Parameter estimate and adaptive control of DARMA systems with uniform quantized output data

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Parameter estimate and adaptive control of DARMA systems with uniform quantized output data

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Abstract

This paper is concerned with parameter estimate and adaptive control problems of deterministic autoregressive moving average (DARMA) systems on the basis of quantized data of system output signals which are generated by a kind of uniform quantizer. By designing system input signals, the extended least-squares (ELS) algorithm with uniform output observations is proved to yield bounded estimation errors under some mild assumptions. Moreover, the adaptive tracking controller under inaccuracy observations are also designed. To analyse the properties of tracking error, I use the expanded form of ELS and research the properties of quantization noise. In addition, I give the expression of tracking error and show how it depends on the size of quantization step when the quantization step satisfies some conditions. A numerical example is supplied to demonstrate the theoretical results.

KEYWORDS
parameter estimate, adaptive control, discrete-time linear time-invariant systems, quantized data

1 | INTRODUCTION

It is of importance to study parameter estimation algorithm and design adaptive controller with quantized data in the fields of adaptive control systems and signal processing. These issues have been extensively researched over the past two decades due to the wide use in practice. For example,[1] proposed a quantized method and studied the parameter estimate problem in genetic associate model.[2] used a quantized model to represent the relation between the feature vector and the authenticity of the radar target and gave the recognition criteria based on quantized parameter estimation.[3] did some research on credit scoring with the help of quantized identification method. The appearance of applications brings new requirements for parameter estimate and adaptive controller design in theory, which are the focus this paper.

Generally speaking, set-valued data, especially binary data, and uniform data are two hotspots in this research field. And numerous papers (see e.g.[4]-[29]) on parameter estimate and system control based on these two kinds of quantized data have been made. Specifically,[4] used Bayesian framework and Markov Chain Monte Carlo methods to estimate the parameters of linear systems with set-valued output data. And the parameters were be estimated by the proposed sampling techniques.[7] proposed a variational approximation of the likelihood function and got the consistent estimates when the output data are integers (a special type of set-valued data).[8] gave two recursive algorithms and got strongly consistent estimator via a binary sensor.[13] presented a new algorithm for multi-input and multi-output (MIMO) finite impulse response (FIR) systems with set-valued output data and showed the comparisons with other estimation algorithms. What's more, a new approach to parameter estimate based on binary output data by using original weighted least-squares criteria was proposed in[14]. The authors also illustrated a simple choice for the weights and the asymptotical properties of the criterion.[15] considered the identification problem of autoregressive moving average (ARMA) systems with binary output data, and the estimates were proved to be convergent to the true values. Based on set-valued output data,[19] proposed a recursive estimator of stochastic approximation type and obtained two accelerated recursive estimators using the Newton-based and averaging techniques.[24],[25] studied the adaptive control problem for linear systems with set-valued output data and showed the adaptive tracking control algorithm is asymptotically optimal. As for parameter estimate with uniform output data,[22,23,28] considered deterministic autoregressive moving average (DARMA) systems and[27] focused on stochastic autoregressive exogenous input (ARX) systems.

More often than not, the parameter estimate methods with set-valued output data and uniform quantized output data are pretty different. This is mainly reflected in the need to analyse quantization noise. In the case of using set-valued data, due to the number of quantized output data is limited, the difference between the real value of system output signal and its observation (set-valued data) may be unbounded, especially the discrete-time unstable linear systems. And this makes it meaningless to study the specific form of quantization noise. Furthermore, using the distribution function of stochastic system noise to design parameter estimate algorithm is one common method under this situation. In the case of using uniform data, because of the difference in quantization approach between set-valued data and uniform data, the quantization noise can be bounded when using uniform quantized output data in discrete-time linear systems. It always appears in the proceeding of expanding estimation algorithm and will affect the accuracy of parameter estimate. So, the properties of quantization noise are always considered in this case. It has been shown in[22,23,28] that the main difficulty in parameter estimate using uniform quantized output data of DARMA systems is analysing the effect of quantization noise. Different from frequently-used assumptions on stochastic system noises, the quantization noise can not be assumed to ba a martingale difference sequence with respect to a nondecreasing family of σ-algebras. Consequently, some useful statistical properties are not applicable.

The purpose of this note is to research parameter estimate problem with uniform output quantized data by using ELS algorithm and to design the adaptive controller based on the estimation algorithm. As mentioned earlier, the introduce of quantization noise brings difficulties to parameter estimation. And it is mainly reflected in two aspects. First, the form of matrix composed by regressor vectors becomes more complex which makes the establishment of excitation condition even more difficult. Second, the recursion of estimate algorithm becomes more complicated and the sound structure of ELS was affected. Actually, I design input signals and explore the properties of quantization step so as to deal with these two issues.

The rest of this paper is organized as follows. Section 2 describes the system model and the form of uniform quantizer. Section 3 shows the estimate algorithm of quantized DARMA systems and the properties of parameter estimation error. Section 4 gives the adaptive controller and analyzes the properties of tracking error. Section 5 uses a numerical example to demonstrate the main theoretical results. Section 6 presents concluding remarks.

Notation: In this paper, $\mathbb{R}$ denotes real number field. For a given vector or matrix x, $x^\top$ denotes the transpose of x; $\|x\|$ denotes the Euclidean norm for vector case and the corresponding induced norm for matrix case. $\lambda_{min}(C)$ denotes the minimum eigenvalue of matrix C.

