The explicit formula for solution of wave differential equation with fractional derivatives in the multi-dimensional space

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April 05, 2024

Abstract

This paper devoted to the obtaining the explicit solution of n-dimensional wave equation with Gerasimov–Caputo fractional derivative in the infinite domain with non-zero initial condition and vanishing condition at infinity. It is shown that this equation can be derived from the classical homogeneous hyperbolic integro-differential equation with memory in which the kernel is $t^{1-\alpha}E_{2-\alpha, 2-\alpha}(-t^{2-\alpha}), \ \alpha\in(1, 2),$ where $E_{\alpha, \beta}$ is the Mittag-Liffler function. Based on Laplace and Fourier transforms the properties of the Fox H-function and convolution theorem, explicit solution for the solution of the considered problem is obtained.

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The explicit formula for solution of wave differential equation with fractional derivatives in the multi-dimensional space

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Abstract. This paper devoted to the obtaining the explicit solution of n-dimensional wave equation with Gerasimov–Caputo fractional derivative in the infinite domain with non-zero initial condition and vanishing condition at infinity. It is shown that this equation can be derived from the classical homogeneous hyperbolic integro-differential equation with memory in which the kernel is $t^{1-\alpha}E_{2-\alpha,2-\alpha}(-t^{2-\alpha}),\ \alpha \in (1,2)$, where $E_{\alpha,\beta}$ is the Mittag-Liffler function. Based on Laplace and Fourier transforms the properties of the Fox H-function and convolution theorem, explicit solution for the solution of the considered problem is obtained.

Keywords: fractional wave equation; Gerasimov–Caputo fractional derivative; Laplace transform; Fourier transform; convolution theorem; explicit solution.

1 Introduction to the problem and its setting

The rapid development of fractional differential equations with various fractional derivatives was largely due to the discovered practical applications of fractional calculus, primarily in the physics of complex inhomogeneous media. Fractional differential equations are ideally suited for modeling anomalous processes occurring in systems with a fractal structure or having a power-law memory.

The study of dynamical systems that have fractal properties or power memory have important theoretical and practical value. The presence of memory in a dynamic system indicates the dependence of its the current state from a finite number of its previous states. It leads to nonlocal properties of dynamical systems, for example, in mechanics when describing the effect of aftereffect is known in viscoelastic media [1, 2], in materials science - fatigue of materials, characterized by the gradual accumulation of defects under the action of stresses, which leads to the destruction of the material [3], in the economy - the effects of dynamic memory in economic theory [4] and even in medicine [5].

Hereditary processes or processes with memory are dedicated to hereditary mechanics in the description of viscoelastic media and materials These processes characterize a state of a mechanical system that depends on its previous conditions. The mathematical apparatus for describing hereditary mechanics is the apparatus of integro-differential equations with a convolution integral terms, in which kernels are called functions of memory [6]. In papers [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] (see also the list of references in them), a wide class of inverse problems of determining these kernels from hyperbolic integro-differential equations was studied on the basis of an additional condition on the solution of the direct problem.

If the memory functions are given and are power-law, then we can go to other types of equations that are based on derivatives of fractional orders, properties of which are considered in books on fractional calculus [5, 20, 21]. The solvability of Cauchy problems and initial-boundary value problems for various types of linear fractional differential equations were investigated in the works [20, 22, 23, 24, 25, 26, 27]. To construct a solution of linear fractional differential equations of diffusion type, various methods and algorithms based on the Green's function, Fourier, Laplace, and Mellin integral transforms, a generalization of the method of separation of variables, reduction to Volterra-type integral equations, and several others were proposed. At the same time, there are practically no methods for obtaining analytical solutions of fractional wave differential equations with fractional derivatives.

In this paper, we consider the following $n-$dimensional integro-differential equation of the (modified) fractional wave equation

$$u_{tt}(x,t) + {}_0^C D_t^{\alpha} u - \Delta u(x,t) = f(x,t), \tag{1.1}$$

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which satisfies the initial and boundary conditions

$$u(x,0)=\varphi(x),\; u_t(x,0)=\psi(x), \quad \lim_{|x|\to\infty}\left(u,\nabla u\right)(x,t)=0, \quad t>0,\; x=(x_1,x_2,...,x_n)\in\mathbb{R}^n, \tag{1.2}$$

where the Gerasimov–Caputo fractional differential operator ${}_0^C D_t^\alpha$ of order $\alpha\in(1,2)$ is defined by [28]

$${}_0^C D_t^\alpha u(x,t) := \frac{1}{\Gamma(2-\alpha)}\int\limits_0^t \frac{u_{\tau\tau}(x,\tau)}{(t-\tau)^{\alpha-1}}d\tau,$$

Δ is the n-dimensional Laplace operator with respect to x and $\nabla=\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)$.

The main goal of this article is to obtain an analytical formula that gives a solution to problem (1.1) - (1.2).

Remark 1. *In Section 4, it will be shown that, under certain conditions, all the equations considered in the works [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] are reduced to the equation of the form (1.1) with $f(x,t)=0$.*

2 Preliminaries

In this section, we present well known definitions, lemmas and theorems that will be used for proof of main results.

