The normalized Laplacians spectrum and characteristic parameters of a class of irregular networks

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Abstract

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Keywords: Irregular network; Möbius graphs; Normalized Laplacian; Degree-Kirchhoff index; Complexity

1. Introduction

It is well established that networks can be represented by graphs. The graphs we consider in this paper are simple, undirected and connected. Let's first recall some definitions commonly used in graph theory. Suppose G represents a simple undirected graph with $|V_G| = n$ and $|E_G| = m$. For more notation, one can be referred to [1].

Note that $D(G) = diag\{d_1, d_2, \cdots, d_n\}$ represents a degree matrix, where d_p is the degree of v_p. $A(G)$ is the adjacency matrix of G. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. The (p,q)-entry of the normalized Laplacian matrix is given by

$$(\mathcal{L}(G))_{pq} = \begin{cases} 1, & p = q; \\ -\frac{1}{\sqrt{d_p d_q}}, & p \neq q \ and \ v_p \backsim v_q; \\ 0, & otherwise. \end{cases} \tag{1.1}$$

As a matter of fact, there are many parameters that can be used to describe the structure and properties of molecular graphs in graph networks. One of the parameters based on resistance distance is defined as Wiener index [2, 3], which is

$$W(G) = \sum_{i<j} d_{ij},$$

where $d_{ij} = d_G(v_i, v_j)$ represents the length of the shortest path between two vertices v_i and v_j in G. Wiener index is widely used in chemical and mathematical research. For details, see [4–7].

The parameter of resistance distance was first proposed by Klein and Randić [8] in 1993. It means that if every edge of a graph G is regarded as a unit resistance, then the distance between any two vertices i and j in G is called resistance distance, which is denoted as r_{ij}. Similar to Wiener index, we give the

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expression of the Kirchhoff index [9, 10] according to the resistance distance, namely

$$Kf(G) = \sum_{i<j} r_{ij} = n \sum_{i=2}^{n} \frac{1}{\mu_i}.$$

In 2007, Chen and Zhang [11] proposed that the eigenvalues and eigenvectors of normalized Laplacian spectrum can be used to describe the resistance distance, and an observation that prompted a new characteristic parameter, called the degree-Kirchhoff index, which is a kind of structural descriptor [11]. However, it is very difficult to calculate the degree-Kirchhoff index from the complexity division of graphs, so it is very important to find the explicit expression of degree-Kirchhoff index. In recent years, many scholars have devoted themselves to the study of degree-Kirchhoff index of various graphs. Huang and Li et al. [12, 13] proved the degree-Kirchhoff index of linear hexagonal chains and linear polyomino chains, successively. H. Bian et al. [14] determined the normalized Laplacians and degree-Kirchhoff index of cylinder phenylene chain. Zhao and Liu et al. [15] described the normalized Laplacian and degree-Kirchhoff index of linear octagonal-quadrilateral networks. For more excellent results, please refer to [16–21]. After learning the excellent works of scholars, in this paper, we use the correlation properties of normalized Laplace matrix to calculate the degree-Kirchhoff index and the complexity of Möbius graph of linear octagonal-quadrilateral networks.

The investigation of complex graph and irregular network has gone through a spectacular development in the past decades. Especially in organic chemistry, more and more attention has been paid to the application of polyomino in polycyclic aromatic compounds. Many scholars are interested in the study of linear octagonal networks and related molecular graphs. As we all know, linear octagonal network is an octagonal system without branch compression. It is constructed by regularly inserting some new points on the straight line of the linear polyomino network. The research on the structure and properties of this kind of natural graph network lays a solid foundation for the advancement of theoretical chemistry, as well as for the development of applied mathematics.

Let $L_n^{8,4}$ be the linear octagonal-quadrilateral networks and octagons and quadrilaterals are connected by a common edge, which depicted in Figure 1. Then the corresponding Möbius graph $MQ_3(8,4)$ of octagonal-quadrilateral networks is obtained by the reverse identification of the opposite edge by $L_n^{8,4}$, see Figure 2. Obviously, we can obtained that $|V_{MQ_n}(8,4)| = 8n$, $|E_{MQ_n}(8,4)| = 10n$.

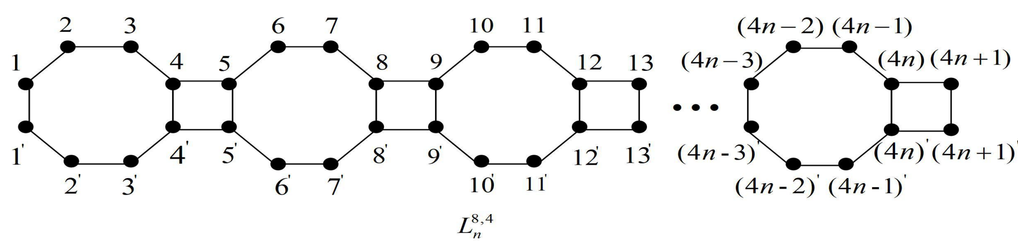

Figure 1: Linear octagonal-quadrilateral networks.

The rest of the paper will be divided into the following several sections: In Section 2, we put forward some basic notation and related lemmas. In Section 3, we determine the normalized Laplacian spectrum of $MQ_n(8,4)$. In Section 4, we committed to give the Kemeny$'$s constant, the degree-Kirchhoff index and the complexity of $MQ_n(8,4)$.

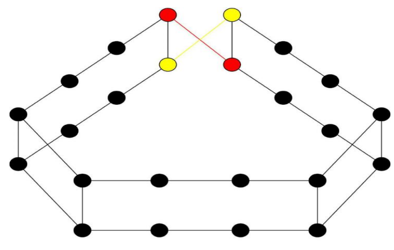

Figure 2: Graph $MQ_3(8,4)$.

2. Preliminary

In this section, we introduce some common symbols and related calculation methods [1], which are applied to the rest of the article.

The characteristic polynomial of matrix R of order n is defined as $P_R(x) = det(xI - R)$. It's not difficult to find that π is an automorphism of G, we can write the product of disjoint 1-cycles and transposition, namely

$$\pi = (\bar{1})(\bar{2})\cdots(\bar{m})(1,1')(2,2')\cdots(k,k').$$

Then one has $|V(G)| = m + 2k$, let $v_0 = \{\bar{1}, \bar{2}, \cdots \bar{m}\}$, $v_1 = \{1, 2 \cdots k\}$, $v_2 = \{1', 2' \cdots k'\}$. Thus the Laplacians matrix can be expressed in the form of block matrix, that is

$$\mathcal{L}(G) = \begin{pmatrix} \mathcal{L}_{V_0V_0} & \mathcal{L}_{V_0V_1} & \mathcal{L}_{V_0V_2} \\ \mathcal{L}_{V_1V_0} & \mathcal{L}_{V_1V_1} & \mathcal{L}_{V_1V_2} \\ \mathcal{L}_{V_2V_0} & \mathcal{L}_{V_2V_1} & \mathcal{L}_{V_2V_2} \end{pmatrix},$$

where

$$\mathcal{L}_{V_0V_1} = \mathcal{L}_{V_0V_2},\ \mathcal{L}_{V_1V_2} = \mathcal{L}_{V_2V_1},\ and\ \mathcal{L}_{V_1V_1} = \mathcal{L}_{V_2V_2}.$$

