# MONTGOMERY IDENTITY AND OSTROWSKI TYPE INEQUALITIES FOR A NEW QUANTUM INTEGRAL

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Abstract. In this paper, quantum version of Montgomery identity by applying $q^b$-integral is obtained. By employing the established identity, some new quantum Ostrowski type inequalities are proved. Several special cases of our main results are presented. It is also shown that the results presented in this paper generalize several other well known inequalities given in the existing literature on this subject.

## 1. Introduction

Various types of integral inequalities have attracted the attention of several mathematicians over the last decades. These inequalities play a significant role in the study of various classes of equations such as integro differential equations and impulse differential equations. That is why, a vast amount of research activities is being carried out on this subject.

The following is classical integral inequality associated with the differentiable mappings ( [29]) :

**Theorem 1.** *If the mapping $F:[\mu,\nu]\to\mathbb{R}$ is differentiable on $(\mu,\nu)$ and integrable on $[\mu,\nu]$. Then following inequality holds:*

$$\left|F(\varkappa)-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}F(s)ds\right|\leq\left[\frac{1}{4}+\frac{\left(\varkappa-\frac{\mu+\nu}{2}\right)^{2}}{(\nu-\mu)^{2}}\right](\nu-\mu)\left\|F'\right\|_{\infty},$$

*for all $\varkappa\in[\mu,\nu]$, where $\left\|F'\right\|_{\infty}=\sup\limits_{s\in(\mu,\nu)}|F'(s)|<+\infty$. Moreover, $\frac{1}{4}$ is the best possible constant.*

**Theorem 2.** *[13] Suppose that $F:[\mu,\nu]\to\mathbb{R}$ is a differentiable on $(\mu,\nu)$ and integrable on $[\mu,\nu]$. If $|F'(\varkappa)|\leq M$, for every $\varkappa\in[\mu,\nu]$. Then following inequality*

$$\left|F(\varkappa)-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}F(s)\,ds\right|\leq\frac{M}{\nu-\mu}\left[\frac{(\varkappa-\mu)^{2}+(\nu-\varkappa)^{2}}{2}\right]\tag{1.1}$$

holds.

Following is the well known Montgomery identity:

**Lemma 1.** *[24] If the mapping $F:[\mu,\nu]\to\mathbb{R}$ is differentiable on $(\mu,\nu)$ and integrable on $[\mu,\nu]$. Then following identity holds:*

$$F(\varkappa)-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}F(s)\,ds=\int_{\mu}^{\varkappa}\frac{s-\mu}{\nu-\mu}F'(s)\,ds+\int_{\varkappa}^{\nu}\frac{s-\nu}{\nu-\mu}F'(s)\,ds.\tag{1.2}$$

By changing the variables, we can rewrite (1.2) in the following way:

**Lemma 2.** *[30, Lemma 1] Suppose that $F:[\mu,\nu]\to\mathbb{R}$ is differentiable on $(\mu,\nu)$ and integrable on $[\mu,\nu]$. Then the following equality holds:*

$$F(\varkappa)-\frac{1}{\nu-\mu}\int_{\mu}^{\nu}F(s)\,ds=(\nu-\mu)\left[\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}}sF'(s\mu+(1-s)\nu)\,ds+\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}(s-1)F'(s\mu+(1-s)\nu)\,ds\right].\tag{1.3}$$

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On the other hand, Quantum calculus (shortly, $q$-calculus) deals with the concepts of calculus without limits, where the classical mathematical results are obtained on taking limit as $q \to 1$.

The study of $q$-calculus was initiated in the early 20th century after the work of Jackson (1910) who defined an integral later known as the $q$-Jackson integral ( [11, 14, 18, 20]). In $q$-calculus, the classical derivative is replaced by the $q$-difference operator in order to deal with non-differentiable functions, For more discussion on this subject, we refer to [5, 12]. Applications of $q$-calculus can be found in the various disciplines of mathematics and physics ( [8, 17, 31, 34]).

Many well known integral inequalities such as Hölder inequality, Hermite-Hadamard inequality, Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss-Cebysev and other integral inequalities in classical analysis have been proved and applied in the setup of $q$-calculus using the classical concept of convexity. For more results in this direction, we refer to [1–4, 10, 16, 19, 23, 26–28, 32, 35].

The purpose of this paper is to study Ostrowski type inequalities for convex functions by applying newly defined concept of $q^{\nu}$-integral. We also discuss the relation of the results obtained herein with comparable results in the existing literature.

The organization of this paper is as follows: In Section 2, we summarize the concept of $q$-calculus and some related work in this setup is given. In Section 3, the proof of Montgomery identity for $q^{\nu}$-integral is given. Using the Montgomery identity for $q^{\nu}$-integral, Ostrowski type inequalities are obtained. Some special cases of our main results are presented in Section 4. The relation of the obtained results with the comparable results in the existing literature is also discussed. Section 5 contains some conclusions and further directions for the future research. We believe that the study initiated in this paper may provide a good source of inspiration to the researchers working on integral inequalities and its applications.