Definition 1. *The Fox $H-$function is a generalized hypergeometric function, defined by means of the Mellin-Barnes type contour integral [29]*

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,A_j)_1^p\\(b_j,B_j)_1^q\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_\Omega \mathcal{H}_{p,q}^{m,n}(s)z^{-s}ds, \tag{2.1}$$

where

$$\mathcal{H}_{p,q}^{m,n}(s)=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_js)\prod\limits_{i=1}^{n}\Gamma(1-a_i-A_is)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod\limits_{i=n+1}^{p}\Gamma(a_i+A_is)}$$

with complex variable $z\neq 0$ and a contour Ω in the complex domain; the orders (m,n,p,q) are non-negative integers so that $0\le m\le q$, $0\le n\le p$, the parameters $A_i>0$, $B_j>0$ are positive and a_i, b_j, $i=1,...,p$; $j=1,...,q$ are arbitrary complex such that

$$A_i(b_j+l)\neq B_j(a_i-l'-1), \quad l,l'=0,1,2,...,\; i=1,...,n,\; j=1,...,m. \tag{2.2}$$

The details on the properties of the Fox's $H-$function and types of contour Ω can be found in [37], where its behavior is described in terms of the following parameter:

$$\kappa := \sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i.$$

Theorem 1. *Let κ is given and let the condition (2.2) be satisfied. If $\kappa\le 0$, then the $H-$function has the asymptotic expansion at infinity given by [29]*

$$H_{p,q}^{m,n}(z)=O(z^d), \quad |z|\to\infty,$$

where

$$d:=\min_{1\le j\le n}\left[\frac{\Re(a_j)-1}{A_j}\right].$$

$\Re(a_j)-$*denote the real part of the complex number* a_j.

The Mittag-Leffler functions $E_\alpha(z)$ and $E_{\alpha,\beta}(z)$ are defined by the following series:

$$E_\alpha(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}=:E_{\alpha,1}(z) \quad \text{and} \quad E_{\alpha,\beta}(z):=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},$$

respectively, where $\alpha, z, \beta\in\mathbb{C}$, $\Re(\alpha)>0$. These functions are natural extensions of the exponential, hyperbolic and trigonometric functions, since

$$E_1(z)=e^z,\quad E_2(z^2)=\cosh z,\quad E_2(-z^2)=\cos z,\quad E_{1,2}(z)=\frac{e^z-1}{z},\quad E_{2,2}(z^2)=\frac{\sinh z}{z}.$$

The three-parameter Mittag-Leffler function or Prabhakar function is [30]:

$$E^{\gamma}_{\alpha,\beta}:=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!}, \tag{2.3}$$

where $\alpha,\beta,\gamma,z\in\mathbb{C}$, $\Re(\alpha)>0$ and $(\gamma)_n$ denotes the Pochammer symbol or the shifted factorial defined by

$$(\gamma)_0=1, \qquad (\gamma)_n=\gamma(\gamma+1)...(\gamma+n-1), \qquad \gamma\neq 0.$$

Also we can write $(\gamma)_n\equiv\Gamma(\gamma+n)/\Gamma(\gamma)$, where $\Gamma(\gamma)$ is the Gamma function. We have following special cases: $E^1_{\alpha,\beta}(z)=E_{\alpha,\beta}(z)$ and $E^1_{\alpha,1}=E_\alpha(z)$.

Recall that the function (2.3) can be rewritten in terms of the Fox H-function as [29, 30]:

$$E^{\gamma}_{\alpha,\beta}(z)=\frac{1}{\Gamma(\gamma)}H^{1,1}_{1,2}\left[-z\left|^{(1-\gamma,1)}_{(0,1),(1-\beta,\alpha)}\right.\right],\quad \Re(\gamma)>0. \tag{2.4}$$

We define the integral operator $\mathcal{E}^{\gamma}_{\alpha,\beta,\omega;a+}$ as follows [29, 31]:

$$\left(\mathcal{E}^{\gamma}_{\alpha,\beta,\omega;a+}\varphi\right)(t):=\left(t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega t^\alpha)\right)*\varphi(t)=\int_a^t(t-\tau)^{\beta-1}E^{\gamma}_{\alpha,\beta}(\omega(t-\tau)^\alpha)\varphi(\tau)d\tau. \tag{2.5}$$

Note the integral operator (2.5) is nowadays known in literature as Prabhakar fractional integral.

Lemma 1. *The following Laplace transform of a three-parameter Mittag-Leffler function is true [29, 32]:*

$$L\left[t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\pm\omega t^\alpha)\right](s)=\int_0^\infty e^{-st}t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\pm\omega t^\alpha)dt=\frac{s^{\alpha\gamma-\beta}}{(s^\alpha\mp\omega)^\gamma},$$

where $|\omega/s^\alpha|<1$.

Lemma 2. *The Laplace transform of* $e^{-\lambda t}t^{\beta-1}E^{\gamma}_{\alpha,\beta}(\pm\omega t^\alpha)$ *is given by the following formula [32]:*

$$L\left[e^{-\lambda t}t^{\beta-1}E^{\gamma}_{\alpha\beta}(\pm\omega t^\alpha)\right](s)=\frac{(s+\lambda)^{\alpha\gamma-\beta}}{((s+\lambda)^\alpha\mp\omega)^\gamma},$$

where $\lambda\geq 0$, $|\omega/(s+\lambda)^\alpha|<1$.

In the case $\lambda=0$, Lemma 2 coincides with Lemma 1.

