Uncertain optimal control problem of production and inventory under time-varying customer demand

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Abstract

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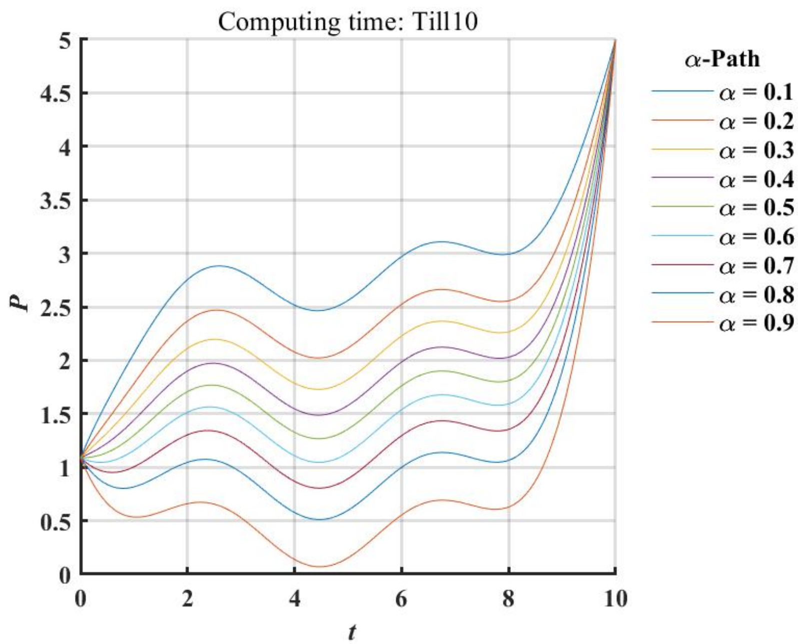
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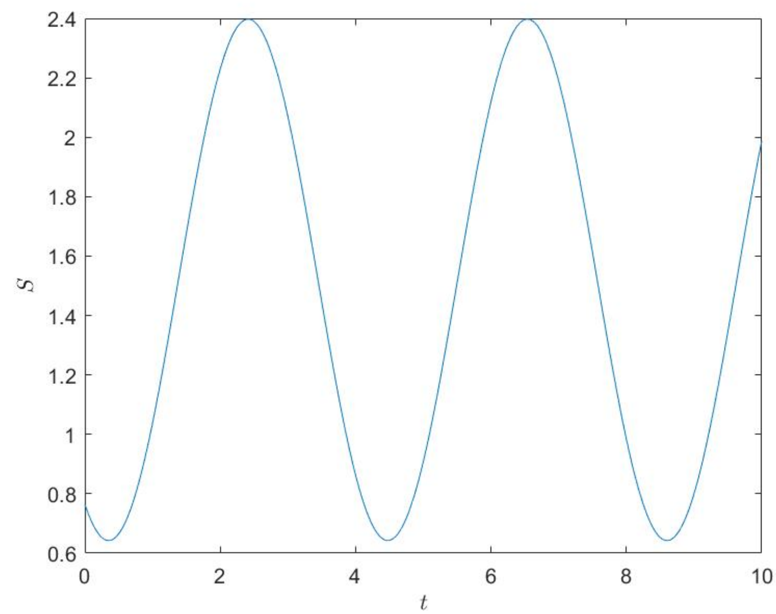
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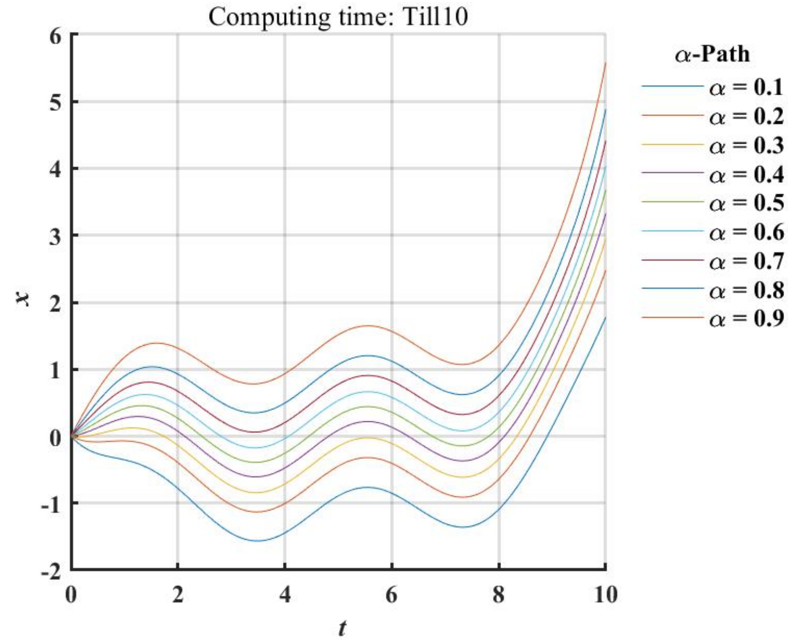
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x
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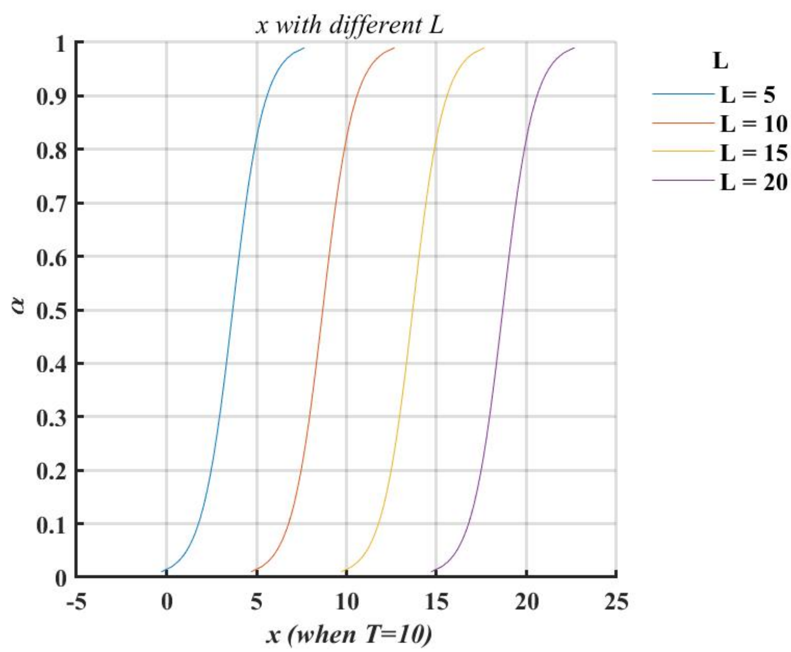
x with different L
α
x (when T=10)
L
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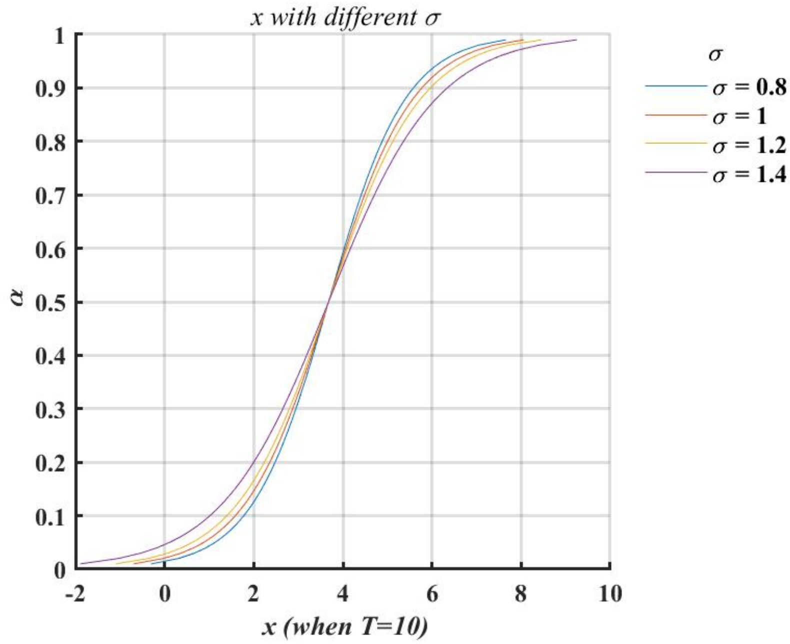
x with different σ
σ
σ = 0.8
σ = 1
σ = 1.2
σ = 1.4
α
x (when T=10)

