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ARTICLE TYPE

Boundary-Domain Integral Equation Systems to the Dirichlet and Neumann Problems for Compressible Stokes Equations with Variable Viscosity in 2D †

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Summary

In this paper, the Dirichlet and Neumann boundary value problems for the steady-state Stokes system of partial differential equations for a compressible viscous fluid with variable viscosity coefficient is considered in two-dimensional bounded domain. Using an appropriate parametrix, this problem is reduced to a system of direct segregated boundary-domain integral equations (BDIEs). The BDIEs in the two-dimensional case have special properties in comparison with the three dimension because of the logarithmic term in the parametrix for the associated partial differential equations. Consequently, we need to set conditions on the function spaces or on the domain to ensure the invertibility of corresponding parametrix-based hydrodaynamic single layer and hypersingular potentials and hence the unique solvability of BDIEs. Equivalence of the BDIE systems to the Dirichlet and Neumann BVPs and the invertibility of the corresponding boundary-domain integral operators in appropriate Sobolev spaces are shown.

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1 | INTRODUCTION

The Stokes equations of partial differential equation(PDE) is derived from the linearised steady-state Navier-Stokes equations. The study of the Stokes equations is useful in itself; it also gives us an opportunity to introduce several tools necessary for a treatment of the full Navier-Stokes equations, see, e.g.,[1] Chapter I, which is a well-known model for laminar viscous fluid flows. In addition to its importance in applications, this system of PDEs has attracted the attention of numerical analysts.

Boundary integral equations and the hydrodynamic potential theory for the Stokes equations with constant viscosity have been extensively studied by numerous authors, e.g.,[2,3,4,5,6,7]. Boundary-domain integral equation systems for the incompressible and compressible Stokes system with variable viscosity in three dimensional space have been investigated in[8,9], but BDIE systems in 2D, following a similar approach as in[10,11] have not yet been studied. In the case of constant viscosity, fundamental solutions for both velocity and pressure are available in analytical form. However, such fundamental solutions are not available for PDEs with variable viscocity. Therefore, the parametrix (Levi function), see, e.g.,[8,9] is used in order to derive and investigate the BDIE systems for the corresponding variable-coefficient boundary value problems(BVPs). In[12,10,11], authors derived and investigated

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BDIE systems for BVP with variable-coefficient scallar elliptic PDE defined on a bounded domain whereas in [8,9], reduced mixed BVP with variable coefficient for Stokes problem defined on a bounded domain to BDIE systems for their further analysis. In this paper, we shall derive and investigate BDIE systems to the Dirichlet and Neumann BVPs for compressible Stokes equations with variable viscosity coefficient in two-dimensional bounded domain in appropriate Sobolev-Slobodetski (Bessel potential) spaces.

2 | PRELIMINARIES

Let $\Omega = \Omega^+$ be an open bounded and simply-connected two-dimensional region of $\mathbb{R}^2$ and let $\Omega^- = \mathbb{R}^2 \setminus \overline{\Omega^+}$. The boundary $\partial\Omega$ be closed and infinitely smooth curve.

Let $\mathbf{v}$ be the velocity vector field, p the pressure scalar field and $\mu \in C^\infty(\Omega)$ be the variable kinematic viscosity of the fluid such that $\mu(\mathbf{x}) > c > 0$. For compressible fluid the stress tensor operator, σ_{ij} , for an arbitrary couple $(p, \mathbf{v})$ is defined as

$$\sigma_{ij}(p, \mathbf{v})(\mathbf{x}) := -\delta_i^j p + \mu(\mathbf{x}) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \alpha \delta_i^j \operatorname{div} \mathbf{v}(\mathbf{x}) \right),$$

and the Stokes operator is defined as

$$\begin{aligned} \mathcal{A}_j(p, \mathbf{v})(\mathbf{x}) :=& \frac{\partial}{\partial x_i} \sigma_{ij}(p, \mathbf{v})(\mathbf{x}) \\ =& \frac{\partial}{\partial x_i} \left(\mu(\mathbf{x}) \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \alpha \delta_i^j \operatorname{div} \mathbf{v}(\mathbf{x}) \right) \right) - \frac{\partial p}{\partial x_j}, \quad j, i \in \{1, 2\}, \end{aligned} \tag{1}$$

where $\alpha = 1$ or $\alpha = \frac{2}{3}$ and δ_i^j is Kronecker symbol. Here and henceforth we assume the Einstein summation in repeated indices from 1 to 2. We denote the Stokes operator as $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_j\}_{j=1}^2$ and $\mathring{\boldsymbol{\mathcal{A}}} := \boldsymbol{\mathcal{A}}|_{\mu=1}$. We will also use the following notation for derivative operators: $\partial_j = \partial_{x_j} := \frac{\partial}{\partial x_j}$ with $j = 1, 2$; $\nabla := (\partial_1, \partial_2)$.

We consider the Stokes PDE, which for sufficiently smooth $(p, \mathbf{v})$ has the following form,

$$\boldsymbol{\mathcal{A}}(p, \mathbf{v})(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \operatorname{div} \mathbf{v} = g(\mathbf{x}), \ \mathbf{x} \in \Omega, \tag{2}$$

where $(p, \mathbf{v})$ is an unknown function and $\mathbf{f}$ and g is a given function in Ω.

In what follows $H^s(\Omega) = H_2^s(\Omega)$, $H^s(\partial\Omega)$ are the Bessel potential spaces, where s is a real number (see, e.g., [13,14]). We recall that H^s coincide with the Sobolev-Slobodetski spaces W_2^s for any non-negative s. We denote by $\widetilde{H}^s(\Omega)$ the subspace of $H^s(\mathbb{R}^2)$, $\widetilde{H}^s(\Omega) = \{g : g \in H^s(\mathbb{R}^2), \operatorname{supp}(g) \subset \overline{\Omega}\}$; similarly, $\widetilde{H}^s(S_1) = \{g : g \in H^s(\partial\Omega), \operatorname{supp}(g) \subset \overline{S_1}\}$, $L_*^2(\Omega) = L^2(\Omega)/\mathbb{R} = \{q \in L^2(\Omega) : \int_\Omega q \, dx = 0\}$. We will also use the notations $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^2$, $\mathbf{L}^2(\Omega) = \left[L^2(\Omega)\right]^2$, $\boldsymbol{\mathcal{D}}(\Omega) = [\mathscr{D}(\Omega)]^2$ for two-dimensional vector space. We will also make use of the following space (see, e.g., [15,12,9]).

$$\mathbf{H}^{s,0}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \mathbf{v}) \in H^{s-1}(\Omega) \times \mathbf{H}^s(\Omega) : \boldsymbol{\mathcal{A}}(p, \mathbf{v}) \in \mathbf{L}^2(\Omega)\}$$

endowed with the norm

$$\|(p, v)\|^2_{\mathbf{H}^{s,0}(\Omega;\boldsymbol{\mathcal{A}})} := \|p\|^2_{H^{s-1}(\Omega)} + \| \mathbf{v} \|^2_{\mathbf{H}^s(\Omega)} + \| \boldsymbol{\mathcal{A}}(p, \mathbf{v}) \|^2_{\mathbf{L}^2(\Omega)} .$$

Let us define a space

$$\mathbf{H}_*^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \mathbf{v}) \in L_*^2(\Omega) \times \mathbf{H}^1(\Omega) : \boldsymbol{\mathcal{A}}(p, \mathbf{v}) \in \mathbf{L}^2(\Omega)\}$$

endowed with the norm

$$\|(p, \mathbf{v})\|^2_{\mathbf{H}_*^{1,0}(\Omega;\boldsymbol{\mathcal{A}})} := \|p\|^2_{L_*^2(\Omega)} + \| \mathbf{v} \|^2_{\mathbf{H}^1(\Omega)} + \| \boldsymbol{\mathcal{A}}(p, \mathbf{v}) \|^2_{\mathbf{L}^2(\Omega)} .$$

We define also the space $\mathbf{H}^1_{\mathcal{R}}(\Omega) = \mathbf{H}^1(\Omega)/\mathcal{R} = \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \int_\Omega \mathbf{v} \cdot \mathbf{w} dx = 0, \text{ for all } \mathbf{w} \in \mathcal{R}\}$, where $\mathcal{R} = \{a + b(-x_2, x_1)^T; a, b \text{ are constant vector and scalar respectively}\}$ is the space of rigid body motions and $\mathcal{R} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right\} = \operatorname{span}\{\mathbf{w}_k\}_{k=1}^3$. Define the $\mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \boldsymbol{\mathcal{A}})$, the subspace of $\mathbf{H}^1_{\mathcal{R}}(\Omega)$ as;

$$\mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \boldsymbol{\mathcal{A}}) := \{(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1_{\mathcal{R}}(\Omega) : \boldsymbol{\mathcal{A}}(p, \mathbf{v}) \in \mathbf{L}^2(\Omega)\}$$

endowed with the norm

$$\|(p, \mathbf{v})\|^2_{\mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})} := \|p\|^2_{L^2(\Omega)} + \| \mathbf{v} \|^2_{\mathbf{H}^1_{\mathcal{R}}(\Omega)} + \| \boldsymbol{\mathcal{A}}(p, \mathbf{v}) \|^2_{\mathbf{L}^2(\Omega)} .$$

The operator $\mathcal{A}$ acting on $(p, \mathbf{v})$ is well defined in the weak sense provided $\mu(\mathbf{x}) \in L^\infty(\Omega)$ as

$$\langle \mathcal{A}(p, \mathbf{v}), \mathbf{u}\rangle_\Omega := -\mathcal{E}((p, \mathbf{v}), \mathbf{u}), \qquad \forall \mathbf{u} \in \widetilde{\mathbf{H}}^1(\Omega),$$

where the form $\mathcal{E} : \left[L^2(\Omega) \times \mathbf{H}^1(\Omega)\right] \times \widetilde{\mathbf{H}}^1(\Omega) \to \mathbb{R}$ is defined as

$$\mathcal{E}\left((p, \mathbf{v}), \mathbf{u}\right) := \int\limits_\Omega E\left((p, \mathbf{v}), \mathbf{u}\right)(\mathbf{x})\, d\mathbf{x},$$

and the function $E\left((p, \mathbf{v}), \mathbf{u}\right)$ given by

$$E((p, \mathbf{v}), \mathbf{u})(\mathbf{x}) := \frac{\mu(\mathbf{x})}{2}\left(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i}\right)\left(\frac{\partial v_i(x)}{\partial x_j} + \frac{\partial v_j(x)}{\partial x_i}\right) - \alpha\mu(x)\,\mathrm{div}\,\mathbf{v}(\mathbf{x})\,\mathrm{div}\,\mathbf{u}(\mathbf{x}) - p(\mathbf{x})\,\mathrm{div}\,\mathbf{u}(\mathbf{x}).$$

For sufficiently smooth functions $(p, \mathbf{v}) \in H^{s-1}(\Omega^\pm) \times \mathbf{H}^s(\Omega^\pm)$ with $s > 3/2$, we can define the classical traction operators, $\mathbf{T}^{c\pm} = \{T_j^{c\pm}\}_{j=1}^2$ on the boundary $\partial\Omega$ as

$$T_j^{c\pm}(p, \mathbf{v})(\mathbf{x}) := \left[\gamma^\pm \sigma_{ij}(p, \mathbf{v})(\mathbf{x})\right] n_i(\mathbf{x}), \tag{3}$$

where $n_i(\mathbf{x})$ denote components of the unit outward normal vector $\mathbf{n(x)}$ to the boundary $\partial\Omega$ of the domain and $\gamma^\pm$ is the trace operator from inside and outside Ω, see, e.g.,[8,9].

Traction operator (3) can be continuously extended to the canonical traction operator $\mathbf{T}^\pm : \mathbf{H}^{1,0}(\Omega^\pm; \mathcal{A}) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ defined in the weak form similar as in[8,9],

$$\langle \mathbf{T}^\pm(p, \mathbf{v}), \mathbf{w}\rangle_{\partial\Omega} := \pm \int\limits_{\Omega^\pm} \left[\mathcal{A}(p, \mathbf{v})(\gamma^{-1}\mathbf{w}) + E((p, \mathbf{v}), \gamma^{-1}\mathbf{w})\right] d\mathbf{x}, \qquad (p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega^\pm; \mathcal{A}), \forall \mathbf{w} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega).$$

Here the operator $\gamma^{-1} : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^1(\mathbb{R}^2)$ denotes a continuous right inverse of the trace operator $\gamma^+ : \mathbf{H}^1(\mathbb{R}^2) \to \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. In addition, for $(p, \mathbf{v}) \in \mathbf{H}_*^{1,0}(\Omega; \mathcal{A})$ the traction operator $\mathbf{T}^\pm$ are also defined.

Furthermore, if $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \mathcal{A})$ and $\mathbf{u} \in \mathbf{H}^1(\Omega)$, the following first Green identity holds, (see, e.g.,[15,12,16,8] and[9]),

$$\langle \mathbf{T}^+(p, \mathbf{v}), \gamma^+\mathbf{u}\rangle_{\partial\Omega} := \int\limits_\Omega \left[\mathcal{A}(p, \mathbf{v})\mathbf{u} + E((p, \mathbf{v}), \mathbf{u})(\mathbf{x})\right] d\mathbf{x}. \tag{4}$$

Equation (4) is also defined for $(p, \mathbf{v}) \in \mathbf{H}_*^{1,0}(\Omega; \mathcal{A})$ and $\mathbf{u} \in \mathbf{H}^1(\Omega)$. Applying the identity (4) to the pairs $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \mathcal{A})$ and $(q, \mathbf{u}) \in \mathbf{H}^{1,0}(\Omega; \mathcal{A})$ with exchanged roles and subtracting the one from the other, we arrive at the second Green identity, (see, e.g.,[14,16,8,9]),

$$\int\limits_\Omega \left[\mathcal{A}_j(p, \mathbf{v})u_j - \mathcal{A}_j(q, \mathbf{u})v_j + q\,\mathrm{div}\,\mathbf{v} - p\,\mathrm{div}\,\mathbf{u}\right] dx = \int\limits_{\partial\Omega} \left[T_j(p, \mathbf{v})u_j - T_j(q, \mathbf{u})v_j\right] dS_{\mathbf{x}}. \tag{5}$$

Equation (5) is also defined for $(p, \mathbf{v}) \in \mathbf{H}_*^{1,0}(\Omega; \mathcal{A})$ and $(q, \mathbf{u}) \in \mathbf{H}_*^{1,0}(\Omega; \mathcal{A})$.