Lemma 3. *For arbitrary* $\alpha>0$, β *is an arbitrary complex number,* $\mu>0$ *and* $a\in\mathbb{R}$, *the following formula is valid [33]:*

$$\int_{\mathbb{R}^n}e^{i\xi\cdot x}E^{(m)}_{\alpha,\beta}(-a|\xi|^\mu)d\xi=(2\pi)^{n/2}|x|^{1-n/2}\int_0^\infty|\xi|^{n/2}E^{(m)}_{\alpha,\beta}(-a|\xi|^\mu)J_{\frac{n}{2}-1}(|x||\xi|)d|\xi|.$$

Here $J_{\frac{n}{2}-1}(\cdot)$ *is a Bessel function and* $E^{(m)}_{\alpha,\beta}(z)$ *denotes m-th derivatives of the Mittag-Leffler function. m-th derivatives of the Mittag-Leffler function can be expressed in terms of the Fox H-function as*

$$E^{(m)}_{\alpha,\beta}(z)=H^{1,1}_{1,2}\left[-z\left|^{(-m,1)}_{(0,1),(1-(\alpha m+\beta),\alpha)}\right.\right].$$

For solvability of an integral equation of the Volterra type with difference kernel it is true the following assertion [34]:

Lemma 4. *If* $\{k(t), r(t)\} \in L_1[0,T]$ *for a fixed* $T > 0$ *and* $k(t), r(t)$ *are connected by the integral equation*

$$r(t) = k(t) + \int_0^t k(t-\tau) r(\tau) d\tau, \qquad t \in [0,T], \tag{2.6}$$

then the solution of the integral equation

$$\varphi(t) = \int_0^t k(t-\tau)\varphi(\tau) d\tau + f(t), \qquad f(t) \in L_1[0,T] \tag{2.7}$$

is expressed by formula

$$\varphi(t) = \int_0^t r(t-\tau) f(\tau) d\tau + f(t). \tag{2.8}$$

Now, we present definitions of multidimensional Fourier, one-dimensional Laplace transforms and Riemann-Lebesgue lemma.

Definition 2. *The* $n-$*dimensional Fourier transform of a function* $f(x,\cdot)$ *of* $x \in \mathbb{R}^n$ *is defined by [20]*

$$F[f(x,\cdot)](\xi) = \tilde{f}(\xi,\cdot) := \int_{\mathbb{R}^n} f(x,\cdot) e^{-i\xi\cdot x} dx, \quad \xi \in \mathbb{R}^n$$

where

$$x \in \mathbb{R}^n, \qquad \xi \in \mathbb{R}^n, \qquad \xi \cdot x = \sum_{j=1}^n \xi_i \cdot x_i, \qquad dx = dx_1 dx_2 ... dx_n$$

while the corresponding inverse Fourier transform is given by the formula

$$F^{-1}[f(\xi,\cdot)](x) = \frac{1}{2\pi}\tilde{f}(-x,\cdot) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(\xi,\cdot) e^{i\xi\cdot x} d\xi, \quad x \in \mathbb{R}^n.$$

Definition 3. *The Laplace transform of a function* $u(\cdot,t)$ *with respect to the variable* $t \in \mathbb{R}_+ := (0,\infty)$ *is defined by [20]*

$$L[u(\cdot,t)](s) = \hat{u}(\cdot,s) := \int_0^\infty e^{-st} u(\cdot,t) dt, \qquad s \in \mathbb{C}.$$

The inverse Laplace transform is given for $t \in \mathbb{R}_+$ *by the formula*

$$L^{-1}[u(\cdot,s)](t) := \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} u(\cdot,s) ds, \quad \gamma = \Re(s) > \gamma_0 =: \inf|s|.$$

where $d\xi = d\xi_1 d\xi_2 ... d\xi_n$.

Lemma 5. *If* f *is* L_1 *integrable on* $\mathbb{R}^n$, *then the Fourier transform of* f *satisfies [20]*

$$\tilde{f}(\xi,\cdot) := \int_{\mathbb{R}^n} f(x,\cdot) e^{-i\xi\cdot x} dx \to 0, \quad as \quad |\xi| \to \infty.$$

We will use these above notations everywhere in this paper.

3 Explicit solution of the problem (1.1) and (1.2)

The unknown function $u(x,t)$ is required to be sufficiently well behaved to be treated with its derivatives $u_t(x,t), u_{tt}(x,t), u_{x_ix_i}(x,t), i=1,\dots,n$ by technique of Laplace (in t) and Fourier (in x) transforms. The given functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ are also assumed to have such properties, in addition, they are such that the integrals and the series in (3.1) converge.

Theorem 2. *The explicit solution of the problem (1.1) and (1.2) can be expressed by formula*

$$u(x,t)=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}\sum_{j=0}^{\infty}(-1)^j\left(\mathcal{E}^{j+1}_{2,\,(2-\alpha)j+2,\,-|\xi|^2;\,0+}\tilde{f}\right)(\xi,t)e^{i\xi\cdot x}d\xi+$$

$$+\int\limits_{\mathbb{R}^n}G_0(x-\xi,t)\varphi(\xi)d\xi+\int\limits_{\mathbb{R}^n}G_1(x-\xi,t)\psi(\xi)d\xi,, \tag{3.1}$$

where the Green functions $G_k(x,t)$, $k=0,1,$ are given by

$$G_0(x,t)=\frac{1}{2\pi^{n/2}|x|^n}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\Bigg(t^{(2-\alpha)j}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{matrix}(1+(2-\alpha)j,1)\\(n/2,1/2),(1+j,1/2)\end{matrix}\right]$$

$$+t^{(2-\alpha)(j+1)}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{matrix}(1+(2-\alpha)(j+1),1)\\(n/2,1/2),(1+j,1/2)\end{matrix}\right]\Bigg),$$

$$G_1(x,t)=\frac{1}{2\pi^{n/2}|x|^n}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\Bigg(t^{(2-\alpha)j+1}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{matrix}(2+(2-\alpha)j,1)\\(n/2,1/2),(1+j,1/2)\end{matrix}\right]$$

$$+t^{(2-\alpha)(j+1)+1}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{matrix}(2+(2-\alpha)(j+1),1)\\(n/2,1/2),(1+j,1/2)\end{matrix}\right]\Bigg).$$

where $x\in\mathbb{R}^n$, $\xi\in\mathbb{R}^n$.