## 2. Preliminaries of $q$-Calculus and Some Inequalities

In this section, we first present some known definitions and related inequalities in $q$-calculus. Set the following notation( [20]):

$$[r]_q = \frac{1-q^r}{1-q} = 1 + q + q^2 + ... + q^{r-1}, \quad q \in (0,1).$$

Jackson [18] defined the $q$-Jackson integral of a given function $F$ from 0 to $\nu$ as follows:

$$\int_0^{\nu} F(\varkappa) \ d_q\varkappa \ = (1-q)\,\nu \sum_{n=0}^{\infty} q^n F(\nu q^n), \text{ where } 0 < q < 1 \tag{2.1}$$

provided that the sum converges absolutely.

Jackson [18] defined the $q$-Jackson integral of a given function over the interval $[\mu, \nu]$ as follows:

$$\int_{\mu}^{\nu} F(\varkappa) \ d_q\varkappa \ = \int_0^{\nu} F(\varkappa) \ d_q\varkappa \ - \int_0^{\mu} F(\varkappa) \ d_q\varkappa \ .$$

**Theorem 3.** *(Hölder's inequality, [7, p. 604]) Suppose that $\varkappa > 0$, $0 < q < 1$, $p_1 > 1$. If $\frac{1}{p_1} + \frac{1}{r_1} = 1$. Then*

$$\int_0^{\varkappa} |F(\varkappa) g(\varkappa)| \, d_q\varkappa \leq \left( \int_0^{\varkappa} |F(\varkappa)|^{p_1} d_q\varkappa \right)^{\frac{1}{p_1}} \left( \int_0^{\varkappa} |g(\varkappa)|^{r_1} d_q\varkappa \right)^{\frac{1}{r_1}}.$$

**Definition 1.** *[33] Let $F : [\mu, \nu] \to \mathbb{R}$ be a continuous function. The $q_{\mu}$-derivative of $F$ at $\varkappa \in [\mu, \nu]$ is identified by the following expression*

$$_{\mu}D_q F(\varkappa) \ = \frac{F(\varkappa) - F(q\varkappa + (1-q)\mu)}{(1-q)(\varkappa - \mu)}, \ \varkappa \neq \mu. \tag{2.2}$$

*Since $F : [\mu, \nu] \to \mathbb{R}$ is a continuous function, we have $_{\mu}D_q F(\mu) \ = \lim_{\varkappa \to \mu} \ _{\mu}D_q F(\varkappa)$ . The function $F$ is said to be $q$- differentiable on $[\mu, \nu]$ if $_{\mu}D_q F(\varkappa)$ exists for all $\varkappa \in [\mu, \nu]$. If we take $\mu = 0$ in (2.2),* *then we have* ${}_0D_qF(\varkappa)=D_qF(\varkappa)$ *, where* $D_qF(\varkappa)$ *is a known q-derivative of F at* $\varkappa\in[\mu,\nu]$ *in ( [20]) given by*

$$D_qF(\varkappa)=\frac{F(\varkappa)-F(q\varkappa)}{(1-q)\varkappa},\ \varkappa\neq 0.$$

**Definition 2.** *[33] Let* $F:[\mu,\nu]\to\mathbb{R}$ *be a continuous function. The* $q_\mu$*-definite integral on* $[\mu,\nu]$ *is defined by*

$$\int_\mu^\nu F(\varkappa)\ {}_\mu d_q\varkappa=(1-q)(\nu-\mu)\sum_{n=0}^{\infty}q^nF(q^n\nu+(1-q^n)\mu)=(\nu-\mu)\int_0^1 F((1-s)\mu+s\nu)\ d_qs\ .$$

Kunt et al. [22] obtained the following Montgomery identity for $q_\mu$-definite integrals:

**Lemma 3.** *If* $F:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ *is a q-differentiable function on* $(\mu,\nu)$ *such that* ${}_\mu D_qF$ *is continuous and integrable on* $[\mu,\nu]$*, then we have:*

$$F(\varkappa)-\frac{1}{\nu-\mu}\int_\mu^\nu F(s)\ {}_\mu d_qs=(\nu-\mu)\int_0^1\Lambda_q(s)\ {}_\mu D_qF(s\nu+(1-s)\mu)\ {}_0d_qs\ ,$$

*where*

$$\Lambda_q(s)=\begin{cases} qs, & s\in\left[0,\frac{\varkappa-\mu}{\nu-\mu}\right]\\ qs-1, & s\in\left(\frac{\varkappa-\mu}{\nu-\mu},1\right]\end{cases}$$

*and* $0<q<1$.