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Abstract

In the dynamic landscape of global trade and logistics, the optimization of inventory control has risen as a crucial topic of academic discourse. Acknowledging that inventory management functions as the intersection of production and sales, these domains often manifest significant uncertainty, which consequently poses formidable challenges to the inventory management optimization. This paper leverages uncertainty theory in a novel approach to articulate the optimal production strategy for navigating time-varying demand with inherent fluctuations. And the inventory state equation, characterized by fluctuations, is proposed as a constraint condition. By factoring in the residual value of terminal inventory, a production inventory model catering to time-varying demand is devised, and the optimal production strategy is ascertained. Simultaneously, solutions along the α- Path are introduced to procure more intuitive numerical results. Finally, a case study of Chinese clothing sales was harnedded to substantiate the reliability of the model conclusion. This research amplifies the application of uncertainty theory to the optimization of production strategies, offering a novel perspective on inventory control in the ambit of uncertain factors. It provides a theoretical paradigm for companies with uncertain customer needs to orchestrate production, thereby bolstering the operational efficiency and profitability of the company.

Keywords: Optimization of Inventory Control, Uncertainty Theory, Cycle customer demand, Optimality Equation, α-Path

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1 Introduction

With the emergence of trade globalization and the widespread adoption of global logistics, optimal inventory control has ascended to become one of the most compelling academic topics today. Optimal control pertains to the pursuit of a control under given constraints to realize a maximum (or minimum) of a given system performance index. It is categorized into deterministic optimal control, stochastic optimal control, and uncertain optimal control, predicated on the consideration of uncertain factors. The optimal inventory control resides in the middle stage of actual production and maintains a strong correlation with upstream production and downstream sales. There are inherently many uncertain factors in the production and sales process, which consequently leads to significant uncertainty in optimal inventory control. This uncertainty can be perceived as a disturbance to the production inventory system, and this disturbance will induce some changes and effects on the production inventory plan. Therefore, it is imperative to consider uncertainty when studying optimal inventory control.

The current research on inventory control predominantly focuses on the dynamic inventory and pricing of perishable products. In contrast to the optimal control of random models, the uncertain optimal control eradicates the dependence on confidence. The consideration of external disturbances as random variables necessitates the use of probability theory tools, while the underlying assumption of using probability theory is that the obtained probability distribution is sufficiently aligned with the true frequency. However, in the scenario of inadequate data support, relying solely on experience to estimate confidence is untenable, which makes uncertainty theory emerges as a more reliable tool.

The salient contributions of this article encompass the following aspects: It contemplates time-varying customer demand, which exhibits temporal variations and typically possesses characteristics of instantaneity and periodicity. It delves into the optimal productivity of a company predicated on customer demand when the demand rate fluctuates periodically. This article provides a robust framework for companies to strategically devise production strategies to cater to diverse customer needs, mitigate storage and production management costs, and realize enhanced profits.

The composition of this article is articulated as follows: Chapter 1 furnishes an overview of the research purpose, methods, and content of this article. Chapter 2 undertakes a review of extant literature on optimal inventory control. Chapter 3 proffers a preliminary introduction to the uncertain optimal control model. Chapter 4 formulates an uncertain optimal control model and Chapter 5 elucidates the processes to decipher it. Chapter 6 employs Chinese clothing sales data to validate the proposed model. Chapter 7 recapitulates and summarizes the entire text.

2 Literature Review

There has been a wealth of literature on inventory control in the past, and many scholars such as Pooya and Balata have conducted detailed research on inventory control[1][2].This chapter aims to summarize existing literature from three aspects: deterministic optimal control, stochastic optimal control, and uncertain optimal control.

Athans (1966) first focused on organizing past research on deterministic optimal control and provided suggestions for future research directions[3]. Bertsekas considered the deterministic discrete-time optimal control problem in the infinite time domain in 1979[4], and later expanded this research to the terminal state set in 2015[5]. Reissig (2016) reconsidered the problem in uncertain or prior bounded time[6]. Dupuis (2001) considered a numerical approximation method based on Markov chains for a class of deterministic nonlinear optimal control problems[7].

Soravia (2008) defined the viscous solution of the Aronsson equation and proved that the corresponding value function for determining the optimal control problem is the solution of the bilateral viscous solution of the Hamilton-Jacobi-Bellman equation[8]. In the same year, Akian proposed another finite element MAX-PLUS method to solve such problems[9]. Gonz ¨¢ lez Aribas (2018) introduced a framework based on optimal control to solve robust and efficient trajectory planning problems under wind prediction uncertainty[10]. Lefebvre (2022) obtained a random search algorithm suitable for trajectory optimization by estimating expected values through Monte Carlo sampling[11].

Porteus first proposed the concept of random inventory theory in his book Foundations of Stochastic Inventory Theory in 2002[12]. Afterwards, Gao et al. (2013) considered the general dynamic resource allocation problem within a stochastic optimal control framework[13]. Later Nie et al. (2014) studied the theory of optimizing stationary random vibration disturbances in the context of wave energy collection[14]. Chen et al. proposed and analyzed a multi-level weighted reduction basis method for solving Stokes equation constrained stochastic optimal control problems in 2015, which improved computational efficiency in high-dimensional situations[15]. In the same year, Malikopoulos considered the problem of minimizing the long-term expected average cost of a complex system composed of interactive subsystems and provided a framework to demonstrate that the control strategy for generating Pareto optimal solutions minimizes the average cost criterion of the system[16]. Lesniewski (2020) studied the optimal control problem of stochastic SIR model in the context of COVID-19[17]. Vlasenko et al. (2020) studied the optimal control problem of singular systems under the evolution description of Ito differential equations[18]. De Vecchi et al. (2021) derived an extension of Noether's theorem in stochastic optimal control problems[19].

Liu founded the uncertainty theory in the book Uncertain Optimal Control in 2008[20], and later extended it to fuzzy mathematics and financial mathematics theory[21]. Zhu et al. combined uncertainty theory with optimal control in 2010 and studied the expected value model of uncertain optimal control problems in 2013[22] [23], providing the optimality principle of the model. Later, in the same year, he used the Hurwicz criterion and uncertain differential equations to solve the problem again[24] [25]. Subsequently, in 2016, uncertain optimal control problems were solved in uncertain linear systems with multiple input delays and linear quadratic conditions, respectively[26]. In 2018, Sheng et al. described a production inventory model under uncertain dynamic systems. They attribute the change in inventory to sudden fluctuations in sales. In addition, they use threshold criteria to consider decision makers' risk attitudes[27]. Subsequently, In 2020, research on the optimal control problem of multi-level dynamic systems was completed[28]. In addition, Yao (2013)[29], Kaya (2017)[30], Duan (2018)[31], Shi (2022) and others have conducted detailed research on this aspect[32]. Specifically, Yao's (2013) study introduced the concept of uncertain differential equations α- Path and its related numerical solutions provide a very effective method and approach for solving uncertain differential equations, and also play a highly guiding role in solving the model in this paper.

Upon synthesizing existing literature, the novelty of this article manifests as follows: It employs uncertainty theory to decipher the optimal production strategy under time-varying demand when the demand rate exhibits periodic variations; In the process of solving the model, the analytical solution is inherently complex, prompting the consideration of introducing a numerical solution using the α-path to procure more intuitive numerical results.