3 | PARAMETRIX AND PARAMETRIX-BASED HYDRODYNAMIC POTENTIALS

3.1 | Parametrix and Remainder

The operator $\mathcal{A}$ becomes the constant-coefficient Stokes operator $\mathring{\mathcal{A}}$ when $\mu = 1$. The fundamental solution defined by the pair of distributions $(\mathring{q}^k, \mathring{\mathbf{u}}^k)$, where $\mathring{u}_j^k$ represent components of the incompressible velocity fundamental solution and $\mathring{q}^k$ represent the components of the pressure fundamental solution, (see, e.g.,[3,2,4,5]). So for $r_0 > 0$, $\mathring{u}_j^k$ and $\mathring{q}^k$ will have the form:

$$\begin{aligned}
\mathring{u}_j^k(\mathbf{x}, \mathbf{y}) &= \frac{1}{4\pi}\left(\delta_j^k \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)\\
\mathring{q}^k(\mathbf{x}, \mathbf{y}) &= \frac{-(x_k - y_k)}{2\pi|\mathbf{x}-\mathbf{y}|^2}
\end{aligned}$$

with $(\mathring{q}^k, \mathring{\mathbf{u}}^k)$ satisfying the relations

$$\frac{\partial}{\partial x_k}\mathring{q}^k(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^2 \frac{\partial^2}{\partial x_k^2}\left(-\frac{1}{2\pi}\log|\mathbf{x}-\mathbf{y}|\right) = -\delta(\mathbf{x}-\mathbf{y}) \tag{6}$$

$$\mathring{\mathcal{A}}_j(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{2}\frac{\partial^2 \mathring{u}_j^k}{\partial x_i^2} - \frac{\partial \mathring{q}^k}{\partial x_j} = \delta_j^k\delta(\mathbf{x}-\mathbf{y}), \qquad \operatorname{div}_{\mathbf{x}}\mathring{\mathbf{u}}^k(\mathbf{x},\mathbf{y}) = 0 \tag{7}$$

Let us denote $\mathring{\sigma}_{ij}(p,\mathbf{v}) := \sigma_{ij}(p,\mathbf{v})|_{\mu=1}$. Then in particular case, for $\mu = 1$ and the fundamental solution $(\mathring{q}^k,\mathring{\mathbf{u}}^k)_{k=1,2}$ of the operator $\mathring{\boldsymbol{\mathcal{A}}}$, the stress tensor $\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})$ is

$$\mathring{\sigma}_{ij}(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}.$$

Indeed,

$$\begin{aligned}
\mathring{\sigma}_{ij}(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) =& -\mathring{q}^k\delta_{ij} + \left(\frac{\partial \mathring{u}_i^k}{\partial x_j} + \frac{\partial \mathring{u}_j^k}{\partial x_i}\right)\\
=& \frac{x_k-y_k}{2\pi|\mathbf{x}-\mathbf{y}|^2}\delta_{ij} + \left[\frac{\partial}{\partial x_i}\left(\frac{1}{4\pi}\left(\delta_j^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)\right)\right.\\
&\left. + \frac{\partial}{\partial x_j}\left(\frac{1}{4\pi}\left(\delta_i^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_i-y_i)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)\right)\right].
\end{aligned}$$

Since $\mathring{\sigma}_{ij}(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}$ the boundary traction becomes

$$\mathring{T}_j^c(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x},\mathbf{y}) := \mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{u}^k)(\mathbf{x}-\mathbf{y})n_i(\mathbf{x}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^4}n_i(\mathbf{x}).$$

Let us define a pair of functions $(q^k,\mathbf{u}^k)_{k=1,2}$ similar as in[8,9],

$$u_j^k(\mathbf{x},\mathbf{y}) = \frac{1}{\mu(\mathbf{y})}\mathring{u}_j{}^k(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi\mu(\mathbf{y})}\left(\delta_j^k\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right), \tag{8}$$

$$q^k(\mathbf{x},\mathbf{y}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{q}^k(\mathbf{x},\mathbf{y}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\frac{y_k-x_k}{2\pi|\mathbf{x}-\mathbf{y}|^2}, \quad j,k\in\{1,2\}. \tag{9}$$

Then

$$\begin{aligned}
\sigma_{ij}(\mathbf{x};q^k,\mathbf{u}^k)(\mathbf{x}-\mathbf{y}) =& -\delta_i^j q^k + \mu(\mathbf{x})\left(\frac{\partial u_i^k}{\partial x_j} + \frac{\partial u_j^k}{\partial x_i} - \alpha\delta_i^j\operatorname{div}u^k(\mathbf{x})\right)\\
=& -\delta_i^j\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{q}^k + \mu(\mathbf{x})\left(\frac{\partial(\frac{1}{\mu(\mathbf{y})}\mathring{u}_i^k)}{\partial x_j} + \frac{\partial(\frac{1}{\mu(\mathbf{y})}\mathring{u}_j^k)}{\partial x_i} - \alpha\delta_i^j\operatorname{div}\left(\frac{1}{\mu(y)}\mathring{u}^k(\mathbf{x})\right)\right)\\
=& \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\left(-\delta_i^j\mathring{q}^k + \left(\frac{\partial \mathring{u}_i^k}{\partial x_j} + \frac{\partial \mathring{u}_j^k}{\partial x_i} - \alpha\delta_i^j\operatorname{div}\mathring{\mathbf{u}}^k(\mathbf{x})\right)\right) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

Thus,

$$\sigma_{ij}(x;q^k,\mathbf{u}^k)(\mathbf{x}-\mathbf{y}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})$$

and

$$T_j(x;q^k,\mathbf{u}^k)(\mathbf{x},\mathbf{y}) := \sigma_{ij}(\mathbf{x};q^k,\mathbf{u}^k)(\mathbf{x}-\mathbf{y})n_i(\mathbf{x}) = \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{T}_j(\mathbf{x};\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x},\mathbf{y}) \tag{10}$$

substituting (8)-(9) into Stokes system (1) with variable coefficients, we get

$$\begin{aligned}
\mathcal{A}_j(\mathbf{x};q^k;\mathbf{u}^k)(\mathbf{x},\mathbf{y}) =& \frac{\partial}{\partial x_i}\left(\sigma_{ij}(\mathbf{x};q^k,\mathbf{u}^k)(\mathbf{x}-\mathbf{y})\right) = \frac{\partial}{\partial x_i}\left(\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\right)\\
=& \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\frac{\partial}{\partial x_i}\left(\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\right) + \frac{\partial}{\partial x_i}\left(\frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\right)\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\\
=& \frac{\mu(\mathbf{x})}{\mu(\mathbf{y})}\mathring{\mathcal{A}}_j(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}) + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})\\
=& \frac{\mu(\mathbf{x})\delta(\mathbf{x}-\mathbf{y})\delta_j^k}{\mu(\mathbf{y})} + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k,\mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}
\end{aligned}$$

$$= \frac{\mu(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\delta_j^k}{\mu(\mathbf{y})} + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})$$

$$= \delta_j^k\delta(\mathbf{x}-\mathbf{y} + \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y})$$

Thus,

$$\mathcal{A}_j(\mathbf{x}; q^k; \mathbf{u}^k)(\mathbf{x},\mathbf{y}) = \delta_j^k\delta(\mathbf{x}-\mathbf{y}) + R_{kj}(\mathbf{x},\mathbf{y}), \tag{11}$$

where

$$R_{kj}(\mathbf{x},\mathbf{y}) = \frac{1}{\mu(\mathbf{y})}\frac{\partial(\mu(\mathbf{x}))}{\partial x_i}\mathring{\sigma}_{ij}(\mathring{q}^k, \mathring{\mathbf{u}}^k)(\mathbf{x}-\mathbf{y}) = \mathcal{O}(|\mathbf{x}-\mathbf{y}|^{-1})$$

is a weakly singular remainder. This implies that $(q^k, \mathbf{u}^k)$ is a parametrix of the operator $\boldsymbol{\mathcal{A}}$.

3.2 | Volume and Surface Potentials

Let ρ and $\boldsymbol{\rho}$ be sufficiently smooth scalar and vector function on Ω. The parametrix-based Newton-type and the Remainder vector potential operators are defined as

$$[\boldsymbol{\mathcal{U}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{U}_{kj}\rho_j(\mathbf{y}) := \int_\Omega u_j^k(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})d\mathbf{x}, \quad [\boldsymbol{\mathcal{R}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{R}_{kj}\rho_j(\mathbf{y}) := \int_\Omega R_{kj}(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})d\mathbf{x}, \quad \mathbf{y}\in\mathbb{R}^2$$

for the velocity $\mathbf{v}$, and the scalar Newton-type and remainder potentials for the pressure,

$$[\boldsymbol{\mathcal{Q}}\rho]_j(\mathbf{y}) = \mathcal{Q}_j\rho(\mathbf{y}) := -\int_\Omega q^j(\mathbf{x},\mathbf{y})\rho(\mathbf{x})d\mathbf{x}, \tag{12}$$

$$\mathcal{Q}\boldsymbol{\rho}(\mathbf{y}) = \boldsymbol{\mathcal{Q}}\cdot\boldsymbol{\rho}(\mathbf{y}) = \mathcal{Q}_j\rho_j(\mathbf{y}) := -\int_\Omega q^j(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})d\mathbf{x}, \tag{13}$$

$$\boldsymbol{\mathcal{R}}^{\bullet}\boldsymbol{\rho}(\mathbf{y}) = -2\langle\partial_i\mathring{q}^j(.,\mathbf{y}), \rho_i\partial_j\mu\rangle_\Omega - 2\rho_i(\mathbf{y})\partial_i\mu(\mathbf{y}) = -2v.p.\int_\Omega \frac{\partial\mathring{q}^j(\mathbf{x},\mathbf{y})}{\partial x_i}\frac{\partial\mu(\mathbf{x})}{\partial x_i}\rho_j(\mathbf{x})d\mathbf{x} - \rho_j(\mathbf{y})\frac{\partial\mu(\mathbf{y})}{\partial y_j}, \tag{14}$$

for $\mathbf{y}\in\mathbf{R}^2$. The integral in (14) is understood as a 2D strongly singular integral in the Cauchy sense, (see, e.g.,[8,9]).

For the velocity, the parametrix-based single layer and double layer potentials are defined for $\mathbf{y}\notin\partial\Omega$ as :

$$[\mathbf{V}\boldsymbol{\rho}]_k(\mathbf{y}) = V_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} u_j^k(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \quad [\mathbf{W}\boldsymbol{\rho}]_k(\mathbf{y}) = W_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} T_j^+(\mathbf{x}; q^k, \mathbf{u}^k)(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}},$$

and for pressure in the variable coefficient Stokes system, the single layer and double layer potentials are defined for $\mathbf{y}\notin\partial\Omega$ as:

$$\Pi^s\boldsymbol{\rho}(\mathbf{y}) = \Pi_j^s\rho_j(\mathbf{y}) := \int_{\partial\Omega}\mathring{q}^j(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \quad \Pi^d\boldsymbol{\rho}(\mathbf{y}) = \Pi_j^d\rho_j(\mathbf{y}) := 2\int_{\partial\Omega}\frac{\partial\mathring{q}^j(\mathbf{x},\mathbf{y})}{\partial\mathbf{n}(\mathbf{x})}\mu(\mathbf{x})\rho_j(\mathbf{x})dS_{\mathbf{x}}.$$

The corresponding boundary integral (pseudo-differential) operators of direct surface values of the single layer potential and the double layer potential, the traction of the single layer potential and the double layer potential are

$$[\boldsymbol{\mathcal{V}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{V}_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} u_j^k(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \quad [\boldsymbol{\mathcal{W}}\boldsymbol{\rho}]_k(\mathbf{y}) = \mathcal{W}_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} T_j^+(\mathbf{x}; q^k, \mathbf{u}^k)(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \quad \mathbf{y}\in\partial\Omega,$$

$$\left[\boldsymbol{\mathcal{W}}'\boldsymbol{\rho}\right]_k(\mathbf{y}) = \mathcal{W}'_{kj}\rho_j(\mathbf{y}) := -\int_{\partial\Omega} T_j^+(\mathbf{y}; q^k, \mathbf{u}^k)(\mathbf{x},\mathbf{y})\rho_j(\mathbf{x})dS_{\mathbf{x}}, \quad \mathbf{y}\in\partial\Omega, \quad \boldsymbol{\mathcal{L}}^{\pm}\boldsymbol{\rho}(\mathbf{y}) := \mathbf{T}^{\pm}(\Pi^d\boldsymbol{\rho}, \mathbf{W}\boldsymbol{\rho})(\mathbf{y}), \quad \mathbf{y}\in\partial\Omega,$$

where $\mathbf{T}^{\pm}$ are the traction operators (see, e.g.,[8,9]).