Proof. Let $F[u(x,t)]:=\tilde{u}(\xi,t)$ be the Fourier transform of $u(x,t)$ with respect to variable x, and $L[u(x,t)]:=\hat{u}(x,s)$ be the Laplace transform of $u(x,t)$ with respect to variable t. In sequence, applying to the equation (1.1) the Laplace transform with respect to the time variable t and the Fourier transform with respect to the spatial variable x, we obtain the following equation:

$$s^2\hat{\tilde{u}}(\xi,s)-\tilde{u}(\xi,0)s-\tilde{u}_t(\xi,0)+s^{\alpha-2}\left[s^2\hat{\tilde{u}}(\xi,0)-\tilde{u}(\xi,0)s-\tilde{u}_t(\xi,0)\right]$$

$$=-|\xi|^2\hat{\tilde{u}}(\xi,s)+\hat{\tilde{f}}(\xi,s),\ \xi\in\mathbb{R}^n.$$

Taking into account the initial conditions from (1.2), this equation yields

$$\hat{\tilde{u}}(\xi,s)=\frac{1}{s^2+s^\alpha+|\xi|^2}\hat{\tilde{f}}(\xi,s)+\frac{s+s^{\alpha-1}}{s^2+s^\alpha+|\xi|^2}\tilde{\varphi}(\xi)+\frac{1+s^{\alpha-2}}{s^2+s^\alpha+|\xi|^2}\tilde{\psi}(\xi)$$

$$=:F(\xi,s)+\Phi(\xi,s)+\Psi(\xi,s). \tag{3.2}$$

We calculate the inverse Laplace and Fourier transforms of the function $\hat{\tilde{u}}(\xi,s)$ defined by (3.2). First, these operations we carry out for $F(\xi,s)$. It may be performed by using the equality

$$\frac{1}{s^2+s^\alpha+|\xi|^2}=\frac{s^{-\alpha}}{s^{2-\alpha}+1}\cdot\frac{1}{1+\frac{|\xi|^2s^{-\alpha}}{s^{2-\alpha}+1}} \tag{3.3}$$

and expanding the second factor on the right side of this expression into an infinitely decreasing geometric series:

$$\frac{1}{1+\frac{|\xi|^2s^{-\alpha}}{s^{2-\alpha}+1}}=\sum_{n=0}^{\infty}(-|\xi|^2)^n\frac{s^{-\alpha n}}{(s^{2-\alpha}+1)^n}$$

for $\left|\frac{|\xi|^2 s^{-\alpha}}{s^{2-\alpha}+1}\right| < 1$. On bases of (3.3) from last equality we have

$$\frac{1}{s^2+s^\alpha+|\xi|^2} = \sum_{n=0}^{\infty}(-|\xi|^2)^n \frac{s^{-\alpha(n+1)}}{(s^{2-\alpha}+1)^{n+1}}. \tag{3.4}$$

Then, according to Lemma 1, we note

$$\frac{s^{-\alpha(n+1)}}{(s^{2-\alpha}+1)^{(n+1)}} = L\left[t^{2n+1}E^{n+1}_{2-\alpha,2(n+1)}(-t^{2-\alpha})\right] \qquad \text{and}$$

$$\frac{1}{s^2+s^\alpha+|\xi|^2} = L\left[\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2n+1}E^{n+1}_{2-\alpha,2n+1}(-t^{2-\alpha})\right].$$

Taking these formulae into account, eventually $F(\xi,s)$ can be expressed as

$$F(\xi,s) = L\left[\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2n+1}E^{n+1}_{2-\alpha,2(n+1)}(-t^{2-\alpha})\right] L\left[\tilde{f}(\xi,t)\right]. \tag{3.5}$$

We now transform the functions $\Phi(\xi,s)$ and $\Psi(\xi,s)$. For this we note that the fractions at these functions as seen from (3.2) differ from (3.3) only by numerators. Because of this

$$\frac{s+s^{\alpha-1}}{s^2+s^\alpha+|\xi|^2} = \sum_{n=0}^{\infty}(-|\xi|^2)^n\left[\frac{s^{-\alpha(n+1)+1}}{(s^{2-\alpha}+1)^{n+1}} + \frac{s^{-\alpha n-1}}{(s^{2-\alpha}+1)^{n+1}}\right],$$

$$\frac{1+s^{\alpha-2}}{s^2+s^\alpha+|\xi|^2} = \sum_{n=0}^{\infty}(-|\xi|^2)^n\left[\frac{s^{-\alpha(n+1)}}{(s^{2-\alpha}+1)^{n+1}} + \frac{s^{-\alpha n-2}}{(s^{2-\alpha}+1)^{n+1}}\right].$$

In view of the last relations, applying Lemma 1 to the functions $\Phi(\xi,s)$ and $\Psi(\xi,s)$, we obtain

$$\Phi(\xi,s) = L\left[\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2n}E^{n+1}_{2-\alpha,2n+1}(-t^{2-\alpha})\right](s)\tilde{\varphi}(\xi)+$$

$$+L\left[\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2(n+1)-\alpha}E^{n+1}_{2-\alpha,2n+3-\alpha}(-t^{2-\alpha})\right](s)\tilde{\varphi}(\xi); \tag{3.6}$$

$$\Psi(\xi,s) = L\left[\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2n+1}E^{n+1}_{2-\alpha,2(n+1)}(-t^{2-\alpha})\right](s)\tilde{\psi}(\xi)+$$

$$+L\left[\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{(2n+3-\alpha)-\alpha}E^{n+1}_{2-\alpha,2(n+2)-\alpha}(-t^{2-\alpha})\right](s)\tilde{\psi}(\xi). \tag{3.7}$$

Further, in accordance with the Mittag-Leffler function definition (2.3), from equation (3.4) we get

$$\sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2n+1}E^{n+1}_{2-\alpha,2(n+1)}(-t^{2-\alpha}) = \sum_{n=0}^{\infty}(-|\xi|^2)^n t^{2n+1}\sum_{j=0}^{\infty}\frac{(n+1)_j}{\Gamma((2-\alpha)j+2(n+1))}\frac{(-t^{2-\alpha})^j}{j!}$$