2 | MODEL AND QUANTIZER

Consider the DARMA system, described by

$$A(z)y_{n+1} = B(z)u_n, \quad n \geq 0, \tag{1}$$

where y_n and u_n are the output signal and input signal. For simplicity, we suppose $y_n = u_n = 0, \forall n < 0$.

$$A(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_p z^p,$$
$$B(z) = b_1 + b_2 z + \cdots + b_q z^{q-1},$$

where a_i and b_j are unknown parameters to be estimated, z is the shift-back operator and the orders p, q are assumed known.

One of the aim of this paper is to estimate the following parameter vector by using system inputs and quantized outputs.

$$\theta = \left[-a_1, \cdots, -a_p, b_1, \cdots, b_q\right]^\top.$$

For the convenience of proof, the model (1) can be rewritten as follows:

$$y_{n+1} = \theta^\top \varphi_n, \tag{2}$$

where

$$\varphi_n = \left[y_n, \cdots, y_{n-p+1}, u_n, \cdots, u_{n-q+1}\right]^\top.$$

For a constant $\varepsilon > 0$, the quantizer used here is of the following uniform form:

$$s_n = \varepsilon \left\lfloor \frac{y_n}{\varepsilon} + \frac{1}{2} \right\rfloor . \tag{3}$$

We can call ε the quantization step and s_n is the quantized output.

From (3) we know that

$$s_{n+1} = \theta^\top \psi_n + \epsilon_{n+1}, \tag{4}$$

where

$$\psi_n = \left[s_n, \cdots, s_{n-p+1}, u_n, \cdots, u_{n-q+1}\right]^\top .$$

From (2), (4) we know that

$$\begin{aligned}
\left|\epsilon_{n+1}\right| &= \left|s_{n+1} - \theta^\top \psi_n\right| \\
&= \left|s_{n+1} - y_{n+1} + \theta^\top \left(\varphi_n - \psi_n\right)\right| \\
&\leq \left|s_{n+1} - y_{n+1}\right| + \left|\theta^\top \left(\varphi_n - \psi_n\right)\right| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \left(\left|a_1\right| + \left|a_2\right| + \cdots + \left|a_{p_0}\right|\right) \\
&= \frac{\varepsilon}{2} \left(\left|a_1\right| + \left|a_2\right| + \cdots + \left|a_{p_0}\right| + 1\right) .
\end{aligned} \tag{5}$$

We call ϵ_n the quantization noise.

3 | PARAMETER ESTIMATE

3.1 | Assumptions

We begin the discussion with assumptions about model (1).

Assumption 1. $A(z)$ and $B(z)$ are coprime, $a_p \neq 0$.

Assumption 2. $u_n = v_n$, $\{v_n\}$ is a sequence of independent and identically distributed (i.i.d.) variables and v_n satisfies uniform distribution in $[-\delta, \delta]$, $\delta > 0$.

Then, we need some definitions used in the lemmas. For any $x \in \mathbb{R}^{p+q}$, $\|x\| = 1$, define

$$x := \left[x_1, x_2, \cdots, x_{p+q}\right]^\top ,$$

$$H_x(z) := x_1 B(z) z + \cdots + x_p B(z) z^p + x_{p+1} A(z) + \cdots + x_{p+q} z^{q-1} A(z) = \sum_{i=0}^{p+q-1} g_i(x) z^i ,$$

$$L_x(z) := \sum_{i=1}^{p} x_i z^{i-1}$$

and

$$g(x) := \left[g_0(x), g_1(x), \cdots, g_{p+q-1}(x)\right]^\top .$$

Then, by Assumption 1 and Lemma 1 of[30] we know that

$$\min_{\|x\|=1} \left|\left|g(x)\right|\right|^2 > 0.$$

Next, we can get the theoretical results of parameter estimate section.

Lemma 1. *Suppose Assumption 2 is satisfied. Then, as $n \to \infty$, there is a constant $c > 0$ such that*

$$\lambda_{min}\left(\sum_{i=0}^{n} U_i U_i^\top\right) \geq c\,(n+1)\,, \quad a.s., \tag{6}$$

where $U_i = \left[u_i, u_{i-1}, \cdots, u_{i-p-q+1}\right]^\top$.

Proof. For a sufficient large positive integer N and $n > N$, from Assumption 2, we know there is a positive constant c such that

$$\begin{aligned}\lambda_{min}\left(\sum_{i=0}^{n} U_i U_i^\top\right) =& \inf_{\|x\|=1} x^\top \left(\sum_{i=0}^{n} U_i U_i^\top\right) x \\ =& \sum_{i=0}^{n}\left(x_1 v_i + x_2 v_{i-1} \cdots + x_{p+q} v_{i-p-q+1}\right)^2 \\ =& \sum_{i=0}^{n}\left(x_1^2 v_i^2 + x_2^2 v_{i-1}^2 \cdots + x_{p+q}^2 v_{i-p-q+1}^2\right) + o\,(n+1) \\ \geq& c\,(n+1)\,, \quad a.s.\end{aligned}$$

This completes the proof. □

Remark 1. Lemma 1 means that system input signals $\{u_n\}$ satisfies the persistent excitation condition in the form of matrix. And this is a common condition of parameter estimate problem in many researches and papers.