The parametrix-based integral operators depending on the variable coefficient, $\mu(\mathbf{x})$, can be expressed in terms of the corresponding integral operators for the constant coefficient case, $\mu = 1$, see ([8,9] for 3D case).

$$\boldsymbol{\mathcal{U}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}, \tag{15}$$

$$[\boldsymbol{\mathcal{R}}\boldsymbol{\rho}]_k = -\frac{1}{\mu}\left[\frac{\partial}{\partial y_j}\mathring{\mathcal{U}}_{ki}(\rho_j\partial_i\mu)(\mathbf{y}) + \frac{\partial}{\partial y_i}\mathring{\mathcal{U}}_{kj}(\rho_j\partial_i\mu) - \mathring{\mathcal{Q}}_k(\rho_j\partial_j\mu)\right], \tag{16}$$

$$\boldsymbol{\mathcal{Q}}\rho = \frac{1}{\mu}\mathring{\boldsymbol{\mathcal{Q}}}(\mu\rho), \quad \mathcal{R}^{\bullet}\boldsymbol{\rho} = -2\frac{\partial}{\partial y_i}\mathring{\mathcal{Q}}_j(\rho_j\partial_i\mu) - \rho_j\frac{\partial\mu}{\partial y_i}, \tag{17}$$

$$\mathbf{V}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{V}}\boldsymbol{\rho}, \quad \mathbf{W}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathbf{W}}(\mu\boldsymbol{\rho}), \tag{18}$$

$$\boldsymbol{\mathcal{V}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathcal{V}}\boldsymbol{\rho}, \quad \boldsymbol{\mathcal{W}}\boldsymbol{\rho} = \frac{1}{\mu}\mathring{\mathcal{W}}(\mu\boldsymbol{\rho}), \tag{19}$$

$$\Pi^s\boldsymbol{\rho} = \mathring{\Pi}^s\boldsymbol{\rho}, \quad \Pi^d\boldsymbol{\rho} = \mathring{\Pi}^d(\mu\boldsymbol{\rho}), \tag{20}$$

$$\left[\boldsymbol{\mathcal{W}}'\boldsymbol{\rho}\right]_k = \left[\mathring{\boldsymbol{\mathcal{W}}}'\boldsymbol{\rho}\right]_k - \left(\frac{\partial_i\mu}{\mu}\left[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}\right]_k + \frac{\partial_k\mu}{\mu}\left[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}\right]_i - \alpha\delta_i^k\frac{\partial_j\mu}{\mu}\left[\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}\right]_j\right)n_i, \tag{21}$$

$$\widehat{\boldsymbol{\mathcal{L}}}(\boldsymbol{\tau}) := \mathring{\boldsymbol{\mathcal{L}}}(\mu\boldsymbol{\tau}). \tag{22}$$

Note that the constant-coefficient velocity potentials $\mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}$, $\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}$ and $\mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\rho}$ are divergence-free in $\Omega^{\pm}$, the corresponding potentials $\boldsymbol{\mathcal{U}}\boldsymbol{\rho}$, $\boldsymbol{\mathcal{V}}\boldsymbol{\rho}$ and $\boldsymbol{\mathcal{W}}\boldsymbol{\rho}$ are not divergence-free for the variable coefficient $\mu(\mathbf{y})$,(see e.g.,[9]). Note also that by 9 and 12,

$$\mathring{\mathcal{Q}}_j\rho = \partial_j P_\Delta\rho \tag{23}$$

where

$$P_\Delta\rho(\mathbf{y}) = -\frac{1}{2\pi}\int_\Omega \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0}\rho(\mathbf{x})d\mathbf{x}$$

is the harmonic Newton potential. Hence

$$\text{div }\mathring{\boldsymbol{\mathcal{Q}}}\rho = \partial_j\mathring{\mathcal{Q}}_j\rho = \Delta P_\Delta\rho = -\rho. \tag{24}$$

Moreover, for the constant-coefficient potentials we have the following well-known relations,

$$\mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^s\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\rho}) = 0, \quad \mathring{\boldsymbol{\mathcal{A}}}(\mathring{\Pi}^d\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{W}}}\boldsymbol{\rho}) = 0, \quad \mathring{\boldsymbol{\mathcal{A}}}(\mathring{\boldsymbol{\mathcal{Q}}}\boldsymbol{\rho}, \mathring{\boldsymbol{\mathcal{U}}}\boldsymbol{\rho}) = \boldsymbol{\rho}. \tag{25}$$

In addition, by (23) and (24),

$$\begin{aligned}\mathring{\mathcal{A}}_j((2-\alpha)\rho, -\mathring{\boldsymbol{\mathcal{Q}}}\rho) &= -\partial_i\left(\partial_i\mathring{\mathcal{Q}}_j\rho + \partial_j\mathring{\mathcal{Q}}_i\rho - \alpha\delta_i^j\text{ div }\mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - (2-\alpha)\partial_j\rho \\ &= -\left(\Delta\mathring{\mathcal{Q}}_j\rho + \partial_j\text{ div }\mathring{\boldsymbol{\mathcal{Q}}}\rho - \alpha\partial_j\text{ div }\mathring{\boldsymbol{\mathcal{Q}}}\rho\right) - (2-\alpha)\partial_j\rho = 0\end{aligned} \tag{26}$$

Theorem 1. Let $s \in \mathbb{R}$, the following operators are continuous:

$$\Pi^s : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to L^2(\Omega), \qquad \Pi^d : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to L^2(\Omega), \tag{27}$$

$$\Pi^s : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to L^2_*(\Omega), \qquad \Pi^d : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to L^2_*(\Omega), \tag{28}$$

$$\mathbf{V} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+\frac{3}{2}}(\Omega), \qquad \mathbf{W} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+\frac{1}{2}}(\Omega), \tag{29}$$

$$\boldsymbol{\mathcal{V}} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega), \qquad \boldsymbol{\mathcal{W}} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega), \tag{30}$$

$$\boldsymbol{\mathcal{L}}^{\pm} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s-1}(\partial\Omega), \qquad \boldsymbol{\mathcal{W}}' : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^{s+1}(\partial\Omega), \tag{31}$$

$$(\Pi^s, \mathbf{V}) : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}), \quad (\Pi^d, \mathbf{W}) : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \tag{32}$$

$$(\Pi^s, \mathbf{V}) : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}), \quad (\Pi^d, \mathbf{W}) : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}). \tag{33}$$

Moreover, the following operators are compact,

$$\boldsymbol{\mathcal{V}} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega), \tag{34}$$

$$\boldsymbol{\mathcal{W}} : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega), \tag{35}$$

$$\boldsymbol{\mathcal{W}}' : \mathbf{H}^s(\partial\Omega) \to \mathbf{H}^s(\partial\Omega). \tag{36}$$

Proof. The continuity of the operators for the constant coefficient case is proved in[3] section 5.6.4. Consequently, from the relations (15)-(21) follows the continuity of variable coefficient operators (27) - (31) as well and the continuity of the operators (32) and (33) can be proved similar to[9] Theorem 4.3. The compactness of operators (34) - (36) is implied by the Rellich compactness embedding theorem (see,[14] Theorem 3.27) for scalar case. □

Theorem 2. Let Ω be a bounded open region $\mathbb{R}^2$ with closed, infinitely smooth boundary $\partial\Omega$. The following operators are continuous:

$$\mathcal{U} : \widetilde{\mathbf{H}}^s(\Omega) \to \mathbf{H}^{s+2}(\Omega), \quad s \in \mathbb{R}, \tag{37}$$
$$\mathcal{U} : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+2}(\Omega), \quad s > -\frac{1}{2}, \tag{38}$$
$$\mathcal{R} : \widetilde{\mathbf{H}}^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{39}$$
$$\mathcal{R} : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{40}$$
$$\mathcal{Q} : \widetilde{H}^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{41}$$
$$\mathcal{Q} : H^s(\Omega) \to \mathbf{H}^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{42}$$
$$\mathcal{Q} : \widetilde{\mathbf{H}}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R}, \tag{43}$$
$$\mathcal{Q} : \mathbf{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2}, \tag{44}$$
$$\mathcal{R}^{\bullet} : \widetilde{\mathbf{H}}^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}, \tag{45}$$
$$\mathcal{R}^{\bullet} : \mathbf{H}^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2}. \tag{46}$$
$$(\mathring{\mathcal{Q}}, \mathcal{U}) : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+2,0}(\Omega; \mathcal{A}), \quad s \geq 0, \tag{47}$$
$$((2-\alpha)\mu I, -\mathcal{Q}) : H^{s-1}(\Omega) \to \mathbf{H}^{s,0}(\Omega; \mathcal{A}), \quad s \geq 1, \tag{48}$$
$$(\mathcal{R}^{\bullet}, \mathcal{R}) : \mathbf{H}^s(\Omega) \to \mathbf{H}^{s+1,0}(\Omega; \mathcal{A}), \quad s \geq 1 \tag{49}$$

Proof. We use similar procedure as in[9] Theorem 4.1. Since the surface $\partial\Omega$ is infinitely differentiable, the operators $\mathcal{U}$ and $\mathcal{Q}$ are respectively pseudodifferential operators of order -2 and -1[[3], section 9.1.3]. Then, the continuity of (37) and (41) immediately follows by virtue of the mapping properties of the pseudodifferential operators. Alternatively, these mapping properties are well studied for the constant coefficient case, i.e. operators $\mathring{\mathcal{U}}$ and $\mathring{\mathcal{Q}}$, see, e.g.,[3].Then continuity of operator (43) immediately follows from representation (13) and continuity of operator (41). Consequently, the respective mapping properties for the remainder operators (39) and (45) immediately follow by considering the relation (16).

For the remaining part of the proof, we shall first assume that $s \in (-\frac{1}{2}, \frac{1}{2})$. In this case, $H^s(\Omega)$ is identified with $\widetilde{H}^s(\Omega)$. Hence, the continuity of the operator (38) immediately follows from the continuity of (37).

To prove the case $s \in (\frac{1}{2}, \frac{3}{2})$, we consider $\mathbf{g} = (g_1, g_2)$, $\mathbf{g} \in \mathbf{H}^s(\Omega)$ and by using divergence theorem and the relation $\frac{\partial}{\partial x_i}\mathring{u}_j^k(x,y) = -\frac{\partial}{\partial y_i}\mathring{u}_j^k(x,y)$ we obtain,

$$\begin{aligned}
\mathring{\mathcal{U}}_{kj}(\partial_i g_j)(\mathbf{y}) &= \int_{\Omega} \mathring{u}_j^k(\mathbf{x},\mathbf{y})(\frac{\partial}{\partial x_i} g_j)(\mathbf{x}) d\mathbf{x} \\
&= \int_{\partial\Omega} \mathring{u}_j^k(\mathbf{x},\mathbf{y})\gamma^+ g_j(\mathbf{x}) n_i d\mathbf{x} - \int_{\Omega} g_j(\mathbf{x}) \frac{\partial}{\partial x_i}\mathring{u}_j^k(\mathbf{x},\mathbf{y}) d\mathbf{x} = \int_{\partial\Omega} \mathring{u}_j^k(\mathbf{x},\mathbf{y})\gamma^+ g_j(\mathbf{x}) n_i d\mathbf{x} + \frac{\partial}{\partial y_i}(\int_{\Omega} \mathring{u}_j^k(\mathbf{x},\mathbf{y}) g_j(\mathbf{x}) d\mathbf{x}) \\
&= -\mathring{V}_{kj}(\gamma^+ g_j n_i)(\mathbf{y}) + \frac{\partial}{\partial y_i}(\mathring{\mathcal{U}}_{kj} g_j(\mathbf{y}))
\end{aligned}$$

that is,

$$\partial_i \mathring{\mathcal{U}}_{kj} g_j = \mathring{\mathcal{U}}_{kj}(\partial_i g_j) + \mathring{V}_{kj}(\gamma^+ g_j n_i), \; i,j,k \in \{1,2\} \tag{50}$$

where n_i denotes the components of the normal vector to the surface $\partial\Omega$ directed outwards the domain. It is well known that $\partial_i g_j \in H^{s-1}(\Omega)$ and $\gamma^+ \mathbf{g} \in \mathbf{H}^{s-\frac{1}{2}}(\partial\Omega)$ due to the continuity of the operator ∂_i and the trace theorem.

Due to the mapping properties of $\mathring{\mathbf{V}} : \mathbf{H}^{s-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{s+1}(\Omega)$ in Theorems 1 and $\mathring{\mathcal{U}} : \mathbf{H}^{s-1}(\Omega) \to \mathbf{H}^{s+1}(\Omega)$ in the previous paragraph, we deduce that $\partial_i \mathring{\mathcal{U}}\mathbf{g} \in \mathbf{H}^{s+1}(\Omega)$ is continuous for $i \in \{1,2\}$. Consequently, from relations (15) and (18), for $s \in (\frac{1}{2}, \frac{3}{2})$, immediately follows the continuity of the operator (38). Furthermore, by induction on $k \in \mathbb{N}$, using the representation in identity (50) and one can prove by induction that the operator (38) is also continuous for $s \in (k - \frac{1}{2}, k + \frac{1}{2})$, where k is an arbitrary nonnegative integer. The continuity of the operator (38) for the cases $s = k + \frac{1}{2}$ is proved by applying the theory of interpolation of Bessel potential spaces, (see, e.g.[17] , Chapter 4). Continuity of the operator (42) and hence (44) can be proved following a similar argument. Continuity of the remainder operators (40) and (46)) immediately follows from the continuity of operators (38) and (42) by relations (16) and (17). Also the Continuity of the operator (47), (48) and (49) can be proved similar as in[9] Theorem 4.1. □

Theorem 3. Let $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and $\boldsymbol{\rho} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Then, the following jump relations hold

$$\gamma^{\pm}\mathbf{V}\boldsymbol{\rho} = \boldsymbol{\mathcal{V}}\boldsymbol{\rho}, \qquad \gamma^{\pm}\mathbf{W}\boldsymbol{\tau} = \mp\frac{1}{2}\boldsymbol{\tau} + \boldsymbol{\mathcal{W}}\boldsymbol{\tau} \tag{51}$$

$$\mathbf{T}^{\pm}(\Pi^{s}\boldsymbol{\rho}, \mathbf{V}\boldsymbol{\rho}) = \pm\frac{1}{2}\boldsymbol{\rho} + \boldsymbol{\mathcal{W}}'\boldsymbol{\rho}, \tag{52}$$

Proof. For constant constant coefficient case, $\mu = 1$, the jump properties for the corresponding operators are proved in[3] Lemma 5.6.5. Due to relations (18) and (21), the theorem holds for (51) and (52) as well. □

Theorem 4. Let $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Then the following jump relation holds

$$(\boldsymbol{\mathcal{L}}_k^{\pm} - \widehat{\boldsymbol{\mathcal{L}}}_k)\boldsymbol{\tau} = -\gamma^{\pm}\left[(\partial_i\mu)W_k(\boldsymbol{\tau}) + (\partial_k\mu)W_i(\boldsymbol{\tau}) - \alpha\delta_i^k(\partial_j\mu)W_j(\boldsymbol{\tau})\right] n_i \tag{53}$$

Proof. The proof follows word for word the corresponding proof in 3D case in[9] , Theorem 4.6. □

Proposition 1. Let $s > \frac{1}{2}$. The following operators are compact,

$$\begin{aligned} \boldsymbol{\mathcal{R}} &: \mathbf{H}^{s}(\Omega) \to \mathbf{H}^{s}(\Omega), \qquad \boldsymbol{\mathcal{R}}^{\bullet} : \mathbf{H}^{s}(\Omega) \to H^{s-1}(\Omega),\ s \in \mathbb{R} \\ \gamma^{+}\boldsymbol{\mathcal{R}} &: \mathbf{H}^{s}(\Omega) \to \mathbf{H}^{s-\frac{1}{2}}(\partial\Omega), \quad \mathbf{T}^{\pm}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) : \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega), \\ \mathbf{T}^{\pm}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) &: \mathbf{H}_{*}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega). \end{aligned}$$

Proof. The proof of the compactness for the operators $\boldsymbol{\mathcal{R}}$, $\gamma^{+}\boldsymbol{\mathcal{R}}$ and $\mathcal{R}^{\bullet}$ immediately follows from Theorem 2 and the trace theorem along with the Rellich compact embedding theorem. To prove the compactness of the operator $\mathbf{T}^{\pm}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})$ we consider a function $\mathbf{g} \in \mathbf{H}^{1}(\Omega)$. Then, $(\mathcal{R}^{\bullet}\mathbf{g}, \boldsymbol{\mathcal{R}}\mathbf{g})$ in $H^{1}(\Omega) \times \mathbf{H}^{2}(\Omega)$ and hence, $(\mathcal{R}^{\bullet}\mathbf{g}, \boldsymbol{\mathcal{R}}\mathbf{g}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})(or\ \mathbf{H}_{*}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}))$.

