# On the exact solutions and numerical simulation of the non-homogeneous Burgers equations

**Latifa Ait Mahiout**[1], **Yassine Benia**[2]

[1] Laboratory of EDP&HM, Departement of Mathematics, E.N.S., P.O Box 92, Kouba 16050, Algiers, Algeria.
E-mail: latifaaitmahiout@gmail.com
[2] Departement of Mathematics and Informatics, university of Ben Youcef Benkhedda (Alger 1), 16000, Algiers, Algeria.
E-mail: a.benia@univ-alger.dz

**Abstract**

In this paper, special attention is paid to developing a general method to calculate an exact solution for nonhomogeneous Burgers equations in one dimensional space. For this purpose, generalized Hopf-Cole method is using. Then, the model is transformed into the linear heat equation and separable nonlinear differential equation which allows us to calculate an exact solution. We also present exact solutions for differents models. Furthermore, to consolidate the theoretical calculation, we make numerical analysis, considering a finite element discretization in space on a viscous flow model. Then, we use a special tretment on the nonlinear term. Finally, we add a numerical simulation using a finite element code FreeFem++.

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# 1 Introduction

Nonlinear evolution equations are of current interest because they play a crucial role in the study of various physical, chemical, biological and engineering problems.

The homogeneous Burgers equation $u_t+uu_x-\nu u_{xx}=0$, where $\nu$ represents the viscosity, is a model that has been solved explicitly, but the nonhomogeneous Burgers equation $u_t+uu_x-\nu u_{xx}=f(x,t)$ for specific second members $f(x,t)$ very few models have been solved. If the second member depends only on time $f(x,t)=G(t)$, this equation can be transformed into a homogeneous Burgers equation in [7], the problem with $f(x,t)=kx$, $f(x,t)=\dfrac{kx}{(2\beta t+1)^2}$ and with an elastic forcing term $f(x,t)=-k^2x+f(t)$ are discussed and analytical solutions are obtained in [11], [9] and [6], and later the problem with $f(x,t)=G(t)x$ has been resolved in [3]. In [10] different types of solution of the forced Burgers equation with variable coefficients such as shock solitary wave, triangular wave, N -wave and rational function solutions are found and discussed.

The nonlinear Burgers equation can be transformed to the linear heat equation and thus explicitly solved [5]. The linearization of the Burgers equation appeared in the twentieth century. She was discovered by Eberhard Hopf and Julian Cole, and she was named the Hopf-Cole Transformation in their honor. This transformation provides an interesting method for solving the viscous Burgers equation, and has also opened other doors for solving higher-order partial differential equations using similar methods.

In previous works (see [1, 2]) we have studied Burgers equation $\partial_t u + u\partial_x u - \partial_x^2 u = f$ (with Dirichlet boundary conditions) in a non rectangular domain $\Omega \subset \mathbb{R}^2$. When the right-hand side $f$ lies in the Lebesgue space $L^2(\Omega)$, and the initial condition is in the space $H_0^1(\Gamma_0)$, we have established the existence of a unique solution in $H^{1,2}(\Omega)$.

In this work, we consider a nonhomogeneous Burgers equation of the form

$$u_t + uu_x - u_{xx} = f(t)x + g(t), \tag{1.1}$$

where $f$ and $g$ are arbitrary functions that depend on $t$ and we obtain a new exact solutions thanks to a generalized Hopf-Cole transformation. We also present exact solutions when the right-hand side of (1.1) is $f(t)$, $g(x)$ and $e^{\alpha x+\beta t}$.

The paper is organized as follows. In the next section we present the well known Hopf-Cole transformation. The exat solution for a forced Burges equation whith $f(x,t) = f(t)x + g(t)$ will be presented in Section 3, under a particular choice of the right hand side, the solutions of Burgers equations are given. In Section 4, we expose a numerical method to calculate approximation solution. Finally, in section 5 we give an example of forced Burger equation with numerical simulation.