Let

$$P = \begin{pmatrix} I_m & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}I_k & \frac{1}{\sqrt{2}}I_k \\ 0 & \frac{1}{\sqrt{2}}I_k & -\frac{1}{\sqrt{2}}I_k \end{pmatrix},$$

then

$$P'\mathcal{L}(G)P = \begin{pmatrix} \mathcal{L}_A(G) & 0 \\ 0 & \mathcal{L}_S(G) \end{pmatrix},$$

noted that P' is the transposition of P, where

$$\mathcal{L}_A = \begin{pmatrix} \mathcal{L}_{V_0V_0} & \sqrt{2}\mathcal{L}_{V_0V_1} \\ \sqrt{2}\mathcal{L}_{V_1V_0} & \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2} \end{pmatrix},\ \mathcal{L}_S = \mathcal{L}_{V_1V_1} - \mathcal{L}_{V_1V_2}.$$

Lemma 2.1. [12] *Let $\mathcal{L}(L_n)(G)$, $\mathcal{L}_A(G)$, $\mathcal{L}_S(G)$ are determined as above, then*

$$P_{\mathcal{L}(L_n)}(G) = P_{\mathcal{L}_A}(G)P_{\mathcal{L}_S}(G).$$

Lemma 2.2. *Let G is a graph with $|V_G| = n$ and $|E_G| = m$, and $0 = \mu_1 < \mu_2 \le \cdots \le \mu_n (n \ge 2)$ are the eigenvalues of $\mathcal{L}(G)$. Then we can quickly confirm that the following formulas holds.*

(a) [22] *The Kemeny's constant of G can be denoted*

$$K_c(G) = \sum_{i=2}^{n} \frac{1}{\mu_i}.$$

(b) [11] *The degree-Kirchhoff index of G is defined as*

$$Kf^*(G) = 2m\sum_{k=2}^{n}\frac{1}{\mu_k}.$$

(c) [1] *The number of spanning trees of G can also be called the complexity of G. Then the complexity of G is*

$$\prod_{i=1}^{n} d_i \sum_{k=2}^{n}\lambda_k = 2m\tau(G).$$

3. The normalized Laplacian spectrum of $MQ_n(8,4)$

In this section, we focus on obtain the normalized Laplacian spectrum of $MQ_n(8,4)$ by lemma 2.1.

Given an $n \times n$ matrix T, and put deleting the $p_1th,\ p_2th,\ \cdots\ p_kth$ rows and columns of T are expressed as $T[\{p_1, p_2, \cdots p_k\}]$. With a suitable labeling, the vertices of $MQ_n(8,4)$ show in Figure 2. Apparently, $\pi = (1,1')(2,2')\cdots(4n,(4n)')$ is an automorphism of $MQ_n(8,4)$. Then $v_0 = \emptyset$, $v_1 = \{1,2,3,\cdots,4n\}$ *and* $v_2 = \{1',2',3',\cdots,(4n)'\}$. Besides, we express $\mathcal{L}_A(MQ_n(8,4))$ and $\mathcal{L}_S(MQ_n(8,4))$ as $\mathcal{L}_A$ and $\mathcal{L}_S$. Then one can get

$$\mathcal{L}_A = \mathcal{L}_{V_1V_1} + \mathcal{L}_{V_1V_2},\ \mathcal{L}_S = L_{V_1V_1} - \mathcal{L}_{V_1V_2}.$$

In views of Equation (1.1), we have

$$\mathcal{L}_{V_1V_1} = \begin{pmatrix}
1 & \frac{-1}{\sqrt{6}} & & & & & & & & & \\
\frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & & \\
 & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & \\
 & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{3} & & & & & & \\
 & & & \frac{-1}{3} & 1 & \frac{-1}{\sqrt{6}} & & & & & \\
 & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{3} & 1 & \frac{-1}{\sqrt{6}} & & \\
 & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\
 & & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\
 & & & & & & & & & \frac{-1}{\sqrt{6}} & 1
\end{pmatrix}_{(4n)\times(4n)},$$

and

$$\mathcal{L}_{V_1V_2} = \begin{pmatrix}
\frac{-1}{3} & & & & & & & & & & \frac{-1}{3} \\
 & 0 & & & & & & & & & \\
 & & 0 & & & & & & & & \\
 & & & \frac{-1}{3} & & & & & & & \\
 & & & & 0 & & & & & & \\
 & & & & & 0 & & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & & \frac{-1}{3} & & & \\
 & & & & & & & & 0 & & \\
 & & & & & & & & & 0 & \\
\frac{-1}{3} & & & & & & & & & & \frac{-1}{3}
\end{pmatrix}_{(4n)\times(4n)}.$$

Hence,

$$\mathcal{L}_A = \begin{pmatrix} \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & & & \frac{-1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & & \\ & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & & & & \\ & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & \\ & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\ & & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\ \frac{-1}{3} & & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} \end{pmatrix}_{(4n)\times(4n)},$$

and

$$\mathcal{L}_S = \begin{pmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & & & \frac{-1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & & & \\ & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & & \\ & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & & & \\ & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & \\ & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & \\ & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\ & & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\ \frac{-1}{3} & & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} \end{pmatrix}_{(4n)\times(4n)}.$$

Assuming that $0 = \eta_1 < \eta_2 \leq \eta_3 \leq \cdots \leq \eta_{4n}$ are the roots of $P_{\mathcal{L}_A}(x) = 0$, and $0 < \varphi_1 \leq \varphi_2 \leq \varphi_3 \leq \cdots \leq \varphi_{4n}$ are the roots of $P_{\mathcal{L}_S}(x) = 0$, respectively. Then according to lemma 2.1 we can get the spectrum of $MQ_n(8,4)$ is just $\eta_1, \eta_2, ..., \eta_{4n}, \ \varphi_1, \varphi_2, ..., \varphi_{4n}$, and it is directly to check that $\eta_1 = 0, \eta_p > 0 (p = 2, 3, ..., 4n)$, and $\varphi_q > 0 (q = 1, 2, ..., 4n)$.

Nextly, we will committed to calculate some main results of $MQ_n(8,4)$ related to the normalized Laplacian spectrum.

4. The degree-Kirchhoff index and the complexity of $MQ_n(8,4)$

In this section, we first introduce some theorems, which are obtained by describing the eigenvalues and eigenvectors of normalized Laplacian matrix. Then obtained the Kemeny$'$s constant, the degree-Kirchhoff index and the complexity of $MQ_n(8,4)$ based on these theorems.

Theorem 4.1.

$$\sum_{p=2}^{4n} \frac{1}{\eta_p} = \frac{200n^2 - 11}{60}.$$

Proof. Let

$$\begin{aligned} P_{L_S}(x) &= det(xI - \mathcal{L}_A) = x^{4n} + a_1 x^{4n-1} + \cdots + a_{4n-1}x + a_{4n} \\ &= x(x^{4n-1} + a_1 x^{4n-2} + \cdots + a_{4n-2}x + a_{4n-1}), \ a_{4n-1} \neq 0. \end{aligned}$$

Then we can exactly get $\eta_1, \eta_2, ..., \eta_{4n}$ are the roots of the following equation

$$x^{4n-1} + a_1 x^{4n-2} + \cdots + a_{4n-2}x + a_{4n-1} = 0.$$

Based on the Vieta$^{'}$s theorem of $P_{\mathcal{L}_A}(x)$, it$^{'}$s easy to get

$$\sum_{p=2}^{4n} \frac{1}{\eta_p} = \frac{(-1)^{4n-2}a_{4n-2}}{(-1)^{4n-1}a_{4n-1}}. \tag{4.2}$$