3 Preliminary Study on Uncertain Optimal Control

C_s represents the prototypical Liu process, and we consider the following uncertain expected value optimization problem

$$\begin{cases} J(0,x_0) \equiv \sup\limits_{u_t \in U} E[\int_0^T f(s,u_s,X_s)ds + G(T,X_T)] \\ \text{s.t.} \\ \mathrm{d}X_s = v\left(s,u_s,X_s\right)\mathrm{d}s + \sigma\left(s,u_s,X_s\right)\mathrm{d}C_s \text{ and } X_0 = x_0, \end{cases} \tag{3.1}$$

where X_s represents the state variable, and U_s represents the decision variable (function $U_s(s,X_s)$ is represented by time s and state X_s), whose value resides in the set U. f represents the profit objective function, and G represents the terminal reward function. For any given u_s, X_s is provided by the constraint equation in model (3.1), where v and σ are two functions of time s, decision variable U_s, and state variable X_s. The function $J(0,x_0)$ represents the optimal expected return that can be obtained in the time interval $[0,T]$, with the initial condition being in the initial state x_0 at time 0. For any $0 < t < T$, $J(t,x)$ is the expected best return that can be obtained in $[t,T]$, assuming that at time t we are in state $X_t = x$, which corresponds to model (3.2).

$$\begin{cases} J(t,x) \equiv \sup\limits_{u_t} E\left[\int_t^T f\left(s,u_s,X_s\right)\mathrm{d}s + G\left(T,X_T\right)\right] \\ \text{s.t.} \\ \mathrm{d}X_s = v\left(s,u_s,X_s\right)\mathrm{d}s + \sigma\left(s,u_s,X_s\right)\mathrm{d}C_s \end{cases} \tag{3.2}$$

3.1 Principle of optimality

Now we propose the optimality principle for uncertain optimal control. For $\forall (t,x) \in [0,T) \times R$, we stipulate that $\Delta t > 0$,$t + \Delta t < T$, and we establish

$$J(t,x) = \sup_{u_t} E\left[f\left(t,u_t,X_t\right)\Delta t + J\left(t+\Delta t, x+\Delta X_t\right) + o(\Delta t)\right], \tag{3.3}$$

where $x + \Delta X_t = X_{t+\Delta t}$.

3.2 Optimality Equation

In the context of the uncertain optimal control problem (3.2), we present a fundamental result known as the optimality equation in uncertain optimal control.

(Optimality equation) Let $J(t,x)$ be differentiable twice on $[0,T]\times\mathbb{R}$, it follows that

$$-J_t(t,x)=\sup_{u_t}\left\{f\left(t,u_t,x\right)+J_x(t,x)v\left(t,u_t,x\right)\right\}. \tag{3.4}$$

where, $J_t(t,x)$ and $J_x(t,x)$ denote the partial derivatives of function $J(t,x)$ with respect to t and x respectively.

4 Model description and formulation

As customer demand fluctuates, the company's production and inventory plans necessitate adaptive adjustments. The initiation of production incurs associated production costs, and if the manufactured product does not immediately find a market, additional storage costs accrue. To optimize profitability, companies must adeptly manage both production and storage costs, making production control paramount. With demand as a determining factor, diverse production strategies are devised to align with customer needs. Moreover, by implementing these production strategies, companies can aspire to attain peak profitability.

We define the following symbols:

Table 1: Symbol description

Symbol	Explanation
I_t	inventory level at time t(state variable)
P_t	Productivity of time t(control variable)
S_t	The demand rate for time t and $S>0$
T	Cycle length
I_0	Initial inventory level
L	Residual value per unit of inventory at time T
C_t	Typical Liu process
σ	Constant diffusion coefficient.

To optimize production, it is imperative to establish an optimization control model. P is the control variable, I is the state variable, and the state equation is given by the inventory flow differential equation $\dot{I}=P(t)-S(t)$. In addition, there are many uncertain factors in the production and storage processes, which can lead to fluctuations in inventory products. Therefore, we add uncertainty process C_t to the original equation to represent this fluctuation and obtain the final state equation $\mathrm{d}I_t=\left(P_t-S_t\right)\mathrm{d}t+\sigma\mathrm{d}C_t$. To maximize operational efficiency, it is imperative to maximize the residual inventory value at the terminal time. Moreover, the minimization of production and storage costs over the entire cycle T is crucial. Specifically, this corresponds to maximizing the reciprocal of the sum of production and storage costs throughout the entire cycle. Assuming that production and storage costs are quadratic in terms of production and inventory, it can be understood as the maximum value of "terminal residual value minus production and inventory costs assumed to be quadratic".

If $S = A\sin(\pi Bt + C) + D$, indicating cyclical customer demand. We recognize that certain products exhibit periodic demand, exemplified by items like ice cream, with diminished demand in winter and escalated demand in summer, exhibiting seasonal fluctuations. For the sake of clarity, we denote x as I and derive model(4.1).

$$\begin{cases} J(0,x_0) = \max E\{Lx_T - \int_0^T ({P_t}^2 + {x_t}^2)\mathrm{d}t\} \\ \text{s.t.} \\ \mathrm{d}x_t = (P_t - A\sin(\pi Bt + C) - D)\mathrm{d}t + \sigma \mathrm{d}C_t \end{cases} \tag{4.1}$$

Let $H_{x,P} = \int_t^T ({P_s}^2 + {x_s}^2)\mathrm{d}s - Lx_T$, and we consider $H_{x,P}$ has a regular uncertainty distribution $\Psi(x)$. When $t = 0$, the model(4.1) can be transformed into model(4.1').

$$\begin{cases} J(0,x_0) = \min E\{\int_0^T ({P_t}^2 + {x_t}^2)\mathrm{d}t - Lx_T\} \\ \text{s.t.} \\ \mathrm{d}x_t = (P_t - A\sin(\pi Bt + C) - D)\mathrm{d}t + \sigma \mathrm{d}C_t \end{cases} \tag{4.1'}$$

5 Optimal production policy under cycle customer demand

In this model, $J(t,x)$ represents the value function in optimal control. Derived the uncertain optimal equation (3.4), we have

$$\max\left[-\left(P^2 + x^2\right) + J_t + J_x(P - A\sin(\pi Bt + C) - D)\right] = 0, \tag{5.1}$$

where, J_x and J_t represent the partial derivatives of J with respect to x and t, respectively. In addition, In addition, we denote x_t and P_t as x and P to prevent confusion between J_x, J_t and x_t, P_t. Let $F = J_t + J_x(P - S) - (P^2 + x^2)$ and taking the derivative of P and making it zero, we can obtain that

$$P^* = \frac{J_x}{2}. \tag{5.2}$$

Here we introduce a lemma and a theorem

Lemma 5.1 *(Ting-Yang)* *The value function* $J(t,x) = \min E[H_{x,P}]$ *is equivalent to the value function* $\widetilde{J(t,x)} = \min\{\mu H_{x,P_{\sup}}(\eta) + (1-\mu)H_{x,P_{\inf}}(\eta)\}$ *, where* $\mu \in [0,1], \eta \in (0,1), H_{x,P_{\sup}}(\eta) = \Psi^{-1}(1-\eta), H_{x,P_{\inf}}(\eta) = \Psi^{-1}(\eta)$*, and* $\Psi^{-1}(x)$ *is the inverse uncertainty distribution of* $H_{x,P}$.

Theorem 5.1 *(Andy-Vivian)* $J(t,x) = \min E[H_{x,P}]$ *is strictly convex to* $\forall t \in [0,T], x \in \mathbb{R}$ *about* x.

We acknowledge that the optimal P defined by equation(5.2) on $[0,T]$ exists, supported by the convexity of $J(t,x)$ as stated in the $Andy-Vivian$ theorem. The $Ting-Yang$ lemma furnishes theoretical underpinnings for the proof of the $Andy-Vivian$ theorem. The detailed proofs can be found in the appendix.

Substituting (5.2) into equation(5.1) results in

$$\frac{J_x^2}{4} - x^2 + J_t - (A\sin(\pi Bt + C) + D) \cdot J_x = 0. \tag{5.3}$$

According to (4.1), we can assume the the following form:

$$J(t,x) = q(t)x^2 + r(t)x + m(t), \tag{5.4}$$

taking the partial derivative of J yields

$$J_x = 2qx + r, J_t = \dot{q}x^2 + \dot{r}x + \dot{m}. \tag{5.4'}$$

Substituting into equation(5.3) results in

$$\left(q^2 + \dot{q} - 1\right)x^2 + [qr + \dot{r} - 2q(A\sin(\pi Bt + C) + D)]x + \frac{1}{4}r^2 + \dot{m} - rS = 0. \tag{5.5}$$

Given that the above equation holds for any x and satisfies the boundary condition $J(T,x) = Lx$, we can obtain the coefficients by comparing them:

$$\begin{cases} q^2 + \dot{q} - 1 = 0 \\ q(T) = 0 \end{cases} \tag{5.6}$$

$$\begin{cases} qr + \dot{r} - 2q(A\sin(\pi Bt + C) + D) = 0 \\ r(T) = L \end{cases} \tag{5.7}$$

$$\begin{cases} \frac{1}{4}r^2 + \dot{m} - r(A\sin(\pi Bt + C) + D) = 0 \\ m(T) = 0. \end{cases} \tag{5.8}$$

Note that $P^* = \frac{J_x}{2} = qx + \frac{1}{2}r$ is not related to m, thus, there is no need to solve the differential equations(5.8).