On the other hand, Bermudo et al. [9] gave the following definitions:

**Definition 3.** *[9] Let* $F:[\mu,\nu]\to\mathbb{R}$ *be a continuous function. The* $q^\nu$*-derivative of F at* $\varkappa\in[\mu,\nu]$ *is given by*

$${}^\nu D_qF(\varkappa)=\frac{F(q\varkappa+(1-q)\nu)-F(\varkappa)}{(1-q)(\nu-\varkappa)},\ \varkappa\neq\nu.$$

**Definition 4.** *[9] Let* $F:[\mu,\nu]\to\mathbb{R}$ *be a continuous function. The* $q^\nu$*-definite integral on* $[\mu,\nu]$ *is given by*

$$\int_\mu^\nu F(\varkappa)\ {}^\nu d_q\varkappa=(1-q)(\nu-\mu)\sum_{n=0}^{\infty}q^nF(q^n\mu+(1-q^n)\nu)=(\nu-\mu)\int_0^1 F(s\mu+(1-s)\nu)\ d_qs\ .$$

## 3. Montgomery Identity and Ostrowski Type Inequalities for quantum Integrals

In this section we first prove quantum Montgomery identity for $q^\nu$-definite integrals. Then by using this identity, we establish some Ostrowski type inequalities.

Let's start with the following useful Lemma which is a Montgomery identity for $q^\nu$-integral.

**Lemma 4** (Quantum Montgomery Identity)**.** *If* $F:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ *is a q-differentiable function on* $(\mu,\nu)$ *such that* ${}^\nu D_qF$ *is continuous and integrable on* $[\mu,\nu]$*, then we have*

$$\frac{1}{\nu-\mu}\int_\mu^\nu F(s)\ {}^\nu d_qs-F(\varkappa)=(\nu-\mu)\int_0^1\Psi_q(s)\ {}^\nu D_qF(s\mu+(1-s)\nu)\,d_qs, \tag{3.1}$$

*where*

$$\Psi_q(s)=\begin{cases} qs, & s\in\left[0,\frac{\nu-\varkappa}{\nu-\mu}\right]\\ qs-1, & s\in\left(\frac{\nu-\varkappa}{\nu-\mu},1\right]\end{cases}$$

*and* $0<q<1$.

*Proof.* It follows from the Definition 3 that

$$^{\nu}D_qF\left(s\mu+(1-s)\nu\right)=\frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)(\nu-\mu)s}.$$

Note that

$$\begin{aligned}
&(\nu-\mu)\int_0^1 \Psi_q(s)\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)d_qs \qquad (3.2)\\
=&\ (\nu-\mu)\left[\int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs+\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (qs-1)\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs\right]\\
=&\ (\nu-\mu)\left[\int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs+\int_0^1 (qs-1)\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs\right.\\
&\left.-\int_0^{\frac{\nu-\varkappa}{\nu-\mu}} (qs-1)\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs\right]\\
=&\ (\nu-\mu)\left[\int_0^{\frac{\nu-\varkappa}{\nu-\mu}} {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs+\int_0^1 (qs-1)\ {}^{\nu}D_qF\left(s\mu+(1-s)\nu\right)\ d_qs\right]\\
=&\ \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} \frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)s}d_qs\\
&+q\int_0^1 \frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)}d_qs\\
&-\int_0^1 \frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)s}d_qs.
\end{aligned}$$

By the equality (2.1), we obtain

$$\begin{aligned}
&\int_0^{\frac{\nu-\varkappa}{\nu-\mu}} \frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)s}d_qs \qquad (3.3)\\
=&\ \sum_{n=0}^{\infty}F\left(q^{n+1}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)\mu+\left(1-q^{n+1}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)\right)\nu\right)\\
&-\sum_{n=0}^{\infty}F\left(q^{n}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)\mu+\left(1-q^{n}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)\right)\nu\right)\\
=&\ F(\nu)-F\left(\left(\frac{\nu-\varkappa}{\nu-\mu}\right)\mu+\left(1-\left(\frac{\nu-\varkappa}{\nu-\mu}\right)\right)\nu\right)\\
=&\ F(\nu)-F(\varkappa)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)s}d_qs \qquad (3.4)\\
=&\ \sum_{n=0}^{\infty}F\left(q^{n+1}\mu+\left(1-q^{n+1}\right)\nu\right)-\sum_{n=0}^{\infty}F\left(q^{n}\mu+\left(1-q^{n}\right)\nu\right)\\
=&\ F(\nu)-F(\mu).
\end{aligned}$$