Before calculating $(-1)^{4n-2}a_{4n-2}$ and $(-1)^{4n-1}a_{4n-1}$, we must determine pth order principal submatrices, R_p^0, R_p^1, R_p^2 and R_p^3, which consists of the first p rows and columns of the following matrices $\mathcal{L}_A^0$,$\mathcal{L}_A^1$,$\mathcal{L}_A^2$ and $\mathcal{L}_A^3$, respectively, $p = 1, 2, ..., 4n$. Let

$$\mathcal{L}_A^0 = \begin{pmatrix}
\frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & & \\
\frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & & \\
 & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & \\
 & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & & & \\
 & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & \\
 & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & \\
 & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\
 & & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\
 & & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3}
\end{pmatrix}_{(4n)\times(4n)},$$

$$\mathcal{L}_A^1 = \begin{pmatrix}
1 & \frac{-1}{2} & & & & & & & & & \\
\frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & & \\
 & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & & & & \\
 & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & \\
 & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\
 & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\
 & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & \\
 & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} \\
 & & & & & & & & & \frac{-1}{3} & \frac{2}{3}
\end{pmatrix}_{(4n)\times(4n)},$$

$$\mathcal{L}_A^2 = \begin{pmatrix} 1 & \frac{-1}{\sqrt{6}} & & & & & & & & & & \\ \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & & & & & & \\ & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & \\ & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & \\ & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & \\ & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & \\ & & & & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & \\ & & & & & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{3} & \\ & & & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \end{pmatrix}_{(4n)\times(4n)},$$

and

$$\mathcal{L}_A^3 = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} & & & & & & & & & & \\ \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & & \\ & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & & \\ & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & \\ & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & & & & \\ & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & & & & \\ & & & & & & \ddots & & & & & \\ & & & & & & \frac{-1}{\sqrt{6}} & \frac{2}{3} & \frac{-1}{3} & & & \\ & & & & & & & \frac{-1}{3} & \frac{2}{3} & \frac{-1}{\sqrt{6}} & & \\ & & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\ & & & & & & & & & \frac{-1}{2} & 1 & \end{pmatrix}_{(4n)\times(4n)}.$$

In this way, we can get four facts.

Fact 1. For $1 \leq p \leq 4n$,

$$r_p^0 = \begin{cases} (p+1)\left(\frac{1}{36}\right)^{\frac{p}{4}}, & if\ p \equiv 0(mod4); \\ \frac{1}{3}(p+1)\left(\frac{1}{36}\right)^{\frac{p-1}{4}}, & if\ p \equiv 1(mod4); \\ \frac{1}{6}(p+1)\left(\frac{1}{36}\right)^{\frac{p-2}{4}}, & if\ p \equiv 2(mod4); \\ \frac{1}{12}(p+1)\left(\frac{1}{36}\right)^{\frac{p-3}{4}}, & if\ p \equiv 3(mod4). \end{cases}$$

Fact 2. For $1 \leq p \leq 4n$,

$$r_p^1 = \begin{cases} (p+1)\left(\frac{1}{36}\right)^{\frac{p}{4}}, & if\ p \equiv 0(mod4); \\ \frac{1}{2}(p+1)\left(\frac{1}{36}\right)^{\frac{p-1}{4}}, & if\ p \equiv 1(mod4); \\ \frac{1}{4}(p+1)\left(\frac{1}{36}\right)^{\frac{p-2}{4}}, & if\ p \equiv 2(mod4); \\ \frac{1}{12}(p+1)\left(\frac{1}{36}\right)^{\frac{p-3}{4}}, & if\ p \equiv 3(mod4). \end{cases}$$

Fact 3. For $1 \le p \le 4n$,

$$r_p^2 = \frac{1}{2} r_{p-2}^0 - \frac{1}{9} r_{p-3}^1.$$

Fact 4. For $1 \le p \le 4n$,

$$r_p^3 = \frac{2}{3} r_{p-1}^0 - \frac{1}{9} r_{p-2}^1.$$

Proof of Fact 1. Take $r_p^0 = \det R_p^0,\ r_p^1 = \det R_p^1,\ r_p^2 = \det R_p^2$ and $r_p^3 = \det R_p^3$. By a straightforward calculation, one can get the following values, see Table 1.

Table 1: Initial value.

r_p^0	$Value$	r_p^0	$Value$	r_p^0	$Value$	r_p^0	$Value$
r_1^0	$\frac{2}{3}$	r_2^0	$\frac{1}{2}$	r_3^0	$\frac{1}{3}$	r_4^0	$\frac{5}{36}$
r_5^0	$\frac{3}{54}$	r_6^0	$\frac{7}{216}$	r_7^0	$\frac{1}{54}$	r_8^0	$\frac{1}{144}$

For $4 \le p \le 4n-1$, we can get expending $\det R_p^0$ with respect to its last row yields

$$r_p^0 = \begin{cases} \frac{2}{3} r_{p-1}^0 - \frac{1}{6} r_{p-2}^0, & if\ p \equiv 0 (mod 4); \\ \frac{2}{3} r_{p-1}^0 - \frac{1}{9} r_{p-2}^0, & if\ p \equiv 1 (mod 4); \\ r_{p-1}^0 - \frac{1}{6} r_{p-2}^0, & if\ p \equiv 2 (mod 4); \\ r_{p-1}^0 - \frac{1}{4} r_{p-2}^0, & if\ p \equiv 3 (mod 4). \end{cases}$$

For $1 \le p \le n-1$, let $a_p = r_{4p}^0;\ 0 \le p \le n-1,\ b_p = r_{4p+1}^0,\ c_p = r_{4p+2}^0,\ d_p = r_{4p+3}^0$. Then we can get $a_1 = \frac{5}{36},\ b_0 = \frac{2}{3},\ c_0 = \frac{1}{2},\ d_0 = \frac{5}{36},\ b_1 = \frac{3}{54},\ c_1 = \frac{7}{216},\ d_1 = \frac{1}{54}$, and for $p \ge 2$, we have

$$\begin{cases} a_p = \frac{2}{3} d_{p-1} - \frac{1}{6} c_{p-1}; \\ b_p = \frac{2}{3} a_p - \frac{1}{9} d_{p-1}; \\ c_p = b_p - \frac{1}{6} a_p; \\ d_p = c_p - \frac{1}{4} b_p. \end{cases}$$

Then it$'$s no difficult to obtain that

$$\begin{cases} a_p = 18 c_p - 24 d_p; \\ b_p = 4 c_p - 4 d_p; \\ c_p = \frac{1}{18} c_{p-1} - \frac{1}{1296} c_{p-2}; \\ d_p = \frac{1}{18} d_{p-1} - \frac{1}{1296} d_{p-2}. \end{cases} \tag{4.3}$$

According to the equation of d_p in (4.3) is $x^2 - \frac{1}{18} x + \frac{1}{1296} = 0$, and its two roots are $\frac{1}{36}$ and $\frac{1}{36}$. Therefore, $d_p = (x_p + y)(\frac{1}{36})^p$ is the general solution. Then we can get $x = \frac{1}{3}$ and $y = \frac{1}{3}$.

Thus, we can obtained $d_p = \frac{1}{3}(p+1)(\frac{1}{36})^p (p \ge 1)$. Similarly ,we have $c_p = (\frac{2p}{3} + \frac{1}{2})(\frac{1}{36})^p (p \ge 1);\ a_p = (4p+1)(\frac{1}{36})^p (p \ge 1)$ and $b_p = \frac{2}{3}(2p+1)(\frac{1}{36})^p (p \ge 1)$.

The result as desired. ■

By the similar consideration, Facts 2 is available. Then based on the conclusion of Facts 1 and 2, we quickly get Facts 3 and 4.

Now, we will further calculate $(-1)^{4n-1} a_{4n-1}$ and $(-1)^{4n-2} a_{4n-2}$ in equation (4.2). For the sake of discussion, it is assumed that $r_0 = 1$.