Next, solve equations(5.6) and(5.7). Equation(5.6) is a simple ordinary differential equation. By separating variables we can derive the solution as follows:

$$q = \frac{e^{2(t-T)} - 1}{e^{2(t-T)} + 1}. \tag{5.9}$$

Substituting (5.9) into (5.7) results in

$$\dot{r} = 2(A\sin(\pi Bt + C) + D) \cdot \frac{e^{2(t-T)} - 1}{e^{2(t-T)} + 1} - r \cdot \frac{e^{2(t-T)} - 1}{e^{2(t-T)} + 1}, \tag{5.10}$$

which is a differential equation concerning r. Note that the solution of r varies with the variation of S, leading to diverse yields for P.

In addition, the requirements S differ among various products. In this article, we consider the demand function $S = A\sin(\pi Bt + C) + D$ that exhibits periodic variations over time.

Equation (5.10) and (5.7) are transformed into

$$\begin{cases} \frac{\mathrm{d}r}{\mathrm{d}t} = -r \cdot \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1} + 2(A\sin(\pi Bt + C) + D) \cdot \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1} \\ r(T) = L. \end{cases} \tag{5.7'}$$

The ordinary differential equation (5.7') is solved using the method of constant variation as outlined below

$$r = e^{\int -\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\mathrm{d}t}\left(\int 2(A\sin(\pi Bt + C) + D) \cdot \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1} \cdot e^{\int \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\mathrm{d}t} + \widetilde{C}\right). \tag{5.11}$$

and

$$\int \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\mathrm{d}t = \int \frac{e^{2(t-T)}}{e^{2(t-T)}+1}\mathrm{d}t - \int \frac{1}{e^{2(t-T)}+1}\mathrm{d}t, \tag{5.12}$$

Let $e^{2(t-T)} = v$, then $\mathrm{d}t = \frac{1}{2v}\mathrm{d}v, (v > 0)$. Solve the equation(5.12) we can obtain

$$\begin{aligned} &\int \frac{v}{v+1} \cdot \frac{1}{2v}\mathrm{d}v - \int \frac{1}{v+1} \cdot \frac{1}{2v}\mathrm{d}v \\ &= \frac{1}{2}\ln(1+v) - \frac{1}{2}\ln\frac{v}{v+1} \\ &= \frac{1}{2}\ln\left(1+e^{2(t-T)}\right) - \frac{1}{2}\ln\frac{e^{2(t-T)}}{e^{2(t-T)}+1} \\ &= \ln(1+e^{2(t-T)}) - (t-T). \end{aligned} \tag{5.13}$$

Substituting equation (5.13) into (5.11) results in

$$r = \frac{e^{t-T}}{e^{2(t-T)}+1}\left[\int 2(A\sin(\pi Bt + C) + D) \cdot \frac{e^{2(t-T)}-1}{e^{t-T}}\mathrm{d}t + \widetilde{C}\right]. \tag{5.14}$$

while

$$\begin{aligned} &\int 2(A\sin(\pi Bt + C) + D) \cdot \frac{e^{2(t-T)}-1}{e^{t-T}}\mathrm{d}t \\ &= \int 2A\sin(\pi Bt + C) \cdot \frac{e^{2(t-T)}-1}{e^{t-T}}\mathrm{d}t + \int 2D \cdot \frac{e^{2(t-T)}-1}{e^{t-T}}\mathrm{d}t \qquad (5.15) \\ &= \int 2A\sin(\pi Bt + C)\left(e^{t-T} - e^{T-t}\right)\mathrm{d}t + 2D\left(\int e^{t-T}\mathrm{d}t - \int e^{T-t}\mathrm{d}t\right) \\ &= 2A\left(\int \sin(\pi Bt + C) \cdot e^{t-T}\mathrm{d}t - \int \sin(\pi Bt + C) \cdot e^{T-t}\mathrm{d}t\right) + 2D(e^{t-T} + e^{-(t-T)}), \end{aligned}$$

Utilizing the method of integration by parts, we can obtain that:

$$\begin{aligned}
&\int \sin(\pi Bt + C) \cdot e^{t-T} \mathrm{d}t \\
&= \sin(\pi Bt + C) \cdot e^{t-T} - \pi B \cos(\pi Bt + C) \cdot e^{t-T} - \pi^2 B^2 \int \sin(\pi Bt + C) \cdot e^{t-T} \mathrm{d}t
\end{aligned} \tag{5.16}$$

Namely

$$\int \sin(\pi Bt + C) \cdot e^{t-T} \mathrm{d}t = \frac{\sin(\pi Bt + C) \cdot e^{t-T} - \pi B \cos(\pi Bt + C) \cdot e^{t-T}}{1 + \pi^2 B^2}. \tag{5.17}$$

Applying the integration by parts method again, we can obtain that:

$$\int \sin(\pi Bt + C) \cdot e^{T-t} \mathrm{d}t = -\frac{\sin(\pi Bt + C) \cdot e^{T-t} + \pi B \cos(\pi Bt + C) \cdot e^{T-t}}{1 + \pi^2 B^2}. \tag{5.18}$$

Combining the two equations above, we can conclude that:

$$\begin{aligned}
&\int 2A \sin(\pi Bt + C) \cdot \frac{e^{2(t-T)} - 1}{e^{t-T}} \mathrm{d}t \\
&= 2A[\frac{\sin(\pi Bt + C) \cdot e^{t-T} - \pi B \cos(\pi Bt + C) \cdot e^{t-T}}{1 + \pi^2 B^2} \\
&+ \frac{\sin(\pi Bt + C) \cdot e^{T-t} + \pi B \cos(\pi Bt + C) \cdot e^{T-t}}{1 + \pi^2 B^2}] \\
&= 2A \cdot \frac{\sin(\pi Bt + C) \cdot \left[e^{t-T} + e^{-(t-T)}\right] + \pi B \cos(\pi Bt + C) \cdot (e^{-(t-T)} - e^{t-T})\big]}{1 + \pi^2 B^2}.
\end{aligned} \tag{5.19}$$

Substituting Equation (5.19) into Equation (5.15) results in

$$\begin{aligned}
r = \frac{e^{t-T}}{e^{2(t-T)} + 1} &\left[2A \cdot \frac{\sin(\pi Bt + C) \cdot \left[e^{t-T} + e^{-(t-T)}\right] + \pi B \cos(\pi Bt + C) \cdot \left(e^{-(t-T)} - e^{t-T}\right)}{1 + \pi^2 B^2} \right. \\
&\left. + 2D \left(e^{t-T} + e^{-(t-T)}\right) + \widetilde{C} \right].
\end{aligned} \tag{5.20}$$

Substitute $r(T) = L$ into the calculation to obtain

$$\widetilde{C} = 2L - 4D - \frac{4A \sin(\pi BT + C)}{1 + \pi^2 B^2}. \tag{5.21}$$

Here, we can determine the solution to the ordinary differential equation (3.8) as

$$\begin{aligned}
r = \frac{e^{t-T}}{e^{2(t-T)} + 1} \cdot &\left[\frac{2A \sin(\pi Bt + C) \left(e^{t-T} + e^{-(t-T)}\right) + 2AB\pi \cos(\pi Bt + C) \left(e^{-(t-T)} - e^{t-T}\right)}{1 + \pi^2 B^2} \right. \\
&\left. + 2D \left(e^{t-T} + e^{-(t-T)}\right) + 2L - 4D - \frac{4A \sin(\pi BT + C)}{1 + \pi^2 B^2} \right].
\end{aligned} \tag{5.22}$$

