Iq-HERMITE-HADAMARD INCLUSIONS FOR THE INTERVAL-VALUED FUNCTIONS OF TWO VARIABLES

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Abstract. In this work, we introduce the concept of double quantum integrals for the interval-valued functions of two variables. We offer several new inclusions of the Hermite-Hadamard type for co-ordinated convex interval-valued functions using the newly defined integrals. Moreover, we prove trapezoidal type inequalities for interval-valued functions of two variables using the ideas of the Pompeiu–Hausdorff distance between the intervals. It is also revealed that the results offered in this work are the generalization of several existing results.

1. Introduction

Many studies have recently been carried out in the field of q-analysis, starting with Euler due to a high demand for mathematics that models quantum computing q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other sciences quantum theory, mechanics, and the theory of relativity [13–16, 18]. Apparently, Euler was the founder of this branch of mathematics, by using the parameter q in Newton's work of infinite series. Later, Jackson was the first to develop q-calculus that known without limits calculus in a systematic way [13]. In 1908-1909, Jackson defined the general q-integral and q-difference operator [16]. In 1969, Agarwal described the q-fractional derivative for the first time [1]. In 1966-1967, Al-Salam introduced a q-analogues of the Riemann-Liouville fractional integral operator and q-fractional integral operator [2]. In 2004, Rajkovic gave a definition of the Riemann-type q-integral which was generalized of Jackson q-integral. In 2013, Tariboon introduced ${}_{\alpha}D_q$-difference operator [6].

Many integral inequalities well known in classical analysis such as Hölder inequality, Simpson's inequality, Newton's inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev, and other integral inequalities have been proved and applied for q-calculus using classical convexity. Many mathematicians have done studies in q-calculus analysis, the interested reader can check [9–11, 17, 20, 22, 23, 28–31].

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1. *A function $\Phi : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ, for all $(x, u), (y, v) \in \Delta$ and $\tau, \sigma \in [0, 1]$, if it satisfies the following inequality:*

$$\begin{aligned} &\Phi(\tau x + (1-\tau)\ y, \sigma u + (1-\sigma)\ v) \\ \leq\ & \tau\sigma\ \Phi(x,u) + \tau(1-\sigma)\Phi(x,v) + \sigma(1-\tau)\Phi(y,u) + (1-\tau)(1-\sigma)\Phi(y,v). \end{aligned} \tag{1.1}$$

The mapping Φ is a co-ordinated concave on Δ if the inequality (1.1) holds in reversed direction for all $\tau, \sigma \in [0, 1]$ and $(x, u), (y, v) \in \Delta$.

In [12], Dragomir proved the following inequalities which are Hermite-Hadamard type inequalities for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$.

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Theorem 1. *Suppose that $\Phi:\Delta\to\mathbb{R}$ is co-ordinated convex, then we have the following inequalities:*

$$
\begin{aligned}
(1.2)\qquad \Phi\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}\right) &\leq \frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi\left(u,\frac{\gamma+\delta}{2}\right)du+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi\left(\frac{\alpha+\beta}{2},v\right)dv\right] \\
&\leq \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ dvdu \\
&\leq \frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(u,\gamma)du+\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(u,\delta)du\right. \\
&+ \left.\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi(\alpha,v)dv+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi(\beta,v)dv\right] \\
&\leq \frac{\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)+\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{4}.
\end{aligned}
$$

The above inequalities are sharp. The inequalities in (1.2) hold in reverse direction if the mapping Φ is a co-ordinated concave mapping.

2. Interval Calculus

We give notation and preliminary information about the interval analysis in this section. Let the space of all closed intervals of $\mathbb{R}$ denoted by I_γ and K be a bounded element of I_γ, we have the representation

$$K=\left[\underline{k},\overline{k}\right]=\left\{\tau\in\mathbb{R}:\underline{k}\ \leq\tau\leq\overline{k}\right\}$$

where $\underline{k},\overline{k}\in\mathbb{R}$ and $\underline{k}\leq\overline{k}$. The length of the interval $K=\left[\underline{k},\overline{k}\right]$ can be stated as $L(K)=\overline{k}-\underline{k}$. The numbers $\underline{k}$ and $\overline{k}$ are called the left and the right endpoints of interval K, respectively. When $\overline{k}=\underline{k}$, the interval K is said to be degenerate and we use the form $K=k=[k,k]$. Also, we can say that K is positive if $\underline{k}>0$, or we can say that K is negative if $\overline{k}<0$. The sets of all closed positive intervals of $\mathbb{R}$ and closed negative intervals of $\mathbb{R}$ are denoted by I_γ^+ and I_γ^-, respectively. The Pompeiu–Hausdorff distance between the intervals K and M is defined by