# 2 Hopf-Cole transformation

Consider the forced Burgers equation

$$u_t + uu_x - \nu u_{xx} = f(x,t), \quad x \in \mathbb{R}, t > 0, \nu > 0, \tag{2.2}$$

with the initial data

$$u(x,0) = u_0(x). \tag{2.3}$$

To begin, we review the association of the Burgers equation with the heat equation. The Hopf-Cole transformation, is defined by

$$u = -2\nu \frac{\varphi_x}{\varphi}. \tag{2.4}$$

So, it follows

$$u_t = \frac{2\nu(\varphi_t\varphi_x - \varphi\varphi_{xt})}{\varphi^2},$$
$$uu_x = \frac{4\nu^2\varphi_x(\varphi\varphi_{xx} - \varphi_x^2)}{\varphi^3},$$
$$\nu u_{xx} = -\frac{2\nu^2(2\varphi_x^3 - 3\varphi\varphi_x\varphi_{xx} + \varphi^2\varphi_{xxx})}{\varphi^3}.$$

By substituting in (2.2), we obtain

$$\frac{2\nu(-\varphi\varphi_{xt} + \varphi_x(\varphi_t - \nu\varphi_{xx}) + \nu\varphi\varphi_{xxx})}{\varphi^3} = f(x,t),$$

then

$$\varphi_x\left[\varphi_t - \nu\varphi_{xx} + F(x,t)\frac{\varphi}{2\nu}\right] = \varphi\left[\varphi_t - \nu\varphi_{xx} + F(x,t)\frac{\varphi}{2\nu}\right]_x$$

where

$$F(x,t) = \int f(x,t)dx + c(t).$$

Therefore, if $\varphi$ solves the equation

$$\varphi_t - \nu\varphi_{xx} = -F(x,t)\frac{\varphi}{2\nu} \tag{2.5}$$

then $u$ solves Equation (2.2).

To completely transform the problem, we still have to work with the initial condition function. To do this, note that (2.4) can be written according to

$$u = -2\nu\big[\log\varphi\big]_x,$$

then

$$\varphi(x,t) = e^{\left(-\int_0^x \frac{u(y,t)}{2\nu}\mathrm{d}y\right)},$$

and

$$\varphi(x,0) = \varphi_0(x) = e^{\left(-\int_0^x \frac{u_0(y)}{2\nu}\mathrm{d}y\right)}$$

In summary, we have reduced the problem (2.2) with the initial data (2.3) to this one

$$\begin{cases} \varphi_t - \nu\varphi_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad \nu > 0, \\ \varphi(x,0) = \varphi_0(x) = e^{\left(-\int_0^x \frac{u_0(y)dy}{2\nu}\right)}, \quad x \in \mathbb{R}. \end{cases}$$

## 3 The explicit solution of the Equation (1.1)

We obtain new exact solutions for the forced Burgers equation

$$u_t + uu_x - \nu u_{xx} = f(t)x + g(t), \ t > 0, x \in \mathbb{R}, \tag{3.6}$$

with initial condition

$$u(x,0) = u_0(x), \tag{3.7}$$

where $f$, $g$ are arbitrary functions that depend on $t$, using the following transformation

$$\begin{aligned} u(x,t) &= -2\nu\alpha(t)\frac{\partial_z v(\tau,z)}{v(\tau,z)} + b(t)x + c(t), \\ \tau = \tau(t), &\quad z = \alpha(t)x + \beta(t). \end{aligned} \tag{3.8}$$

The functions $\alpha(t), \beta(t), \tau(t)$ and $z(x,t)$ are to be determine.
We have

$$\begin{aligned} v_t(\tau,z) &= \tau'(t)v_\tau(\tau,z) + (\alpha'(t)x + \beta'(t))v_z(\tau,z), \\ (v_t)_z(\tau,z) &= \tau'(t)(v_\tau)_z(\tau,z) + (\alpha'(t)x + \beta'(t))v_{zz}(\tau,z), \\ v_x(\tau,z) &= \alpha(t)v_z(\tau,z), \\ (v_x)_z(\tau,z) &= \alpha(t)v_{zz}(\tau,z), \\ (v^2)_{xx}(\tau,z) &= 2\alpha(t)vv_z(\tau,z), \\ ((v_z)_x)^2 &= 2\alpha(t)(v_z)(v_{zz}), \end{aligned}$$