Lemma 2. *Suppose Assumptions 1 and 2 are satisfied, then there is a suitable ε such that $\left|\left(H_x(z)u_i\right)\left(L_x(z)\epsilon_i\right)\right| \leq \frac{1}{3}\min_{\|x\|=1}\left|\left|g(x)\right|\right|^2 c$, for any $x \in \mathbb{R}^{p+q}$, $\|x\| = 1$.*

Proof. From Assumption 1, we know that $\min_{\|x\|=1}\left|\left|g(x)\right|\right|^2 > 0$. Since $\|x\| = 1$, the coefficients of $H_x(z)$ and $L_x(z)$ are bounded. From (5) and Assumption 2 we know that $\left|\epsilon_i\right|$ and $|u_i|$ are bounded. So, there exists a ε such that

$$\left|\left(H_x(z)u_i\right)\left(L_x(z)\epsilon_i\right)\right| \leq \frac{1}{3}\min_{\|x\|=1}\left|\left|g(x)\right|\right|^2 c.$$

And this completes the proof. □

Lemma 3. *Suppose Assumptions 1 and 2 are satisfied for a suitable ε, then*

$$\lambda_{min}\left(\sum_{i=0}^{n} \psi_i \psi_i^\top\right) \geq c_1\,(n+1)\,, \quad a.s., \quad n \to \infty, \tag{7}$$

where $c_1 > 0$ is a constant.

Proof. Let

$$\phi_n = A(z)\psi_n. \tag{8}$$

Then we have

$$\phi_n = \left[\left(zB(z)u_n + \epsilon_n\right), \cdots, \left(z^p B(z)u_n + \epsilon_{n-p+1}\right), A(z)u_n, \cdots, z^{q-1}A(z)u_n\right]^\top. \tag{9}$$

From (8), for any $x \in \mathbb{R}^{p+q}$, $\|x\| = 1$, we get

$$
\begin{aligned}
x^\top \left(\sum_{i=0}^{n} \phi_i\phi_i^\top\right) x =& \sum_{i=0}^{n} \left(x^\top \phi_i\right)^2 \\
=& \sum_{i=0}^{n} \left(\sum_{j=0}^{p} a_j x^\top \psi_{i-j}\right)^2 \\
\leq& \sum_{j=0}^{p} a_j^2 \sum_{i=0}^{n}\sum_{j=0}^{p} \left(x^\top \psi_{i-j}\right)^2 \\
\leq& (p+1)\sum_{j=0}^{p} a_j^2 \left(x^\top \sum_{i=0}^{n} \psi_i\psi_i^\top x\right),
\end{aligned} \tag{10}
$$

where $a_0 = 1$.

From (10), we have

$$
\lambda_{min}\left(\sum_{i=0}^{n} \psi_i\psi_i^\top\right) \geq \frac{1}{(p+1)\sum_{j=0}^{p} a_j^2}\lambda_{min}\left(\sum_{i=0}^{n} \phi_i\phi_i^\top\right). \tag{11}
$$

Therefore, from (6), (9) and Lemma 2 it follows that

$$
\begin{aligned}
x^\top \sum_{i=0}^{n} \phi_i\phi_i^\top x =& \sum_{i=0}^{n} (H_x(z)u_i + L_x(z)\epsilon_i)^2 \\
=& g^\top(x)\sum_{i=0}^{n} U_iU_i^\top g(x) + 2\sum_{i=0}^{n} (H_x(z)u_i)(L_x(z)\epsilon_i) + \sum_{i=0}^{n} (L_x(z)\epsilon_i)^2 \\
\geq& \min_{\|x\|=1} \left|\left|g(x)\right|\right|^2 \lambda_{min}\left(\sum_{i=0}^{n} U_iU_i^\top\right) + 2\sum_{i=0}^{n} (H_x(z)u_i)(L_x(z)\epsilon_i) \\
\geq& \min_{\|x\|=1} \left|\left|g(x)\right|\right|^2 \lambda_{min}\left(\sum_{i=0}^{n} U_iU_i^\top\right) - \frac{2}{3}\min_{\|x\|=1} \left|\left|g(x)\right|\right|^2 c\,(n+1) \\
=& \frac{1}{3}\min_{\|x\|=1} \left|\left|g(x)\right|\right|^2 c\,(n+1),
\end{aligned}
$$

which implies

$$
\lambda_{min}\left(\sum_{i=0}^{n} \phi_i\phi_i^\top\right) \geq \frac{1}{3}\min_{\|x\|=1} \left|\left|g(x)\right|\right|^2 c\,(n+1). \tag{12}
$$

From (11) and (12), let $c_1 = \frac{\min_{\|x\|=1}\|g(x)\|^2 c}{3(p+1)\sum_{j=0}^{p} a_j^2}$. This completes the proof. □