According to equation (5.2), the optimal vaule of$P$ is obtained as

$$
\begin{aligned}
&P=\frac{J_x}{2}=qx+\frac{r}{2}\\
&=\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\cdot x+\frac{e^{t-T}}{e^{2(t-T)}+1}\cdot\\
&\Bigg[\frac{A\sin(\pi Bt+C)(e^{t-T}+e^{-(t-T)})+AB\pi\cos(\pi Bt+C)(e^{-(t-T)}-e^{t-T})}{1+\pi^2B^2}\\
&+D\left(e^{t-T}+e^{-(t-T)}\right)+L-2D-\frac{2A\sin(\pi BT+C)}{1+\pi^2B^2}\Bigg]\\
&=\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\cdot x+(\frac{A\sin(\pi Bt+C)}{1+\pi^2B^2}+D)-\frac{e^{t-T}-e^{-(t-T)}}{e^{t-T}+e^{-(t-T)}}\cdot\frac{AB\pi\cos(\pi Bt+C)}{1+\pi^2B^2}\\
&+\frac{e^{t-T}}{e^{2(t-T)}+1}(L-2D-\frac{2A\sin(\pi BT+C)}{1+\pi^2B^2})\\
&=\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\left[x-\frac{AB\pi\cos(\pi Bt+C)}{1+\pi^2B^2}\right]+\frac{e^{t-T}}{e^{2(t-T)}+1}(L-2D-\frac{2A\sin(\pi BT+C)}{1+\pi^2B^2})\\
&+(\frac{A\sin(\pi Bt+C)}{1+\pi^2B^2}+D)\ . \qquad (5.23)
\end{aligned}
$$

Substituting (5.23) into the constraint conditions of (4.1) results in

$$
\begin{aligned}
\mathrm{d}x_t=&(\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}x_t-\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\cdot\frac{AB\pi\cos(\pi Bt+C)}{1+\pi^2B^2}+\frac{e^{t-T}}{e^{2(t-T)}+1}(L-2D-\frac{2A\sin(\pi BT+C)}{1+\pi^2B^2})\\
&-\frac{\pi^2B^2}{1+\pi^2B^2}A\sin(\pi Bt+C))\mathrm{d}t+\sigma\mathrm{d}C_t. \qquad (5.24)
\end{aligned}
$$

Let $G(t)=\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}$, $f(t)=-\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\cdot\frac{AB\pi\cos(\pi Bt+C)}{1+\pi^2B^2}+\frac{e^{t-T}}{e^{2(t-T)}+1}(L-2D-\frac{2A\sin(\pi BT+C)}{1+\pi^2B^2})-\frac{\pi^2B^2}{1+\pi^2B^2}A\sin(\pi Bt+C)$, then (5.24) deforms to

$$\mathrm{d}x_t=(G(t)x_t+f(t))\mathrm{d}t+\sigma\mathrm{d}C_t. \qquad (5.25)$$

Whose α-path is

$$\mathrm{d}x_t^\alpha=(G(t)x_t^\alpha+f(t))\mathrm{d}t+|\sigma|\Phi^{-1}(\alpha)\mathrm{d}t, \qquad (5.26)$$

namely

$$\frac{\mathrm{d}x_t^\alpha}{\mathrm{d}t}=G(t)x_t^\alpha+f(t)+|\sigma|\Phi^{-1}(\alpha), \qquad (5.26')$$

where $\Phi^{-1}(\alpha)=\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}(\alpha\in(0,1))$. Subsequently, if $F(t)=f(t)+|\sigma|\Phi^{-1}(\alpha)$ is set, then (5.26') becomes

$$\frac{\mathrm{d}x_t^\alpha}{\mathrm{d}t}=G(t)x_t^\alpha+F(t). \qquad (5.27)$$

n accordance with Yao's (2013) research, we are aware that the initial condition for (5.27) is $x_0^\alpha=x_0$. Next, we proceed to solve the ordinary differential equation (5.28).

$$\begin{cases}\frac{\mathrm{d}x_t^\alpha}{\mathrm{d}t}=G(t)x_t^\alpha+F(t)\\ x_0^\alpha=x_0.\end{cases} \qquad (5.28)$$

Applying the method of constant variation, we can obtain that:

$$x_t^\alpha = e^{\int G(t)\mathrm{d}t}(\int F(t)e^{-\int G(t)\mathrm{d}t} + \widetilde{C_1}), \tag{5.29}$$

Further calculations yield:

$$\begin{aligned}
x_t^\alpha =& \frac{e^{2(t-T)}+1}{e^{t-T}}(\int F(t)\frac{e^{t-T}}{e^{2(t-T)}+1}\mathrm{d}t + \widetilde{C_1}) \\
=& \frac{e^{2(t-T)}+1}{e^{t-T}}(\int f(t)\frac{e^{t-T}}{e^{2(t-T)}+1}\mathrm{d}t + |\sigma|\Phi^{-1}(\alpha)\mathrm{arctan}(e^{t-T}) + \widetilde{C_1}) \\
=& \frac{e^{2(t-T)}+1}{e^{t-T}}(-\frac{AB\pi}{1+\pi^2B^2}\int\frac{(e^{2(t-T)}-1)e^{t-T}\mathrm{cos}(\pi Bt+C)}{(e^{2(t-T)}+1)^2}\mathrm{d}t- \\
&\frac{A\pi^2B^2}{1+\pi^2B^2}\int\mathrm{sin}(\pi Bt+C)\frac{e^{t-T}}{e^{2(t-T)}+1}\mathrm{d}t \\
&-\frac{1}{2(e^{2(t-T)}+1)}(L-2D-\frac{2A\mathrm{sin}(\pi BT+C)}{1+\pi^2B^2}) + |\sigma|\Phi^{-1}(\alpha)\mathrm{arctan}(e^{t-T}) + \widetilde{C_1}).
\end{aligned} \tag{5.30}$$

Upon revisiting the constraint conditions of (4.1), we observe that its α-path is:

$$\mathrm{d}x_t^\alpha = (P_t - A\mathrm{sin}(\pi Bt+C) - D)\mathrm{d}t + |\sigma|\Phi^{-1}(\alpha)\mathrm{d}t, \tag{5.31}$$

namely,

$$x_t^\alpha = \int P_t\mathrm{d}t - \int[A\mathrm{sin}(\pi Bt+C)+D]\mathrm{d}t + |\sigma|\Phi^{-1}(\alpha)t. \tag{5.32}$$

Substituting (5.30) into (5.32) results in:

$$P_t = \frac{\mathrm{d}x_t^\alpha}{\mathrm{d}t} + A\mathrm{sin}(\pi Bt+C) + D - |\sigma|\Phi^{-1}(\alpha), \tag{5.33}$$

The result of further calculations for the optimal value of P in (5.23) is:

$$\begin{aligned}
P =& G(t)x_t^\alpha + F(t) + A\mathrm{sin}(\pi Bt+C) + D - |\sigma|\Phi^{-1}(\alpha) \\
=& \frac{e^{2(t-T)}-1}{e^{t-T}}(-\frac{AB\pi}{1+\pi^2B^2}\int\frac{(e^{2(t-T)}-1)e^{t-T}\mathrm{cos}(\pi Bt+C)}{(e^{2(t-T)}+1)^2}\mathrm{d}t - \frac{A\pi^2B^2}{1+\pi^2B^2}\int\mathrm{sin}(\pi Bt+C)\frac{e^{t-T}}{e^{2(t-T)}+1}\mathrm{d}t \\
&+\widetilde{C_1}) + \frac{e^{2(t-T)}-1}{e^{t-T}}|\sigma|\Phi^{-1}(\alpha)\mathrm{arctan}(e^{t-T}) + \frac{A}{1+\pi^2B^2}\mathrm{sin}(\pi Bt+C) + D - \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}\cdot\frac{AB\pi\mathrm{cos}(\pi Bt+C)}{1+\pi^2B^2} \\
&+\frac{1}{2e^{t-T}}(L-2D-\frac{2A\mathrm{sin}(\pi BT+C)}{1+\pi^2B^2}).
\end{aligned} \tag{5.34}$$

6 Numerical example and management insights

To demonstrate the model's effectiveness, this section conducts empirical tests using real-world cases. Taking into account the seasonal variations and periodic shifts in clothing sales, this study investigates the demand for clothing in China from June 2017 to September 2019.