$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)j+1}\frac{(j+1)_n}{\Gamma((2-\alpha)j+2(n+1))}\frac{(-|\xi|^2t^2)^n}{n!} = \sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)j+1}E^{j+1}_{2,(2-\alpha)j+2}(-|\xi|^2t^2).$$

By virtue of this fact we continue converting of the function $F(\xi,s)$ as

$$F(\xi,s) = L\left[\sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)j+1}E^{j+1}_{2,(2-\alpha)j+2}(-|\xi|^2t^2)\right]$$

$$\times L\left[\tilde{f}(\xi,t)\right] = L\left[\left(\sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)j+1} E^{j+1}_{2,(2-\alpha)j+2}(-|\xi|^2t^2)\right) * \tilde{f}(\xi,t)\right].$$

Taking into consideration the convolution property of the Laplace transform and the definition of integral operator $\mathcal{E}^{\gamma}_{\alpha,\beta,\omega;a+}\varphi$ by (2.4), the inverse Laplace transform of the function $F(\xi,s)$ from last relations can be obtained as follows:

$$L^{-1}\left[F(\xi,s)\right](t) = \sum_{j=0}^{\infty}(-1)^j\left(\mathcal{E}^{j+1}_{2,\,(2-\alpha)j+2,\,-|\xi|^2;\,0+}\tilde{f}\right)(\xi,t). \tag{3.8}$$

Analogically, the inverse Laplace transform of functions $\Phi(\xi,s)$ and $\Psi(\xi,s)$ by virtue of (3.6) and (3.7) can be expressed as

$$L^{-1}\left[\Phi(\xi,s)\right](t) =$$

$$= \sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)j} E^{j+1}_{2,\,(2-\alpha)j+1}(-|\xi|^2t^2)\tilde{\varphi}(\xi) + \sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)(j+1)} E^{j+1}_{2,\,(2-\alpha)(j+1)+1}(-|\xi|^2t^2)\tilde{\varphi}(\xi),$$

$$L^{-1}\left[\Psi(\xi,s)\right](t) =$$

$$= \sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)j+1} E^{j+1}_{2,\,(2-\alpha)j+2}(-|\xi|^2t^2)\tilde{\varphi}(\xi) + \sum_{j=0}^{\infty}(-1)^j t^{(2-\alpha)(j+1)+1} E^{j+1}_{2,\,(2-\alpha)(j+1)+2}(-|\xi|^2t^2)\tilde{\psi}(\xi).$$

Considering the relationship between the generalized Mittag-Leffler function and the Fox-H function (2.4), the last equalities can be rewritten in the form [29]

$$L^{-1}\left[\Phi(\xi,s)\right] = \sum_{j=0}^{\infty}(-1)^j\left(t^{(2-\alpha)j}\frac{1}{\Gamma(j+1)}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j,\,2)}\right.\right]\right.$$

$$\left.+t^{(2-\alpha)(j+1)}\frac{1}{\Gamma(j+1)}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1),2)}\right.\right]\right)\tilde{\varphi}(\xi)$$

$$= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j,\,2)}\right.\right] + t^{(2-\alpha)(j+1)}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1),2)}\right.\right]\right)$$

$$\times\tilde{\varphi}(\xi) =: \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j}\tilde{\Phi}_{1j}(\xi,t) + t^{(2-\alpha)(j+1)-1}\tilde{\Phi}_{2j}(\xi,t)\right)\tilde{\varphi}(\xi); \tag{3.8}$$

$$L^{-1}\left[\Psi(\xi,s)\right] = \sum_{j=0}^{\infty}(-1)^j\left(t^{(2-\alpha)j+1}\frac{1}{\Gamma(j+1)}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j-1,\,2)}\right.\right]\right.$$

$$\left.+t^{(2-\alpha)(j+1)+1}\frac{1}{\Gamma(j+1)}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1)-1,2)}\right.\right]\right)\tilde{\psi}(\xi)$$

$$= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j+1}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j-1,\,2)}\right.\right] + t^{(2-\alpha)(j+1)+1}\times\right.$$

$$\left.\times H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1)-1,\,2)}\right.\right]\right)\tilde{\psi}(\xi) =$$

$$= \sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j+1}\tilde{\Psi}_{1j}(\xi,t) + t^{(2-\alpha)(j+1)+1}\tilde{\Psi}_{2j}(\xi,t)\right)\tilde{\psi}(\xi). \tag{3.9}$$

In (3.8), (3.9) we introduced the following notations:

$$\tilde{\Phi}_{1j}(\xi,t) = H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j,\,2)}\right.\right],\ \tilde{\Phi}_{2j}(\xi,t) = H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1),2)}\right.\right], \tag{3.10}$$

$$\tilde{\Psi}_{1j}(\xi,t)=H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j-1,\,2)}\right.\right],\ \tilde{\Psi}_{2j}(\xi,t)=H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1)-1,\,2)}\right.\right]. \quad (3.11)$$

We also denote

$$\tilde{G}_0(\xi,t):=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j}\tilde{\Phi}_{1j}(\xi,t)+t^{(2-\alpha)(j+1)}\tilde{\Phi}_{2j}(\xi,t)\right), \quad (3.12)$$

$$\tilde{G}_1(\xi,t):=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j+1}\tilde{\Psi}_{1j}(\xi,t)+t^{(2-\alpha)(j+1)+1}\tilde{\Psi}_{2j}(\xi,t)\right). \quad (3.13)$$