3.2 | Parameter estimate algorithm

For θ, we use the following estimation algorithm:

$$
\theta_{n+1} = \left(\sum_{i=0}^{n} \psi_i\psi_i^\top\right)^{-1} \sum_{i=0}^{n} \psi_i s_{i+1} = P_{n+1}\sum_{i=0}^{n} \psi_i s_{i+1}, \tag{13}
$$

where

$$
P_{n+1} = \left(P_0^{-1} + \sum_{i=0}^{n} \psi_i\psi_i^\top\right)^{-1} = \left(P_n^{-1} + \psi_n\psi_n^\top\right)^{-1} = P_n - d_nP_n\psi_n\psi_n^\top P_n, \tag{14}
$$

$$d_n = \left(1 + \psi_n^\top P_n \psi_n\right)^{-1}. \tag{15}$$

From (13)-(15) it follows that

$$\begin{aligned}
\theta_{n+1} &= \left(P_n - d_n P_n \psi_n \psi_n^\top P_n\right)\left(\sum_{i=0}^{n-1} \psi_i s_{i+1} + \psi_n s_{n+1}\right) \\
&= \theta_n - d_n P_n \psi_n \psi_n^\top \theta_n + P_n \psi_n s_{n+1} - d_n P_n \psi_n \psi_n^\top P_n \psi_n s_{n+1} \\
&= \theta_n - d_n P_n \psi_n \psi_n^\top \theta_n + P_n \psi_n \left(1 - d_n \psi_n^\top P_n \psi_n\right) s_{n+1} \\
&= \theta_n - d_n P_n \psi_n \psi_n^\top \theta_n + d_n P_n \psi_n s_{n+1} \\
&= \theta_n + d_n P_n \psi_n \left(s_{n+1} - \psi_n^\top \theta_n\right).
\end{aligned} \tag{16}$$

So, we have obtained the recursive algorithm for the LS estimation.

We set $P_0 = I$, and take θ_0 arbitrarily. Denote by $\lambda_{min}(n)$ the smallest eigenvalue of P_{n+1}^{-1}.

And the property of algorithm is researched in the following theorem.

Theorem 1. *For (4), suppose Assumptions 1-2 hold for a suitable ε which satisfies $0 < \varepsilon < \frac{1}{2(1+\sum_{i=1}^{p}|a_i|)}$. Then, we have*

$$\left|\left|\tilde{\theta}_{n+1}\right|\right| \leq c_2 \left(\sqrt{\frac{1}{n+1} + \varepsilon}\right), \quad a.s., \quad n \to \infty, \tag{17}$$

where

$$\tilde{\theta}_n = \theta - \theta_n \tag{18}$$

is the parameter estimation error, and c_2 is a positive constant independent of n and ε.

Proof. Noticing $P_{n+1}^{-1} \geq \lambda_{min}(n) I$, we see that

$$\left|\left|\tilde{\theta}_{n+1}\right|\right|^2 \leq \frac{1}{\lambda_{min}(n)} \tilde{\theta}_{n+1}^\top P_{n+1}^{-1} \tilde{\theta}_{n+1}. \tag{19}$$

Firstly, we need to prove there exist constants c_3, c_4 independent of n and ε such that

$$\tilde{\theta}_{n+1}^\top P_{n+1}^{-1} \tilde{\theta}_{n+1} \leq c_3 + c_4 \varepsilon (n+1). \tag{20}$$

From (15)-(16) it can be seen that

$$\begin{aligned}
s_{n+1} - \psi_n^\top \theta_{n+1} &= s_{n+1} - \psi_n^\top \left(\theta_n + d_n P_n \psi_n \left(s_{n+1} - \psi_n^\top \theta_n\right)\right) \\
&= \left(1 - d_n \psi_n^\top P_n \psi_n\right)\left(s_{n+1} - \psi_n^\top \theta_n\right) \\
&= d_n \left(s_{n+1} - \psi_n^\top \theta_n\right).
\end{aligned} \tag{21}$$

Hence, by (4), (18) and (21), we can rewrite (16) as

$$\begin{aligned}
\tilde{\theta}_{n+1} &= \tilde{\theta}_n - P_n \psi_n \left(s_{n+1} - \psi_n^\top \theta_n\right) \\
&= \tilde{\theta}_n - P_n \psi_n \left(s_{n+1} - \psi_n^\top \theta_{n+1}\right) \\
&= \tilde{\theta}_n - P_n \psi_n \left(\tilde{\theta}_{n+1}^\top \psi_n + \epsilon_{n+1}\right).
\end{aligned} \tag{22}$$

We expand $\tilde{\theta}_{k+1}^{\top} P_{k+1}^{-1} \tilde{\theta}_{k+1}$ by using (14) and (22)

$$
\begin{aligned}
\tilde{\theta}_{k+1}^{\top} P_{k+1}^{-1} \tilde{\theta}_{k+1} =& \tilde{\theta}_{k+1}^{\top} \left(P_k^{-1} + \psi_k \psi_k^{\top}\right) \tilde{\theta}_{k+1} \\
=& \left[\tilde{\theta}_k - P_k \psi_k \left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)\right]^{\top} P_k^{-1} \left[\tilde{\theta}_k - P_k \psi_k \left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)\right] + \left(\tilde{\theta}_{k+1}^{\top} \psi_k\right)^2 \\
=& \left(\tilde{\theta}_{k+1}^{\top} \psi_k\right)^2 - 2\left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right) \tilde{\theta}_k^{\top} \psi_k + \psi_k^{\top} P_k \psi_k \left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)^2 + \tilde{\theta}_k^{\top} P_k^{-1} \tilde{\theta}_k \\
=& \tilde{\theta}_k^{\top} P_k^{-1} \tilde{\theta}_k - 2\left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right) \left[\tilde{\theta}_{k+1} + P_k \psi_k \left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)\right]^{\top} \psi_k \\
&+ \psi_k^{\top} P_k \psi_k \left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)^2 + \left(\tilde{\theta}_{k+1}^{\top} \psi_k\right)^2 \\
=& \tilde{\theta}_k^{\top} P_k^{-1} \tilde{\theta}_k + \left(\tilde{\theta}_{k+1}^{\top} \psi_k\right)^2 - \psi_k^{\top} P_k \psi_k \left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)^2 - 2\left(\tilde{\theta}_{k+1}^{\top} \psi_k + \epsilon_{k+1}\right)\left(\tilde{\theta}_{k+1}^{\top} \psi_k\right) \\
\leq& \tilde{\theta}_k^{\top} P_k^{-1} \tilde{\theta}_k - \left(\tilde{\theta}_{k+1}^{\top} \psi_k\right)^2 - 2\epsilon_{k+1} \tilde{\theta}_{k+1}^{\top} \psi_k.
\end{aligned} \tag{23}
$$

