1} is complete in $L_2(\mathbb{R})$. Theorem 1.6 is proved. □

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Author contributions

Mussakan Muratbekov: supervision, conceptualization, formulation of the problem, methodology, investigation, writing original draft, funding acquisition, project administration.
Madi Muratbekov: investigation, methodology, some computations, writing and editing original manuscript, project administration, bibliography.

Conflict of interest

The authors declare no potential conflict of interests.

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