then, we get

$$\begin{aligned} u_t = &-2\nu\alpha'(t)\frac{v_z}{v} - 2\nu\alpha(t)\frac{(\alpha'x+\beta')\,vv_{zz} + \tau'v(v_\tau)_z - (\alpha'x+\beta')\,(v^2)_z\tau'v_\tau v_z}{v^2} \\ &+ b'(t)x + c'(t), \end{aligned}$$

$$u_x = -2\nu\alpha^2(t)\frac{vv_{zz} - (v_{zz})^2}{v^2} + b(t),$$

$$\begin{aligned} uu_x = &\, 4\nu\alpha^3(t)\frac{vv_z(v)_{zz} - v_z(v^2)_z}{v^3} - 2\nu\alpha^2(t)(b(t)x + c(t))\frac{vv_{zz} - (v^2)_z}{v^2} \\ &- 2\nu\alpha(t)b(t)\frac{v_z}{v} + b^2(t)x + b(t)c(t), \end{aligned}$$

$$u_{xx} = -2\nu\alpha^3(t)\left(\frac{v_{zzz}}{v} - 3\frac{v_zv_{zz}}{v^2} + 2\frac{(v_z)^3}{v^3}\right)$$

So, equation (3.6) can be written as follows:

$$-2\nu\alpha(t)\tau'(t)\frac{(v_\tau)_z}{v}+2\nu^2\alpha^3(t)\frac{v_{zzz}}{v}-2\nu^2\alpha^3(t)\frac{v_zv_{zz}}{v^2}+2\nu\alpha(t)\tau'(t)\frac{v_\tau v_z}{v^2}-2\nu(\alpha'(t)+\alpha(t)b(t))\frac{v_z}{v}$$
$$+2\nu\alpha(t)((\alpha(t)b(t)+\alpha'(t))x+\alpha(t)c(t)+\beta'(t))\frac{(v_z)^2}{v^2}-2\nu\alpha(t)((\alpha(t)b(t)+\alpha'(t))x$$
$$+\alpha(t)c(t)+\beta'(t))\frac{v_{zz}}{v}+(b^2(t)+b'(t)-f(t))x+b(t)c(t)+c'(t)-g(t)=0,$$

because $\dfrac{v_z(v^2)_z}{v^3}=\dfrac{(v_z)^3}{v^3}$.

By considering the following conditions

$$\begin{gathered}\tau'(t)=\nu\alpha^2(t),\\ \alpha'(t)+\alpha(t)b(t)=0,\\ \beta'(t)+\alpha(t)c(t)=0,\\ b(t)c(t)+c'(t)-g(t)=0,\\ b^2(t)+b'(t)-f(t)=0,\end{gathered}$$

we get

$$2\nu\alpha'(t)\tau'(t)\left[\frac{(v_\tau)_z}{v}-\frac{v_{zzz}}{v}+\frac{v_zv_{zz}}{v^2}-\frac{v_\tau v_z}{v^2}\right]=0.$$

Then $v$ satisfies the equation

$$v(v_\tau-v_{zz})-v_z(v_\tau-v_{zz})=0,$$

wich yields to the linear heat equation

$$v_\tau-v_{zz}=0.$$

The unknown functions satisfy the following system of ordinary differential equation

$$\begin{cases}\tau'(t)=\nu\alpha^2(t),\\ \alpha'(t)=-\alpha(t)b(t),\\ \beta'(t)=-\alpha(t)c(t),\\ c'(t)=g(t)-b(t)c(t).\end{cases}$$

To solve this system we start with the third equation, and we obtain

$$\alpha(t)=c_1\exp\left(-\int b(t)\mathrm{d}t\right), \tag{3.9}$$

$$\beta(t) = c_2 - \int \alpha(t)c(t)\mathrm{d}t, \tag{3.10}$$

$$\tau(t) = c_3 + \nu \int \alpha^2(t)\mathrm{d}t \tag{3.11}$$

$$c(t) = \alpha(t) \int \frac{g(t)}{\alpha(t)}\mathrm{d}t - c_4\alpha(t), \tag{3.12}$$

where $c_1, c_2, c_3$ and $c_4$ are arbitrary constants.