Claim 1. $(-1)^{4n-1}a_{4n-1} = 40n^2(\frac{1}{36})^n$.
Proof of Claim 1. Since the $(-1)^{4n-1}a_{4n-1}$ is the total of all the principal minors of order $4n-1$ of $\mathcal{L}_A$, we have

$$\begin{aligned}(-1)^{4n-1}a_{4n-1} &= \sum_{p=1}^{4n} det\mathcal{L}_A[p] = \sum_{p=4,p\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p] + \sum_{p=1,p\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p] \\ &\quad + \sum_{p=2,p\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p] + \sum_{p=3,p\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p].\end{aligned}$$

where

$$\begin{aligned}\sum_{p=4,p\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p] &= \sum_{p=4,p\equiv 0(mod4)}^{4n} (r^0_{p-1}r^0_{4n-p} - \frac{1}{9}r^1_{p-2}r^0_{4n-p-1});\\ \sum_{p=1,p\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p] &= \sum_{p=1,p\equiv 1(mod4)}^{4n-3} (r^0_{p-1}r^1_{4n-p} - \frac{1}{9}r^1_{p-2}r^1_{4n-p-1});\\ \sum_{p=2,p\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p] &= \sum_{p=2,p\equiv 2(mod4)}^{4n-2} (r^0_{p-1}r^2_{4n-p} - \frac{1}{9}r^1_{p-2}r^2_{4n-p-1});\\ \sum_{p=3,p\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p] &= \sum_{p=3,p\equiv 3(mod4)}^{4n-1} (r^0_{p-1}r^3_{4n-p} - \frac{1}{9}r^1_{p-2}r^3_{4n-p-1}).\end{aligned}$$

By Fact 1-2, we have

$$\begin{aligned}\sum_{p=4,p\equiv 0(mod4)}^{4n} (r^0_{p-1}r^0_{4n-p} - \frac{1}{9}r^1_{p-2}r^0_{4n-p-1}) &= \sum_{p=4,p\equiv 0(mod4)}^{4n} \Big[\frac{p}{12}\Big(\frac{1}{36}\Big)^{\frac{p-4}{4}}(4n-p+1)\frac{1}{12}\Big(\frac{1}{36}\Big)^{\frac{4n-p}{4}} \\ &\quad -\frac{1}{9}(p-1)\frac{1}{4}\Big(\frac{1}{36}\Big)^{\frac{p-4}{4}}\frac{1}{12}(4n-p)\Big(\frac{1}{36}\Big)^{\frac{4n-p-4}{4}}\Big] \\ &= \sum_{p=4,p\equiv 0(mod4)}^{4n} 12n\Big(\frac{1}{36}\Big)^n \\ &= 12n^2\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Similarly, by Fact 1-4 we can get

$$\sum_{p=1,p\equiv 1(mod4)}^{4n-3} (r^0_{p-1}r^1_{4n-p} - \frac{1}{9}r^1_{p-2}r^1_{4n-p-1}) = 12n^2\Big(\frac{1}{36}\Big)^n;$$

$$\sum_{p=2,p\equiv 2(mod4)}^{4n-2} (r^0_{p-1}r^2_{4n-p} - \frac{1}{9}r^1_{p-2}r^2_{4n-p-1}) = 8n^2\Big(\frac{1}{36}\Big)^n;$$

$$\sum_{p=3,p\equiv 3(mod4)}^{4n-1} (r^0_{p-1}r^3_{4n-p} - \frac{1}{9}r^1_{p-2}r^3_{4n-p-1}) = 8n^2\Big(\frac{1}{36}\Big)^n.$$

Hence, according to the above results, we have

$$(-1)^{4n-1}a_{4n-1} = \sum_{p=1}^{4n} det\mathcal{L}_A[p] = 40n^2\Big(\frac{1}{36}\Big)^n.$$

The proof of Claim 1 completed. ∎

Claim 2. $(-1)^{4n-2}a_{4n-2} = \frac{2}{3}(200n^4 - 11n^2)\Big(\frac{1}{36}\Big)^n$.

Proof of Claim 2. It$'$s not hard to see that $(-1)^{4n-2}a_{4n-2}$ is the total of the those principal minors $\mathcal{L}_A$, which have (4n-2) rows and columns. Thus we have

$$(-1)^{4n-2}a_{4n-2} = \sum_{1\le i<j\le 4n} det\mathcal{L}_A[p,q]. \tag{4.4}$$

By equation (4.4), it can be seen that the change of i and j values will lead to different $det\mathcal{L}_A[p,q]$ results. Therefore, we will choose different p and q to list the following equations.

$$\begin{aligned}
\sum_{1\le p<q\le 4n} det\mathcal{L}_A[p,q] = & \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 3(mod4)}^{4n-2}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
& + \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q].
\end{aligned}$$

where by Fact 1-4, we can compute the following results
Case 1.

$$\begin{aligned}
\sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] &= \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 0(mod4)}^{4n} (r^0_{p-1}r^0_{q-p-1}r^0_{4n-q} - \frac{1}{9}r^1_{p-2}r^0_{q-p-1}r^0_{4n-q-1}) \\
&= \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 0(mod4)}^{4n} 9(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n \\
&= 12(n^4 - n^2)\Big(\frac{1}{36}\Big)^n.
\end{aligned}$$

Case 2.

$$\sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] = \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 1(mod4)}^{4n-3} (r^0_{p-1}r^0_{q-p-1}r^1_{4n-q} - \frac{1}{9}r^1_{p-2}r^0_{q-p-1}r^1_{4n-q-1})$$
$$= \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 1(mod4)}^{4n-3} 9(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n$$
$$= \frac{3}{2}(8n^4-12n^3+n^2+3n)\Big(\frac{1}{36}\Big)^n.$$

Case 3.

$$\sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] = \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 2(mod4)}^{4n-2} (r^0_{p-1}r^0_{q-p-1}r^2_{4n-q} - \frac{1}{9}r^1_{p-2}r^0_{q-p-1}r^2_{4n-q-1})$$
$$= \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 2(mod4)}^{4n-2} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n$$
$$= (8n^4+8n^3+4n^2+4n)\Big(\frac{1}{36}\Big)^n.$$

Case 4.

$$\sum_{p\equiv 0(m0d4)}^{4n-4}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] = \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 3(mod4)}^{4n-1} (r^0_{p-1}r^0_{q-p-1}r^3_{4n-q} - \frac{1}{9}r^1_{p-2}r^0_{q-p-1}r^3_{4n-q-1})$$
$$= \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 3(mod4)}^{4n-1} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n$$
$$= (8n^4-4n^3+n^2-5n)\Big(\frac{1}{36}\Big)^n.$$

Case 5.

$$\sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] = \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 0(mod4)}^{4n} (r^0_{p-1}r^1_{q-p-1}r^0_{4n-q} - \frac{1}{9}r^1_{p-2}r^1_{q-p-1}r^0_{4n-q-1})$$
$$= \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 0(mod4)}^{4n} 9(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n$$
$$= \frac{3}{2}(8n^4+12n^3+n^2-3n)\Big(\frac{1}{36}\Big)^n.$$

Case 6.

$$\sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] = \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 1(mod4)}^{4n-3} (r^0_{p-1}r^1_{q-p-1}r^1_{4n-q} - \frac{1}{9}r^1_{p-2}r^1_{q-p-1}r^1_{4n-q-1})$$
$$= \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 1(mod4)}^{4n-3} 9(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n$$
$$= 12(n^4-n^2)\Big(\frac{1}{36}\Big)^n.$$

Case 7.