The traction operator $\mathbf{T}^{\pm}$ is the composite of a differential operators, with respect to the first variable and with respect to the second variable, and the trace operator $\gamma^{\pm}$ which reduces the regularity by $1/2$ according to the Trace Theorem. Therefore, $\mathbf{T}^{\pm}(\mathcal{R}^{\bullet}\mathbf{g}, \boldsymbol{\mathcal{R}}\mathbf{g}) \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Then, its compactness follows from the Rellich compact embedding $\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \subset \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. □

4 | INVERTIBILITY OF THE HYDRODYNAMIC SINGLE LAYER POTENTIAL OPERATOR IN 2D

Suppose that $\boldsymbol{\rho} = \mathbf{T}^{+}(p, \mathbf{v})$ where $(p, \mathbf{v}) \in \mathbf{H}_{*}^{1,0}(\Omega)$. The single layer potential operator is a Fredholm of index zero. In 3D case, for $\boldsymbol{\rho} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$, if $\mathbf{V}\boldsymbol{\rho}(\mathbf{y}) = 0$, $\mathbf{y} \in \Omega$, then $\boldsymbol{\rho} = 0$. But this is not generally true for 2D case.

It is well known[7] p.696 that in $\mathbb{R}^2$ the single layer operator fail to be invertible. So that for some 2D domains the kernel of the operator $\mathring{\boldsymbol{\mathcal{V}}} : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is non-zero, which is by the first relation in (19) implies that also $\ker\boldsymbol{\mathcal{V}} \neq \{0\}$ for some domains. The following example is in[18] , Lemma 1 which illustrates this fact.

Example 1. Take the density function $\rho_j^m = \delta_{jm}$ and $\Omega = B(0, R)$ to be a disc of radius R centered at the origin and $\partial\Omega = \partial B(0, R)$ be the circular boundary of the disc. We want to show that

$$\mu(\mathbf{y})\mathcal{V}_{kj}\rho_j^m(\mathbf{y}) = \mathring{\mathcal{V}}_{kj}\rho_j^m(\mathbf{y}) = -\frac{R}{2}\delta_{km}(2\log\frac{R}{r_0} - 1), \quad |\mathbf{y}| \leq R, \quad k, j, m \in \{1, 2\}.$$

Remark 1. If we set $r_0 = Re^{-\frac{1}{2}}$ in Example 1, with $\mu(\mathbf{y}) \neq 0$, we get, $[\mathbf{V}\boldsymbol{\rho}]_k\,(\mathbf{y}) = 0$ in $\overline{\Omega}$.

In order to have invertibility for the single layer potential operator in 2D, we define the subspace $\mathbf{H}_{**}^{s}(\partial\Omega)$ of the space $\mathbf{H}^{s}(\partial\Omega)$, see e.g.,[7] Appendix A, in particular $s = -\frac{1}{2}$ and $\frac{1}{2}$,

$$\mathbf{H}_{**}^{s}(\partial\Omega) := \{\boldsymbol{\rho} \in \mathbf{H}^{s}(\partial\Omega) : \langle\rho_i, 1\rangle_{\partial\Omega} = 0 \quad \text{for} \quad i = 1, 2\}, \tag{54}$$

where the norm in $\mathbf{H}_{**}^{s}(\partial\Omega)$ is induced norm of $\mathbf{H}^{s}(\partial\Omega)$.

Theorem 5. If $\boldsymbol{\Psi} \in \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega)$ satisfies $\boldsymbol{\mathcal{V}}\boldsymbol{\Psi} = 0$ on $\partial\Omega$, then $\boldsymbol{\Psi} = 0$.

Proof. Let us proof by using similar procedure as in[14] Corollary 8.11. The single layer potential $(\mathring{p}, \mathring{\mathbf{v}}) = (\mathring{\Pi}^s\boldsymbol{\Psi}, \mathring{\mathbf{V}}\boldsymbol{\Psi})$ satisfies

$$\Delta\mathring{\mathbf{v}} - \nabla\mathring{p} = 0 \quad \text{in} \quad \Omega^{\pm}, \tag{55}$$

$$\operatorname{div}\mathring{\mathbf{v}} = 0 \quad \text{in} \quad \Omega^{\pm}, \tag{56}$$

$$\gamma^{\pm}\mathring{\mathbf{v}} = 0 \quad \text{on} \quad \partial\Omega. \tag{57}$$

For the exterior problem , we use the following growth conditions at infinity,

$$\mathring{\mathbf{v}}(\mathbf{x}) = A\log\frac{|\mathbf{x}|}{r_0} + \mathcal{O}(1), \quad \mathring{p} = \mathcal{O}(|\mathbf{x}|^{-1}) \quad \text{as} \quad |\mathbf{x}| \to \infty,$$

where $A = \int_{\partial\Omega}\boldsymbol{\Psi}ds_{\mathbf{x}}$, see, e.g.,[3] section 2.3.1. Since $\boldsymbol{\Psi} \in \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega)$,i.e., $\int_{\partial\Omega}\boldsymbol{\Psi}ds_{\mathbf{x}} = 0$, it follows that $\mathring{\mathbf{v}} = 0$ and $\mathring{p} = 0$ in Ω^-. For the interior problem, using first Green identity and Dirichlet condition, we get, $\mathring{\mathbf{v}} = 0$ and using interior part of (55), we have that $\nabla\mathring{p} = 0$ in Ω. Since $p \in L^2_*(\Omega)$, then $p = 0$. Consequently, $\boldsymbol{\Psi} = \mathring{\mathbf{T}}^+(\mathring{\Pi}^s\boldsymbol{\Psi}, \mathring{\mathbf{V}}\boldsymbol{\Psi}) - \mathring{\mathbf{T}}^-(\mathring{\Pi}^s\boldsymbol{\Psi}, \mathring{\mathbf{V}}\boldsymbol{\Psi}) = 0$. Thus, $\boldsymbol{\Psi} = 0$. That is, from $\mathring{\boldsymbol{\mathcal{V}}}\boldsymbol{\Psi} = 0$ follows that $\boldsymbol{\Psi} = 0$ and relation (19) implies for the operator $\boldsymbol{\mathcal{V}}$ as well. ☐

Theorem 6. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Then the single layer potential $\boldsymbol{\mathcal{V}} : \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega) \to \mathbf{H}_{**}^{\frac{1}{2}}(\partial\Omega)$ is invertible.

Proof. Due to[7] Lemma A.2 the operator $\mathring{\boldsymbol{\mathcal{V}}}$ is Fredholm of index zero and the first relation in (19) implies that so is operator $\boldsymbol{\mathcal{V}}$. Theorem 5 implies the injectivity of operetor $\boldsymbol{\mathcal{V}}$ and hence the invertibility of operator $\boldsymbol{\mathcal{V}}$. ☐

To prove the $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- ellipticity of the single-layer potential operator for the Stokes system by setting the condition on the domain, for $r_0 > 0$, consider the fundamental solution

$$\mathring{u}_j^k(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi}\left(\delta_j^k \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right)$$

.

$$\mathring{\mathcal{V}}_j^k w_j(\mathbf{x}, \mathbf{y}) = -\int\limits_{\partial\Omega} \frac{1}{4\pi}\left(\delta_j^k \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j - y_j)(x_k - y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right) w_j(\mathbf{x})dS_{\mathbf{x}}.$$

Due to[19] Appendix, the single layer potential operator $\mathring{\boldsymbol{\mathcal{V}}}$ is positive , that is,

$$\langle\mathring{\boldsymbol{\mathcal{V}}}\widetilde{\mathbf{w}}, \widetilde{\mathbf{w}}\rangle_S > 0 \tag{58}$$

for a non-zero $\widetilde{\mathbf{w}}$ that satisfy $\int_S \widetilde{\mathbf{w}}dS = 0$ where S is the boundary of the domain and follows the theorem.

Consider the following basis of the space of rigid body translations in plane: $\mathbf{e}^1 = [1, 0]^T$, $\mathbf{e}^2 = [0, 1]^T$.

Theorem 7. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\partial\Omega$ is contained in the interior of a circular disk with a radius R. If $r_0 \geq Re^{-\frac{1}{2}}$, then $\boldsymbol{\mathcal{V}}$ is $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- elliptic.

Proof. First we show the positivity of $\mathring{\boldsymbol{\mathcal{V}}}$ and we use a similar procedure as in[20] Proposition 2. Let ∂B denote the boundary of the disk with radius R containing $\partial\Omega$. The operator $\mathring{\boldsymbol{\mathcal{V}}}$ is positive by (58). So that

$$\langle[\mathring{\boldsymbol{\mathcal{V}}}\widetilde{\mathbf{w}}]_j, \widetilde{w}_j\rangle_{(\partial\Omega\cup\partial B)} > 0 \tag{59}$$

for non-zero $\widetilde{\mathbf{w}} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega \cup \partial B)$ satisfying

$$\int\limits_{\partial\Omega\cup\partial B} \widetilde{w}_j(\mathbf{x})dS_{\mathbf{x}} = 0. \tag{60}$$

Let us take $\widetilde{\mathbf{w}}$ in the form $\widetilde{\mathbf{w}} = \begin{cases} \mathbf{w} & \text{on} \quad \partial\Omega, \\ \sum_{k=1}^2 \omega_k\mathbf{e}^k & \text{on} \quad \partial B, \end{cases}$ with ω_k chosen so that (60) is satisfied. Let $c_j = \int_{\partial\Omega} w_j(\mathbf{x})dS_{\mathbf{x}}$. Condition (60) gives $0 = \int_{\partial\Omega\cup\partial B}\widetilde{w}_j(\mathbf{x})dS_{\mathbf{x}} = \int_{\partial\Omega} w_j(\mathbf{x})dS_{\mathbf{x}} + \int_{\partial B}\sum_{k=1}^2 \omega_k e_j^k dS_{\mathbf{x}} = c_j + 2\pi R\omega_j$. But

$$\begin{aligned}\langle[\mathring{\boldsymbol{\mathcal{V}}}\mathbf{w}]_j, w_j\rangle_{(\partial\Omega\cup\partial B)} &= \langle -\int\limits_{\partial\Omega\cup\partial B} \mathring{u}_{jk}(\mathbf{x}, \mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{(\partial\Omega\cup\partial B)} \\ &= \langle -\int\limits_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x}, \mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{(\partial\Omega\cup\partial B)} + \langle -\int\limits_{\partial B} \mathring{u}_{jk}(\mathbf{x}, \mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{(\partial\Omega\cup\partial B)}\end{aligned}$$

$$
\begin{aligned}
&= \langle -\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial\Omega} + \langle -\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial\Omega} \\
&+ \langle -\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B} + \langle -\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B} \\
&= \langle [\mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} + \langle -\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial\Omega} \\
&+ \langle -\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B} + \langle -\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle -\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B} &= \sum_{j,k=1}^{2}\int_{\partial\Omega} w_j(\mathbf{x})\left[-\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k dS_{\mathbf{y}}\right]dS_{\mathbf{x}} \\
&= \sum_{j,k=1}^{2}(-\int_{\partial\Omega} w_j(\mathbf{x})\int_{\partial B}\frac{1}{4\pi}(\delta_j^k \log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - \frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2})w_k dS_{\mathbf{y}}dS_{\mathbf{x}}) \\
&= -\sum_{j,k=1}^{2}\int_{\partial\Omega} w_j(\mathbf{x})[\int_{\partial B}\frac{1}{4\pi}(\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0})w_j dS_{\mathbf{y}}]dS_{\mathbf{x}} - \sum_{j,k=1}^{2}\int_{\partial\Omega} w_j(\mathbf{x})[\int_{\partial B}\frac{1}{4\pi}(-\frac{(x_k-y_k)^2}{|\mathbf{x}-\mathbf{y}|^2})w_j dS_{\mathbf{y}}]dS_{\mathbf{x}} \\
&= \sum_{j=1}^{2}(-\int_{\partial\Omega} w_j(\mathbf{x})[\int_{\partial B}\frac{1}{4\pi}(2\log\frac{|\mathbf{x}-\mathbf{y}|}{r_0} - 1)w_j dS_{\mathbf{y}}]dS_{\mathbf{x}}) \\
&= \sum_{j=1}^{2}(-\frac{1}{4\pi}(2\log\frac{R}{r_0} - 1)\int_{\partial\Omega} w_j(\mathbf{x})dS_{\mathbf{x}}\int_{\partial B} w_j(\mathbf{y})dS_{\mathbf{y}}) = -\sum_{j=1}^{2}\frac{1}{4\pi}(2\log\frac{R}{r_0} - 1)(-c_j^2) \\
&= -\frac{1}{4\pi}(-2\log\frac{R}{r_0} + 1)(c_1^2 + c_2^2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\langle -\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{\partial B} &= -\frac{1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2), \\
\langle -\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}, w_j\rangle_{(\partial B)} &= \frac{1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2).
\end{aligned}
$$

Therefore, the integral (59) yields

$$
\begin{aligned}
0 < \langle [\mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} &+ \int_{\partial\Omega} w_j(\mathbf{x})\left[-\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k dS_{\mathbf{y}}\right]dS_{\mathbf{x}} \\
&+ \int_{\partial B} w_j\left[-\int_{\partial\Omega} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(y)dS_{\mathbf{y}}\right]dS_{\mathbf{x}} \\
&+ \int_{\partial B} w_j(\mathbf{x})\left[-\int_{\partial B} \mathring{u}_{jk}(\mathbf{x},\mathbf{y})w_k(\mathbf{y})dS_{\mathbf{y}}\right]dS_{\mathbf{x}}. \qquad (61)
\end{aligned}
$$

Hence, equation (61) becomes

$$
0 < [\langle \mathring{\mathcal{V}}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} - \frac{1}{4\pi}\left[-2\log\frac{R}{r_0} + 1\right](c_1^2 + c_2^2). \qquad (62)
$$

Also equation (62) can be written as

$$\frac{1}{4c\pi}\left[-2\log\frac{R}{r_0}+1\right](c_1^2+c_2^2) < \langle[\mathcal{V}\mathbf{w}]_j, w_j\rangle_{\partial\Omega}. \tag{63}$$

Then the relation $\langle[\mathcal{V}\mathbf{w}]_j, w_j\rangle_{\partial\Omega} > 0$ is always true for $r_0 \geq Re^{-\frac{1}{2}}$, therefore (63) must be positive for any non-zero $\mathbf{w}$. From[3], Theorem 5.6.13, eq.5.6.50 and[19], Eq.(A.15) satisfy Gårding inequality. Thus from positivity and Gårding inequality implies that $\mathcal{V}$ is $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- elliptic due to Lemma 5.2.5 in[3]. □

Theorem 8. Let $\Omega \subset \mathbb{R}^2$. If $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\mathrm{diam}(\Omega)$, then the operator $\mathcal{V}$ has a bounded inverse on $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$.