Then, solving equation (3.6) is equivalent to solve the following two equations

$$v_\tau = v_{zz},\ b'(t) = -b^2(t) + f(t). \tag{3.13}$$

## 3.1 Burgers equation with other second member

In this section we present exact solutions for the following forced Burgers equations

$$u_t + uu_x - \nu u_{xx} = f(t), \tag{3.14}$$
$$u_t + uu_x - \nu u_{xx} = g(x), \tag{3.15}$$
$$u_t + uu_x - \nu u_{xx} = \alpha e^{\alpha x+\beta t}, \tag{3.16}$$

where $f$, is a function that depends on $t$, and $g$ is a function that depends on $x$.

By the Hopf-Cole transformation $u = -2\nu\dfrac{\partial_x\varphi}{\varphi}$ we find that if $\varphi$ solve equations

$$\varphi - \nu\varphi_{xx} + \frac{1}{2\nu}xf(t)\varphi = 0, \tag{3.17}$$

$$\varphi - \nu\varphi_{xx} + G(x)\varphi = 0, \quad (\text{where } G(x) = \int g(x)\,\mathrm{d}x) \tag{3.18}$$

$$\varphi - \nu\varphi_{xx} + \frac{1}{2\nu}e^{\alpha x+\beta t}\varphi = 0, \tag{3.19}$$

then $u$ solve Equations (3.14), (3.15) and (3.16).

For equation (3.17), we can use the transformation

$$\varphi(x,t) = w(z,t)\exp\left(xF(t) + \nu\int F^2(t)dt\right),$$
$$z = x + 2\nu\int F(t)dt,$$

where $F(t) = -\dfrac{1}{2\nu}\displaystyle\int f(t)dt$.

Then

$$\begin{aligned}\varphi_t &= \left(w_t + w_z z_t + \left(-\frac{1}{2\nu}xf(t) + \nu F^2(t)\right) w\right)\exp\left(xF(t) + \nu\int F^2(t)dt\right)\\ &= \left(w_t + 2\nu F(t)w_z + \left(-\frac{1}{2\nu}xf(t) + \nu F^2(t)\right) w\right)\exp\left(xF(t) + \nu\int F^2(t)dt\right),\end{aligned}$$

$$\begin{aligned}\varphi_x &= (w_z z_x + F(t)w)\exp\left(xF(t) + \nu\int F^2(t)dt\right)\\ &= (w_z + F(t)w)\exp\left(xF(t) + \nu\int F^2(t)dt\right),\end{aligned}$$

and

$$\begin{aligned}\varphi_{xx} &= (w_{zz} + F(t)w_z + F(t)\left(w_z + F(t)w\right))\exp\left(xF(t) + \nu\int F^2(t)dt\right)\\ &= \left(w_{zz} + 2F(t)w_z + F^2(t)w\right)\exp\left(xF(t) + \nu\int F^2(t)dt\right).\end{aligned}$$

Submiting the precedent calculations into (3.17), we get

$$\begin{aligned}& w_t + 2\nu F(t)w_z - \frac{1}{2\nu}xf(t)w + \nu F^2(t)w\\ &- \nu w_{zz} - 2\nu F(t)\partial_z w - \nu F^2(t)w + \frac{1}{2\nu}xf(t)w = 0,\end{aligned}$$

which leads to the heat equation

$$w_t - \nu w_{zz} = 0.$$

Equation (3.18) has particular solutions in the form

$$\varphi(x,t) = e^{\lambda t}w(x),$$

where $\lambda$ is an arbitrary constant and $w$ is solutiono of an ordinary differential equation.