Summing up both sides of (23) from 0 to n and letting $c_3 = \tilde{\theta}_0^{\top} P_0^{-1} \tilde{\theta}_0$, we get

$$
\begin{aligned}
\tilde{\theta}_{n+1}^{\top} P_{n+1}^{-1} \tilde{\theta}_{n+1} \leq& \tilde{\theta}_0^{\top} P_0^{-1} \tilde{\theta}_0 - \sum_{i=0}^{n} \left(\tilde{\theta}_{i+1}^{\top} \psi_i\right)^2 - 2\sum_{i=0}^{n} \epsilon_{i+1} \tilde{\theta}_{i+1}^{\top} \psi_i \\
=& c_3 - \sum_{i=0}^{n} \left(\tilde{\theta}_{i+1}^{\top} \psi_i\right)^2 - 2\sum_{i=0}^{n} \epsilon_{i+1} \tilde{\theta}_{i+1}^{\top} \psi_i,
\end{aligned}
$$

or equivalently,

$$
\tilde{\theta}_{n+1}^{\top} P_{n+1}^{-1} \tilde{\theta}_{n+1} + \sum_{i=0}^{n} \left(\tilde{\theta}_{i+1}^{\top} \psi_i\right)^2 \leq c_3 + \left|2\sum_{i=0}^{n} \epsilon_{i+1} \tilde{\theta}_{i+1}^{\top} \psi_i\right|. \tag{24}
$$

From (5) and $0 < \varepsilon < \frac{1}{2\left(1+\sum_{i=1}^{p} |a_i|\right)}$, we have

$$
\begin{aligned}
\left|2\sum_{i=0}^{n} \epsilon_{i+1} \tilde{\theta}_{i+1}^{\top} \psi_i\right| \leq& 2\sum_{i=0}^{n} |\epsilon_{i+1}| \left|\tilde{\theta}_{i+1}^{\top} \psi_i\right| \\
\leq& \varepsilon \left(1 + \sum_{i=1}^{p} |a_i|\right) \sum_{i=0}^{n} \left|\tilde{\theta}_{i+1}^{\top} \psi_i\right| \\
\leq& \varepsilon \left(1 + \sum_{i=1}^{p} |a_i|\right) \sum_{i=0}^{n} \left(\left|\tilde{\theta}_{i+1}^{\top} \psi_i\right|^2 + 1\right) \\
=& \varepsilon \left(1 + \sum_{i=1}^{p} |a_i|\right) \sum_{i=0}^{n} \left|\tilde{\theta}_{i+1}^{\top} \psi_i\right|^2 \\
&+ \left(1 + \sum_{i=1}^{p} |a_i|\right) \varepsilon (n+1) \\
<& \frac{1}{2} \sum_{i=0}^{n} \left|\tilde{\theta}_{i+1}^{\top} \psi_i\right|^2 + \left(1 + \sum_{i=1}^{p} |a_i|\right) \varepsilon (n+1).
\end{aligned} \tag{25}
$$

From (24), (25) we know that there is a positive constant c_4 independent of n and ε such that

$$
\begin{aligned}
\tilde{\theta}_{n+1}^{\top} P_{n+1}^{-1} \tilde{\theta}_{n+1} + \sum_{i=0}^{n} \left(\tilde{\theta}_{i+1}^{\top} \psi_i\right)^2 \leq& c_3 + \frac{1}{2} \sum_{i=0}^{n} \left|\tilde{\theta}_{i+1}^{\top} \psi_i\right|^2 + \left(1 + \sum_{i=1}^{p} |a_i|\right) \varepsilon (n+1) \\
\leq& c_3 + \frac{1}{2} \sum_{i=0}^{n} \left|\tilde{\theta}_{i+1}^{\top} \psi_i\right|^2 + c_4 \varepsilon (n+1).
\end{aligned}
$$

Thus, we have

$$\begin{aligned}\tilde{\theta}_{n+1}^{\top}P_{n+1}^{-1}\tilde{\theta}_{n+1} \leq & \tilde{\theta}_{n+1}^{\top}P_{n+1}^{-1}\tilde{\theta}_{n+1} + \frac{1}{2}\sum_{i=0}^{n}\left(\tilde{\theta}_{i+1}^{\top}\psi_i\right)^2 \\ \leq & c_3 + c_4\varepsilon\,(n+1)\,. \end{aligned} \tag{26}$$

So, (20) is proved.