$$d_H(K,M)=d_H\left(\left[\underline{k},\overline{k}\right],\left[\underline{m},\overline{m}\right]\right)=\max\left\{\left|\underline{k}-\underline{m}\right|,\left|\overline{k}-\overline{m}\right|\right\}. \tag{2.1}$$

(I_γ,d) is known to be a complete metric space (see, [7]).
The absolute value of K is denoted by $|K|$, is the maximum of the absolute values of its endpoints:

$$|K|=\max\left\{\left|\underline{k}\right|,\left|\overline{k}\right|\right\}.$$

Now, we mention the definitions of fundamental interval arithmetic operations for the intervals K and M as follows:

$$
\begin{aligned}
K+M &= \left[\underline{k}+\underline{m},\overline{k}+\overline{m}\right], \\
K-M &= \left[\underline{k}-\overline{m},\overline{k}-\underline{m}\right], \\
K\cdot M &= [\min U,\max U]\ \text{ where } U=\left\{\underline{k}\ \underline{m},\underline{k}\ \overline{m},\ \overline{k}\underline{m},\overline{k}\ \overline{m}\right\}, \\
K/M &= [\min V,\max V]\ \text{ where } V=\left\{\underline{k}/\underline{m},\underline{k}/\overline{m},\overline{k}/\underline{m},\overline{k}/\overline{m}\right\}\ \text{ and } 0\notin M.
\end{aligned}
$$

Scalar multiplication of the interval K is defined by

$$\mu K=\mu\left[\underline{k},\overline{k}\right]=\begin{cases}\left[\mu\underline{k},\mu\overline{k}\right], & \mu>0;\\ \{0\}, & \mu=0;\\ \left[\mu\overline{k},\mu\underline{k}\right], & \mu<0,\end{cases}$$

where $\mu\in\mathbb{R}$.
The opposite of the interval K is

$$-K:=(-1)K=[-\overline{k},-\underline{k}],$$

where $\mu=-1$.
The subtraction is given by

$$K-M=K+(-M)=[\underline{k}-\overline{m},\overline{k}-\underline{m}].$$

In general, $-K$ is not additive inverse for K, i.e. $K-K\neq 0$.
The definitions of operations cause a great many algebraic features which allows I_γ to be quasilinear space (see, [26]). These properties can be listed as follows (see, [7, 24–27]):

(1) (Associativity of addition) $(K+M)+N=K+(M+N)$ for all $K,M,N\in I_\gamma$,
(2) (Additivity element) $K+0=0+K=K$ for all $K\in I_\gamma$,
(3) (Commutativity of addition) $K+M=M+K$ for all $K,M\in I_\gamma$,
(4) (Cancellation law) $K+N=M+N\Longrightarrow K=M$ for all $K,M,N\in I_\gamma$,
(5) (Associativity of multiplication) $(K\cdot M)\cdot N=K\cdot(M\cdot N)$ for all $K,M,N\in I_\gamma$,
(6) (Commutativity of multiplication) $K\cdot M=M\cdot K$ for all $K,M\in I_\gamma$,
(7) (Unity element) $K\cdot 1=1\cdot K$ for all $K\in I_\gamma$,
(8) (Associativity law) $\lambda(\mu K)=(\lambda\mu)K$ for all $K\in I_\gamma$ and all $\lambda,\mu\in\mathbb{R}$,
(9) (First distributivity law) $\lambda(K+M)=\lambda K+\lambda M$ for all $K,M\in I_\gamma$ and all $\lambda\in\mathbb{R}$,
(10) (Second distributivity law) $(\lambda+\mu)K=\lambda K+\mu K$ for all $K\in I_\gamma$ and all $\lambda,\mu\in\mathbb{R}$.