Now by (2.1) and Definition 4, we have

$$
\begin{aligned}
(3.5)\qquad & \int_0^1 \frac{F\left(qs\mu+(1-qs)\nu\right)-F\left(s\mu+(1-s)\nu\right)}{(1-q)}d_qs \\
= & \sum_{n=0}^{\infty} q^n F\left(q^{n+1}\mu+\left(1-q^{n+1}\right)\nu\right)-\sum_{n=0}^{\infty} q^n F\left(q^n\mu+\left(1-q^n\right)\nu\right) \\
= & \frac{1}{q}\sum_{n=0}^{\infty} q^{n+1} F\left(q^{n+1}\mu+\left(1-q^{n+1}\right)\nu\right)-\sum_{n=0}^{\infty} q^n F\left(q^n\mu+\left(1-q^n\right)\nu\right) \\
= & \frac{1}{q}\sum_{n=0}^{\infty} q^n F\left(q^n\mu+\left(1-q^n\right)\nu\right)-\frac{1}{q}F(\mu)-\sum_{n=0}^{\infty} q^n F\left(q^n\mu+\left(1-q^n\right)\nu\right) \\
= & \left(\frac{1}{q}-1\right)\sum_{n=0}^{\infty} q^n F\left(q^n\mu+\left(1-q^n\right)\nu\right)-\frac{1}{q}F(\mu) \\
= & \frac{1}{q(\nu-\mu)}\int_\mu^\nu F(s)\ {}^{\nu}d_qs-\frac{1}{q}F(\mu).
\end{aligned}
$$

Using (3.3), (3.4) and (3.5) in (3.2), we have the required identity (3.1). □

**Remark 1.** *On taking limit as $q\to 1^-$ in the Lemma 4, identity (3.1) reduces to (1.3).*

**Theorem 4.** *Suppose that $F:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ is a $q$-differentiable function on $(\mu,\nu)$ and ${}^{\nu}D_qF$ is continuous and integrable on $[\mu,\nu]$. If $\left|{}^{\nu}D_qF\right|^{p_1}$ is convex on $[\mu,\nu]$, where $p_1\geq 1$, then we have following inequality*

$$
\begin{aligned}
& \left|\frac{1}{\nu-\mu}\int_\mu^\nu F(s)\ {}^{\nu}d_qs-F(\varkappa)\right| \\
\leq\ & (\nu-\mu)\Big[A_1^{1-\frac{1}{p_1}}(\mu,\nu,q,\varkappa)\left(\left|{}^{\nu}D_qF(\mu)\right|^{p_1}A_2(\mu,\nu,q,\varkappa)+\left|{}^{\nu}D_qF(\nu)\right|^{p_1}A_3(\mu,\nu,q,\varkappa)\right)^{\frac{1}{p_1}} \\
& +A_4^{1-\frac{1}{p_1}}(\mu,\nu,q,\varkappa)\left(\left|{}^{\nu}D_qF(\mu)\right|^{p_1}A_5(\mu,\nu,q,\varkappa)+\left|{}^{\nu}D_qF(\nu)\right|^{p_1}A_6(\mu,\nu,q,\varkappa)\right)^{\frac{1}{p_1}}\Big],
\end{aligned}
$$

*where*

$$
\begin{aligned}
A_1(\mu,\nu,q,\varkappa) &= \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs\,d_q\varkappa=\frac{q}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^2, \\
A_2(\mu,\nu,q,\varkappa) &= \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs^2\,d_q\varkappa=\frac{q}{1+q+q^2}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^3, \\
A_3(\mu,\nu,q,\varkappa) &= \int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs\,d_q\varkappa-\int_0^{\frac{\nu-\varkappa}{\nu-\mu}} qs^2\,d_q\varkappa=A_1(\mu,\nu,q,\varkappa)-A_2(\mu,\nu,q,\varkappa), \\
A_4(\mu,\nu,q,\varkappa) &= \int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (1-qs)\,d_qs=\int_0^1(1-qs)\,d_qs-\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}(1-qs)\,d_qs \\
&= \frac{1-q}{1+q}\left(\frac{\varkappa-\mu}{\nu-\mu}\right)+\frac{q}{1+q}\left(\frac{\varkappa-\mu}{\nu-\mu}\right)^2, \\
A_5(\mu,\nu,q,\varkappa) &= \int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 \left(s-qs^2\right)d_qs=\int_0^1\left(s-qs^2\right)d_qs-\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}\left(s-qs^2\right)d_qs \\
&= \frac{1}{(1+q)\left(1+q+q^2\right)}-\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^2+\frac{q}{1+q+q^2}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^3,
\end{aligned}
$$

*and*

$$\begin{aligned}
A_6(\mu,\nu,q,\varkappa) &= \int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}(1-s)(1-qs)\,d_qs \\
&= \int_0^1(1-s)(1-qs)\,d_qs-\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}(1-s)(1-qs)\,d_qs \\
&= \int_0^1(1-qs)\,d_qs-\int_0^1\left(s-qs^2\right)d_qs-\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}(1-qs)\,d_qs+\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}\left(s-qs^2\right)d_qs \\
&= A_4(\mu,\nu,q,\varkappa)-A_5(\mu,\nu,q,\varkappa),
\end{aligned}$$

*where* $0<q<1$.