Proof. By Theorem 7 the operator $\mathcal{V}$ is $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$- elliptic and due to Theorem 1 it is also continuous, that is, bounded. Hence the Lax-Milgram Lemma implies $\mathcal{V}$ has a bounded inverse. □

5 | THE THIRD GREEN IDENTITIES

Theorem 9. For any $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\mathcal{A})$(*or* $\mathbf{H}_*^{1,0}(\Omega;\mathcal{A})$) the following third Green identities hold

$$\mathbf{v} + \mathcal{R}\mathbf{v} - \mathbf{V}\mathbf{T}^+(p,\mathbf{v}) + \mathbf{W}\gamma^+\mathbf{v} = \mathcal{U}\mathcal{A}(p,\mathbf{v}) - \mathcal{Q}\ \mathrm{div}\ \mathbf{v} \ \text{ in } \Omega, \tag{64}$$

$$p + \mathcal{R}^\bullet\mathbf{v} - \Pi^s\mathbf{T}^+(p,\mathbf{v}) + \Pi^d\gamma^+\mathbf{v} = \mathring{\mathcal{Q}}\mathcal{A}(p,\mathbf{v}) + (2-\alpha)\mu\ \mathrm{div}\ \mathbf{v} \ \text{ in } \Omega. \tag{65}$$

Proof. The proof is similar to the corresponding proof in[9] 3D case. □

If the couple $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\mathcal{A})$(*or* $\mathbf{H}_*^{1,0}(\Omega;\mathcal{A})$) is a solution of the Stokes PDE (2) with variable coefficient, then (64) and (65) give

$$\mathbf{v} + \mathcal{R}\mathbf{v} - \mathbf{V}\mathbf{T}^+(p,\mathbf{v}) + \mathbf{W}\gamma^+\mathbf{v} = \mathcal{U}\mathbf{f} - \mathcal{Q}g, \ \text{ in } \Omega \tag{66}$$

$$p + \mathcal{R}^\bullet\mathbf{v} - \Pi^s\mathbf{T}^+(p,\mathbf{v}) + \Pi^d\gamma^+\mathbf{v} = \mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g, \ \text{ in } \Omega \tag{67}$$

We will also need the trace and traction of the third Green identities (66) and (67) on $\partial\Omega$.

$$\frac{1}{2}\gamma^+\mathbf{v} + \mathcal{R}^+\mathbf{v} - \mathcal{V}\mathbf{T}^+(p,\mathbf{v}) + \mathcal{W}\gamma^+\mathbf{v} = \gamma^+\mathcal{U}\mathbf{f} - \gamma^+\mathcal{Q}g \tag{68}$$

$$\frac{1}{2}\mathbf{T}^+(p,\mathbf{v}) + \mathbf{T}^+(\mathcal{R}^\bullet,\mathcal{R})\mathbf{v} - \mathcal{W}'\mathbf{T}^+(p,\mathbf{v}) + \mathcal{L}^+\gamma^+\mathbf{v} = \mathbf{T}^+(\mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g,\ \mathcal{U}\mathbf{f} - \mathcal{Q}g) \tag{69}$$

One can prove the following two assertions that are instrumental for proof of equivalence of the BDIEs and the original PDE.

Lemma 1. Let $\mathbf{v} \in \mathbf{H}^1(\Omega), p \in L^2(\Omega)($ or $L^2_*(\Omega)), g \in L^2(\Omega), \mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{\Psi} \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega), \mathbf{\Phi} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ satisfy equations,

$$\mathbf{v} + \mathcal{R}\mathbf{v} - \mathbf{V}\mathbf{\Psi} + \mathbf{W}\mathbf{\Phi} = \mathcal{U}\mathbf{f} - \mathcal{Q}g, \ \text{ in } \Omega, \tag{70}$$

$$p + \mathcal{R}^\bullet\mathbf{v} - \Pi^s\mathbf{\Psi} + \Pi^d\mathbf{\Phi} = \mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g, \ \text{ in } \Omega. \tag{71}$$

Then $(p,\mathbf{v}) \in \mathbf{H}^{1,0}(\Omega;\mathcal{A})$(*or* $\mathbf{H}_*^{1,0}(\Omega;\mathcal{A})$) and solve the equations

$$\mathcal{A}(\mathbf{y};p,\mathbf{v}) = \mathbf{f}, \quad \mathrm{div}\ \mathbf{v} = g. \tag{72}$$

Moreover, the following relations hold true:

$$\mathbf{V}(\mathbf{\Psi} - \mathbf{T}^+(p,\mathbf{v}))(\mathbf{y}) - \mathbf{W}(\mathbf{\Phi} - \gamma^+\mathbf{v})(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega, \tag{73}$$

$$\Pi^s(\mathbf{\Psi} - \mathbf{T}^+(p,\mathbf{v}))(\mathbf{y}) - \Pi^d(\mathbf{\Phi} - \gamma^+\mathbf{v})(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega. \tag{74}$$

Proof. The proof is similar to the corresponding proof in[9] 3D case . □

Lemma 2. (i) Let either $\mathbf{\Psi}^* \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\mathrm{diam}(\Omega)$ or $\mathbf{\Psi}^* \in \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega)$. If

$$\mathbf{V}\mathbf{\Psi}^*(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega, \tag{75}$$

then $\mathbf{\Psi}^* = 0$

(ii) Let $\mathbf{\Phi}^* \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. If
$$\mathbf{W\Phi}^*(\mathbf{y}) = 0, \quad \mathbf{y} \in \Omega, \tag{76}$$
then $\mathbf{\Phi}^* = 0$.

Proof. We will use similar procedures as in [10].

(i) Taking the trace of (75) on $\partial\Omega$ and using jump relation (51). Then we have
$$\mathcal{V}\mathbf{\Psi}^* = 0 \quad \text{on } \partial\Omega$$
If $\mathbf{\Psi}^* \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\text{diam}(\Omega)$, then the result follows from the invertability of the single layer potential given by Theorem 8. On the other hand, if $\mathbf{\Psi}^* \in \mathbf{H}_{**}^{-\frac{1}{2}}(\partial\Omega)$, then the result is implied by Theorem 6.

(ii) Taking the trace of (76) and then by (51) gives
$$-\frac{1}{2}\mathbf{\Phi}^* + \mathcal{W}\mathbf{\Phi}^* = 0 \quad \text{on } \partial\Omega,$$
due to (19), $-\frac{1}{2}\widehat{\mathbf{\Phi}}^* + \mathring{\mathcal{W}}\widehat{\mathbf{\Phi}}^* = 0$ on $\partial\Omega$, where $\widehat{\mathbf{\Phi}}^* = \mu\mathbf{\Phi}^*$. Due to the contraction property of the operator $-\frac{1}{2}\mathbf{I} + \mathring{\mathcal{W}}$, then $\widehat{\mathbf{\Phi}}^*$ is uniquely solvable and $\mu(\mathbf{y}) \neq 0$,
$$\widehat{\mathbf{\Phi}}^* = 0 \text{ implies } \mathbf{\Phi}^* = 0.$$

□

6 | INVERTIBILITY OF THE HYPERSINGULAR OPERATOR IN 2D

The kernel of the traction of the double layer potential $\mathring{\mathcal{L}}^+$ is not trivial. This can be seen by taking $p = 0$ and $\mathbf{v} = \mathbf{w}$ in Ω, where $\mathbf{w} \in \mathcal{R}$ and inserting in to the integral equation (69) for the case $\mu = 1$, we obtain $\mathring{\mathcal{L}}^+\gamma^+\mathbf{v} = 0$ on $\partial\Omega$. In order to have invertibility for the hypersingular operator, we define the following subspace of the space $\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$,
$$\mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega) = \{\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega) : \langle \mathbf{v}, \mathbf{w}\rangle_{\partial\Omega} = 0 \text{ for all } \mathbf{w} \in \mathcal{R}\}.$$

Theorem 10. Let $\partial\Omega$ be an infinitely smooth boundary curve. The pseudo-differential operator ,
$$\widehat{\mathcal{L}} : \mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{77}$$
is invertible. The operator
$$\mathcal{L}^+ - \widehat{\mathcal{L}} : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{78}$$
is bounded and compact.

Proof. For $\boldsymbol{\tau} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ using the jump relation (53), one can obtain the relation, $\widehat{\mathcal{L}}_k\boldsymbol{\tau} = \mathring{\mathcal{L}}_k^+(\mu\boldsymbol{\tau}) = \mathcal{L}_k^+\boldsymbol{\tau} + \gamma^+\left[(\partial_i\mu)W_k(\boldsymbol{\tau}) + (\partial_k\mu)W_i(\boldsymbol{\tau}) - \alpha\delta_i^k(\partial_j\mu)W_j(\boldsymbol{\tau})\right]n_i$. The hypersingular boundary integral operator $\mathring{\mathcal{L}} : \mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ is bounded by Theorem 1. Moreover, from [3] , Theorem 5.6.13 satisfy the inequality
$$\langle \mathring{\mathcal{L}}\boldsymbol{\tau}, \boldsymbol{\tau}\rangle_{\partial\Omega} + \sum_{k=1}^{3}\langle \boldsymbol{\tau}, \mathbf{w}_k\rangle_{\partial\Omega}^2 \geq c \parallel \boldsymbol{\tau} \parallel^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}.$$
But $\boldsymbol{\tau} \in \mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)$ we have that $\langle \mathring{\mathcal{L}}\boldsymbol{\tau}, \boldsymbol{\tau}\rangle_{\partial\Omega} \geq c \parallel \boldsymbol{\tau} \parallel^2_{\mathbf{H}^{\frac{1}{2}}(\partial\Omega)}$ for all $\boldsymbol{\tau} \in \mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)$ which implies that $\mathring{\mathcal{L}}$ is $\mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)$-elliptic. Then the Lax-Milgram lemma implies the $\mathbf{H}_{\mathcal{R}}^{\frac{1}{2}}(\partial\Omega)$ invertibility of $\mathring{\mathcal{L}}$. Hence the invertibility of (77) follows. The operator $\mathcal{W}_k, \mathcal{W}_i, \mathcal{W}_j : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{3}{2}}(\partial\Omega)$ are continuous and since $H^{\frac{3}{2}}(\partial\Omega)$ is continuously embedded in $H^{\frac{1}{2}}(\partial\Omega)$, using the relation
$$\mathcal{L}_k^+ - \widehat{\mathcal{L}}_k = -\frac{\partial\mu}{\partial n_i}\left((-\frac{1}{2}I + \mathcal{W}_k) + \delta_i^k(-\frac{1}{2}I + \mathcal{W}_i) + \alpha\delta_j^k(-\frac{1}{2}I + \mathcal{W}_j)\right),$$
we obtain continuity of the operator $\mathcal{L}^+ - \widehat{\mathcal{L}} : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. The embedding $\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \subset \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ is compact ,which implies that the operator $\mathcal{L}^+ - \widehat{\mathcal{L}} : \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \to \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ is compact. □

7 | BDIE SYSTEMS FOR THE DIRICHLET PROBLEM

We shall derive and investigate the BDIE systems for the following Dirichlet boundary value problem. *Given the functions* $\boldsymbol{\varphi}_0 \in \boldsymbol{H}^{\frac{1}{2}}(\partial\Omega)$*,* $g \in L^2(\Omega)$ *and* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$*, find a couple of functions* $(p, \boldsymbol{v}) \in \boldsymbol{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$ *satisfying,*

$$\boldsymbol{\mathcal{A}}(p, \mathbf{v})(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{79a}$$

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{79b}$$

$$\gamma^+\mathbf{v}(\mathbf{x}) = \boldsymbol{\varphi}_0(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{79c}$$

Theorem 11. The Dirichlet BVP (79a)-(79c) has a unique solution in the space $\mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$.