Now we compute the inverse Fourier transform of relations (3.8) and (3.9). For this applying the inverse transform F^{-1} to equalities (3.10) and (3.11), we have

$$\Phi_{1j}(x,t)=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j,\,2)}\right.\right]e^{i\xi\cdot x}d\xi, \quad (3.14)$$

$$\Phi_{2j}(x,t)=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1),2)}\right.\right]e^{i\xi\cdot x}d\xi; \quad (3.15)$$

$$\Psi_{1j}(x,t)=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j-1,\,2)}\right.\right]e^{i\xi\cdot x}d\xi, \quad (3.16)$$

$$\Psi_{2j}(x,t)=\frac{1}{(2\pi)^n}\int\limits_{\mathbb{R}^n}H^{1,1}_{1,2}\left[|\xi|^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1)-1,\,2)}\right.\right]e^{i\xi\cdot x}d\xi. \quad (3.17)$$

Using Lemma 3, we obtain the following results from formulae (3.14)-(3.17)

$$\Phi_{1j}(x,t)=\frac{1}{(2\pi)^{n/2}}|x|^{1-\frac{n}{2}}\int\limits_0^{\infty}y^{n/2}H^{1,1}_{1,2}\left[y^2t^2\left|^{(-j,1)}_{(0,1),(-(2-\alpha)j,\,2)}\right.\right]\mathfrak{J}_{\frac{n}{2}-1}(|x|y)dy,$$

$$\Phi_{2j}(x,t)=\frac{1}{(2\pi)^{n/2}}|x|^{1-\frac{n}{2}}\int\limits_0^{\infty}y^{n/2}H^{1,1}_{1,2}\left[y^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1),2)}\right.\right]\mathfrak{J}_{\frac{n}{2}-1}(|x|y)dy;$$

$$\Psi_{1j}(x,t)=\frac{1}{(2\pi)^{n/2}}|x|^{1-\frac{n}{2}}\int\limits_0^{\infty}y^{n/2}H^{1,1}_{1,2}\left[y^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)j-1,2)}\right.\right]\mathfrak{J}_{\frac{n}{2}-1}(|x|y)dy,$$

$$\Psi_{2j}(x,t)=\frac{1}{(2\pi)^{n/2}}|x|^{1-\frac{n}{2}}\int\limits_0^{\infty}y^{n/2}H^{1,1}_{1,2}\left[y^2t^2\left|^{(-j,1)}_{(0,1),\,(-(2-\alpha)(j+1)-1,\,2)}\right.\right]\mathfrak{J}_{\frac{n}{2}-1}(|x|y)dy.$$

Taking into account a Hankel transform [29](p. 57) and the properties Fox-H function [29](pp. 11-13), the first function of the last ones can be written as

$$\Phi_{1j}(x,t)=\frac{1}{\pi^{n/2}|x|^n}H^{2,1}_{2,3}\left[\frac{|x|^2}{4t^2}\left|\begin{array}{c}(1,1),(1+(2-\alpha)j,2)\\(n/2,1),(1+j,1),(1,1)\end{array}\right.\right]=$$

$$=\frac{1}{\pi^{n/2}|x|^n}H^{2,0}_{1,2}\left[\frac{|x|^2}{4t^2}\left|\begin{array}{c}(1+(2-\alpha)j,2)\\(n/2,1),(1+j,1)\end{array}\right.\right]=$$

$$=\frac{1}{2\pi^{n/2}|x|^n}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\left|\begin{array}{c}(1+(2-\alpha)j,1)\\(n/2,1/2),(1+j,1/2)\end{array}\right.\right]. \quad (3.18)$$

By same arguments for Φ_{2j}, Ψ_{1j} and Ψ_{2j} we have

$$\Phi_{2j}(x,t)=\frac{1}{2\pi^{n/2}|x|^n}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(1+(2-\alpha)(j+1),1)\\(n/2,1/2),(1+j,1/2)\end{array}\right], \tag{3.19}$$

$$\Psi_{1j}(x,t)=\frac{1}{2\pi^{n/2}|x|^n}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(2+(2-\alpha)j,1)\\(n/2,1/2),(1+j,1/2)\end{array}\right], \tag{3.20}$$

$$\Psi_{2j}(x,t)=\frac{1}{2\pi^{n/2}|x|^n}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(2+(2-\alpha)(j+1),1)\\(n/2,1/2),(1+j,1/2)\end{array}\right]. \tag{3.21}$$

Now, applying the inverse Fourier transform to both sides of (3.12) and (3.13), and substituting into resulting equalities formulae (3.18)-(3.21), we get

$$G_0(x,t)=\frac{1}{2\pi^{n/2}|x|^n}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(1+(2-\alpha)j,1)\\(n/2,1/2),(1+j,1/2)\end{array}\right]+\right.$$
$$\left.+t^{(2-\alpha)(j+1)}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(1+(2-\alpha)(j+1),1)\\(n/2,1/2),(1+j,1/2)\end{array}\right]\right), \tag{3.22}$$

$$G_1(x,t)=\frac{1}{2\pi^{n/2}|x|^n}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\left(t^{(2-\alpha)j+1}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(2+(2-\alpha)j,1)\\(n/2,1/2),(1+j,1/2)\end{array}\right]+\right.$$
$$\left.+t^{(2-\alpha)(j+1)+1}H^{2,0}_{1,2}\left[\frac{|x|}{2t}\,\middle|\,\begin{array}{c}(2+(2-\alpha)(j+1),1)\\(n/2,1/2),(1+j,1/2)\end{array}\right]\right). \tag{3.23}$$

Continuing to convert the equalities (3.6) and (3.7) we can write formally

$$L^{-1}\left[\Phi(\xi,s)+\Psi(\xi,s)\right]=L^{-1}\Big(F\left[G_0(x,s)\right](\xi)\tilde{\varphi}(\xi)+F\left[G_1(x,s)\right](\xi)\tilde{\psi}(\xi)\Big). \tag{3.24}$$

In view of (3.8) and (3.24), applying an inverse Laplace transform to equation (3.2) we finally obtain

$$\tilde{u}(\xi,t)=\sum_{j=0}^{\infty}(-1)^j\left(\mathcal{E}^{j+1}_{2,(2-\alpha)j+2,-|\xi|^2;0+}\tilde{f}\right)(\xi,t)+F\left[G_0(x,t)\right](\xi)\varphi(\xi)+F\left[G_1(x,t)\right](\xi)\psi(\xi). \tag{3.25}$$

To equation (3.25) can be further applied inverse Fourier transform and Fourier convolution property in sequence. Accordingly, the Theorem 2 is proven.