In addition to all these features, the distributive law is not always true for intervals. As an example, $K=[1,2], M=[2,3]$ and $N=[-2,-1]$.

$$K\cdot(M+N)=[0,4],$$

whereas

$$K\cdot M+K\cdot N=[-2,5].$$

Definition 2. *[23] For the intervals K and M, we state the $g\mathcal{H}$-difference of K and M as the interval T such that*

$$K\ominus_g M=T\Leftrightarrow\begin{cases}K=M+T,\\ \quad or\\ T=K+(-M).\end{cases}$$

It looks beyond dispute that

$$K\ominus_g M=\begin{cases}\left[\underline{k}-\underline{m},\overline{k}-\overline{m}\right], \text{ if } L(K)\geq L(M),\\ \left[\overline{k}-\overline{m},\underline{k}-\underline{m}\right], \text{ if } L(K)<L(M).\end{cases}$$

Particularly, if $M=m\in\mathbb{R}$ is a constant, we have

$$K\ominus_g M=\left[\underline{k}-m,\overline{k}-m\right].$$

Moreover, another set feature is the inclusion $\subseteq$ that is defined by

$$K\subseteq M\Longleftrightarrow \underline{k}\leq\underline{m} \text{ and } \overline{k}\leq\overline{m}.$$

Throughout this paper, $0<q,q_1,q_2<1$ and $\Delta=[\alpha,\beta]\times[\gamma,\delta]\subseteq\mathbb{R}^2$. For condensation, interval valued quantum calculus denoted by Iq-calculus.

In [35], Zhao et al. gave the notions about the co-ordinated convex interval-valued functions and inclusions of Hermite-Hadamard type.

Definition 3. *[35] A function $\Phi=\left[\underline{\Phi},\overline{\Phi}\right]:\Delta\to I_{\gamma}^{+}$ is said to be co-ordinated convex interval-valued function, if the following inclusion holds:*

$$\begin{aligned}&\Phi(\tau x+(1-\tau)y,\sigma u+(1-\sigma)w)\\ \supseteq\;&\tau\sigma\Phi(x,u)+\tau(1-\sigma)\Phi(x,w)+\sigma(1-\tau)\Phi(y,u)+(1-\sigma)(1-\tau)\Phi(y,w),\end{aligned}$$

for all $(x,y),(u,w)\in\Delta$ and $\sigma,\tau\in[0,1]$.

Lemma 1. *[35] A function $\Phi:\Delta\to I_{\gamma}^{+}$ is an interval-valued convex on co-ordinates if and only if there exist two functions $\Phi_x:[\gamma,\delta]\to I_{\gamma}^{+},\ \Phi_x(w)=\Phi(x,w)$ and $\Phi_y:[\alpha,\beta]\to I_{\gamma}^{+},\ \Phi_y(u)=\Phi(y,u)$ are interval-valued convex.*

It is easy to prove that an interval-valued convex function is an interval-valued co-ordinated convex, but the converse may not be true. For this, we can see the following example.

Example 1. *An interval-valued function $\Phi:[0,1]^2\to I_{\gamma}^{+}$ defined as $\Phi(x,y)=[xy,(6-e^{x})(6-e^{y})]$ is an interval-valued convex on co-ordinates but it is not an interval-valued convex on $[0,1]^2$.*

Theorem 2. *[35] If $\Phi=\left[\underline{\Phi},\overline{\Phi}\right]:\Delta\to I_{\gamma}^{+}$ is a co-ordinated convex interval-valued function on Δ, then the following inclusions hold:*

$$\begin{aligned}&\Phi\left(\frac{\alpha+\beta}{2},\frac{\gamma+\delta}{2}\right)\\ \supseteq\;&\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi\left(u,\frac{\gamma+\delta}{2}\right)d^{I}u+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi\left(\frac{\alpha+\beta}{2},v\right)d^{I}v\right]\\ \supseteq\;&\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)d^{I}vd^{I}u\\ \supseteq\;&\frac{1}{4}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(u,\gamma)d^{I}u+\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi(u,\delta)d^{I}u\right.\\ &\left.+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi(\alpha,v)d^{I}v+\frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi(\beta,v)d^{I}v\right]\\ \supseteq\;&\frac{\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)+\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{4}.\end{aligned}\tag{2.2}$$

For more recent inclusions of Hermite-Hadamard type for co-ordinated convex interval-valued functions one can read [19].

3. Notations and Preliminaries of q-Calculus

In this section, we review some necessary definitions and related inequalities about q-calculus.