Proof. Let $(p_1, \mathbf{v}_1)$ and $(p_2, \mathbf{v}_2)$ are in $\mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$ that satisfy the BVP (79a)-(79c). Then $(p, \mathbf{v}) := (p_2, \mathbf{v}_2) - (p_1, \mathbf{v}_1)$ also belongs to $\mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$ and satisfy the following homogeneous Dirichlet BVP

$$\boldsymbol{\mathcal{A}}(p, \mathbf{v})(\mathbf{x}) = \mathbf{0}, \quad x \in \Omega, \tag{80a}$$

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{80b}$$

$$\gamma^+\mathbf{v}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega \tag{80c}$$

The first Green identity (4) holds for any $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and for any pair $(p, \mathbf{v}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$. Then due to (80a)-(80c) we have, $0 = \int_\Omega E(\mathbf{v}, \mathbf{u})(\mathbf{x}) d\mathbf{x} := \mathcal{E}(\mathbf{v}, \mathbf{u})$, that is

$$\mathcal{E}(\mathbf{v}, \mathbf{u}) = \int_\Omega E(\mathbf{v}, \mathbf{u})(\mathbf{x}) d\mathbf{x} = \int_\Omega \frac{\mu(\mathbf{x})}{2} \left(\frac{\partial u_i(\mathbf{x})}{\partial x_j} + \frac{\partial u_j(\mathbf{x})}{\partial x_i} \right) \left(\frac{\partial v_i(\mathbf{x})}{\partial x_j} + \frac{\partial v_j(\mathbf{x})}{\partial x_i} \right) d\mathbf{x} = 0.$$

Now if we choose $\mathbf{u} = \mathbf{v}$, then we get, $\mathcal{E}(\mathbf{v}, \mathbf{v}) = 0$. As $\mu(\mathbf{x}) > 0$, the only possibility is that $\mathbf{v}(\mathbf{x}) = \mathbf{a} + b(-x_2, x_1)^T$, i.e, $\mathbf{v}$ is a rigid movement, see ,[14] , Lemma 10.5 for 3D case and[3] eq.2.2.11. Taking into account the Dirichlet condition (80c), we deduce that $\mathbf{v} \equiv \mathbf{0}$. Hence, $\mathbf{v}_1 = \mathbf{v}_2$.

Considering now $\mathbf{v} = \mathbf{0}$ and keeping in mind equation (80a), we have $\boldsymbol{\mathcal{A}}(p, \mathbf{v})(\mathbf{x}) = \mathbf{0}$ and then we get $\nabla p = 0$. Since $p \in L^2_*(\Omega)$, we get $p = 0$. □

7.1 | BDIE formulations for the Dirichlet Problem

We aim to obtain a segregated boundary-domain integral equation systems for Dirichlet BVP (79a)-(79c). We will use similar procedures as in[10]. Let us denote the unknown traction as $\boldsymbol{\psi} = \mathbf{T}^+(p, \mathbf{v}) \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and will further consider $\boldsymbol{\psi}$ as formally independent on p and $\mathbf{v}$. Assuming that the function $(p, \mathbf{v})$ satisfies system of PDEs (79a)-(79b), by substituting the Dirichlet condition into the third Green identities (64),(65) and either into its trace (68) or into its traction (69) on $\partial\Omega$, we can reduce the BVP (79a)- (79c) to two different systems of Boundary-Domain Integral Equations for the unknowns $(p, \mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$.

BDIE System (D1) From the equations (66), (67) and(68) we obtain

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\psi} = F_0 \;\text{ in } \Omega, \tag{81a}$$

$$\mathbf{v} + \mathcal{R}\mathbf{v} - \mathbf{V}\boldsymbol{\psi} = \mathbf{F} \;\text{ in } \Omega, \tag{81b}$$

$$\gamma^+\boldsymbol{\mathcal{R}}\mathbf{v} - \boldsymbol{\mathcal{V}}\boldsymbol{\psi} = \gamma^+\mathbf{F} - \boldsymbol{\varphi}_0 \quad \text{on } \partial\Omega, \tag{81c}$$

where

$$F_0 := \mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g - \Pi^d\boldsymbol{\varphi}_0, \qquad \mathbf{F} := \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g - \mathbf{W}\boldsymbol{\varphi}_0 \tag{82}$$

Using theorems 1 and 2 we have, $(F_0, \mathbf{F}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$.

We denote the right hand side of BDIE system (81a)-(81c) as

$$\boldsymbol{\mathcal{F}}^1 := [F_0, \mathbf{F},\ \gamma^+\mathbf{F} - \boldsymbol{\varphi}_0]^T, \tag{83}$$

which implies $\boldsymbol{\mathcal{F}}^1 \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$.

Note that BDIE system (81a)-(81c) can be split into the BDIE system (**D1**), of two vector equations (81b), (81c)) for two vector unknowns, $\mathbf{v}$ and $\boldsymbol{\psi}$,and the scalar equation (81a) that can be used after solving the system to obtain the pressure, p.

The system (**D1**) given by equations (81a)-(81c) can be written using matrix notation as

$$\mathcal{D}^1\mathcal{X} = \mathcal{F}^1, \tag{84}$$

where $\mathcal{X}$ represents the vector containing the unknowns of the system $\mathcal{X} = (p, \mathbf{v}, \boldsymbol{\psi}) \in L^2_*(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. The matrix operator $\mathcal{D}^1$ is defined by

$$\mathcal{D}^1 = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s \\ 0 & I+\mathcal{R} & -\mathbf{V} \\ 0 & \gamma^+\mathcal{R} & -\mathcal{V} \end{bmatrix}$$

Remark 2. The term $\mathcal{F}^1 = 0$ if and only if $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = 0$.

Suppose $\mathcal{F}^1 = 0$, then $[F_0, \mathbf{F},\ \gamma^+\mathbf{F} - \boldsymbol{\varphi}_0]^T = 0$. Now multiplying the second equation of (82) by μ and applying Stokes operator with $\mu = 1$ to these two equations (82), by (25)and (26) we obtain $\mathbf{f} = 0$.

In addition $\gamma^+\mathbf{F} - \boldsymbol{\varphi}_0 = 0$ implies $\boldsymbol{\varphi}_0 = 0$. Therefore, we obtain that $\boldsymbol{\varphi}_0 = 0$ on $\partial\Omega$ and by first equation of (82) we obtain $g = 0$. On the other hand assume that $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = 0$. Then immediately we have $\mathcal{F}^1 = 0$.

BDIE System (D2) From the equations (66), (67) and (69) we obtain

$$p + \mathcal{R}^{\bullet}\mathbf{v} - \Pi^s\boldsymbol{\psi} = F_0 \ \text{ in } \Omega, \tag{85a}$$

$$\mathbf{v} + \mathcal{R}\mathbf{v} - \mathbf{V}\boldsymbol{\psi} = \mathbf{F} \ \text{ in } \Omega, \tag{85b}$$

$$\frac{1}{2}\boldsymbol{\psi} + \mathbf{T}^+(\mathcal{R}^{\bullet}, \mathcal{R})\mathbf{v} - \mathcal{W}'\boldsymbol{\psi} = \mathbf{T}^+(F_0, \mathbf{F}) \ \text{ on } \partial\Omega, \tag{85c}$$

where F_0 and $\mathbf{F}$ are given by (82). System (**D2**) can be written in the matrix form as $\mathcal{D}^2\mathcal{X} = \mathcal{F}^2$, where

$$\mathcal{D}^2 = \begin{bmatrix} I & \mathcal{R}^{\bullet} & -\Pi^s \\ 0 & \mathbf{I}+\mathcal{R} & -\mathbf{V} \\ 0 & \mathbf{T}^+(\mathcal{R}^{\bullet}, \mathcal{R}) & \frac{1}{2}I - \mathcal{W}' \end{bmatrix}, \quad \mathcal{F}^2 = \begin{bmatrix} F_0 \\ \mathbf{F} \\ \mathbf{T}^+(F_0,\ \mathbf{F}) \end{bmatrix}$$

Note that BDIE system (85a)-(85c) can be split in to the BDIE system (**D2**), of 2 vector equations (85b), (85c)) for two vector unknowns, $\mathbf{v}$ and $\boldsymbol{\psi}$,and the scalar equation (85a) that can be used, after solving the system, to obtain the pressure, p.

Remark 3. The term $\mathcal{F}^2 = 0$ if and only if $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = 0$.

Indeed, it is evident that $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = 0$ implies $\mathcal{F}^2 = 0$. Let now $\mathcal{F}^2 = 0$. Lemma 1 with $F_0 = 0$ for p and $\mathbf{F} = \mathbf{0}$ for $\mathbf{v}$ applying to equation (82) implies that $\mathbf{f} = 0$, $g = 0$ and $\Pi^d\boldsymbol{\varphi}_0 = 0$, $\mathbf{W}\boldsymbol{\varphi}_0 = 0$ in Ω. Therefore, by Lemma 2(ii) $\boldsymbol{\varphi}_0 = 0$ on $\partial\Omega$.

In the following theorem we shall prove the equivalence of the the boundary-domain integral equation systems to original Dirichlet boundary value problem.

7.2 | Equivalence and Invertibility Theorems

Theorem 12 (Equivalence Theorem). Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$ and $\boldsymbol{\varphi}_0 \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$

(i) If some $(p, \mathbf{v}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$ solve the Dirichlet BVP (79a)-(79c), then $(p, \mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$, where

$$\boldsymbol{\psi} = \mathbf{T}^+(p, \mathbf{v}) \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{86}$$

solves the BDIE systems (**D1**) and (**D2**) .

(ii) If $(p, \mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$ solves the BDIE system (**D1**) , then $(p, \mathbf{v})$ solves the BDIE system (**D2**) and BVP (79a) -(79c), this solution is unique, and $\boldsymbol{\psi}$ satisfies (86).

(iii) If $(p, \mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (**D2**) , then $(p, \mathbf{v})$ solves the BDIE system (**D1**) and BVP (79a) -(79c), this solution is unique, and $\boldsymbol{\psi}$ satisfies (86).

Proof. (i) Let $(p, \mathbf{v}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$ be a solution of the BVP. Let us define the function $\boldsymbol{\psi}$ by (86). Taking into account the Green identities (66)- (68), we immediately obtain that $(p, \mathbf{v}, \boldsymbol{\psi})$ solves BDIE systems (**D1**) and (**D2**).

We note that if $(p, \mathbf{v}, \boldsymbol{\psi}) \in L^2_*(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ solves BDIE systems (**D1**) and (**D2**). Due to the first two equations in the BDIE systems, the hypotheses of Lemma 1 are satisfied implying that $(p, \mathbf{v}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})$ and solves PDEs (79a)-(79b) in Ω and also satisfying

$$\mathbf{V}(\boldsymbol{\psi} - \mathbf{T}^+(p, \mathbf{v})) - \mathbf{W}(\boldsymbol{\varphi}_0 - \gamma^+\mathbf{v}) = 0 \tag{87}$$

(ii) let $(p, \mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (**D1**). If we take the trace of the second equation in (**D1**)and subtracting the third equation from it, we arrive at $\gamma^+\mathbf{v} = \boldsymbol{\varphi}_0$ on $\partial\Omega$. Therefore the Dirichlet boundary is satisfied. Now using Dirichlet condition in (87), we have $\mathbf{V}(\boldsymbol{\psi} - \mathbf{T}^+(p, \mathbf{v})) = 0, \mathbf{y} \in \Omega$, Lemma 2(i) then implies $\boldsymbol{\psi} = \mathbf{T}^+(p, \mathbf{v})$.

(iii) Let $(p, \mathbf{v}, \boldsymbol{\psi}) \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ solve BDIE system (**D2**). If we take the traction of the first and second equations in (**D2**) and subtracting the third equation from it, we arrive at $\boldsymbol{\psi} = \mathbf{T}^+(p, \mathbf{v})$ on $\partial\Omega$. Therefore $\boldsymbol{\psi}$ satisfies (86) . Now putting $\boldsymbol{\psi} = \mathbf{T}^+(p, \mathbf{v})$ in (87),we have $\mathbf{W}(\boldsymbol{\varphi}_0 - \gamma^+\mathbf{v}) = 0, \mathbf{y} \in \Omega$, Lemma 2(ii) then implies $\boldsymbol{\varphi}_0 = \gamma^+\mathbf{v}$. Therefore satisfy the Dirichlet Condition. The uniqueness of the BDIE system solutions follows form Theorem 11.

□

Theorem 13. If $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\mathrm{diam}(\Omega)$ or $\boldsymbol{\psi} \in \mathbf{H}^{-\frac{1}{2}}_{**}(\partial\Omega)$, then the following operators are invertible

$$\boldsymbol{\mathcal{D}}^1 : L^2_*(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow L^2_*(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{88}$$

$$\boldsymbol{\mathcal{D}}^1 : \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega). \tag{89}$$

Proof. Theorem 12(ii) implies that operators 88 and 89 are injective. To see this, let $\boldsymbol{\mathcal{D}}^1\boldsymbol{\mathcal{X}} = \mathbf{0}$, then $\boldsymbol{\mathcal{F}}^1 = \mathbf{0}$, or $[F_0, \mathbf{F}, \ \gamma^+\mathbf{F} - \boldsymbol{\varphi}_0]^T = \mathbf{0}$ by Remark 2, which implies $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = \mathbf{0}$. This means $\boldsymbol{\mathcal{A}}(p, \mathbf{v}) = \mathbf{0}$, $\mathrm{div}\,\mathbf{v} = 0, \boldsymbol{\varphi}_0 = \gamma^+\mathbf{v} = \mathbf{0}$, hence by Theorem 12(ii), $\mathbf{v} = \mathbf{0}, p = 0, \boldsymbol{\psi} = \mathbf{0}$. Therefore, $\boldsymbol{\mathcal{X}} = \mathbf{0}$.

Let us denote

$$\widetilde{\boldsymbol{\mathcal{D}}}^1 = \begin{bmatrix} I & 0 & -\Pi^s \\ \mathbf{0} & \mathbf{I} & -\mathbf{V} \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{\mathcal{V}} \end{bmatrix}.$$

Then $\widetilde{\boldsymbol{\mathcal{D}}}^1 : L^2_*(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow L^2_*(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is continuous which is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : L^2_*(\Omega) \longrightarrow L^2_*(\Omega), \mathbf{I} : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{H}^1(\Omega)$ and $-\boldsymbol{\mathcal{V}} : \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ (see theorem 8). Due to proposition 1 the operator $\boldsymbol{\mathcal{D}}^1 - \widetilde{\boldsymbol{\mathcal{D}}}^1 : L^2_*(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow L^2_*(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ which is

$$\boldsymbol{\mathcal{D}}^1 - \widetilde{\boldsymbol{\mathcal{D}}}^1 = \begin{bmatrix} 0 & \boldsymbol{\mathcal{R}}^{\bullet} & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}} & \mathbf{0} \\ \mathbf{0} & \gamma^+\boldsymbol{\mathcal{R}} & \mathbf{0} \end{bmatrix}$$

is compact, implying that operator (88) is Fredholm operator with zero index (cf.[14], Theorem 2.27) and then the injectivity of operator (88) implies its invertibility.