*Proof.* By Lemma 4 and quantum power mean inequality, we obtain that

$$\begin{aligned}
&\left|\frac{1}{\nu-\mu}\int_\mu^\nu F(s)\ {}^{\nu}d_qs-F(\varkappa)\right| \\
\leq\ & (\nu-\mu)\int_0^1|\Psi_q(s)|\ \left|{}^{\nu}D_qF(s\mu+(1-s)\nu)\right|d_qs \\
\leq\ & (\nu-\mu)\left[\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}qs\ \left|{}^{\nu}D_qF(s\mu+(1-s)\nu)\right|d_qs+\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1(1-qs)\ \left|{}^{\nu}D_qF(s\mu+(1-s)\nu)\right|d_qs\right] \\
\leq\ & (\nu-\mu)\left[\left(\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}qs\ d_qs\right)^{1-\frac{1}{p_1}}\left(\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}qs\left|{}^{\nu}D_qF(s\mu+(1-s)\nu)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1(1-qs)\ d_qs\right)^{1-\frac{1}{p_1}}\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1(1-qs)\left|{}^{\nu}D_qF(s\mu+(1-s)\nu)\right|^{p_1}d_qs\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

As $\left|{}^{\nu}D_qF\right|^{p_1}$ is convex on $[\mu,\nu]$, we have

$$\begin{aligned}
&\left|\frac{1}{\nu-\mu}\int_\mu^\nu F(s)\ {}^{\nu}d_qs-F(\varkappa)\right| \\
\leq\ & (\nu-\mu)\left[\left(\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}qs\ d_qs\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(\int_0^{\frac{\nu-\varkappa}{\nu-\mu}}qs\left(\left|{}^{\nu}D_qF(\mu)\right|^{p_1}s+\left|{}^{\nu}D_qF(\nu)\right|^{p_1}(1-s)\right)d_qs\right)^{\frac{1}{p_1}} \\
&+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1(1-qs)\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1(1-qs)\left(\left|{}^{\nu}D_qF(\mu)\right|^{p_1}s+\left|{}^{\nu}D_qF(\nu)\right|^{p_1}(1-s)\right)d_qs\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= (\nu-\mu)\left[\left(\int_0^{\frac{\varkappa-\mu}{\nu-\mu}} qs\ d_q s\right)^{1-\frac{1}{p_1}}\right. \\
&\quad\times\left(\left|{}^{\nu}D_q F(\mu)\right|^{p_1}\int_0^{\frac{\varkappa-\mu}{\nu-\mu}} qs^2 d_q s+\left|{}^{\nu}D_q F(\nu)\right|^{p_1}\int_0^{\frac{\varkappa-\mu}{\nu-\mu}}\left(qs-qs^2\right)d_q s\right)^{\frac{1}{p_1}} \\
&\quad+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (1-qs)\ d_q s\right)^{1-\frac{1}{p_1}} \\
&\quad\left.\times\left(\left|{}^{\nu}D_q F(\mu)\right|^{p_1}\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 s(1-qs)\,d_q s+\left|{}^{\nu}D_q F(\nu)\right|^{p_1}\int_{\frac{\varkappa-\mu}{\nu-\mu}}^1 (1-s)(1-qs)\,d_q s\right)^{\frac{1}{p_1}}\right] \\
&= (\nu-\mu)\left[A_1^{1-\frac{1}{p_1}}(\mu,\nu,q,\varkappa)\left(\left|{}^{\nu}D_q F(\mu)\right|^{p_1}A_2(\mu,\nu,q,\varkappa)+\left|{}^{\nu}D_q F(\nu)\right|^{p_1}A_3(\mu,\nu,q,\varkappa)\right)^{\frac{1}{p_1}}\right. \\
&\quad\left.+A_4^{1-\frac{1}{p_1}}(\mu,\nu,q,\varkappa)\left(\left|{}^{\nu}D_q F(\mu)\right|^{p_1}A_5(\mu,\nu,q,\varkappa)+\left|{}^{\nu}D_q F(\nu)\right|^{p_1}A_6(\mu,\nu,q,\varkappa)\right)^{\frac{1}{p_1}}\right]
\end{aligned}
$$

which completes the proof. □

**Theorem 5.** *Suppose that $F:[\mu,\nu]\subset\mathbb{R}\to\mathbb{R}$ is a $q$-differentiable on $(\mu,\nu)$ and ${}^{\nu}D_qF$ is continuous and integrable on $[\mu,\nu]$. If $\left|{}^{\nu}D_qF\right|^{p_1}$ is convex on $[\mu,\nu]$ for some $p_1>1$ with $\frac{1}{r_1}+\frac{1}{p_1}=1$, then we have,*