Noticing $\lambda_{min}(n) \geq \lambda_{min}\left(\sum_{i=0}^{n}\psi_i\psi_i^{\top}\right)$, from (7), (19) and (20), it can be seen that as $n \to \infty$,

$$\begin{aligned}\left|\left|\tilde{\theta}_{n+1}\right|\right|^2 \leq & \frac{1}{\lambda_{min}(n)}\tilde{\theta}_{n+1}^{\top}P_{n+1}^{-1}\tilde{\theta}_{n+1} \\ \leq & \frac{c_3 + c_4\varepsilon\,(n+1)}{c_1\,(n+1)} \\ = & c_5\frac{1}{n+1} + c_6\varepsilon \\ \leq & c_7\left(\frac{1}{n+1} + \varepsilon\right), \quad a.s.,\end{aligned}$$

where $c_5 = \frac{c_3}{c_1}$, $c_6 = \frac{c_4}{c_1}$, $c_7 = \max\{c_5, c_6\}$. So,

$$\left|\left|\tilde{\theta}_{n+1}\right|\right| \leq \sqrt{c_7}\left(\sqrt{\frac{1}{n+1}} + \varepsilon\right), \quad a.s., \quad n \to \infty.$$

And let $c_2 = \sqrt{c_7}$. So, (17) is proved. This completes the proof. □

Remark 2. Theorem 1 indicates that the parameter estimation error depends on the quantization step. While $0 < \varepsilon < \frac{1}{2\left(1+\sum_{i=1}^{p}|a_i|\right)}$, the smaller the quantization step, the smaller the value of parameter estimation error.

4 | ADAPTIVE CONTROL

Let $\{y_n^*\}$ be a sequence of bounded deterministic reference signal. The tracking error is of the form $\frac{1}{n+1}\sum_{i=0}^{n}\left(s_{i+1} - y_{i+1}^*\right)^2$. Then we have the following theorem.

Theorem 2. *For (4), suppose u_n could be chosen to satisfy*

$$\theta_n^{\top}\psi_n = y_{n+1}^*. \tag{27}$$

And suppose

$$\sup_n \psi_n^{\top}P_n\psi_n = c_9 < \infty. \tag{28}$$

Then, for a suitable ε which satisfies $0 < \varepsilon < \frac{1}{2\left(1+\sum_{i=1}^{p}|a_i|\right)}$,

$$\limsup_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\left(s_{i+1} - y_{i+1}^*\right)^2 \leq \left(8c_4 + 16c_4c_9^2\right)\varepsilon + \left(2c_8 + 8c_8c_9^2\right)\varepsilon^2. \tag{29}$$

Proof. From (26), we have

$$\frac{1}{2}\sum_{i=0}^{n}\left(\tilde{\theta}_{i+1}^{\top}\psi_i\right)^2 \leq c_3 + c_4\varepsilon\,(n+1)\,.$$

So,

$$\sum_{i=0}^{n} \left(\tilde{\theta}_{i+1}^{\top}\psi_i\right)^2 \leq 2c_3 + 2c_4\varepsilon\,(n+1)\,. \tag{30}$$

Let $c_8 = \frac{(|a_1|+|a_2|+\cdots+|a_{p_0}|+1)^2}{4}$. Then, from (5), it can be seen that

$$\epsilon_{n+1}^2 \leq \frac{\varepsilon^2}{4}\left(|a_1| + |a_2| + \cdots + |a_{p_0}| + 1\right)^2 = c_8\varepsilon^2. \tag{31}$$

From (22), we know that

$$\tilde{\theta}_i^{\top}\psi_i = \tilde{\theta}_{i+1}^{\top}\psi_i + \psi_i^{\top}P_i\psi_i\left(\tilde{\theta}_{i+1}^{\top}\psi_i + \epsilon_{i+1}\right). \tag{32}$$

So, from (28), (30)-(32), we get

$$\begin{aligned}
\sum_{i=0}^{n}\left(\tilde{\theta}_i^{\top}\psi_i\right)^2 \leq& 2\sum_{i=0}^{n}\left(\tilde{\theta}_{i+1}^{\top}\psi_i\right)^2 + 2\sum_{i=0}^{n}\left(\psi_i^{\top}P_i\psi_i\right)^2\left(\tilde{\theta}_{i+1}^{\top}\psi_i + \epsilon_{i+1}\right)^2\\
\leq& 2\sum_{i=0}^{n}\left(\tilde{\theta}_{i+1}^{\top}\psi_i\right)^2 + 4\sum_{i=0}^{n}\left(\psi_i^{\top}P_i\psi_i\right)^2\left(\left(\tilde{\theta}_{i+1}^{\top}\psi_i\right)^2 + \epsilon_{i+1}^2\right)\\
\leq& 4c_3 + 8c_3c_9^2 + \left(4c_4 + 8c_4c_9^2\right)\varepsilon\,(n+1) + 4c_8c_9^2\varepsilon^2\,(n+1)\,.
\end{aligned} \tag{33}$$

So, from (4), (27), (31), (33), we have

$$\begin{aligned}
\sum_{i=0}^{n}\left(s_{i+1} - y_{i+1}^{*}\right)^2 =& \sum_{i=0}^{n}\left(\theta^{\top}\psi_i + \epsilon_{i+1} - \theta_i^{\top}\psi_i\right)^2\\
=& \sum_{i=0}^{n}\left(\tilde{\theta}_i^{\top}\psi_i + \epsilon_{i+1}\right)^2\\
\leq& 2\sum_{i=0}^{n}\left(\tilde{\theta}_i^{\top}\psi_i\right)^2 + 2\sum_{i=0}^{n}\epsilon_{i+1}^2\\
\leq& 8c_3 + 16c_3c_9^2 + \left(\left(8c_4 + 16c_4c_9^2\right)\varepsilon + \left(2c_8 + 8c_8c_9^2\right)\varepsilon^2\right)(n+1)\,.
\end{aligned}$$