To prove the invertibility of the operator (89), consider the solution $\mathcal{X} = (\boldsymbol{\mathcal{D}}^1)^{-1}\boldsymbol{\mathcal{F}}^1$ of the system (84). Here $\boldsymbol{\mathcal{F}}^1 \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is an arbitrary right hand side and $(\boldsymbol{\mathcal{D}}^1)^{-1}$ is the inverse of the operator (88) which exists. Applying Lemma 1 to the first two equations of the system (81a)- (81c), we get that $\mathcal{X} \in \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Consequently, the operator $(\boldsymbol{\mathcal{D}}^1)^{-1}$ is also the continuous inverse of the operator(89). □

Theorem 14. The operators

$$\boldsymbol{\mathcal{D}}^2 : L^2_*(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow L^2_*(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{90}$$

$$\boldsymbol{\mathcal{D}}^2 : \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \longrightarrow \mathbf{H}^{1,0}_*(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{91}$$

are invertible.

Proof. Theorem 12(iii) implies that operators 90 and 91 are injective. To see this, let $\boldsymbol{\mathcal{D}}^2\boldsymbol{\mathcal{X}} = \mathbf{0}$, then $\boldsymbol{\mathcal{F}}^2 = \mathbf{0}$, or $[F_0, \mathbf{F}, \ \mathbf{T}^+(F_0, \mathbf{F})]^T = \mathbf{0}$ by Remark 3, which implies $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = \mathbf{0}$. This means $\boldsymbol{\mathcal{A}}(p, \mathbf{v}) = \mathbf{0}$, $\mathrm{div}\,\mathbf{v} = 0$, $\boldsymbol{\varphi}_0 = \mathbf{0}$, hence by Theorem 12(iii), $\mathbf{v} = \mathbf{0}, p = 0, \boldsymbol{\psi} = \mathbf{0}$. Therefore, $\boldsymbol{\mathcal{X}} = \mathbf{0}$.

Let us denote

$$\widetilde{\mathcal{D}}^2 = \begin{bmatrix} I & 0 & -\Pi^s \\ \mathbf{0} & \mathbf{I} & -\mathbf{V} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{I} \end{bmatrix}$$

Then $\widetilde{\mathcal{D}}^2$ is continuous which is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : L^2_*(\Omega) \longrightarrow L^2_*(\Omega)$ and $\mathbf{I} : \mathbf{H}^1(\Omega) \longrightarrow \mathbf{H}^1(\Omega)$. Due to Theorem 1 and proposition 1, the operator

$$\mathcal{D}^2 - \widetilde{\mathcal{D}}^2 = \begin{bmatrix} 0 & \mathcal{R}^\bullet & 0 \\ 0 & \mathcal{R} & 0 \\ 0 & \mathbf{T}^+(\mathcal{R}^\bullet, \mathcal{R}) & -\mathcal{W}' \end{bmatrix}$$

is compact, implying that operator (90) is Fredholm operator with zero index (see,[14],Theorem 2.27) and then the injectivity of operator (90) implies its invertibility.

To prove the invertibility of the operator (91), consider the solution $\mathcal{X} = (\mathcal{D}^2)^{-1}\mathcal{F}^2$. Here $\mathcal{F}^2 \in \mathbf{H}^{1,0}_*(\Omega; \mathcal{A}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is an arbitrary right hand side and $(\mathcal{D}^2)^{-1}$ is the inverse of operator (90) which exists. Applying Lemma 1 to the first two equations of the system (85a) - (85c), we get that $\mathcal{X} \in \mathbf{H}^{1,0}_*(\Omega; \mathcal{A}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Consequently, operator $(\mathcal{D}^2)^{-1}$ is also the continuous inverse of the operator(91). □

8 | BDIE SYSTEMS FOR THE NEUMANN PROBLEM

We shall derive and investigate the BDIE systems for the following Neumann boundary value problem. *Given the functions* $\boldsymbol{\psi}_0 \in \boldsymbol{H}^{-\frac{1}{2}}(\partial\Omega)$*,* $g \in L^2(\Omega)$ *and* $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$*, find a couple of functions* $(p, v) \in L^2(\Omega) \times \boldsymbol{H}^1(\Omega)$ *satisfying,*

$$\mathcal{A}(p, \mathbf{v})(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{92a}$$

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = g(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{92b}$$

$$\mathbf{T}^+(p(\mathbf{x}), \mathbf{v}(\mathbf{x})) = \boldsymbol{\psi}_0(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega. \tag{92c}$$

Theorem 15.

i. The homogeneous problem corresponding to the BVP (92a)-(92c), admits solutions in $L^2(\Omega) \times \mathbf{H}^1(\Omega)$ spanned by $(p^0, \mathbf{v}^0) = (0, \{w_k\}_{k=1}^3)$.

ii. The non-homogeneous problem (92a)-(92c) is solvable if and only if the following solvability condition is satisfied

$$\langle \mathbf{f}, \mathbf{v}^0 \rangle_{\Omega} - \langle \boldsymbol{\psi}_0, \gamma^+ \mathbf{v}^0 \rangle_{\partial\Omega} = 0 \tag{93}$$

for each rigid body motion $\mathbf{v}^0$ of $\mathcal{R}$ in $\mathbb{R}^2$.

Proof.

i. Consider the following homogeneous Neumann BVP

$$\mathcal{A}(p, \mathbf{v})(\mathbf{x}) = \mathbf{0}, \quad x \in \Omega, \tag{94a}$$

$$\operatorname{div} \mathbf{v}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega, \tag{94b}$$

$$\mathbf{T}^+(p, \mathbf{v})(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \partial\Omega. \tag{94c}$$

The first Green identity (4) holds for any $\mathbf{u} \in \mathbf{H}^1(\Omega)$ and for any pair $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \mathcal{A})$. Then due to (94a)-(94c) we have, $0 = \int_\Omega E(\mathbf{v}, \mathbf{u})(\mathbf{x}) d\mathbf{x} := \mathcal{E}(\mathbf{v}, \mathbf{u})$, that is

$$0 = \int_\Omega E(\mathbf{v}, \mathbf{u})(\mathbf{x}) d\mathbf{x} = \int_\Omega \frac{\mu(\mathbf{x})}{2} \left(\frac{\partial u_i(\mathbf{x})}{\partial x_j} + \frac{\partial u_j(\mathbf{x})}{\partial x_i} \right) \left(\frac{\partial v_i(\mathbf{x})}{\partial x_j} + \frac{\partial v_j(\mathbf{x})}{\partial x_i} \right) d\mathbf{x}.$$

Now if we choose $\mathbf{u} = \mathbf{v}$, then we get, $\mathcal{E}(\mathbf{v}, \mathbf{v}) = 0$. As $\mu(\mathbf{x}) > 0$, the only possibility is that $\mathbf{v} \in \mathcal{R}$, i.e, $\mathbf{v}$ is a rigid movement. Considering now $\mathbf{v}$ is a rigid movement and keeping in mind equation the Neumann condition (94c), we have $p = 0$.

ii. Let $(p, \mathbf{v})$ is a solution to system (92a)-(92a), then $\mathcal{A}(p, \mathbf{v}) - \mathbf{f} = 0$. Now multiplying by $\mathbf{v}^0 \in \mathcal{R}$ and then we arrive at (93).

□

8.1 | BDIE formulations for Neumann problem

We aim to obtain a segregated boundary-domain integral equation systems for Neumann BVP (92a)-(92c). Let us denote the unknown trace as $\boldsymbol{\varphi} = \gamma^{+}\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ and will further consider $\boldsymbol{\varphi}$ as formally independent on p and $\mathbf{v}$. Assuming that the function $(p, \mathbf{v})$ satisfies system of PDEs (92a)-(92b), by substituting the Neumann condition into the third Green identities (64), (65) and either into its trace (68) or into its traction (69) on $\partial\Omega$, we can reduce the BVP (92a)- (92c) to two different systems of Boundary-Domain Integral Equations for the unknowns $(p, \mathbf{v}, \boldsymbol{\varphi}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$.

BDIE System(N1) From the equations (66), (67) and its traction (69) we obtain

$$p + \mathcal{R}^{\bullet}\mathbf{v} + \Pi^{d}\boldsymbol{\varphi} = G_0 \text{ in } \Omega, \tag{95a}$$

$$\mathbf{v} + \boldsymbol{\mathcal{R}}\mathbf{v} + \mathbf{W}\boldsymbol{\varphi} = \mathbf{G} \text{ in } \Omega, \tag{95b}$$

$$\mathbf{T}^{+}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}})\mathbf{v} + \boldsymbol{\mathcal{L}}^{+}\boldsymbol{\varphi} = \mathbf{T}^{+}(G_0, \mathbf{G}) - \boldsymbol{\psi}_0 \text{ on } \partial\Omega, \tag{95c}$$

where

$$G_0 := \mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g + \Pi^{s}\boldsymbol{\psi}_0, \qquad \mathbf{G} := \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g + \mathbf{V}\boldsymbol{\psi}_0 \tag{96}$$

By Theorem 1 and 2, $(G_0, \mathbf{G}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$.

We denote the right hand side of BDIE system (95a)-(95c) as

$$\boldsymbol{\mathcal{G}}^{1} := [G_0, \mathbf{G}, \mathbf{T}^{+}(G_0, \mathbf{G}) - \boldsymbol{\psi}_0]^{T}, \tag{97}$$

which implies $\boldsymbol{\mathcal{G}}^{1} \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}}) \times \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. In matrix form it can be written as $\boldsymbol{\mathcal{N}}^{1}\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{G}}^{1}$, where

$$\boldsymbol{\mathcal{N}}^{1} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & \Pi^{d} \\ \mathbf{0} & \mathbf{I} + \boldsymbol{\mathcal{R}} & \mathbf{W} \\ \mathbf{0} & \mathbf{T}^{+}(\mathcal{R}^{\bullet}, \boldsymbol{\mathcal{R}}) & \boldsymbol{\mathcal{L}}^{+} \end{bmatrix}, \quad \boldsymbol{\mathcal{G}}^{1} = \begin{bmatrix} G_0 \\ \mathbf{G} \\ \mathbf{T}^{+}(G_0,\ \mathbf{G}) - \boldsymbol{\psi}_0 \end{bmatrix}$$

Note that BDIE system (95a)-(95c) can be split in to the BDIE system (**N1**), of two vector equations (95b), (95c)) for two vector unknowns, $\mathbf{v}$ and $\boldsymbol{\varphi}$, and the scalar equation (95a) that can be used after solving the system, to obtain the pressure p.

Remark 4. The term $\boldsymbol{\mathcal{G}}^{1} = \mathbf{0}$ if and only if $(\mathbf{f}, g, \boldsymbol{\psi}_0) = \mathbf{0}$.

Indeed, it is evident that $(\mathbf{f}, g, \boldsymbol{\psi}_0) = \mathbf{0}$ implies $\boldsymbol{\mathcal{G}}^{1} = \mathbf{0}$. Let now $\boldsymbol{\mathcal{G}}^{1} = \mathbf{0}$ which implies $G_0 = 0$, $\mathbf{G} = \mathbf{0}$ and $\mathbf{T}^{+}(G_0,\ \mathbf{G}) - \boldsymbol{\psi}_0 = \mathbf{0}$. Thus $\boldsymbol{\psi}_0 = \mathbf{0}$. From (96) with $\boldsymbol{\psi}_0 = \mathbf{0}$, we have

$$\mathring{\mathcal{Q}}\mathbf{f} + (2-\alpha)\mu g = 0, \qquad \boldsymbol{\mathcal{U}}\mathbf{f} - \boldsymbol{\mathcal{Q}}g = \mathbf{0}.$$

multiplying by μ of the second equation above and applying the Stokes operator , we get $\mathbf{f} = \mathbf{0}$ and then using the first equation in above we also get $g = 0$.

BDIE System(N2) From the equations (66), (67) and(68) we obtain

$$p + \mathcal{R}^{\bullet}\mathbf{v} + \Pi^{d}\boldsymbol{\varphi} = G_0 \text{ in } \Omega, \tag{98a}$$

$$\mathbf{v} + \mathcal{R}\mathbf{v} + \mathbf{W}\boldsymbol{\varphi} = \mathbf{G} \text{ in } \Omega, \tag{98b}$$

$$\gamma^{+}\boldsymbol{\mathcal{R}}\mathbf{v} + \frac{1}{2}\boldsymbol{\varphi} + \boldsymbol{\mathcal{W}}\boldsymbol{\varphi} = \gamma^{+}\mathbf{G} \text{ on } \partial\Omega, \tag{98c}$$

Note that BDIE system (98a)-(98c) can be split into the BDIE system (**N2**), of two vector equations (98b), (98c)) for two vector unknowns, $\mathbf{v}$ and $\boldsymbol{\varphi}$, and the scalar equation (98a) that can be used after solving the system to obtain the pressure p.
The system (**N2**) given by equations (98a)-(98c) can be written using matrix notation as

$$\boldsymbol{\mathcal{N}}^{2}\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{G}}^{2}, \tag{99}$$

where $\boldsymbol{\mathcal{X}}$ represents the vector containing the unknowns of the system

$$\boldsymbol{\mathcal{X}} = (p, \mathbf{v}, \boldsymbol{\varphi}) \in L^{2}(\Omega) \times \mathbf{H}^{1}(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$$

The matrix operator $\boldsymbol{\mathcal{N}}^{2}$ is defined by

$$\boldsymbol{\mathcal{N}}^{2} = \begin{bmatrix} I & \mathcal{R}^{\bullet} & \Pi^{d} \\ \mathbf{0} & \mathbf{I} + \boldsymbol{\mathcal{R}} & \mathbf{W} \\ \mathbf{0} & \gamma^{+}\boldsymbol{\mathcal{R}} & \frac{1}{2}\mathbf{I} + \boldsymbol{\mathcal{W}} \end{bmatrix}, \quad \boldsymbol{\mathcal{G}}^{2} = \begin{bmatrix} G_0 \\ \mathbf{G} \\ \gamma^{+}\ \mathbf{G} \end{bmatrix}$$

Remark 5. Let $\boldsymbol{\psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and $r_0 > \frac{1}{2}\mathrm{diam}(\Omega)e^{-\frac{1}{2}}$. The term $\mathcal{G}^2 = \mathbf{0}$ if and only if $(\mathbf{f}, g, \boldsymbol{\psi}_0) = \mathbf{0}$.