In fact,

$$\lambda e^{\lambda t}w - \nu e^{\lambda t}w'' + G(x)e^{\lambda t}w = 0,$$

then

$$\lambda w'' + (G(x) + \lambda)w = 0.$$

**Remark 1.** *From the above we can obtain a particular solution for forced Burgers equation whith second member $f(t) + g(x)$,* For Equation (3.19), we can use the transformation

$$\varphi(x,t) = w(z,t)e^{\mu x}, \quad z = x + \frac{\beta}{\alpha}t \text{ where } \mu = \frac{\beta}{2\nu\alpha}.$$

We have

$$\begin{aligned}\varphi_t &= (w_z\partial_t z + w_t)\, e^{\mu}\left(z + \frac{\beta}{\alpha}t\right)\\ &= \left(\frac{\beta}{\alpha}w_z + w_t\right) e^{\mu}\left(z + \frac{\beta}{\alpha}t\right),\end{aligned}$$

$$\begin{aligned}\varphi_x &= (w_z z_x + \mu w)\, e^{\mu}\left(z - \frac{\beta}{\alpha}t\right)\\ &= (w_z + \mu w)\, e^{\mu}\left(z - \frac{\beta}{\alpha}t\right),\end{aligned}$$

and

$$\begin{aligned}\varphi_{xx} &= \left(\partial_x(w_z) + \mu w_x + \mu w_x + \mu^2 w\right) e^{\mu}(z - \frac{\beta}{\alpha}t)\\ &= \left(w_{zz} + 2\mu w_z + \mu^2 w\right) e^{\mu}(z - \frac{\beta}{\alpha}t).\end{aligned}$$

Submitting into (3.17), we obtain

$$w_t - \nu\partial_z^2 w + \left(\frac{1}{2\nu}e^{\alpha z} - 3\nu\mu^2\right) w = 0.$$

witch is of the form (3.18) where $G(z) = \dfrac{1}{2\nu}e^{\alpha z} - 3\nu\mu^2$.

# 4 Numerical Analysis

In this section we consider the 1 D equation of forced Burgers equation

$$u_t + uu_x - \nu u_{xx} = f(x,t),$$

and consider a finite element discretization in space. We use a special treatment of the nonlinear term to utilize the fact that we solve the time dependent problem.

## 4.1 The boundary value problem

### 4.1.1 PDE formulation.

We consider the one dimensional forced Burgers equation on the interval $[0, \infty] \times [0, 1]$:

$$u_t + uu_x - \nu u_{xx} = f(t, x), \tag{4.20}$$

with Dirichlet boundary conditions

$$u(t, 0) = g_0(t),\ u(t, 1) = g_1(t),\ t > 0,$$

and initial condition

$$u(0, x) = u_0(x),\ x \in [0, 1],$$

where $g_i(t)$, $i = 1, 2$ and $u_0(x)$ are continuous and they meet continuously at $(0, 0)$ and $(0, 1)$.

For reasons that do not concern us here we rewrite Equation (4.20) in an equivalent form

$$u_t + \frac{1}{2}(u^2)_x - \nu u_{xx} = f(x, t).$$

### 4.1.2 Variational formulation.

To find a weak formulation we first of all need a smooth function that satisfies the bonddary condition

$$G(t, x) = (1 - x)g_0(t) + xg_1(t),$$

and try to find $w$ suche that $u = w + G$ satisfies the differential equation:

$$w_t + G_t + \frac{1}{2}\,(w + G)_x^2 - \nu w_{xx} = f(x, t),$$

or

$$w_t + \frac{1}{2}\,(w + G)_x^2 - \nu w_{xx} = f(x, t) + (g_0)_t - (g_1)_t.$$

Now we look for $w \in H_0^1(0, 1)$ such that $w(0, x) = u_0 - G(0, x)$ and

$$\int_0^1 v(w_t + \frac{1}{2}(w + G)_x^2 - \nu w_{xx})\ \mathrm{d}x = \int_0^1 v(f(t, x) + (g_0)_t - (g_1)_t)\ \mathrm{d}x, \tag{4.21}$$

and this should hold $\forall v \in H_0^1(0, 1)$.