Then, we get

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\left(s_{i+1} - y_{i+1}^{*}\right)^2\\
\leq& \limsup_{n\to\infty}\frac{8c_3 + 16c_3c_9^2}{n+1} + \left(8c_4 + 16c_4c_9^2\right)\varepsilon + \left(2c_8 + 8c_8c_9^2\right)\varepsilon^2\\
=& \left(8c_4 + 16c_4c_9^2\right)\varepsilon + \left(2c_8 + 8c_8c_9^2\right)\varepsilon^2.
\end{aligned}$$

So, (29) is proved. This completes the proof. □

Remark 3. From Theorem 2 we know that the parameter estimation error depends on the quantization step. And the expended structure of ELS is the key to design adaptive controller.

5 | SIMULATION EXAMPLE

In this section, I illustrate the theoretical results with a simulation example.

Consider the following system: $y_n = ay_{n-1} + bu_{n-1}$, where $\theta = [a, b]^T = [0.5, 1]^T$ is the parameter to be estimated, $\theta_0 = [0, 0]^T$. By conditions of Theorems 1 and 2, ε should satisfy $0 < \varepsilon < \frac{1}{2(1+\sum_{i=1}^{p}|a_i|)} = \frac{1}{3}$.

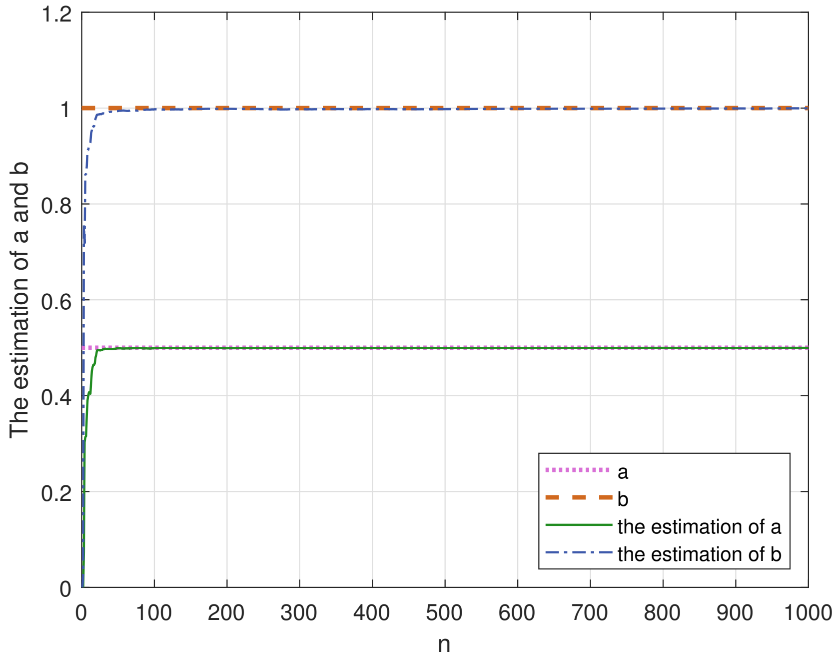

FIGURE 1 The trajectories of the estimation of a and b.

Parameter Estimate: Let y_n be quantized by (3) under $\varepsilon = 0.1$. $\{u_i\}$ satisfies uniform distribution in $[-3, 3]$ ($\delta = 3$), which satisfies Assumption 2. We estimate θ by (13). The simulation results are given in Figure 1 and Figure 2. From them, we can see that the estimate converges to the true value.

Adaptive Control: Let y_n be quantized by (3) under $\varepsilon = 0.1$ and $\varepsilon = 0.3$, respectively. $\{u_i\}$ is defined from (27) where $y_n^* = 10$. The simulation results are given in Figure 3 and Figure 4. From them, we can see that the smaller the quantization step, the smaller the tracking error.

6 | CONCLUSION

This paper researches the parameter estimate and adaptive control problem of DARMA systems by using uniform quantized data. The ELS algorithm is introduced to estimate unknown system parameters. Under some conditions, I prove that the parameter estimation error is tend to zero when the size of the quantization step satisfies some hypotheses. I also design the adaptive controller to track the deterministic signal $\{y_n^*\}$. Besides, I show that the tracking error is affected by quantization step. However, in this paper, I only consider the cases without system noises. For the systems with stochastic noise case, the analysis may be more complex.