Suppose $\mathcal{G}^2 = \mathbf{0}$, then $[G_0, \mathbf{G},\ \gamma^+\mathbf{G}]^T = \mathbf{0}$. Taking into account how the terms $\mathbf{G}$ and G_0 are defined in (96) considering that $G_0 = 0$ and $\mathbf{G} = \mathbf{0}$, we can deduce by applying Lemma 1 to equations (96) that $\mathbf{f} = \mathbf{0}$, $g = 0$ and

$$\Pi^s\boldsymbol{\Psi}_0 = 0, \quad \mathbf{V}\boldsymbol{\psi}_0 = \mathbf{0}$$

Therefore, by Lemma 2(i) we obtain that

$$\boldsymbol{\psi}_0 = \mathbf{0} \text{ on } \partial\Omega$$

.

In the following theorem we shall see the equivalence of the original Neumann boundary value problem to the boundary domain integral equation systems.

8.2 | Equivalence and Invertibility Theorems

Theorem 16 (Equivalence Theorem). Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$, $\boldsymbol{\psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ and satisfy the solvability condition (93)

(i) If some $(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega)$ solves the Neumann BVP (92a)- (92c), then $(p, \mathbf{v}, \boldsymbol{\varphi})$ where

$$\boldsymbol{\varphi} = \gamma^+\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{100}$$

solves BDIE system (**N1**) and (**N2**) .

(ii) If $(p, \mathbf{v}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (**N1**), then $(p, \mathbf{v})$ solves the BDIE system (**N2**) and the Neumann BVP (92a)- (92c) and the function $\boldsymbol{\varphi}$ satisfies (100).

(iii) If $(p, \mathbf{v}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (**N2**) and $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\mathrm{diam}(\Omega)$, then $(p, \mathbf{v})$ solves the BDIE system (**N1**) and the Neumann BVP (92a) - (92c) and the function $\boldsymbol{\varphi}$ satisfies (100).

(iv) The homogeneous BDIE systems (**N1**) and (**N2**) have linearly independent solution $\mathcal{X}^0 = (p^0, \mathbf{v}^0, \boldsymbol{\varphi}^0)^T = (0, \{\mathbf{w}_k\}_{k=1}^3, \{\mathbf{w}_k\}_{k=1}^3)^T$ in $L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Condition (93) is necessary and sufficient for solvability of the nonhomogeneous BDIE systems (**N1**) and (**N2**) in $L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$.

Proof. (i) Let $(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega)$ be a solution of the BVP. Since $\mathbf{f} \in \mathbf{L}^2(\Omega)$, then $(p, \mathbf{v}) \in \mathbf{H}^{1,0}(\Omega; \boldsymbol{\mathcal{A}})$. Let us define the function $\boldsymbol{\varphi}$ by (100). Taking into account the Green identities (66)- (69), we immediately obtain that $(p, \upsilon, \boldsymbol{\varphi})$ solve system (**N1**) and (N2).

(ii) let $(p, \mathbf{v}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (N1). If we take the traction of (95a) and (95b)and subtracting (95c) from it, we arrive at $\boldsymbol{\psi}_0 = \mathbf{T}^+(p, \mathbf{v})$ on $\partial\Omega$. Thus the Neumann condition is satisfied.

Also we note that if $(p, \mathbf{v}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega)$ then $\boldsymbol{\mathcal{A}}(p, \mathbf{v}) = \mathbf{f} \in \mathbf{L}^2(\Omega)$. Due to relations (95a) and (95b) the hypotheses of the Lemma 1 are satisfied. As a result we obtain that $(p, \mathbf{v})$ is a solution of $\boldsymbol{\mathcal{A}}(p, \mathbf{v}) = \mathbf{f}$ satisfying

$$\mathbf{V}(\boldsymbol{\psi}_0 - \mathbf{T}^+(p, \mathbf{v})) - \mathbf{W}(\boldsymbol{\varphi} - \gamma^+\mathbf{v}) = 0 \tag{101}$$

Now inserting $\boldsymbol{\psi}_0 = \mathbf{T}^+(p, \mathbf{v})$ in (101),we have $\mathbf{W}(\boldsymbol{\varphi} - \gamma^+\mathbf{v}) = 0, \mathbf{y} \in \Omega$, Lemma 2(ii) then implies $\boldsymbol{\varphi} = \gamma^+\mathbf{v}$. Therefore $\boldsymbol{\varphi}$ satisfies (100) .

(iii) let$(p, \mathbf{v}, \boldsymbol{\varphi}) \in L^2(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ solve BDIE system (**N2**). If we take the trace of (98b) and subtracting (98c) from it, we arrive at $\boldsymbol{\varphi} = \gamma^+\mathbf{v}$ on $\partial\Omega$. Then inserting $\boldsymbol{\varphi} = \gamma^+\mathbf{v}$ in (101) gives $\mathbf{V}(\boldsymbol{\psi}_0 - \mathbf{T}^+(p, \mathbf{v})) = 0$, Lemma 2(i) then implies $\boldsymbol{\psi}_0 = \mathbf{T}^+(p, \mathbf{v})$ on $\partial\Omega$) . Hence the Neumann condition is satisfied.

(iv) Theorem 15 along with item (i)-(iii) imply the claims of item (iv).

□

If we consider $(p, \mathbf{v}) \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \boldsymbol{\mathcal{A}})$, the Neumann BVP has a unique solution in $\mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \boldsymbol{\mathcal{A}})$ and we have the following equivalence theorem and the invertibility of the operators $\boldsymbol{\mathcal{N}}^1$ and $\boldsymbol{\mathcal{N}}^2$. Note that in this case we use the space $\mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \boldsymbol{\mathcal{A}})$ instead of $\mathbf{H}^{1,0}_{*}(\Omega; \boldsymbol{\mathcal{A}})$ as we saw from section 7.

Remark 6. The Neumann BVP (92a)- (92c) has a unique solution in $\mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})$.

Theorem 17. Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $g \in L^2(\Omega)$ and $\boldsymbol{\psi}_0 \in \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$

(i) If some $(p,\mathbf{v}) \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})$ solves the Neumann BVP (92a)- (92c), then $(p,\mathbf{v},\boldsymbol{\varphi})$ where

$$\boldsymbol{\varphi} = \gamma^+\mathbf{v} \in \mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{102}$$

solves BDIE system (**N1**) and (**N2**) .

(ii) If $(p,\mathbf{v},\boldsymbol{\varphi}) \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (**N1**), then $(p,\mathbf{v})$ solves the BDIE system (**N2**)and the Neumann BVP (92a)- (92c) and the function $\boldsymbol{\varphi}$ satisfies (102).

(iii) If $(p,\mathbf{v},\boldsymbol{\varphi}) \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ solves the BDIE system (**N2**) and $r_0 > \frac{1}{2}e^{-\frac{1}{2}}\mathrm{diam}(\Omega)$, then $(p,\mathbf{v})$ solves the BDIE system (**N1**)and the Neumann BVP (92a)- (92c) and the function $\boldsymbol{\varphi}$ satisfies (102).

(iv) The BDIE systems (**N1**) and (**N2**) are uniquely solvable in $L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega$.

Proof. The procedure of the proof is similar with the above Theorem. □

Theorem 18. The operators

$$\boldsymbol{\mathcal{N}}^1 \;:\; L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \longrightarrow L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{103}$$

$$\boldsymbol{\mathcal{N}}^1 \;:\; \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \longrightarrow \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega) \tag{104}$$

are invertible.

Proof. Theorem 17(ii) implies that operators 103 and 104 are injective. To see this, let $\boldsymbol{\mathcal{N}}^1\boldsymbol{\mathcal{X}} = \mathbf{0}$, then $\boldsymbol{\mathcal{N}}^1 = \mathbf{0}$, or $[G_0, \mathbf{G},\ \mathbf{T}^+(G_0,\mathbf{G}) - \boldsymbol{\psi}_0]^T = \mathbf{0}$ by Remark 4, which implies $(\mathbf{f}, g, \boldsymbol{\psi}_0) = \mathbf{0}$. This means $\boldsymbol{\mathcal{A}}(p,\mathbf{v}) = \mathbf{0}$, $\operatorname{div}\mathbf{v} = 0$, $\boldsymbol{\psi}_0 = \mathbf{0}$, hence by Theorem 17, $\mathbf{v} = \mathbf{0}$, $p = 0$, $\boldsymbol{\varphi} = \mathbf{0}$. Therefore, $\boldsymbol{\mathcal{X}} = \mathbf{0}$.

Let us denote

$$\widetilde{\boldsymbol{\mathcal{N}}}^1 = \begin{bmatrix} I & 0 & \Pi^d \\ \mathbf{0} & \mathbf{I} & \mathbf{W} \\ \mathbf{0} & \mathbf{0} & \hat{\boldsymbol{\mathcal{L}}} \end{bmatrix}.$$

Then $\widetilde{\boldsymbol{\mathcal{N}}}^1 \;:\; L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \longrightarrow L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ is continuous which is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I \;:\; L^2(\Omega) \longrightarrow L^2(\Omega)$, $\mathbf{I} \;:\; \mathbf{H}^1_{\mathcal{R}}(\Omega) \longrightarrow \mathbf{H}^1_{\mathcal{R}}(\Omega)$ and $\hat{\boldsymbol{\mathcal{L}}} \;:\; \mathbf{H}^{\frac{1}{2}}_{\mathcal{R}}(\partial\Omega) \longrightarrow \mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Due to proposition 1 the operator $\boldsymbol{\mathcal{N}}^1 - \widetilde{\boldsymbol{\mathcal{N}}}^1 \;:\; L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \longrightarrow L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ which is

$$\boldsymbol{\mathcal{N}}^1 - \widetilde{\boldsymbol{\mathcal{N}}}^1 = \begin{bmatrix} 0 & \mathcal{R}^\bullet & 0 \\ \mathbf{0} & \boldsymbol{\mathcal{R}} & \mathbf{0} \\ 0 & T^+(\mathcal{R}^\bullet, \boldsymbol{\mathcal{R}}) & \boldsymbol{\mathcal{L}}^+ - \hat{\boldsymbol{\mathcal{L}}} \end{bmatrix}$$

is compact, implying that operator (103) is Fredholm operator with zero index and then the injectivity of operator (103) implies its invertibility.

To prove the invertibility of the operator (104), consider the solution $\mathcal{X} = (\boldsymbol{\mathcal{N}}^1)^{-1}\mathcal{G}^1$ of (N1). Here $\mathcal{G}^1 \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$ is an arbitrary right hand side and $(\boldsymbol{\mathcal{N}}^1)^{-1}$ is the inverse of the operator (103) which exists. Applying Lemma 1 to the first two equations of the system (98a)- (98c), we get that $\mathcal{X} \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Consequently, the operator $(\boldsymbol{\mathcal{N}}^1)^{-1}$ is also the continuous inverse of the operator(104). □

Theorem 19. The operators

$$\boldsymbol{\mathcal{N}}^2 \;:\; L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \longrightarrow L^2(\Omega)\times\mathbf{H}^1_{\mathcal{R}}(\Omega)\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{105}$$

$$\boldsymbol{\mathcal{N}}^2 \;:\; \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \longrightarrow \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega;\boldsymbol{\mathcal{A}})\times\mathbf{H}^{\frac{1}{2}}(\partial\Omega) \tag{106}$$

are invertible.

Proof. Theorem 17(iii) implies that operators 105 and 106 are injective. To see this, let $\mathcal{N}^2\mathcal{X} = 0$, then $\mathcal{G}^2 = 0$, or $[G_0, \mathbf{G}, \gamma^+ G]^T = 0$ by Remark 5, which implies $(\mathbf{f}, g, \boldsymbol{\varphi}_0) = 0$. This means $\mathcal{A}(p, \mathbf{v}) = 0$, $\operatorname{div} \mathbf{v} = 0, \boldsymbol{\psi}_0 = 0$, hence by Theorem 17, $\mathbf{v} = 0, p = 0, \boldsymbol{\varphi} = 0$. Therefore, $\mathcal{X} = 0$.

Let us denote

$$\widetilde{\mathcal{N}}^2 = \begin{bmatrix} I & 0 & \Pi^d \\ \mathbf{0} & \mathbf{I} & \mathbf{W} \\ \mathbf{0} & \mathbf{0} & \frac{1}{2}\mathbf{I} \end{bmatrix}$$

Then $\widetilde{\mathcal{N}}^2$ is continuous which is bounded. It is invertible due to its triangular structure and invertibility of its diagonal operators $I : L^2(\Omega) \longrightarrow L^2(\Omega)$, $\mathbf{I} : \mathbf{H}^1_{\mathcal{R}}(\Omega) \longrightarrow \mathbf{H}^1_{\mathcal{R}}(\Omega)$ and $\mathbf{I} : \mathbf{H}^{\frac{1}{2}} \longrightarrow \mathbf{H}^{\frac{1}{2}}(\Omega)$. Due to proposition 1, the operator

$$\mathcal{N}^2 - \widetilde{\mathcal{N}}^2 = \begin{bmatrix} 0 & \mathcal{R}^{\bullet} & 0 \\ 0 & \boldsymbol{\mathcal{R}} & 0 \\ 0 & \gamma^+\boldsymbol{\mathcal{R}} & \boldsymbol{\mathcal{W}} \end{bmatrix}$$

is compact, implying that operator (105) is Fredholm operator with zero index and then the injectivity of operator (105) implies its invertibility.

To prove the invertibility of the operator (106), consider the solution $\mathcal{X} = (\mathcal{N}^2)^{-1}\mathcal{G}^2$. Here $\mathcal{G}^2 \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \mathcal{A}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$ is an arbitrary right hand side and $(\mathcal{N}^2)^{-1}$ is the inverse of the operator (105) which exists. Applying Lemma 1 to the first two equations of the system (98a) - (98c), we get that $\mathcal{X} \in \mathbf{H}^{1,0}_{\mathcal{R}}(\Omega; \mathcal{A}) \times \mathbf{H}^{\frac{1}{2}}(\partial\Omega)$. Consequently, the operator $(\mathcal{N}^2)^{-1}$ is also the continuous inverse of the operator(106). □

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