Now we see that Equation (4.21) is completely equivalent to the weak formulation: find $u(t, x) \in H^1(0, 1)$, such that $u(0, x) = u_0(x),\ u(t, 0) = g_0(t),\ u(t, 1) = g_1(t)$

and

$$\int_0^1 v\left(u_t + \frac{1}{2}(u_x)^2\right) + \nu v_x u_x \mathrm{d}x = \int_0^1 v f(t, x)\mathrm{d}x,\ \forall v \in H_0^1(0, 1). \tag{4.22}$$

#### 4.1.3 Finite Element formulation.

We shall first formulate a finite element discretization in space and look at the time integration later on. We subdevise the interval $(0, 1)$ into $N$ subintervals $1 \dots N$ with nodes $x_0, x_1, \dots, x_N$ and use piece-wise linear basis functions $\phi_i(x)$, such that $\phi_j(x_i) = 0,\ i \neq j$ and $\phi_i(x_i) = 1$.

We now put $u_h(t, x) = \sum_{k=0}^{n} u_k(t)\phi_k(x)$. The finite dimensional approximation now becomes:

$$\int_0^1 \phi_j \left( (u_h)_t + \frac{1}{2}(u_h^2)_x \right) + \nu\phi_{jx}u_{hx}\ \mathrm{d}x = \int_0^1 \phi_j f_h(t)\ \mathrm{d}x,$$

for all $\phi_1, \phi_2, \dots, \phi_{n-1}$. This gives us a set of $n-1$ differential equations of the form:

$$M(u_h)_t + +\frac{1}{2}D(u_h)^2 + \nu S u_h = M f_h.$$

The matrix $M$ is called mass matrix and its elements are:

$$m_{kj} = \int_0^1 \phi_k\phi_j \mathrm{d}x.$$

The convection term leads to an expression of the form

$$\frac{1}{2}\int_0^1 \phi_j \sum_{k=0}^{n} u_h^2\phi_{kx}\ \mathrm{d}x,$$

which leads to an expression of the form:

$$\frac{1}{2}Du_h^2,$$

with $D$ the difference matrix with elements

$$d_{jk} = \int_0^1 \phi_j\phi_{kx}\ \mathrm{d}x$$

Finally we have the second order term of the form

$$\nu\int_0^1 \phi_{jx} \sum_{k=0}^{n} u_h\phi_{kx}\ \mathrm{d}x,$$

or

$$\nu S u_h,$$

with $S$ the stiffness matrix with elements

$$s_{kj} = \int_0^1 \phi_{kx}\phi_{jx}\ \mathrm{d}x.$$

Let us write those symbolically as

$$\tilde{M}u_h + \frac{1}{2}\tilde{D}(u_h)^2 + \nu\tilde{S}u = Mf_h + b_h, \tag{4.23}$$

where the first and the last component of $b_h$ contain contributions of boundary values, the other components are 0. In the sequel we drop tildes.

## 4.2 Time integration

if $\nu$ is not really small, say $> 0.01$, this system behaves like a real second order system in space and we shall treat it like that. If $\nu$ gets much smaller than that the system becomes a singularity perturbed first order system (really one dimensional Euler) and develops boundary layers that require spacial attention.

### 4.2.1 Implicit or explicit.

Since we have a mass matrix in the time derivative we have to lump the mass matrix to be able to work explicitly, but expecially for not too small $\nu$ (the case we consider rigt now) implicit methods are way better stabilitywise and we shall concentrate on that, because we only have 3-diagonal matrices to deal with and solving those systems takes only $O(N)$ flops.

### 4.2.2 Straight Crank Nicolson.

To integrate a system of the form

$$M(u_h)_t = f_h(t, u_h)$$

over one time step we use the trapezoid rule:

$$\int_{t_k}^{t_{k+1}} M(u_h)_t \, \mathrm{d}t = \int_{t_k}^{tk+1} f_h(t, u_h) \, \mathrm{d}t$$

$$Mu_h^{n+1} = Mu_h^n + \frac{1}{2}\Delta t(f_h\left(t, u_h^n\right) + f_h(t, u_h^{n+1})\big) . \tag{4.24}$$

This is an non linear system of equations in $u_h^{n+1}$ that we have to solve iteratively by a method like Newton.