Utilizing MATLAB for calculations, we obtain the cyclical customer demand $S = 0.8773 \times \sin(\pi \times 0.4849t - 2.105) + 1.52$, i.e. $A = 0.8773, B = 0.4849, C = -2.105, D = 1.52$. The corresponding S-t image is as follows.

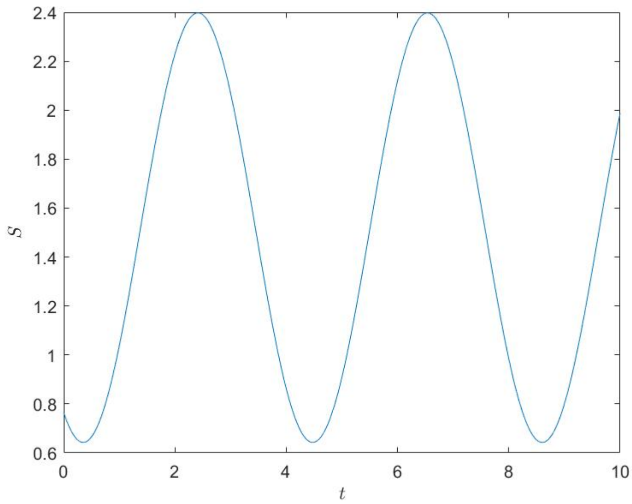

Figure 1: Cycle customer demand function: S-t image

Next, we use the Runge-Kutta method to solve ordinary differential equations (5.28). The optimal value of P can be obtained by substituting the formula (5.34) once x_t^α has been solved. Let $g(t, x_t^\alpha) = G(t)x_t^\alpha + F(t)$, and consider utilizing the fourth-order Runge-Kutta method, which offers higher accuracy for solving. Taking a step size of $h = 0.01$ and substituting $g(t, x_t^\alpha) = G(t)x_t^\alpha + F(t)$ into the fourth-order Runge-Kutta format, we can obtain that:

$$\begin{cases} x_0^\alpha = x_0, \quad h = 0.01, \\ x_{i+1}^\alpha = x_i^\alpha + \dfrac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4), \\ k_1 = G(t_i)x_i^\alpha + F(t_i), \\ k_2 = G(t_i + \dfrac{1}{2}h)(x_i^\alpha + \dfrac{1}{2}hk_1) + F(t_i + \dfrac{1}{2}h), \\ k_3 = G(t_i + \dfrac{1}{2}h)(x_i^\alpha + \dfrac{1}{2}hk_2) + F(t_i + \dfrac{1}{2}h), \\ k_4 = G(t_i + h)(x_i^\alpha + hk_3) + F(t_i + h). \end{cases} \tag{6.1}$$

Where $G(t) = \frac{e^{2(t-T)}-1}{e^{2(t-T)}+1}$, $F(t) = -\frac{e^{2(t-T)}-1}{e^{2(t-T)}+1} \cdot \frac{AB\pi\cos(\pi Bt+C)}{1+\pi^2B^2} + \frac{e^{t-T}}{e^{2(t-T)}+1}(L - 2D - \frac{2A\sin(\pi BT+C)}{1+\pi^2B^2}) - \frac{\pi^2B^2}{1+\pi^2B^2}A\sin(\pi Bt + C) + |\sigma|\frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}(\alpha \in (0, 1))$, and here, we set $T = 10, L = 10, \sigma = 1, x_0 = 0$. The values $A = 0.8773, B = 0.4849, C = -2.105, D = 1.52$ have been fitted. Iterating over α in the range (0, 1) with a step of 0.1, the x-t image and P-t image obtained through programming represent the results of numerical solutions, as illustrated in Figure 2 and Figure 3.

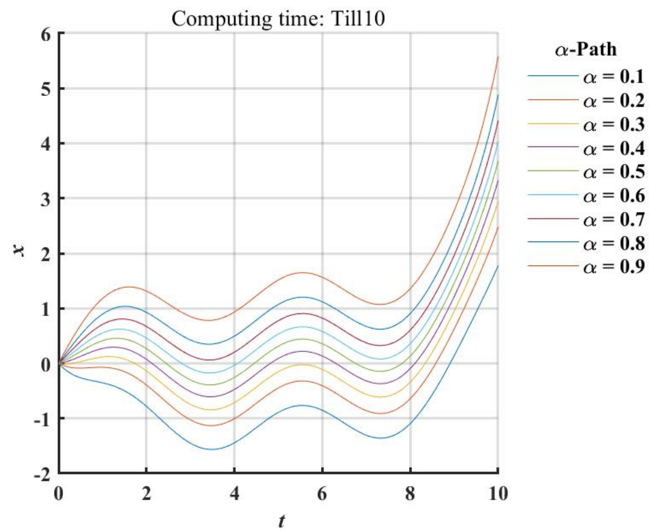

Figure 2: Numerical Solutions 1: x-t image

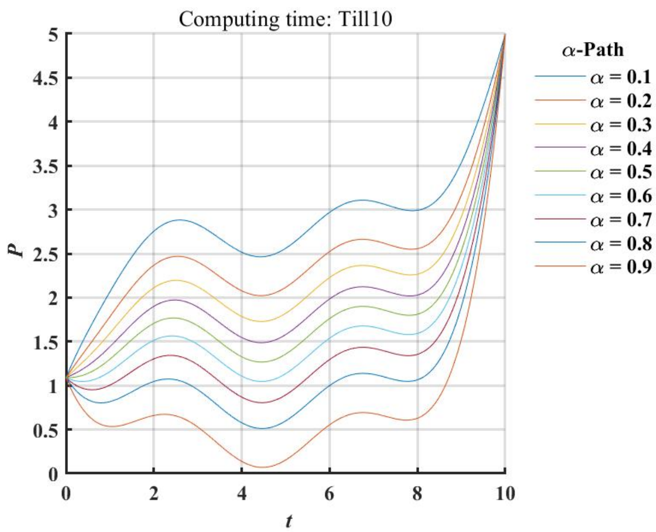

Figure 3: Numerical Solutions 1: P-t image

Upon observing the x-t image and P-t image, it becomes apparent that there is a specific moment when both the P-value of production and the x-value of inventory undergo changes. Before this moment, both P-value and x-value fluctuate up and down with time, but after this moment, both P-value and x-value consistently increase over time. This suggests that, in the context of the current study, the company needs to increase production after a certain period to maximize profits. Additionally, for the same duration, a larger α-value corresponds to a larger x-value and a smaller P-value. This implies a reciprocal feedback control mechanism between production and inventory.

Now, we examine the relationship between the terminal time inventory x_T and α.

First, we keep σ unchanged and vary the value of L. Set L to be 5, 10, 15, and 20, respectively, and present the images of x_T-α under different L as shown in the following figure.

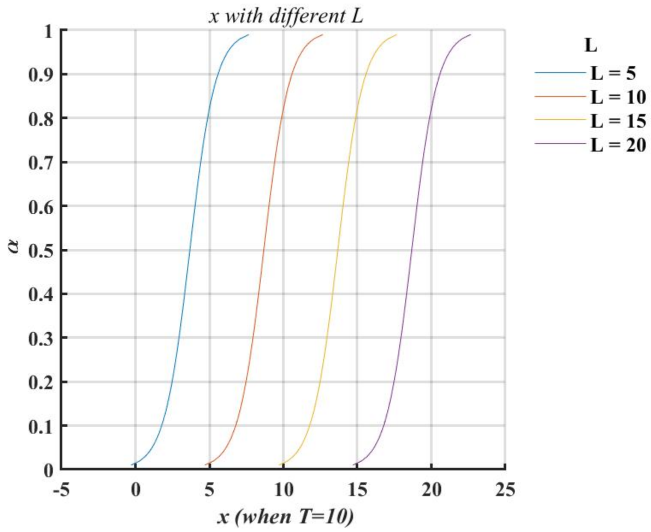

Figure 4: Change L: x_T-α image

Observing the image, we can discern that when L is larger, the corresponding value of x_T under the same value of α is also larger. This indicates that when the residual value of terminal inventory per unit of inventory is higher, the company should control the inventory value at terminal time to be appropriately large in order to maximize profits.