$$
\begin{aligned}
&\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu} F(s)\ {}^{\nu}d_q s-F(\varkappa)\right| \qquad (3.6)\\
&\leq (\nu-\mu)\left[\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^{1+\frac{1}{r_1}}\left(\frac{q}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\right. \\
&\quad\times\left(\left|{}^{\nu}D_q F(\mu)\right|^{p_1}\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^2+\left|{}^{\nu}D_q F(\nu)\right|^{p_1}\left(\frac{\nu-\varkappa}{\nu-\mu}-\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^2\right)\right)^{\frac{1}{p_1}} \\
&\quad+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^1 (1-qs)^{r_1}\,d_q s\right)^{\frac{1}{r_1}} \\
&\quad\times\left(\left|{}^{\nu}D_q F(\mu)\right|^{p_1}\left(\frac{1}{1+q}\left(1-\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^2\right)\right)\right. \\
&\quad\left.\left.+\left|{}^{\nu}D_q F(\nu)\right|^{p_1}\left(\frac{q}{1+q}-\frac{\nu-\varkappa}{\nu-\mu}+\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^2\right)\right)^{\frac{1}{p_1}}\right],
\end{aligned}
$$

*where $0<q<1$.*

*Proof.* On taking the modulus in the Lemma 4 and applying the quantum Hölder's inequality, we obtain that

$$
\begin{aligned}
&\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu} F(s)\ {}^{\nu}d_q s-F(\varkappa)\right| \\
&\leq (\nu-\mu)\int_0^1 |\Psi_q(s)|\left|{}^{\nu}D_q F(s\mu+(1-s)\nu)\right|d_q s
\end{aligned}
$$

$$\leq \left(\nu-\mu\right)\left[\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}} qs\left|\ {}^{\nu}D_{q}F\left(s\mu+\left(1-s\right)\nu\right)\right|d_{q}s+\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}\left(1-qs\right)\left|\ {}^{\nu}D_{q}F\left(s\mu+\left(1-s\right)\nu\right)\right|d_{q}s\right]$$

$$\leq \left(\nu-\mu\right)\left[\left(\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}}\left(qs\right)^{r_{1}}d_{q}s\right)^{\frac{1}{r_{1}}}\left(\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}}\left|{}^{\nu}D_{q}F\left(s\mu+\left(1-s\right)\nu\right)\right|^{p_{1}}d_{q}s\right)^{\frac{1}{p_{1}}}\right.$$
$$\left.+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}\left(1-qs\right)^{r_{1}}d_{q}s\right)^{\frac{1}{r_{1}}}\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}\left|{}^{\nu}D_{q}F\left(s\mu+\left(1-s\right)\nu\right)\right|^{p_{1}}d_{q}s\right)^{\frac{1}{p_{1}}}\right].$$

Using an assumption that $|{}^{\nu}D_{q}F|^{p_{1}}$ is convex on $[\mu,\nu]$, we have

$$\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu}F\left(s\right)\ {}^{\nu}d_{q}s-F\left(\varkappa\right)\right|$$

$$\leq \left(\nu-\mu\right)\left[\left(\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}}\left(qs\right)^{r_{1}}d_{q}s\right)^{\frac{1}{r_{1}}}\left(\left|\ {}^{\nu}D_{q}F\left(\mu\right)\right|^{p_{1}}\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}}sd_{q}s+\left|\ {}^{\nu}D_{q}F\left(\nu\right)\right|^{p_{1}}\int_{0}^{\frac{\nu-\varkappa}{\nu-\mu}}\left(1-s\right)d_{q}s\right)^{\frac{1}{p_{1}}}\right.$$
$$\left.+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}\left(1-qs\right)^{r_{1}}d_{q}s\right)^{\frac{1}{r_{1}}}\left(\left|\ {}^{\nu}D_{q}F\left(\mu\right)\right|^{p_{1}}\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}sd_{q}s+\left|\ {}^{\nu}D_{q}F\left(\nu\right)\right|^{p_{1}}\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}\left(1-s\right)d_{q}s\right)^{\frac{1}{p_{1}}}\right].$$

$$= \left(\nu-\mu\right)\left[\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^{1+\frac{1}{r_{1}}}\left(\frac{q}{\left[r_{1}+1\right]_{q}}\right)^{\frac{1}{r_{1}}}\right.$$
$$\times\left(\left|\ {}^{\nu}D_{q}F\left(\mu\right)\right|^{p_{1}}\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^{2}+\left|\ {}^{\nu}D_{q}F\left(\nu\right)\right|^{p_{1}}\left(\frac{\nu-\varkappa}{\nu-\mu}-\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^{2}\right)\right)^{\frac{1}{p_{1}}}$$
$$+\left(\int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1}\left(1-qs\right)^{r_{1}}d_{q}s\right)^{\frac{1}{r_{1}}}$$
$$\times\left(\left|\ {}^{\nu}D_{q}F\left(\mu\right)\right|^{p_{1}}\left(\frac{1}{1+q}\left(1-\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^{2}\right)\right)\right.$$
$$\left.\left.+\left|\ {}^{\nu}D_{q}F\left(\nu\right)\right|^{p_{1}}\left(\frac{q}{1+q}-\frac{\nu-\varkappa}{\nu-\mu}+\frac{1}{1+q}\left(\frac{\nu-\varkappa}{\nu-\mu}\right)^{2}\right)\right)^{\frac{1}{p_{1}}}\right]$$