Definition 4. *[33] For a continuous function $\Phi:[\alpha,\beta]\to\mathbb{R}$, the q_{α}- derivative of Φ at $u\in[\alpha,\beta]$ is characterized by the expression*

$$_{\alpha}d_{q}\Phi\left(u\right)=\frac{\Phi\left(u\right)-\Phi\left(qu+\left(1-q\right)\alpha\right)}{\left(1-q\right)\left(u-\alpha\right)},\ u\neq\alpha.\tag{3.1}$$

Since $\Phi:[\alpha,\beta]\to\mathbb{R}$ is a continuous function, therefoe, we can state:

$$_{\alpha}d_{q}\Phi\left(\alpha\right)=\lim_{u\to\alpha}{}_{\alpha}d_{q}\Phi\left(u\right).$$

The function Φ is said to be q_{α}- differentiable on $[\alpha,\beta]$ if $_{\alpha}d_{q}\Phi\left(u\right)$ exists for all $u\in[\alpha,\beta]$. If $\alpha=0$ in (3.1), then $_{0}d_{q}\Phi\left(u\right)=d_{q}\Phi\left(u\right)$, where $d_{q}\Phi\left(u\right)$ is the familiar q-derivative of Φ at $u\in[\alpha,\beta]$ defined by the expression (see, [18])

$$d_{q}\Phi\left(u\right)=\frac{\Phi\left(u\right)-\Phi\left(qu\right)}{\left(1-q\right)u},\ u\neq0.\tag{3.2}$$

Definition 5. *[33] Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ be a continuous function. Then, the q_α-definite integral on $[\alpha, \beta]$ is defined as*

$$\int_{\alpha}^{u} \Phi(\tau) \ {}_{\alpha}d_q\tau = (1-q)(u-\alpha)\sum_{n=0}^{\infty} q^n \Phi(q^n u + (1-q^n)\alpha) \tag{3.3}$$

for $u \in [\alpha, \beta]$.

We have to give the following notation which will be used many times in the next sections (see, [18]):

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

In [6], Alp et al. proved the following q_α-Hermite-Hadamard inequalities for convex functions on quantum integral:

Theorem 3. *Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ be a convex differentiable function on $[\alpha, \beta]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities*

$$\Phi\left(\frac{q\alpha + \beta}{[2]_q}\right) \leq \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} \Phi(u) \ {}_{\alpha}d_q u \leq \frac{q\Phi(\alpha) + \Phi(\beta)}{[2]_q}. \tag{3.4}$$

On the other hand, Bermudo et al. gave the following new definition and related Hermite-Hadamard type inequalities:

Definition 6. *[8] Let $\Phi : [\alpha, \beta] \to \mathbb{R}$ be a continuous function. Then, the q^β-definite integral on $[\alpha, \beta]$ is defined as*

$$\int_{u}^{\beta} \Phi(\tau) \ {}^{\beta}d_q\tau = (1-q)(\beta-u)\sum_{n=0}^{\infty} q^n \Phi(q^n u + (1-q^n)\beta)$$

for $u \in [\alpha, \beta]$.

Theorem 4. *[8] If $\Phi : [\alpha, \beta] \to \mathbb{R}$ be a convex differentiable function on $[\alpha, \beta]$ and $0 < q < 1$. Then, q-Hermite-Hadamard inequalities*

$$\Phi\left(\frac{\alpha + q\beta}{[2]_q}\right) \leq \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} \Phi(u) \ {}^{\beta}d_q u \leq \frac{\Phi(\alpha) + q\Phi(\beta)}{[2]_q}. \tag{3.5}$$

In [21], Latif et al. defined $q_{\alpha\gamma}$-integral and derivatives for two variables functions and proved associated inequalities as follows:

Definition 7. *Suppose that $\Phi : \Delta \to \mathbb{R}$ is continuous function. Then, the definite $q_{\alpha\gamma}$-integral on Δ is defined by*

$$\begin{aligned} \int_{\alpha}^{u}\int_{\gamma}^{v} \Phi(\tau, \sigma) \ {}_{\gamma}d_{q_2}\sigma \ {}_{\alpha}d_{q_1}\tau &= (1-q_1)(1-q_2)(u-\alpha)(v-\gamma) \\ &\quad \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(q_1^n u + (1-q_1^n)\alpha, q_2^m v + (1-q_2^m)\gamma) \end{aligned}$$

for $(u, v) \in \Delta$.