We maintain L at a constant value and vary the parameter σ. Set σ to be 0.8, 1, 1.2, and 1.4 respectively, and present the images of x_T-α under different σ as follows.

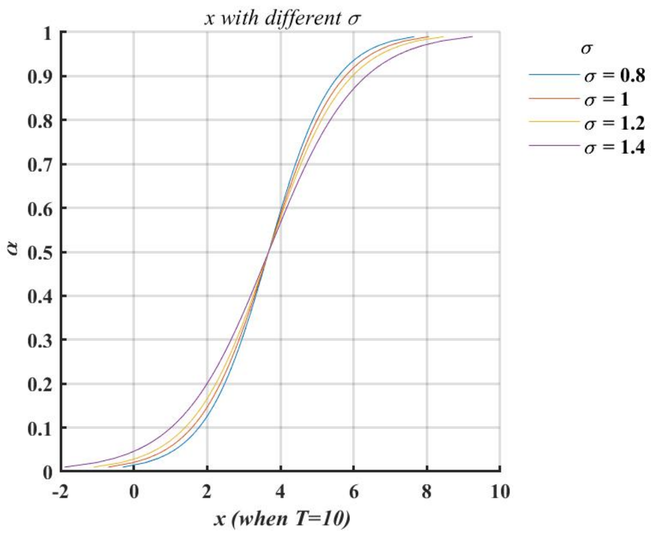

Figure 5: Change σ: x_T-α image

Upon observing the image, we can deduce that when $\alpha = 0.5$, the value of terminal time inventory x_T remains constant regardless of the variation in σ. At this point, the company only needs to control x_T to achieve this specific value for maximizing profit. When $\alpha < 0.5$, the larger the σ, the smaller the corresponding value of x_T. In this scenario, with greater disturbance, the company should aim to control the terminal inventory value to be appropriately small. Conversely, when $\alpha > 0.5$, the greater the σ, the larger the corresponding x_T value. Thus, when facing increased disturbance, the company should aim to control the terminal inventory value to be appropriately large.

7 Conclusion

Based on the principles of 'optimality' and the 'optimality equation' in uncertain optimal control, this paper investigates the uncertain optimal control of production and inventory in the presence of periodic changes in customer demand. Taking into consideration the uncertainties inherent in the production and storage processes, an uncertain *Liu* process C_t is introduced to represent fluctuations. The inventory state equation, accounting for these fluctuations, is formulated as a constraint condition. The production-inventory model under time-varying demand is established by introducing the residual value of terminal inventory and assuming quadratic production and storage costs.

The uncertain optimality equation is utilized to calculate and solve for the optimal P. The $Ting$-$Yang$ lemma provides a crucial equivalent characterization of the original objective function, and the *Andy*-$Vivian$ theorem is employed to prove the existence of the optimal P. Following the derivation of the optimal analytical solution for P through integration and ordinary differential equations, the α-path is employed to obtain the numerical solution, enhancing the visibility of the results.

In our paper, we consider the uncertainties inherent in both the production-inventory system and customer demand, leading to a more detailed and practical mathematical model. The established model can be applied to a wide range of sales scenarios in everyday life, providing valuable insights for merchants to optimize their production and sales plans.

Furthermore, through a meticulous consideration of uncertainties within our mathematical framework, our model becomes an invaluable asset for real-world sales scenarios encountered in daily business operations. By adeptly addressing the intricate interplay between uncertainties in production, inventory, and customer demand, our model serves as a pragmatic tool for merchants to augment their decision-making processes. This resource enables merchants to proactively anticipate and navigate through fluctuations in customer demand, facilitating the optimization of production and sales plans to achieve enhanced efficiency and profitability. The insights derived from this study provide practical guidance for businesses navigating the complexities of dynamic markets, empowering them to make well-informed decisions amidst inherent uncertainties.

8 Acknowledgment

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9 Data Availability

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix

First and foremost, we give the proof of Lemma 5.1 ($Ting$-$Yang$).

Proof:

Since $H_{x,P}$ has a regular uncertainty distribution $\Psi(x)$,

we can obtain $E[H_{x,P}] = \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha$.

And then from the First mean value theorem for definite integrals,

we can get: $\exists \xi \in (0,1)$, s.t. $\int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = \Psi^{-1}(\xi)$.

Let $\Psi^{-1}(x) = f(x), x \in (0,1)$. It follows from the properties of $\Psi(x)$ and the inverse function that $f(x)$ is strictly monotonically increasing and continuous on (0,1).

Due to

$$\begin{aligned}
&\mu H_{x,P_{\sup}}(\eta) + (1-\mu)H_{x,P_{\inf}}(\eta) \\
&= \mu\Psi^{-1}(1-\eta) + (1-\mu)\Psi^{-1}(\eta) \\
&= \mu f(1-\eta) + (1-\mu)f(\eta),
\end{aligned}$$

then we prove(1): $\exists \xi, \eta \in (0,1), \exists \mu \in [0,1]$, s.t. $f(\xi) = \mu f(1-\eta) + (1-\mu)f(\eta)$.

(i) When $\mu = 0$, $J(t,x) = f(\eta)$, only let $\eta = \xi \in (0,1)$, the equation in (1) can be established.

(ii) When $\mu = 1$, $J(t,x) = f(1-\eta)$, only let $\eta = 1-\xi \in (0,1)$, the equation in (1) can also be established.

(iii) When $0 < \mu < 1$, first we determine the value of η from the value of ξ, and require that the interval $(\eta$,$1-\eta)$(or $(1-\eta$,$\eta)$) can contain ξ. We only need to prove: it's certain that $\exists \eta \in (0,1)$, s.t. $\xi \in (\eta, 1-\eta)$(or $\xi \in (1-\eta, \eta)$).

I. If $\xi = \frac{1}{2}$, let $\eta = \frac{1}{2}$, the equation in (1) can be established. And obviously η takes any value on (0,1) except 0.5, there is always $\xi \in (\eta, 1-\eta)$(or $\xi \in (1-\eta, \eta)$).

II. If $0 < \xi < \frac{1}{2}$, from the Density of real numbers, there must be: $\exists \eta$, s.t. $0 < \eta < \xi < 1$. At the same time, there is: $1-\eta > 1-\frac{1}{2} = \frac{1}{2} > \xi$. Therefore, $\xi \in (\eta, 1-\eta)$.

III. If $\frac{1}{2} < \xi < 1$, similarly, we can get: $\exists \eta$, s.t. $\frac{1}{2} < \xi < \eta < 1$. Simultaneously, we have $1-\eta < 1-\frac{1}{2} = \frac{1}{2} < \xi$. So, $\xi \in (1-\eta, \eta)$.

Therefore, we are able to determine the value of η from the value of ξ.

Then we determine the value of μ, s.t. the equation in (1) can be established.

(i) When $f(\eta) = f(1-\eta)$, due to $f(x)$ is strictly monotonically increasing, we can obtain $\eta = \frac{1}{2}$ and $\mu f(1-\eta) + (1-\mu)f(\eta) = f(\frac{1}{2})$. In this case, no matter what value μ takes on (0,1), the equation in (1) can be established only let $\xi = \frac{1}{2}$. This is exactly the case where $\xi = \frac{1}{2}$.

(ii) We only study the case of $f(\eta) < f(1-\eta)$, and the case of $f(\eta) > f(1-\eta)$ can be proved in the same way.

We can get:

$$\begin{aligned}
g(\mu) &= \mu f(1-\eta) + (1-\mu)f(\eta) \\
&= [f(1-\eta) - f(\eta)]\mu + f(\eta), \;\; \mu \in (0,1),
\end{aligned}$$

obviously, $g(\mu)$ is strictly monotonically increasing. Due to $f(x)$ is continuous and properties of continuous functions, we can obtain that $g(\mu)$ is strictly monotonically increasing and continuous on [0,1]. Meanwhile, we have $f(\eta) = g(0) < g(\mu) < g(1) = f(1-\eta)$.