This is fairly cumbersome, but our system offers some possibilities, since we only have a slightly non linear term.

### 4.2.3 Crank Nicolson for Equation (4.23).

We apply (4.24) to (4.23) to obtain dropping the tildes and moving all unknowns to the left hand side:

$$(M+\frac{1}{2}\Delta t\nu S)u_h n+1+\frac{1}{4}\Delta t\ D(u_h^{n+1})^2=(M-\frac{1}{2}\Delta t\ \nu S)u_h^n-\frac{1}{4}\Delta t\ D(u_h^n)^2+\Delta t\ r_h^{n+\frac{1}{2}}, \quad (4.25)$$

where $r_h^{n+\frac{1}{2}}$ is short for all known things on the right hand side at time level $t_n+\frac{1}{2}\Delta t$. We could iterate this as folows:

Let $\mathbf{w_0}=u_h^n$, solve

$$\left(M+\frac{1}{2}\Delta t\nu S\right)\mathbf{w^{k+1}}+\frac{1}{2}\Delta t\ D(\mathbf{w^k})^2=(M-\frac{1}{2}\Delta t\ \nu S)u_h^n-\frac{1}{2}\Delta t\ D(b_h^n)+\Delta t\ r_h^{n+\frac{1}{2}} \quad (4.26)$$

a couple of times to obtain $\mathbf{u}^{n+1}$.

## 4.3 The Ad van Oman trick for quadric terms.

The following clever trick was devised by Ad van Ommen, that does away with all iterations and still maintains the same accuracy as Crank Nicolson, that is $\mathcal{O}(\Delta t^2)$.

Instead of taking

$$\frac{1}{2}\left(D\left((u_h^n)^2+(u_h^{n+1})^2\right)\right),$$

for the quadratic term (trapezoid rule) he takes the rectangle rule

$$D(u_h^{n+\frac{1}{2}})^2,$$

and approximates that by

$$D{u_h}^n u_h^{n+1}.$$

Now all time integration steps only involve the solution of linear tridiagonal systems.

Let $\searrow u^n$ denote the diagonal matrix with elements of $\mathbf{u^n}$ on the diagonal. The system becomes:

$$M+\frac{1}{2}\Delta t\ \left((D\searrow u^n)+\nu S\right)u_h^{n+1}=\left(M-\frac{1}{2}\Delta t\ \nu S\right)u_h^n+\Delta t\ r_h^{n+\frac{1}{2}},$$

and does no longer involve any non linearities.

# 5 Example with numerical simulations.

Taking $f(t) = g(t) = 1$ in the problem (2.2) , we calculate an exact solution of the problem

$$u_t + uu_x - \nu u_{xx} = 1 + x, \ t > 0, x \in \mathbb{R}. \tag{5.27}$$

Using the method described on section 3, we find the following exact solution uex$(x, t)$:

$$\mathrm{uex}(x,t) = -2\nu \frac{1}{1 + x + 1/2e^{-t}} + x + 1 + e^{-t}.$$

Using a finite element code FreeFem++ we compute a numerical solution of the problem (5.27) for $\nu = 0.01$ and for all $x \in ]0, 1[$. We present the graph of approximate solution $u_h$ as a function of $x$ and $t$.