Asymptotic of the function (3.1) at infinity. From the Theorem 1 and Lemma 5 implies following asymptotic expansion at infinity for the solution of the (1.1)-(1.2) Cauchy problem:

$$u(x,t)\sim|x|^{3-\alpha-n}e^{-|x|^{2-\alpha}}, \tag{3.26}$$

where $n\in\mathbb{N}$, $\alpha\in(1,2)$, $t>0$.

4 The integro-differential wave equation with the Mittag-Leffler function in the kernel

In this section we show the equivalence of one integro-differential wave equation with the Mittag-Leffler function in the kernel to the fractional wave equation.

Theorem 2. *The integro-differential wave equation*

$$u_{tt}-\triangle u+\int\limits_0^t k(t-\tau)\triangle u(x,\tau)d\tau=0,\ \ x\in\mathbb{R}^n,\ \ t>0 \tag{4.1}$$

with memory $k(t) = t^{1-\alpha}E_{2-\alpha,\ 2-\alpha}\left(-t^{2-\alpha}\right),\ \alpha \in (1,2)$, *is equivalent to the time-fractional wave equation*

$$u_{tt} + {}_0^C D_t^{\alpha} u - \triangle u(x,t) = 0. \tag{4.2}$$

Proof. Considering equation (4.1) as the Volterra integral equation of the second kind with respect to $\triangle u$ for fixed x and applying Lemma 4, we have

$$\triangle u = u_{tt} + \int\limits_0^t r(t-\tau)u_{\tau\tau}(x,\tau)d\tau, \tag{4.3}$$

where $r(t)$ is resolvent of $k(t)$ and it satisfies the integral equation (2.6)

We apply to both sides of (2.5) the Laplace and denoting by $K(s)$ and $R(s)$ the imagines of origins $k(t)$ and $r(t)$, respectively, obtain

$$R(s) = K(s) + K(s)R(s).$$

From this relation we get

$$K(s) = \frac{R(s)}{1+R(s)} = \frac{1}{s^{2-\alpha}+1},\ \Re(s) > 1.$$

Applying the inverse Laplace transformation to last equality (see [28])

$$k(t) = L^{-1}\left[K(s)\right] = L^{-1}\left[\frac{1}{s^{2-\alpha}+1}\right] = t^{1-\alpha}E_{2-\alpha,\ 2-\alpha}\left(-t^{2-\alpha}\right). \tag{4.4}$$

Thus, if we choose $k(t)$ by formula (4.4) then, (4.3) yields (4.2).

Remark 2. *Thus, the equation (4.1) with memory kernel* $k(t) = t^{1-\alpha}E_{2-\alpha,\ 2-\alpha}\left(-t^{2-\alpha}\right)$ *describes the homogeneous time-fractional wave equation (1.1).*

From this remark it follows that the solution of equation (4.1) with conditions (1.2) can be given by formula (3.1) for $f(x,t) = 0$.

5 Conclusions

In practices, using the different types of a memory kernel $k(t)$ in equation (4.1) it can be described a wide variety of physical phenomena with memory effects. In this work, it is shown that n-dimensional wave equation with Gerasimov–Caputo fractional derivative (1.1) with $f(x,t) = 0$ can be derived from the hyperbolic integro-differential equation (4.1) with memory kernel $t^{1-\alpha}E_{2-\alpha,\ 2-\alpha}\left(-t^{2-\alpha}\right)$. Based on the Laplace transform method to the time variable and Fourier transform to the spatial variable, the analytical explicit solution of initialЏboundary problem for the equation (1.1) is obtained. This solution includes the Prabhakar fractional integral and Fox H-functions.

Список литературы

[1] Boltzmann L. "Zur theorie der elastischen nachwirkung Annalen der Physik, 241:11, (1878), 430–432.

[2] Vronsky A.P. "The phenomenon of aftereffect in a solid"// Academy of Sciences of the USSR. Applied mathematics and mechanics,**5** (1) (1941), 31–56. [in Russian] [Vronskij A. P., "Javlenie posledejstvija v tverdom tele AN SSSR. Prikladnaja matematika i mehanika, **5** (1) (1941), 31–56].

[3] Gerasimov A.N. "Generalization of linear deformation laws and their application to problems of internal friction"// Academy of Sciences of the USSR. Applied Mathematics and Mechanics, **12** (1948), 529–539. [in Russian] [Gerasimov A. N., YObobshhenie linejnyh zakonov deformacii i ih prilozhenie k zadacham vnutrennego trenijaҰ, AN SSSR. Prikladnaja matematika i mehanika, **12** (1948), 529–539].

[4] Tarasova V.V., "Tarasov V.E. Economic Interpretation of Fractional Derivatives"// Progr. Fract. Differ. Appl. **3** (1) (2017) 1–6.

[5] Uchaikin V. V., "Fractional derivatives for physicists and engineers". V. I: Background and theory, Springer, Berlin, 2013, 373 p.