$$\begin{aligned}\sum_{p\equiv1(mod4)}^{4n-3}\sum_{q\equiv2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] &= \sum_{p\equiv1(mod4)}^{4n-3}\sum_{q\equiv2(mod4)}^{4n-2}(r^0_{p-1}r^1_{q-p-1}r^2_{4n-q}-\frac{1}{9}r^1_{p-2}r^1_{q-p-1}r^2_{4n-q-1})\\ &= \sum_{p\equiv1(mod4)}^{4n-3}\sum_{q\equiv2(mod4)}^{4n-2} 9(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= (8n^4+4n^3+n^2+5n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 8.

$$\begin{aligned}\sum_{p\equiv1(mod4)}^{4n-3}\sum_{q\equiv3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] &= \sum_{p\equiv1(mod4)}^{4n-3}\sum_{q\equiv3(mod4)}^{4n-1}(r^0_{p-1}r^1_{q-p-1}r^3_{4n-q}-\frac{1}{9}r^1_{p-2}r^1_{q-p-1}r^3_{4n-q-1})\\ &= \sum_{p\equiv1(mod4)}^{4n-3}\sum_{q\equiv3(mod4)}^{4n-1} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= (8n^4-8n^3+4n^2-4n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 9.

$$\begin{aligned}\sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv0(mod4)}^{4n} det\mathcal{L}_A[p,q] &= \sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv0(mod4)}^{4n}(r^0_{p-1}r^2_{q-p-1}r^0_{4n-q}-\frac{1}{9}r^1_{p-2}r^2_{q-p-1}r^0_{4n-q-1})\\ &= \sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv0(mod4)}^{4n} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= (8n^4+8n^3+4n^2+4n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 10.

$$\begin{aligned}\sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] &= \sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv1(mod4)}^{4n-3}(r^0_{p-1}r^2_{q-p-1}r^1_{4n-q}-\frac{1}{9}r^1_{p-2}r^2_{q-p-1}r^1_{4n-q-1})\\ &= \sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv1(mod4)}^{4n-3} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= (8n^4-4n^3+n^2-5n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 11.

$$\begin{aligned}\sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] &= \sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv2(mod4)}^{4n-2}(r^0_{p-1}r^2_{q-p-1}r^2_{4n-q}-\frac{1}{9}r^1_{p-2}r^2_{q-p-1}r^2_{4n-q-1})\\ &= \sum_{p\equiv2(mod4)}^{4n-2}\sum_{q\equiv2(mod4)}^{4n-2} 4(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= \frac{16}{3}(n^4-n^2)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 12.

$$\begin{aligned}\sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] &= \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 3(mod4)}^{4n-1}(r^0_{p-1}r^2_{q-p-1}r^3_{4n-q}-\frac{1}{9}r^1_{p-2}r^2_{q-p-1}r^3_{4n-q-1})\\ &= \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 3(mod4)}^{4n-1} 4(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= \frac{2}{3}(8n^4+4n^3+n^2+5n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 13.

$$\begin{aligned}\sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 0(mod4)}^{4n}(r^0_{p-1}r^3_{q-p-1}r^0_{4n-q}-\frac{1}{9}r^1_{p-2}r^3_{q-p-1}r^0_{4n-q-1})\\ &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 0(mod4)}^{4n} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= (8n^4+4n^3+n^2+5n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 14.

$$\begin{aligned}\sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 1(mod4)}^{4n-3}(r^0_{p-1}r^3_{q-p-1}r^1_{4n-q}-\frac{1}{9}r^1_{p-2}r^3_{q-p-1}r^1_{4n-q-1})\\ &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 1(mod4)}^{4n-3} 6(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= (8n^4-8n^3+4n^2-4n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 15.

$$\begin{aligned}\sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 2(mod4)}^{4n-2}(r^0_{p-1}r^3_{q-p-1}r^2_{4n-q}-\frac{1}{9}r^1_{p-2}r^3_{q-p-1}r^2_{4n-q-1})\\ &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 2(mod4)}^{4n-2} 4(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= \frac{2}{3}(8n^4-4n^3+n^2-5n)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Case 16.

$$\begin{aligned}\sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 3(mod4)}^{4n-1}(r^0_{p-1}r^3_{q-p-1}r^3_{4n-q}-\frac{1}{9}r^1_{p-2}r^3_{q-p-1}r^3_{4n-q-1})\\ &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 2(mod4)}^{4n-2} 4(q-p)(4n-q+p)\Big(\frac{1}{36}\Big)^n\\ &= \frac{16}{3}(n^4-n^2)\Big(\frac{1}{36}\Big)^n.\end{aligned}$$

Then, according to the value of p, the above sixteen cases can be divided into the following four categories.

$$
\begin{aligned}
F_0 &= \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
&\quad + \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 0(mod4)}^{4n-4}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
&= \frac{1}{2}(80n^4 - 28n^3 - 11n^2 + 7n)\Big(\frac{1}{36}\Big)^n.
\end{aligned}
$$

$$
\begin{aligned}
F_1 &= \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
&\quad + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 1(mod4)}^{4n-3}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
&= \frac{1}{2}(80n^4 + 28n^3 - 11n^2 - 7n)\Big(\frac{1}{36}\Big)^n.
\end{aligned}
$$

$$
\begin{aligned}
F_2 &= \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
&\quad + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 2(mod4)}^{4n-2}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
&= \frac{1}{3}(80n^4 + 20n^3 + n^2 + 7n)\Big(\frac{1}{36}\Big)^n.
\end{aligned}
$$

$$
\begin{aligned}
F_3 &= \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 0(mod4)}^{4n} det\mathcal{L}_A[p,q] + \sum_{p\equiv 3(mod4)}^{4n-2}\sum_{q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_A[p,q] \\
&\quad + \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_A[p,q] + \sum_{p\equiv 3(mod4)}^{4n-1}\sum_{q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_A[p,q] \\
&= \frac{1}{3}(80n^4 - 20n^3 + 10n^2 - 7n)\Big(\frac{1}{36}\Big)^n.
\end{aligned}
$$

Substituting $F_0,\ F_1,\ F_2$, and F_3 to Equation(4.4), one has

$$(-1)^{4n-2}a_{4n-2} = F_0 + F_1 + F_2 + F_3 = \frac{2}{3}(200n^4 - 11n^2)\Big(\frac{1}{36}\Big)^n.$$

The result as desired. ∎

So substituting the results of Claim 1 and 2 into Equation (4.2) yields

$$
\begin{aligned}
\sum_{p=2}^{4n}\frac{1}{\eta_p} &= \frac{(-1)^{4n-2}a_{4n-2}}{(-1)^{4n-1}a_{4n-1}} \\
&= \frac{\frac{2}{3}(200n^4 - 11n^2)(\frac{1}{36})^n}{40n^2(\frac{1}{36})^n} \\
&= \frac{200n^2 - 11}{60}.
\end{aligned}
$$

Theorem 4.2.

$$\sum_{q=1}^{4n}\frac{1}{\varphi_q}=\frac{41n\sqrt{14}}{28}\Big[\frac{\big((15+4\sqrt{14})^n-(15-4\sqrt{14})^n\big)}{\big((15+4\sqrt{14})^n+(15-4\sqrt{14})^n\big)+2}\Big].$$

Proof. Let

$$\begin{aligned}P_{L_S}(x) &= det(xI-\mathcal{L}_S)=x^{4n}+b_1x^{4n-1}+\cdots+b_{4n-1}x+b_{4n}\\ &= x(x^{4n-1}+b_1x^{4n-2}+\cdots+b_{4n-2}x+b_{4n-1}),\ b_{4n-1}\neq 0.\end{aligned}$$