which completes the proof. □

## 4. Some Special Cases

In this section, some special cases of our main results are discussed and several new results in the field of Ostrowski and Midpoint type inequalities are obtained.

**Remark 2.** *If we consider Theorem 4. Then*

*(i) by using* $\left|\ {}^{\nu}D_{q}F\left(\varkappa\right)\right|\leq M$ *and taking* $p_{1}=1$ *in Theorem 4, we have following new quantum Ostrowski type inequality*

(4.1)
$$\left|\frac{1}{\nu-\mu}\int_{\mu}^{\nu}F\left(s\right)\ {}^{\nu}d_{q}s-F\left(\varkappa\right)\right|\leq\frac{M}{\left(1+q\right)\left(\nu-\mu\right)}\left[q\left(\varkappa-\mu\right)^{2}+\left(1-q\right)\left(\varkappa-\mu\right)\left(\nu-\mu\right)+q\left(\nu-\varkappa\right)^{2}\right].$$

*Moreover, on taking limit as* $q\to1^{-}$ *in (4.1), the inequality (4.1) reduces to (1.1).*

*(ii) by choosing* $\varkappa = \frac{\mu+q\nu}{1+q}$*, we have the following new midpoint type inequality*

$$\begin{aligned} (4.2) \quad & \left| \frac{1}{\nu-\mu} \int\limits_{\mu}^{\nu} F(s) \ {}^{\nu}d_q s - F\left(\frac{\mu+q\nu}{1+q}\right) \right| \\ \leq \ & (\nu-\mu) \left[ \left( \frac{q}{(1+q)^3} \right)^{1-\frac{1}{p_1}} \right. \\ & \times \left( \left| {}^{\nu}D_q F(\mu) \right|^{p_1} \left( \frac{q}{(1+q)^3 (1+q+q^2)} \right) + \left| {}^{\nu}D_q F(\nu) \right|^{p_1} \left( \frac{q^2+q^3}{(1+q)^3 (1+q+q^2)} \right) \right)^{\frac{1}{p_1}} \\ & + \left( \frac{q}{(1+q)^3} \right)^{1-\frac{1}{p_1}} \\ & \left. \times \left( \left| {}^{\nu}D_q F(\mu) \right|^{p_1} \left( \frac{2q}{(1+q)^3 (1+q+q^2)} \right) + \left| {}^{\nu}D_q F(\nu) \right|^{p_1} \left( \frac{-q+q^2+q^3}{(1+q)^3 (1+q+q^2)} \right) \right)^{\frac{1}{p_1}} \right]. \end{aligned}$$

*Specifically, on applying the limit as* $q \to 1^-$ *and taking* $p_1 = 1$*, we have following midpoint type inequality in [21, Theorem 2.2]*

$$\left| \frac{1}{\nu-\mu} \int\limits_{\mu}^{\nu} F(s)\, ds - F\left(\frac{\mu+\nu}{2}\right) \right| \leq (\nu-\mu) \left[ \frac{|F'(\mu)| + |F'(\nu)|}{8} \right].$$

*(iii) by taking* $r_1 = 1$ *in Theorem 4, we have the following inequality*

$$\begin{aligned} (4.3) \quad & \left| \frac{1}{\nu-\mu} \int\limits_{\mu}^{\nu} F(s) \ {}^{\nu}d_q s - F(\varkappa) \right| \\ \leq \ & (\nu-\mu) \left\{ \left| {}^{\nu}D_q F(\mu) \right| \left[ A_2(\mu,\nu,q,\varkappa) + A_5(\mu,\nu,q,\varkappa) \right] \right. \\ & \left. + \left| {}^{\nu}D_q F(\nu) \right| \left[ A_3(\mu,\nu,q,\varkappa) + A_6(\mu,\nu,q,\varkappa) \right] \right\}. \end{aligned}$$