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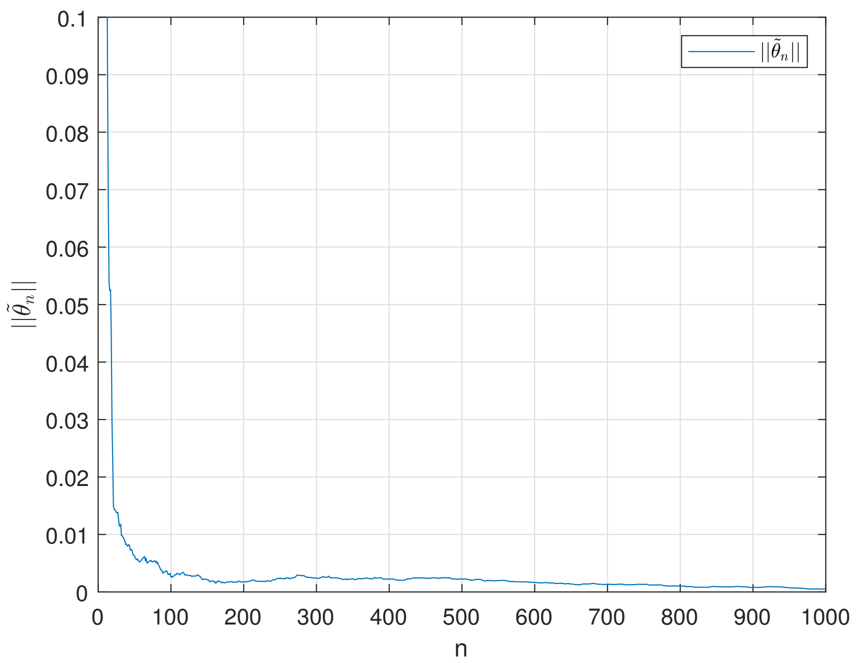

FIGURE 2 The trajectory of $\left\|\tilde{\theta}_n\right\|$.

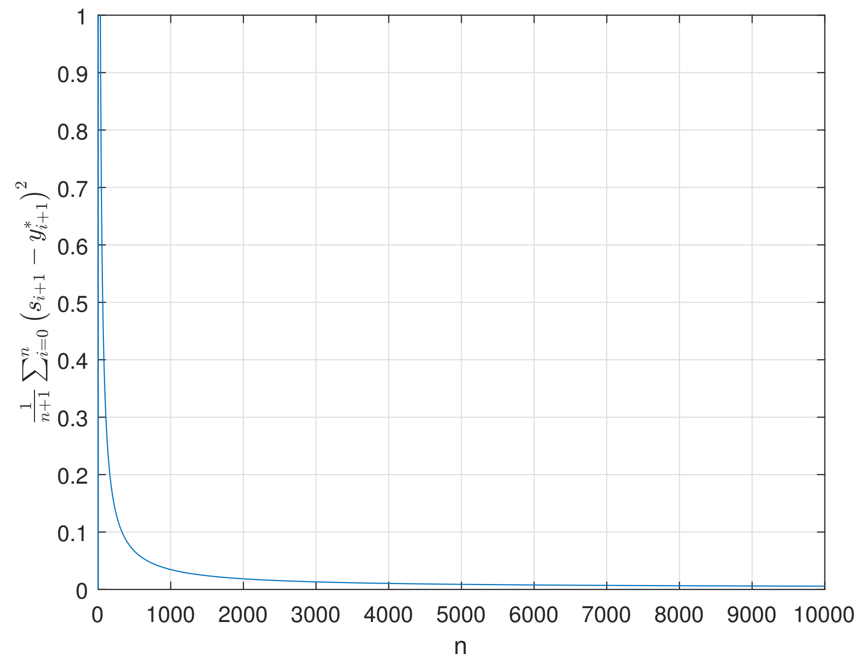

FIGURE 3 The trajectory of $\frac{1}{n+1}\sum_{i=0}^{n}\left(s_{i+1}-y_{i+1}^{*}\right)^{2}$ when $\varepsilon = 0.1$.

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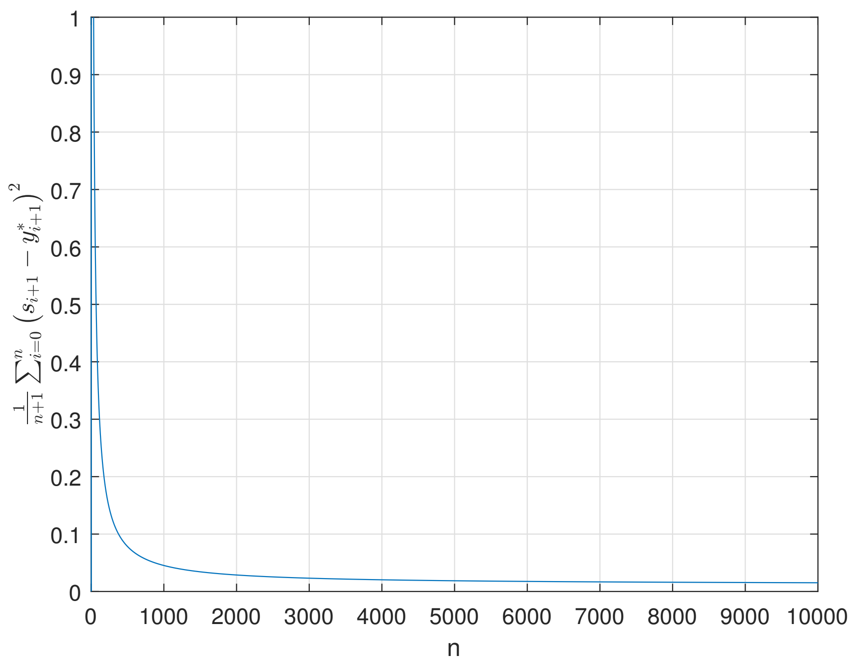

FIGURE 4 The trajectory of $\frac{1}{n+1}\sum_{i=0}^{n}\left(s_{i+1}-y_{i+1}^{*}\right)^{2}$ when $\varepsilon = 0.3$.

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