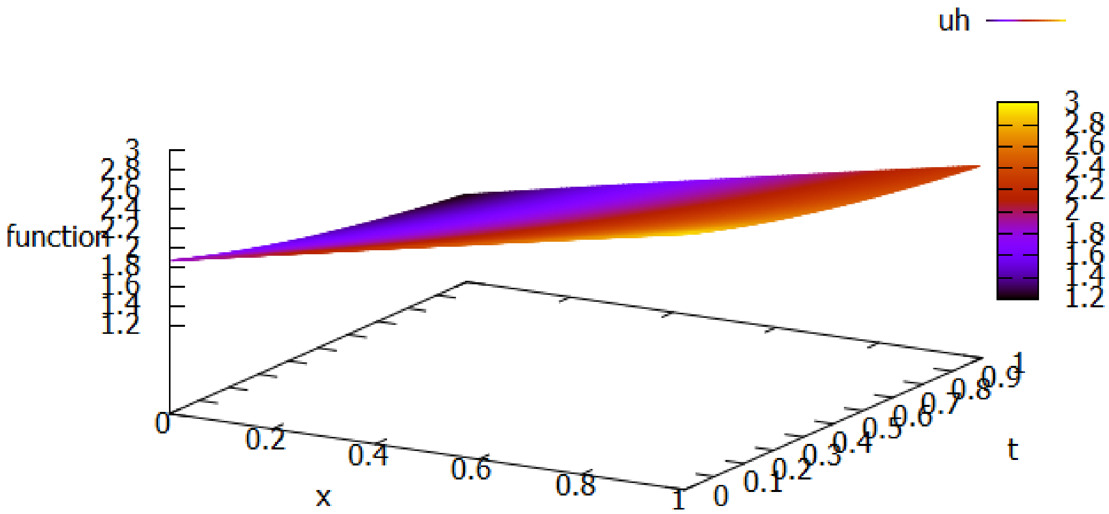


Figure 1: Graph of $u_h$ as a function of $x$ and $t$.

The following graph represent the numerical error in norm of $L^2$ between the exact solution uex and approximate solution $u_h$ as function of $t$.

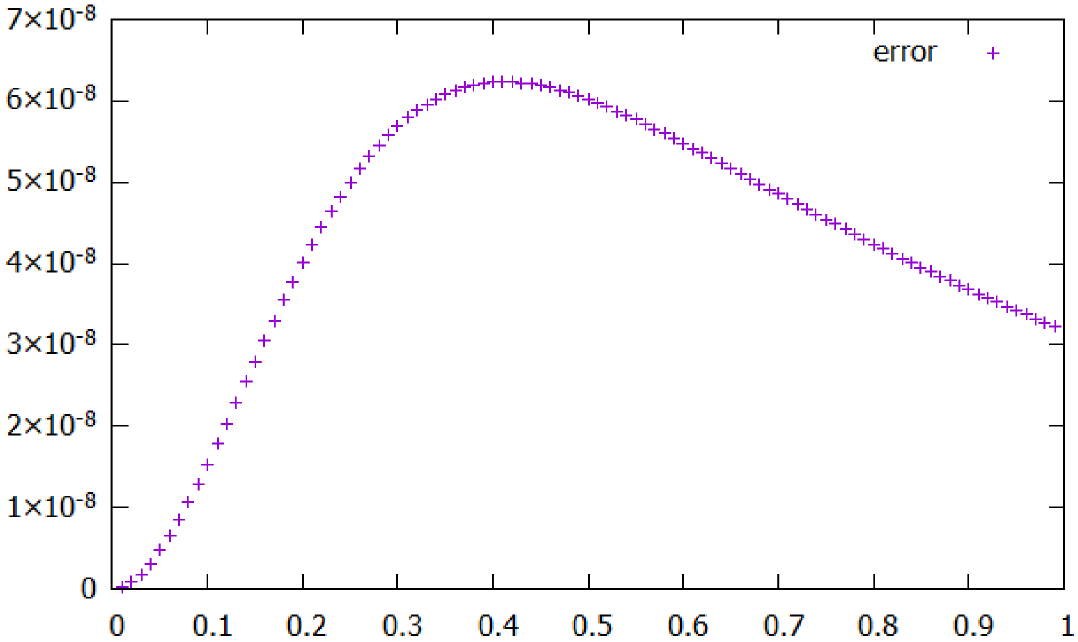


Figure 2: Graph of the numerical error in norm of $L^2$ between uex and $u_h$ as function of $t$.

These results show that the finite element method provides a satisfactory approximation to the exact solution.

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