*(iv) by setting* $r_1 = 1$*,* $\varkappa = \frac{\mu+q\nu}{1+q}$*, we have following new midpoint type inequality in [10],*

$$\begin{aligned} (4.4) \quad & \left| \frac{1}{\nu-\mu} \int\limits_{\mu}^{\nu} F(s) \ {}^{\nu}d_q s - F\left(\frac{\mu+q\nu}{1+q}\right) \right| \\ \leq \ & (\nu-\mu) \left[ \left| {}^{\nu}D_q F(\mu) \right| \left( \frac{3q}{(1+q)^3 (1+q+q^2)} \right) + \left| {}^{\nu}D_q F(\nu) \right| \left( \frac{-q+2q^2+2q^3}{(1+q)^3 (1+q+q^2)} \right) \right]. \end{aligned}$$

**Remark 3.** *If we consider Theorem 5. Then*

*(i) by using* $\left| {}^{\nu}D_q F(\varkappa) \right| \leq M$ *in Theorem 5, we have the following new quantum Ostrowski type inequality*

$$\begin{aligned} (4.5) \quad & \left| \frac{1}{\nu-\mu} \int\limits_{\mu}^{\nu} F(s) \ {}^{\nu}d_q s - F(\varkappa) \right| \\ \leq \ & M(\nu-\mu) \left[ \left( \frac{\nu-\varkappa}{\nu-\mu} \right)^2 \left( \frac{q}{[r_1+1]_q} \right)^{\frac{1}{r_1}} + \left( \int_{\frac{\nu-\varkappa}{\nu-\mu}}^{1} (1-qs)^{r_1} d_q s \right)^{\frac{1}{r_1}} \left( \frac{\varkappa-\mu}{\nu-\mu} \right)^{\frac{1}{p_1}} \right]. \end{aligned}$$

*Specifically on taking limit as $q \to 1^-$, we obtain the following Ostrwoski type inequality given in [6, Theorem 3 with s=1];*

$$\left| \frac{1}{\nu-\mu} \int_{\mu}^{\nu} F(s)\,ds - F(\varkappa) \right| \leq \frac{M(\nu-\mu)}{(r_1+1)^{\frac{1}{r_1}}} \left[ \left( \frac{\varkappa-\mu}{\nu-\mu} \right)^2 + \left( \frac{\nu-\varkappa}{\nu-\mu} \right)^2 \right]$$

*(ii) by choosing $\varkappa = \frac{\mu+q\nu}{1+q}$, we have following new midpoint type inequality*

$$\begin{aligned} (4.6) \quad & \left| \frac{1}{\nu-\mu} \int_{\mu}^{\nu} F(s)\ {}^{\nu}d_q s - F\left( \frac{\mu+q\nu}{1+q} \right) \right| \\ \leq & (\nu-\mu) \left[ \left( \frac{1}{1+q} \right)^{1+\frac{1}{r_1}} \left( \frac{q}{[r_1+1]_q} \right)^{\frac{1}{r_1}} \left( \left| {}^{\nu}D_q F(\mu) \right|^{p_1} \frac{1}{(1+q)^3} + \left| {}^{\nu}D_q F(\nu) \right|^{p_1} \frac{2q+q^2}{(1+q)^3} \right)^{\frac{1}{p_1}} \right. \\ & \left. + \left( \int_{\frac{1}{1+q}}^{1} (1-qs)^{r_1}\, d_q s \right)^{\frac{1}{r_1}} \left( \left| {}^{\nu}D_q F(\mu) \right|^{p_1} \frac{2q+q^2}{(1+q)^3} + \left| {}^{\nu}D_q F(\nu) \right|^{p_1} \frac{q^3+q^2-q}{(1+q)^3} \right) \right]. \end{aligned}$$

*Specifically on taking limit as $q \to 1^-$, we obtain following midpoint type inequality*

$$\begin{aligned} & \left| \frac{1}{\nu-\mu} \int_{\mu}^{\nu} F(s)\,ds - F\left( \frac{\mu+\nu}{2} \right) \right| \\ \leq & \frac{\nu-\mu}{4(r_1+1)^{\frac{1}{r_1}}} \left[ \left( \frac{|F'(\mu)|^{p_1} + 3|F'(\nu)|^{p_1}}{4} \right)^{\frac{1}{p_1}} + \left( \frac{3|F'(\mu)|^{p_1} + |F'(\nu)|^{p_1}}{4} \right)^{\frac{1}{p_1}} \right] \\ \leq & \left( \frac{\nu-\mu}{4} \right) \left( \frac{4}{r_1+1} \right)^{\frac{1}{r_1}} \left[ |F'(\mu)| + |F'(\nu)| \right] \end{aligned}$$

*which was proved by Kırmacı in [21].*

It is worth mentioning that the deduced results (4.1)-(4.6) are also new in the literature on Ostrowski inequalities.

## 5. Conclusions

In this paper, Ostrowski type inequalities for convex functions by applying recently defined $q^{\nu}$-integral are obtained. It is also shown that the results proved in this paper are potential generalization of the existing comparable results in the literature. As future directions, one may finds the similar inequalities through different types of convexities.

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