Then we can exactly get $\varphi_1,\varphi_2,...,\varphi_{4n}$ are the roots of the following equation

$$x^{4n-1}+b_1x^{4n-2}+\cdots+b_{4n-2}x+b_{4n-1}=0.$$

Based on the Vieta$^{'}$s theorem of $P_{\mathcal{L}_S}(x)$, one has

$$\sum_{q=1}^{4n}\frac{1}{\varphi_q}=\frac{(-1)^{4n-1}b_{4n-1}}{(-1)^{4n}b_{4n}}=\frac{(-1)^{4n-1}b_{4n-1}}{det\mathcal{L}_S}. \tag{4.5}$$

Before calculating $(-1)^{4n-1}b_{4n-1}$ and $det\mathcal{L}_S$, we must determine ith order principal submatrices,S_q^0, S_q^1, S_q^2 and S_q^3, which consists of the first q rows and columns of the matrices $\mathcal{L}_S^0$,$\mathcal{L}_S^1$,$\mathcal{L}_S^2$ and $\mathcal{L}_S^3$, respectively, $q=1,2,...,4n$. Let

$$\mathcal{L}_S^0=\begin{pmatrix}
\frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & & \\
\frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & & \\
 & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & \\
 & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & & \\
 & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & \\
 & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & \\
 & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \\
 & & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} \\
 & & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3}
\end{pmatrix}_{(4n)\times(4n)},$$

$$\mathcal{L}_S^1=\begin{pmatrix}
1 & \frac{-1}{2} & & & & & & & & & \\
\frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & & & \\
 & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & & & \\
 & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & \\
 & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & \\
 & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & \\
 & & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & \\
 & & & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} \\
 & & & & & & & & & \frac{-1}{3} & \frac{4}{3}
\end{pmatrix}_{(4n)\times(4n)},$$

$$
\mathcal{L}_S^2 = \begin{pmatrix}
1 & \frac{-1}{\sqrt{6}} & & & & & & & & & \\
\frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & & & & \\
 & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & & \\
 & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & \\
 & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & \\
 & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & \\
 & & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & \\
 & & & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} \\
 & & & & & & & & & \frac{-1}{\sqrt{6}} & 1
\end{pmatrix}_{(4n)\times(4n)},
$$

and

$$
\mathcal{L}_S^3 = \begin{pmatrix}
\frac{4}{3} & \frac{-1}{3} & & & & & & & & & \\
\frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & & & & & \\
 & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & & & & & & & \\
 & & \frac{-1}{2} & 1 & \frac{-1}{\sqrt{6}} & & & & & & \\
 & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & & & & \\
 & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & & & & \\
 & & & & & & \ddots & & & & \\
 & & & & & & \frac{-1}{\sqrt{6}} & \frac{4}{3} & \frac{-1}{3} & & \\
 & & & & & & & \frac{-1}{3} & \frac{4}{3} & \frac{-1}{\sqrt{6}} & \\
 & & & & & & & & \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} \\
 & & & & & & & & & \frac{-1}{2} & 1
\end{pmatrix}_{(4n)\times(4n)}.
$$

In this way, let´s start with the following facts.

Fact 5. For $1 \leq q \leq 4n$,

$$
s_q^0 = \begin{cases}
\left(\frac{4}{8} + \frac{9\sqrt{14}}{56}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q}{4}} + \left(\frac{4}{8} - \frac{9\sqrt{14}}{56}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q}{4}}, & if\ q \equiv 0(mod4); \\
\left(\frac{2}{3} + \frac{31\sqrt{14}}{168}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q-1}{4}} + \left(\frac{2}{3} - \frac{31\sqrt{14}}{168}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q-1}{4}}, & if\ q \equiv 1(mod4); \\
\left(\frac{7}{12} + \frac{53\sqrt{14}}{336}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q-2}{4}} + \left(\frac{7}{12} - \frac{53\sqrt{14}}{336}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q-2}{4}}, & if\ q \equiv 1(mod4); \\
\left(\frac{5}{12} + \frac{25\sqrt{14}}{224}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q-3}{4}} + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q-3}{4}}, & if\ q \equiv 3(mod4).
\end{cases}
$$

Fact 6. For $1 \leq q \leq 4n$,

$$
s_q^1 = \begin{cases}
\left(\frac{1}{2} + \frac{11\sqrt{14}}{56}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q}{4}} + \left(\frac{1}{2} - \frac{11\sqrt{14}}{56}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q}{4}}, & if\ q \equiv 0(mod4); \\
\left(\frac{1}{2} + \frac{17\sqrt{14}}{112}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q-1}{4}} + \left(\frac{1}{2} - \frac{17\sqrt{14}}{112}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q-1}{4}}, & if\ q \equiv 1(mod4); \\
\left(\frac{3}{8} + \frac{23\sqrt{14}}{224}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q-2}{4}} + \left(\frac{3}{8} - \frac{23\sqrt{14}}{224}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q-2}{4}}, & if\ q \equiv 1(mod4); \\
\left(\frac{5}{12} + \frac{25\sqrt{14}}{224}\right)\left(\frac{5}{12} + \frac{\sqrt{14}}{9}\right)^{\frac{q-3}{4}} + \left(\frac{5}{12} - \frac{25\sqrt{14}}{224}\right)\left(\frac{5}{12} - \frac{\sqrt{14}}{9}\right)^{\frac{q-3}{4}}, & if\ q \equiv 3(mod4).
\end{cases}
$$

Fact 7. For $1 \le q \le 4n$,

$$s_q^2 = \frac{7}{6} s_{q-2}^0 - \frac{1}{9} s_{q-3}^1.$$

Fact 8. For $1 \le q \le 4n$,

$$s_q^3 = \frac{4}{3} s_{q-1}^0 - \frac{1}{9} s_{q-2}^1.$$

Proof of Fact 5. Take $s_q^0 = \det S_q^0,\ s_q^1 = \det S_q^1,\ s_q^2 = \det S_q^2$ and $s_q^3 = \det S_q^3$. By direct calculation, it′s no difficult to get the following values, see Table 2.

Table 2: Initial value.

s_q^0	$Value$	s_q^0	$Value$	s_q^0	$Value$	s_q^0	$Value$
s_1^0	$\frac{4}{3}$	s_2^0	$\frac{7}{6}$	s_3^0	$\frac{5}{6}$	s_4^0	$\frac{33}{36}$
s_5^0	$\frac{61}{54}$	s_6^0	$\frac{211}{216}$	s_7^0	$\frac{25}{36}$	s_8^0	$\frac{989}{1296}$

For $4 \le q \le 4n$, we have the expansion-formula of the $detS_q^0$ with respect to its last row yields

$$s_q^0 = \begin{cases} \frac{4}{3} t_{q-1}^0 - \frac{1}{6} s_{q-2}^0, & if\ q \equiv 0(mod4); \\ \frac{4}{3} t_{q-1}^0 - \frac{1}{9} s_{q-2}^0, & if\ q \equiv 1(mod4); \\ s_{q-1}^0 - \frac{1}{6} s_{q-2}^0, & if\ q \equiv 2(mod4); \\ s_{q-1}^0 - \frac{1}{4} s_{q-2}^0, & if\ q \equiv 3(mod4). \end{cases}$$

For $1 \le q \le n$, let $A_q = s_{4q};\ 0 \le q \le n-1,\ B_q = s_{4q+1},\ C_q = s_{4q+2},\ D_q = s_{4q+3}$. Then we may obtain that

$$\begin{cases} A_q = \frac{4}{3} D_{q-1} - \frac{1}{6} C_{q-1}; \\ B_q = \frac{4}{3} A_q - \frac{1}{9} D_{q-1}; \\ C_q = B_q - \frac{1}{6} A_q; \\ D_q = C_q - \frac{1}{4} B_q. \end{cases} \tag{4.6}$$