Definition 8. *[21] Let $\Phi : \Delta \to \mathbb{R}$ be a continuous function of two variables. Then the partial q_1-derivatives, q_2-derivatives and q_1q_2-derivatives at $(u,v)\in\Delta$ can be given as follows:*

$$\begin{aligned}
\frac{{}_{\alpha}\partial_{q_1}\Phi(u,v)}{{}_{\alpha}\partial_{q_1}u} &= \frac{\Phi(q_1u+(1-q_1)\alpha,v)-\Phi(u,v)}{(1-q_1)(u-\alpha)},\ u\neq\beta \\
\frac{{}_{\gamma}\partial_{q_2}\Phi(u,v)}{{}_{\gamma}\partial_{q_2}v} &= \frac{\Phi(u,q_2v+(1-q_2)\gamma)-\Phi(u,v)}{(1-q_2)(v-\gamma)},\ v\neq\gamma \\
\frac{{}_{\alpha,\ \gamma}\partial^2_{q_1,q_2}\Phi(u,v)}{{}_{\alpha}\partial_{q_1}u\ {}_{\gamma}\partial_{q_2}v} &= \frac{1}{(u-\alpha)(v-\gamma)(1-q_1)(1-q_2)}\left[\Phi(q_1u+(1-q_1)\alpha,q_2v+(1-q_2)\gamma)\right. \\
&\quad \left.-\Phi(q_1u+(1-q_1)\alpha,v)-\Phi(u,q_2v+(1-q_2)\gamma)+\Phi(u,v)\right],\ u\neq\alpha,\ v\neq\gamma.
\end{aligned}$$

For more details related to q-integrals for the functions of two variables (see, [21]).
If we set $r=1$ in [21, Theorem 6], then we can deduce the following Theorem.

Theorem 5. *[21] Let $\Phi : \Delta \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ. If the partial q_1q_2-derivative $\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}$ is continuous and integrable on Δ and $\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ is co-ordinated convex on Δ. Then, we have following inequality:*

$$\begin{aligned}
&\left|{}_{\alpha,\gamma}I_{q_1,\ q_2}(\alpha,\beta,\gamma,\delta)(\Phi)\right| \\
\leq\ & \frac{q_1q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[A(q_1)A(q_2)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi(\alpha,\gamma)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|+B(q_1)A(q_2)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi(\alpha,\delta)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right. \\
&\left.+B(q_2)A(q_1)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi(\beta,\gamma)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|+B(q_1)B(q_2)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi(\beta,\delta)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right]
\end{aligned} \tag{3.6}$$

where

$$\begin{aligned}
{}_{\alpha,\gamma}I_{q_1,\ q_2}(\alpha,\beta,\gamma,\delta)(\Phi) &= \frac{q_1q_2\Phi(\alpha,\gamma)+q_1\Phi(\alpha,\delta)+q_2\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}} \\
&\quad +\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}_{\gamma}d_{q_2}v\ {}_{\alpha}d_{q_1}u \\
&\quad -\left[\frac{q_2}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}_{\alpha}d_{q_1}u+\frac{1}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}_{\alpha}d_{q_1}u\right. \\
&\quad \left.+\frac{q_1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}_{\gamma}d_{q_2}v+\frac{1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}_{\gamma}d_{q_1}v\right]
\end{aligned}$$

and

$$A(q)=\frac{q\left(1+3q^2+2q^3\right)}{[3]_q[2]_q^3},\quad B(q)=\frac{q\left(1+4q+q^2\right)}{[3]_q[2]_q^3}.$$

Recently, Budak et al. gave the following definitions of q_{α}^{δ}, q_{β}^{γ} and $q^{\beta\delta}$ integrals:

Definition 9. *[10] Suppose that $\Phi : \Delta \to \mathbb{R}$ is continuous function. Then the following q_{α}^{δ}, q_{γ}^{β} and $q^{\beta\delta}$ integrals on Δ are defined by*

$$\begin{aligned} & \int_{\alpha}^{u} \int_{v}^{\delta} \Phi(\tau,\sigma) \ {}^{\delta}d_{q_2}\sigma \ \ {}_{\alpha}d_{q_1}\tau \\ = \ & (1-q_1)(1-q_2)(u-\alpha)(\delta-v) \\ & \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi\left(q_1^n u + (1-q_1^n)\alpha, q_2^m v + (1-q_2^m)\delta\right), \end{aligned} \tag{3.7}$$

$$\begin{aligned} & \int_{u}^{\beta} \int_{\gamma}^{v} \Phi(\tau,\sigma) \ {}_{\gamma}d_{q_2}\sigma \ \ {}^{\beta}d_{q_1}\tau \\ = \ & (1-q_1)(1-q_2)(\beta-u)(v-\gamma) \\ & \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi\left(q_1^n u + (1-q_1^n)\beta, q_2^m v + (1-q_2^m)\gamma\right) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} & \int_{u}^{\beta} \int_{v}^{\delta} \Phi(\tau,\sigma) \ {}^{\delta}d_{q_2}\sigma \ \ {}^{\beta}d_{q_1}\tau \\ = \ & (1-q_1)(1-q_2)(\beta-u)(\delta-v) \\ & \times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi\left(q_1^n u + (1-q_1^n)\beta, q_2^m v + (1-q_2^m)\delta\right) \end{aligned} \tag{3.9}$$

respectively, for $(u,v) \in \Delta$.

Theorem 6. *[3] Let $\Phi : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ such that partial q_1q_2-derivative $\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\Phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}$ is continuous and integrable on Δ. Then we have following inequality provided that $\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\Phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|$ is co-ordinated convex on Δ*

$$\begin{aligned} & \left|{}^{\beta,\ \delta}I_{q_1,\ q_2}(\alpha,\beta,\gamma,\delta)(\Phi)\right| \\ \leq \ & \frac{q_1q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}} \left[B(q_1)B(q_2)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\Phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right| + B(q_1)A(q_2)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\Phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right| \right. \\ & \left. + A(q_1)B(q_2)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\Phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right| + A(q_1)A(q_2)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2}\Phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right| \right], \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} & {}^{\beta,\ \delta}I_{q_1,\ q_2}(\alpha,\beta,\gamma,\delta)(\Phi) \\ : \ = \ & \frac{\Phi(\alpha,\gamma)+q_1\Phi(\beta,\gamma)+q_2\Phi(\alpha,\delta)+q_1q_2\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}} - \frac{q_1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}^{\delta}d_{q_2}v \\ & -\frac{1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}^{\delta}d_{q_2}v - \frac{q_2}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}^{\beta}d_{q_1}u \\ & -\frac{1}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}^{\beta}d_{q_1}u + \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}^{\beta}d_{q_1}u\ {}^{\delta}d_{q_2}v \end{aligned}$$

and $A(q)$, $B(q)$ are defined in Theorem 5 and $0 < q_1, q_2 < 1$.

Theorem 7. *[3] Let $\Phi : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a twice partially q_1q_2-differentiable function on Δ such that partial q_1q_2-derivative $\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\Phi(\tau,\sigma)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}$ is continuous and integrable on Δ. Then we have following* *inequality provided that* $\left|\frac{{}_{\alpha}^{\delta}\partial_{q_1,q_2}^2\Phi(\tau,\sigma)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|$ *is co-ordinated convex on* Δ

$$\begin{aligned}
&\left|{}_{\alpha}^{\delta}I_{q_1,\ q_2}\left(\alpha,\beta,\gamma,\delta\right)\left(\Phi\right)\right| \\
\leq\ & \frac{q_1q_2\left(\beta-\alpha\right)\left(\delta-\gamma\right)}{[2]_{q_1}[2]_{q_2}}\left[B\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\alpha}^{\delta}\partial_{q_1,q_2}^2\Phi(\beta,\gamma)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|+B\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha}^{\delta}\partial_{q_1,q_2}^2\Phi(\beta,\delta)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|\right. \\
&\left.+A\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\alpha}^{\delta}\partial_{q_1,q_2}^2\Phi(\alpha,\gamma)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|+A\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha}^{\delta}\partial_{q_1,q_2}^2\Phi(\alpha,\delta)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|\right],
\end{aligned}\tag{3.11}$$