From Intermediate value theorem, we can obtain: $\exists \mu \in (0,1)$, s.t. $g(0) < g(\mu) < g(1)$. This guarantees the existence of μ.

Now let $g(\mu) = f(\xi)$, i.e. $[f(1-\eta) - f(\eta)]\mu + f(\eta) = f(\xi)$,

we can obtain $\mu = \frac{f(\xi)-f(\eta)}{f(1-\eta)-f(\eta)} \in (0,1)$.

Thus we determine the value of μ. And here we complete the proof of (1).

(1) indicates that $E[H_{x,P}]$ and $\mu H_{x,P_{\sup}}(\eta) + (1-\mu)H_{x,P_{\inf}}(\eta)$ can be transformed into each other. And then, we can obtain that $J(t,x)$ is equivalent to $\widetilde{J(t,x)}$. $\square$

Next, here is the proof of Theorem 5.1 (*Andy-Vivian*).

Proof:

By *Ting-Yang* lemma, $J(t,x)$ can be converted to $J_1(x) = \min\{H_{x,P_{\sup}}(\eta)\}$, that is, corresponding to the case of $\mu = 1, \eta = 1-\xi \in (0,1)$.

We first prove $J_\eta\{H_{x,P}\} = H_{x,P_{\sup}}(\eta)$ is strictly convex to $\forall t \in [0,T], x \in \mathbb{R}$ about x.

Take $\forall x_1, x_2 \in x_t, \forall P_1, P_2 \in P_t$, and let $\omega \in (0,1)$, $x_\omega = \omega x_1 + (1-\omega)x_2$, $P_\omega = \omega P_1 + (1-\omega)P_2$.

Solve the constraint equation in the model (4.1') and write α-Path accordingly:

$$x_1^\alpha(t) = x_1(T) - \int_t^T (P_1 - S)\mathrm{d}s - \sigma \int_t^T \Phi^{-1}(\alpha)\mathrm{d}s$$
$$x_2^\alpha(t) = x_2(T) - \int_t^T (P_2 - S)\mathrm{d}s - \sigma \int_t^T \Phi^{-1}(\alpha)\mathrm{d}s$$
$$x_\omega^\alpha(t) = x_\omega(T) - \int_t^T (P_\omega - S)\mathrm{d}s - \sigma \int_t^T \Phi^{-1}(\alpha)\mathrm{d}s.$$

And then, we can get $x_\omega^\alpha(t) = \omega x_1^\alpha(t) + (1-\omega)x_2^\alpha(t) = [\omega x_1(t) + (1-\omega)x_2(t)]^\alpha$.

Note that $[x_\omega^2(t)]_{\sup}(\alpha) = \{[\omega x_1(t) + (1-\omega)x_2(t)]^2\}_{\sup}(\alpha)$,

and

$$\begin{aligned}
[\omega x_1(t) + (1-\omega)x_2(t)]^2 &= \omega^2 x_1^2(t) + (1-\omega)^2 x_2^2(t) + 2\omega(1-\omega)x_1(t)x_2(t) \\
&< [\omega^2 + \omega(1-\omega)]x_1^2(t) + [(1-\omega)^2 + \omega(1-\omega)]x_2^2(t) \\
&= \omega x_1^2(t) + (1-\omega)x_2^2(t).
\end{aligned}$$

Similarly we can get $P_\omega^2 < \omega P_1^2 + (1-\omega)P_2^2$.

Due to $x_1(t)$ and $x_2(t)$ are independent of each other, we can obtain:

$$\begin{aligned}
[\omega x_1^2(t) + (1-\omega)x_2^2(t)]_{\sup}(\alpha) &= [\omega x_1^2(t)]_{\sup}(\alpha) + [(1-\omega)x_2^2(t)]_{\sup}(\alpha) \\
&> \{[\omega x_1(t) + (1-\omega)x_2(t)]^2\}_{\sup}(\alpha) \\
&= [x_\omega^2(t)]_{\sup}(\alpha).
\end{aligned}$$

Use ψ_1 and ψ_2 to represent the uncertainty distribution of $x_1^2(t)$ and $x_2^2(t)$ respectively, we can obtain: $\psi_\omega^{-1}(\alpha) < \omega\psi_1^{-1}(\alpha) + (1-\omega)\psi_2^{-1}(\alpha)$.

Due to $\{H_{x_\omega,P_\omega}\}_{\sup}(\eta) = \{\int_t^T ({P_\omega}^2 + {x_\omega}^2)\mathrm{d}s - Lx_T\}_{\sup}(\eta)$

and $\{\int_t^T ({P_\omega}^2 + {x_\omega}^2)\mathrm{d}s - Lx_T \le \int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T\} \supset \{x_\omega^2 \le \psi_\omega^{-1}(\eta)\}$,

we can get:

$\mathcal{M}\{\int_t^T ({P_\omega}^2 + {x_\omega}^2)\mathrm{d}s - Lx_T \le \int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T\} \ge \mathcal{M}\{x_\omega^2 \le \psi_\omega^{-1}(\eta)\} = \eta$.

Similarly we can get:

$\mathcal{M}\{\int_t^T ({P_\omega}^2 + {x_\omega}^2)\mathrm{d}s - Lx_T > \int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T\} \ge \mathcal{M}\{x_\omega^2 > \psi_\omega^{-1}(\eta)\} = 1-\eta$.

From the Axiom of Self-duality, we have:
$\mathcal{M}\{\int_t^T ({P_\omega}^2 + {x_\omega}^2)\mathrm{d}s - Lx_T \leq \int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T\} = \eta$.
Thus, $\int_t^T (P_\omega^2 + x_\omega^2)\mathrm{d}s - Lx_T$ has an inverse uncertainty distribution:
$\widehat{\Psi^{-1}(\eta)} = \int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T$.
And we can also obtain:

$$\{H_{x_\omega,P_\omega}\}_{\sup}(\eta) = \{\int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T\}$$
$$\{H_{x_1,P_1}\}_{\sup}(\eta) = \{\int_t^T ({P_1}^2 + \psi_1^{-1}(\eta))\mathrm{d}s - Lx_T\}$$
$$\{H_{x_2,P_2}\}_{\sup}(\eta) = \{\int_t^T ({P_2}^2 + \psi_2^{-1}(\eta))\mathrm{d}s - Lx_T\}.$$

Therefore,

$$\begin{aligned}
\{H_{x_\omega,P_\omega}\}_{\sup}(\eta) &= \{\int_t^T ({P_\omega}^2 + \psi_\omega^{-1}(\eta))\mathrm{d}s - Lx_T\} \\
&< \{\int_t^T (\omega P_1^2 + \omega\psi_1^{-1}(\eta) + (1-\omega)P_2^2 + (1-\omega)\psi_2^{-1}(\eta))\mathrm{d}s - Lx_T\} \\
&= \omega\{\int_t^T ({P_1}^2 + \psi_1^{-1}(\eta))\mathrm{d}s - Lx_T\} + (1-\omega)\{\int_t^T ({P_2}^2 + \psi_2^{-1}(\eta))\mathrm{d}s - Lx_T\} \\
&= \omega\{H_{x_1,P_1}\}_{\sup}(\eta) + (1-\omega)\{H_{x_2,P_2}\}_{\sup}(\eta).
\end{aligned}$$

Here, the strict convexity of $J_\eta\{H_{x,P}\}$ has been proved.

Now we let P_1^* and P_2^* represent the optimal control of initial states x_1 and x_2, respectively, let $P_\omega^* = \omega P_1^* + (1-\omega)P_2^*$, we can obtain:

$$\begin{aligned}
J_1(x_\omega) &= J_1(\omega x_1 + (1-\omega)x_2) \\
&= \min\{H_{x_\omega,P_{\sup}}(\eta)\} \\
&\leq J_\eta\{H_{x_\omega,\omega P_1^*+(1-\omega)P_2^*}\} \\
&< \omega J_\eta\{H_{x_1,P_1^*}\} + (1-\omega)J_\eta\{H_{x_2,P_2^*}\} \\
&= \omega J_1(x_1) + (1-\omega)J_1(x_2).
\end{aligned}$$

Therefore, the strict convexity of $J(t,x)$ has also been proved. □

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