From the first three equations in (4.6), one can get $A_q = \frac{12}{13} C_q + \frac{1}{78} C_{q-1}$. Next, substituting A_q to the third equation, one has $B_q = \frac{15}{13} C_q + \frac{1}{468} C_{q-1}$. Then substituting B_q to the fourth equation, we have $D_q = \frac{37}{52} C_q - \frac{1}{1872} C_{q-1}$. Finally, Substituting A_q and d_q to the first equation, one has $c_q - 30c_{q-1} + c_{q=2} = 0$. Thus

$$C_q = k_1 \Big(\frac{5}{12} + \frac{\sqrt{14}}{9} \Big)^q + k_2 \Big(\frac{5}{12} - \frac{\sqrt{14}}{9} \Big)^q.$$

In views of $C_0 = \frac{7}{6}, C_1 = \frac{211}{216}$, we have

$$\begin{cases} k_1 + k_2 = \frac{7}{6}; \\ k_1 \Big(\frac{5}{12} + \frac{\sqrt{14}}{9} \Big) + k_2 \Big(\frac{5}{12} - \frac{\sqrt{14}}{9} \Big) = \frac{211}{216}, \end{cases}$$

and

$$\begin{cases} k_1 = \Big(\frac{7}{12} + \frac{53\sqrt{14}}{336} \Big); \\ k_2 = \Big(\frac{7}{12} - \frac{53\sqrt{14}}{336} \Big). \end{cases}$$

Thus it is routine to deduce that

$$\begin{cases} A_q = \left(\frac{4}{8}+\frac{9\sqrt{14}}{56}\right)\left(\frac{5}{12}+\frac{\sqrt{14}}{9}\right)^q+\left(\frac{4}{8}-\frac{9\sqrt{14}}{56}\right)\left(\frac{5}{12}-\frac{\sqrt{14}}{9}\right)^q, & if\ q\equiv 0(mod4); \\ B_q = \left(\frac{2}{3}+\frac{31\sqrt{14}}{168}\right)\left(\frac{5}{12}+\frac{\sqrt{14}}{9}\right)^q+\left(\frac{2}{3}-\frac{31\sqrt{14}}{168}\right)\left(\frac{5}{12}-\frac{\sqrt{14}}{9}\right)^q, & if\ q\equiv 1(mod4); \\ C_q = \left(\frac{7}{12}+\frac{53\sqrt{14}}{336}\right)\left(\frac{5}{12}+\frac{\sqrt{14}}{9}\right)^q+\left(\frac{7}{12}-\frac{53\sqrt{14}}{336}\right)\left(\frac{5}{12}-\frac{\sqrt{14}}{9}\right)^q, & if\ q\equiv 1(mod4); \\ D_q = \left(\frac{5}{12}+\frac{25\sqrt{14}}{224}\right)\left(\frac{5}{12}+\frac{\sqrt{14}}{9}\right)^q+\left(\frac{5}{12}-\frac{25\sqrt{14}}{224}\right)\left(\frac{5}{12}-\frac{\sqrt{14}}{9}\right)^q, & if\ q\equiv 3(mod4). \end{cases}$$

The result as desired. ■

In the same way, we can quickly prove the result of fact 6.

Then we expand $dets_q^2$ and s_q^3 according to the properties of determinant, and we can get facts 7 and 8.

Now by exploiting the property of determinant, we can get

$$\begin{aligned} \det \mathcal{L}_S &= \begin{vmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{-1}{\sqrt{6}} \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{6}} & \frac{4}{3} \end{vmatrix}_{4n} \\ &= \begin{vmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \frac{-1}{\sqrt{6}} \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{6}} & \frac{4}{3} \end{vmatrix}_{4n} + \begin{vmatrix} \frac{4}{3} & \frac{-1}{\sqrt{6}} & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{-1}{\sqrt{6}} & 1 & \frac{-1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{-1}{2} & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & \cdots & 0 & \frac{-1}{\sqrt{6}} & 0 \end{vmatrix}_{4n} \\ &= s_{4n}^0 - \frac{1}{9}s_{4n-2}^1 + 2(\frac{1}{36})^n \end{aligned}$$

Together with Facts 1 and 2, we can obtain one interesting Claim.

Claim 3. $det\mathcal{L}_S = \left(\frac{5}{12}+\frac{\sqrt{14}}{9}\right)^n + \left(\frac{5}{12}-\frac{\sqrt{14}}{9}\right)^n + 2\left(\frac{1}{36}\right)^n.$

Then we're going to concentrate on calculating $(-1)^{4n-1}b_{4n-1}$.

Claim 4. $(-1)^{4n-1}b_{4n-1} = \frac{41n\sqrt{14}}{28}\Big[\frac{\left((15+4\sqrt{14})^n-(15-4\sqrt{14})^n\right)}{\left((15+4\sqrt{14})^n+(15-4\sqrt{14})^n\right)+2}\Big].$

Proof. Since the $(-1)^{4n-1}b_{4n-1}$ is the total of all the principal minors of order $4n-1$ of $\mathcal{L}_S$, we have

$$\begin{aligned} (-1)^{4n-1}b_{4n-1} = \sum_{q=1}^{4n} det\mathcal{L}_S[q] = &\sum_{q=4,q\equiv 0(mod4)}^{4n} det\mathcal{L}_S[q] + \sum_{q=1,q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_S[q] \\ &+ \sum_{q=2,q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_S[q] + \sum_{q=3,q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_S[q]. \end{aligned}$$

where

$$det\mathcal{L}_S[q]=\begin{cases} s^0_{q-1}s^0_{4n-q}-\frac{1}{9}s^1_{q-2}s^0_{4n-q-1}, & if\ q\equiv 0(mod4);\\ s^0_{q-1}s^1_{4n-q}-\frac{1}{9}s^1_{q-2}s^1_{4n-q-1}, & if\ q\equiv 1(mod4);\\ s^0_{q-1}s^2_{4n-q}-\frac{1}{9}s^1_{q-2}s^2_{4n-q-1}, & if\ q\equiv 2(mod4);\\ s^0_{q-1}s^3_{4n-q}-\frac{1}{9}s^1_{q-2}s^3_{4n-q-1}, & if\ q\equiv 3(mod4). \end{cases} \tag{4.7}$$

For$q\equiv 0(mod4)$ *and* $4\leq q\leq 4n-4$, in views of (4.7) and Fact 5-8, one gets

$$\begin{aligned} det\mathcal{L}_S[q] &= s^0_{q-1}s^0_{4n-q}-\frac{1}{9}s^1_{q-2}s^0_{4n-q-1}\\ &= \Big[\Big(\frac{5}{12}+\frac{25\sqrt{14}}{224}\Big)\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{\frac{j-4}{4}}+\Big(\frac{5}{12}-\frac{25\sqrt{14}}{224}\Big)\Big(\frac{5}{12}-\frac{\sqrt{14}}{9}\Big)^{\frac{j-4}{4}}\Big]\\ &\quad\times\Big[\Big(\frac{4}{8}+\frac{25\sqrt{14}}{56}\Big)\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{\frac{4n-j}{4}}+\Big(\frac{4}{8}-\frac{25\sqrt{14}}{56}\Big)\Big(\frac{5}{12}-\frac{\sqrt{14}}{9}\Big)^{\frac{4n-j}{4}}\Big]\\ &\quad-\frac{1}{9}\Big[\Big(\frac{3}{8}+\frac{23\sqrt{14}}{224}\Big)\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{\frac{j-4}{4}}+\Big(\frac{3}{8}-\frac{23\sqrt{14}}{224}\Big)\Big(\frac{5}{12}-\frac{\sqrt{14}}{9}\Big)^{\frac{j-4}{4}}\Big]\\ &\quad\times\Big[\Big(\frac{5}{12}+\frac{25\sqrt{14}}{224}\Big)\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{\frac{4n-j-4}{4}}+\Big(\frac{5}{12}-\frac{25\sqrt{14}}{224}\Big)\Big(\frac{5}{12}-\frac{\sqrt{14}}{9}\Big)^{\frac{4n-j-4}{4}}\Big]\\ &= \frac{15n\sqrt{14}}{56}\Big[\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}-\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}\Big]. \end{aligned}$$