where

$$\begin{aligned}
&{}_{\alpha}^{\delta}I_{q_1,\ q_2}\left(\alpha,\beta,\gamma,\delta\right)\left(\Phi\right) \\
:=\ & \frac{q_1\Phi\left(\alpha,\gamma\right)+q_1q_2\Phi\left(\alpha,\delta\right)+\Phi\left(\beta,\gamma\right)+q_2\Phi\left(\beta,\delta\right)}{[2]_{q_1}[2]_{q_2}}-\frac{q_1}{[2]_{q_1}\left(\delta-\gamma\right)}\int_{\gamma}^{\delta}\Phi\left(\alpha,v\right)\ {}^{\delta}d_{q_2}v \\
&-\frac{1}{[2]_{q_1}\left(\delta-\gamma\right)}\int_{\gamma}^{\delta}\Phi\left(\beta,v\right)\ {}^{\delta}d_{q_2}v-\frac{q_2}{[2]_{q_2}\left(\beta-\alpha\right)}\int_{\alpha}^{\beta}\Phi\left(u,\delta\right)\ {}_{\alpha}d_{q_1}u \\
&-\frac{1}{[2]_{q_2}\left(\beta-\alpha\right)}\int_{\alpha}^{\beta}\Phi\left(u,\gamma\right)\ {}_{\alpha}d_{q_1}u+\frac{1}{\left(\beta-\alpha\right)\left(\delta-\gamma\right)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi\left(u,v\right)\ {}_{\alpha}d_{q_1}u\ {}^{\delta}d_{q_2}v
\end{aligned}$$

and $A\left(q\right),\ B\left(q\right)$ *are defined in Theorem 5 and* $0<q_1,q_2<1$.

Theorem 8. *[3] Let* $\Phi:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ *be a twice partially* q_1q_2*-differentiable function on* Δ *such that partial* q_1q_2*-derivative* $\frac{{}_{\gamma}^{\beta}\partial_{q_1,q_2}^2\Phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}$ *is continuous and integrable on* Δ*. Then we have following inequality provided that* $\left|\frac{{}_{\gamma}^{\beta}\partial_{q_1,q_2}^2\Phi(\tau,\sigma)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ *is co-ordinated convex on* Δ

$$\begin{aligned}
&\left|{}_{\gamma}^{\beta}I_{q_1,\ q_2}\left(\alpha,\beta,\gamma,\delta\right)\left(\Phi\right)\right| \\
\leq\ & \frac{q_1q_2\left(\beta-\alpha\right)\left(\delta-\gamma\right)}{[2]_{q_1}[2]_{q_2}}\left[B\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\gamma}^{\beta}\partial_{q_1,q_2}^2\Phi\left(\alpha,\delta\right)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|+B\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\gamma}^{\beta}\partial_{q_1,q_2}^2\Phi\left(\alpha,\gamma\right)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right. \\
&\left.+A\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\gamma}^{\beta}\partial_{q_1,q_2}^2\Phi\left(\beta,\delta\right)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|+A\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\gamma}^{\beta}\partial_{q_1,q_2}^2\Phi\left(\beta,\gamma\right)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right],
\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}
&{}_{\gamma}^{\beta}I_{q_1,\ q_2}\left(\alpha,\beta,\gamma,\delta\right)\left(\Phi\right) \\
:=\ & \frac{q_2\Phi\left(\alpha,\gamma\right)+\Phi\left(\alpha,\delta\right)+q_1q_2\Phi\left(\beta,\gamma\right)+q_1\Phi\left(\beta,\delta\right)}{[2]_{q_1}[2]_{q_2}}-\frac{q_1}{[2]_{q_1}\left(\delta-\gamma\right)}\int_{\gamma}^{\delta}\Phi\left(\beta,v\right)\ {}_{\gamma}d_{q_2}v \\
&-\frac{1}{[2]_{q_1}\left(\delta-\gamma\right)}\int_{\gamma}^{\delta}\Phi\left(\alpha,v\right)\ {}_{\gamma}d_{q_2}v-\frac{q_2}{[2]_{q_2}\left(\beta-\alpha\right)}\int_{\alpha}^{\beta}\Phi\left(u,\gamma\right)\ {}^{\beta}d_{q_1}u \\
&-\frac{1}{[2]_{q_2}\left(\beta-\alpha\right)}\int_{\alpha}^{\beta}\Phi\left(u,\delta\right)\ {}^{\beta}d_{q_1}u+\frac{1}{\left(\beta-\alpha\right)\left(\delta-\gamma\right)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi\left(u,v\right)\ {}^{\beta}d_{q_1}u\ {}_{\gamma}d_{q_2}v
\end{aligned}$$

and $A\left(q\right),\ B\left(q\right)$ *are defined in Theorem 5 and* $0<q_1,q_2<1$.