[6] Volterra V. "Sur les equations integro-differentielles et leurs applications". // Acta Mathematica. **35**, (1) (1912) 295–356.

[7] Durdiev D.K. "Global solvability of an inverse problem for an integro–differetial equation of electrodynamics // Differential Equations, **44** (7) (2008), 867–873.

[8] Durdiev D.K. "The problem of determining a function of memoty of a medium and the regular part of a pulsed source // Mat. Zametki, **86** (2) (2009), 202–212.

[9] Durdiev D.K., Rahmonov A.A. "Inverse problem for the system integro-differential equation Sh waves in a visco-elastic porous medium: Global solubility // TMF, **195** (3), (2018), 925–940.

[10] Durdiev D.K., Safarov Zh.Sh. "Inverse problem of determining the one-dimentional kernel of the viscoelasticity equation in a bounded domain // Mat. Zametki, **97** (6) (2015), 855–867.

[11] Durdiev D.K., Totieva Z.D. "The problem of determining the one-dimensional matrix kernel of the system of viscoelasticity equations // Mathematical Methods in the Applied Sciences, **41** (17) (2018), 8019–8032.

[12] Durdiev D.K., Totieva Z.D. "The problem of determining the one-dimensional kernel of viscoelasticity equation with a source of explosive type // Journal of Inverse and Ill-posed Problems, **28** (1) (2020), 43–52.

[13] Durdiev D.K., Rahmonov A.A., "A 2D kernel determination problem in a viscoelastic porous medium with a weakly horizontally inhomogeneity // Math Meth Appl Sci. **43** (1) (2020), 8776–8796.

[14] Janno J., von Wolfersdorf L. "An inverse problem for identification of a time- and space-dependent memory kernel in viscoelasticity // Inverse problems, **17** (1), (2001), 13–24.

[15] Janno J. "Recovering a time- and space-dependent kernel in a hyperbolic integro–differential equation from a restricted Dirichlet–to–Neymann operator // Electronic Journal of Differential Equations, **2004** (67),(2004), 1–16.

[16] Durdiev U.D. "A problem of identification of a special 2D memory kernel in an integro-differential hyperbolic equation // Eurasian J. Math. Comp. App., **7** (2) (2019), 4–19.

[17] Durdiev U.D., Totieva Z.D. "A problem of determining a special spatial part of 3D memory kernel in an integro-differential hyperbolic equation // Mathematical methods in Applied Sciences, **42** (18) 2019, 7440–7451.

[18] Lorenzi A. "An identification problem related to a nonlinear hyperbolic integro - differential equation // Nonlinear Analysis: Theory, Methods and Applications, No. 22 (1994), p. 297–321.

[19] Lorenzi A., Paparoni E. "Direct and inverse problems in the theory of materials with memory // Ren. Sem. Math. Univ., No. 87 (1992), 105–138.

[20] Kilbas A.A., Srivastava H.M., Trujillo. J.J. "Theory and Applications of Fractional Differential Equations". Amsterdam: Elsevier, (2006) 523 p.

[21] Shishkina E.L., Sitnik S.M. "Transmutations, singular and fractional differential equations with applications to mathematical physics Elsevier, Amsterdam, 2020.

[22] Kilbas A.A., Pierantozzi T., Trujillo J.J., Vazquez L. "On the solution of fractional evolution equations // J. Phys. A: Math. Gen. **37** (9), (2004) 3271–3283.

[23] Kilbas A.A., Trujillo J.J., Voroshilov A.A, "Cauchy-type problem for difussion-wave equation with the Riemann-Liouville partial derivative // Fract. Calc. Appl. Anal. **9** (2), (2005) 225–239.

[24] Voroshilov A.A, Kilbas A.A. "Cauchy problem for difussion-wave equation with the Caputo partial derivative // Diff. Eq. **42** (5), (2006) 599–609. [in Russian]

[25] Kochubei A.N., Eidelman S.D. "The fundamental solutions for the fractional diffusion-wave equation // J. Differential Equations, **199** (2), (2004) 211–255.

[26] Luchko Yu. "Some uniqueness and existence results for the initial-boundary-value problems for the generalized time-fractional diffusion equation // Comput. Math. Appl. **59** (5), (2010) 1766–1772.

[27] Mainardi F. "The fundamental solutions for the fractional diffusion-wave equation // Appl. Math. Lett. **9** (6), (1996) 25–28.

[28] Gorenflo R., Mainardi F. "Fractional calculus: integral and differential equations of fractional order // Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York, (1997), 223–276.

[29] Mathai A.M., Saxena R.K., Haubold H.J. "The H-function: Theory and Application Springer, Berlin/Heidelberg 2010, 268 p.

[30] Giusti A., Colombaro I., Garra R., at el., "Apractical guide to Prabhakar fractional calculus Fract. Calc. Appl. Anal. **23** (1), (2020) 9–54.

[31] Tomovski Ž., Sandev T. "Effects of a fractional friction with power-law memory kernel on string fibrations', // Comput. Math. Appl. **62**, (2011) 1554–1561.

[32] Tomovski Ž., Hilfer R., Srivastava N.M. "Fractional and operational calculus with generalized fractional derivatives operators and Mittag-Leffler type functions // Integral Transform. Spec. Funct. **21**, (2010) 797–814.

[33] Zhang Y., Benson D.A., Reeves D.M. "Time and space nonlocalities underlying fractional-derivative models: distinction and literature review of field applications // Adv. Water Resour. **32** (4), (2009) 561–581.

[34] Durdiev Durdimurod K., Shishkina Elina L., Sitnik Sergei M. "The explicit formula for solution of anomalous diffusion equation in the multi-dimensional space arXiv:2009.10594v1 [math. CA] 20 Sep 2020.