Similarly, for$q\equiv 1(mod4)$ *and* $1\leq q\leq 4n-3$, we have

$$\begin{aligned} \sum_{q=1,q\equiv 1(mod4)}^{4n-3} det\mathcal{L}_S[q] &= s^0_{q-1}s^1_{4n-q}-\frac{1}{9}s^1_{q-2}s^1_{4n-q-1}\\ &= \frac{15n\sqrt{14}}{56}\Big[\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}-\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}\Big]. \end{aligned}$$

For$q\equiv 2(mod4)$ *and* $2\leq q\leq 4n-2$, we have

$$\begin{aligned} \sum_{q=2,q\equiv 2(mod4)}^{4n-2} det\mathcal{L}_S[q] &= s^0_{q-1}s^2_{4n-q}-\frac{1}{9}s^1_{q-2}s^2_{4n-q-1}\\ &= \frac{13n\sqrt{14}}{28}\Big[\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}-\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}\Big]. \end{aligned}$$

For$q\equiv 3(mod4)$ *and* $3\leq q\leq 4n-1$, we have

$$\begin{aligned} \sum_{q=3,q\equiv 3(mod4)}^{4n-1} det\mathcal{L}_S[q] &= s^0_{q-1}s^3_{4n-q}-\frac{1}{9}s^1_{q-2}s^3_{4n-q-1}\\ &= \frac{13n\sqrt{14}}{28}\Big[\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}-\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}\Big]. \end{aligned}$$

Thus, one has the following equation

$$(-1)^{4n-1}b_{4n-1}=\frac{41n\sqrt{14}}{28}\Big[\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}-\Big(\frac{5}{12}+\frac{\sqrt{14}}{9}\Big)^{n}\Big].$$

Therefore, substituting the results of claim 3 and 4 into (4.5) can be obtained

$$\sum_{q=1}^{4n}\frac{1}{\varphi_q}=\frac{41n\sqrt{14}}{28}\Big[\frac{\big((15+4\sqrt{14})^n-(15-4\sqrt{14})^n\big)}{\big((15+4\sqrt{14})^n+(15-4\sqrt{14})^n\big)+2}\Big].$$

The result as desired. ■

Note that $|E_{MQ_n}(8,4)| = 10n$. Take the results of Theorems 4.1 and 4.2 to Lemma 2.2 (a) and (b), we can immediately get the following two Theorems.

Theorem 4.3. *Let $MQ_n(8,4)$ be a möbius graph with n octagonal and n quadrilateral. Then*

$$\begin{aligned} Kc(MQ_n(8,4)) &= \sum_{p=2}^{4n} \frac{1}{\eta_p} + \sum_{q=1}^{4n} \frac{1}{\varphi_q} \\ &= \frac{200n^2 - 11}{60} + \frac{41n\sqrt{14}}{28}\Big[\frac{\big((15+4\sqrt{14})^n - (15-4\sqrt{14})^n\big)}{\big((15+4\sqrt{14})^n + (15-4\sqrt{14})^n\big)+2}\Big]. \end{aligned}$$

Theorem 4.4. *Let $MQ_n(8,4)$ be a möbius graph with n octagonal and n quadrilateral. Then*

$$\begin{aligned} Kf^*(MQ_n(8,4)) &= 20n\Big(\sum_{p=2}^{4n} \frac{1}{\eta_p} + \sum_{q=1}^{4n} \frac{1}{\varphi_q}\Big) \\ &= 20n\Big(\frac{200n^2 - 11}{60} + \frac{41n\sqrt{14}}{28}\Big[\frac{\big((15+4\sqrt{14})^n - (15-4\sqrt{14})^n\big)}{\big((15+4\sqrt{14})^n + (15-4\sqrt{14})^n\big)+2}\Big]\Big) \\ &= \frac{200n^3 - 11n}{3} + 20n\varrho(n), \end{aligned}$$

where$\varrho(n) = \frac{41n\sqrt{14}}{28}\Big[\frac{\big((15+4\sqrt{14})^n - (15-4\sqrt{14})^n\big)}{\big((15+4\sqrt{14})^n + (15-4\sqrt{14})^n\big)+2}\Big]$.

The degree-Kirchhoff indices of Möbius graph of linear octagonal-quadrilateral networks, see Table 3.

Table 3: The degree-Kirchhoff indices of $MQ_1(8,4), MQ_2(8,4), ..., MQ_{16}(8,4)$.

G	$Kf^*(G)$	G	$Kf^*(G)$	G	$Kf^*(G)$	G	$Kf^*(G)$
$MQ_1(8,4)$	165.50	$MQ_5(8,4)$	11054.43	$MQ_9(8,4)$	57442.75	$MQ_{13}(8,4)$	164937.53
$MQ_2(8,4)$	963.33	$MQ_6(8,4)$	18322.78	$MQ_{10}(8,4)$	77587.71	$MQ_{14}(8,4)$	204359.11
$MQ_3(8,4)$	2775.12	$MQ_7(8,4)$	28210.28	$MQ_{11}(8,4)$	101951.83	$MQ_{15}(8,4)$	249599.85
$MQ_4(8,4)$	6005.23	$MQ_8(8,4)$	41116.93	$MQ_{12}(8,4)$	130935.10	$MQ_{16}(8,4)$	301059.74

Finally, we will concentrate on calculate the complexity of $MQ_n(8,4)$.

Theorem 4.5. *Let $MQ_n(8,4)$ denote a Möbius graph of linear octagonal-quadrilateral networks of length $n \geq 2$. Then*

$$\tau(MQ_n(8,4)) = 4n\Big((15+4\sqrt{14})^n + (15-\sqrt{14})^n + 2\Big).$$

Proof.

By Claim 1, one can get

$$\prod_{p=2}^{4n} \eta_p = (-1)^{4n-1} a_{4n-1} = 40n^2(\frac{1}{36})^n.$$

Similarly, according to Claim 3, we have

$$\prod_{q=1}^{4n} \varphi_q = det\mathcal{L}_S = \Big(\frac{5}{12} + \frac{\sqrt{14}}{9}\Big)^n + \Big(\frac{5}{12} - \frac{\sqrt{14}}{9}\Big)^n + 2\Big(\frac{1}{36}\Big)^n.$$

Note that $\prod_{i=1}^{8n} d_i(MQ_n) = 2^{4n}3^{4n}$ and $|E_{MQ_n}(8,4)| = 10n$. By lemma 2.4, one gets

$$\tau(Q_n(8,4)) = \frac{1}{10n}\prod_{p=2}^{4n}\eta_p \cdot \prod_{q=1}^{4n}\varphi_q = 4n\Big((15+4\sqrt{14})^n + (15-\sqrt{14})^n + 2\Big).$$

This completes the proof. ■

Thus we can get the complexity of $MQ_n(8,4)$, which are listed in Table 4.

Table 4: The complexity of $Q_1, Q_2, ..., Q_{10}$.

G	$\tau(G)$	G	$\tau(G)$
Q_1	128	Q_5	17379554400
Q_2	7200	Q_7	607607778176
Q_3	322944	Q_8	20809093939328
Q_4	12902464	Q_9	701525710449792
Q_5	483303040	Q_{10}	23358178980900000

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