4. Iq_1q_2-integrals for the functions of two variables

In this section, we recall some necessary notions and integral inclusions about Iq-calculus. Furthermore, we are interested to give the notions Iq_1q_2-integrals for the functions of two variables.

Definition 10. *[23] For a continuous interval-valued function* $\Phi=\left[\underline{\Phi},\ \overline{\Phi}\right]:[\alpha,\beta]\to I_{\gamma}$*, the* Iq_{α}*-derivative of* Φ *at* $u\in[\alpha,\beta]$ *is defined by*

$$ {}_{\alpha}D_q\Phi\left(u\right)=\frac{\Phi\left(u\right)\ominus_g\Phi\left(qu+\left(1-q\right)\alpha\right)}{\left(1-q\right)\left(u-\alpha\right)},\ u\neq\alpha. \tag{4.1}$$

Since $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma$ is continuous function, we can state as:

$$_\alpha D_q \Phi(\alpha) = \lim_{u \to \alpha} {}_\alpha D_q \Phi(u).$$

The function Φ is said to be Iq-differentiable on $[\alpha, \beta]$, if $_\alpha D_q \Phi(u)$ exist for all $u \in [\alpha, \beta]$. If we set $\alpha = 0$ in (4.1), then $_0 D_q \Phi(\alpha) = D_q \Phi(\alpha)$, where $D_q \Phi(\alpha)$ is called Iq-Jackson derivative of Φ at $u \in [\alpha, \beta]$ defined by the expression:

$$D_q \Phi(u) = \frac{\Phi(u) \ominus_g \Phi(qu)}{(1-q)\, u}.$$

Definition 11. *[23] For a continuous interval-valued function $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma$, the Iq_α-definite integral is defined by*

$$\int_\alpha^u \Phi(\sigma)\ {}_\alpha d_q^I \sigma = (1-q)(u-\alpha) \sum_{n=0}^{\infty} q^n \Phi(q^n u + (1-q^n)\alpha) \tag{4.2}$$

for all $u \in [\alpha, \beta]$.

Remark 1. *If we set $\alpha = 0$ in (4.2), then we have Iq-Jackson integral defined by the following equation:*

$$\int_0^u \Phi(\sigma)\ {}_0 d_q^I \sigma = (1-q)\, u \sum_{n=0}^{\infty} q^n \Phi(q^n u)$$

for all $u \in [0, \infty)$.

Theorem 9. *[23] Let $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma^+$ be a Iq_α-differentiable and convex on $[\alpha, \beta]$. Then, the Iq_α-Hermite-Hadamard inclusions are expressed as:*

$$\Phi\left(\frac{q\alpha + \beta}{[2]_q}\right) \supseteq \frac{1}{\beta - \alpha} \int_\alpha^\beta \Phi(u)\ {}_\alpha d_q^I u \supseteq \frac{q\Phi(\alpha) + \Phi(\beta)}{[2]_q}. \tag{4.3}$$

In [4], Alp et al. gave the definition of Iq^β-integral and proved inclusions of Hermite-Hadamard type for interval-valued convex functions by using Iq^β-integral.

Definition 12. *For a continuous interval-valued function $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma$, the Iq^β-definite integral is defined by*

$$\int_u^\beta \Phi(\sigma)\ {}^\beta d_q^I \sigma = (1-q)(\beta - u) \sum_{n=0}^{\infty} q^n \Phi(q^n u + (1-q^n)\beta) \tag{4.4}$$

for all $u \in [\alpha, \beta]$.

Theorem 10. *Let $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma^+$ be an interval-valued convex on $[\alpha, \beta]$. Then, the Iq^β-Hermite-Hadamard inclusions are expressed as:*

$$\Phi\left(\frac{\alpha + q\beta}{[2]_q}\right) \supseteq \frac{1}{\beta - \alpha} \int_\alpha^\beta \Phi(u)\ {}^\beta d_q^I u \supseteq \frac{\Phi(\alpha) + q\Phi(\beta)}{[2]_q}. \tag{4.5}$$

Corollary 1. *Let $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma^+$ be an interval-valued convex on $[\alpha, \beta]$. Then we have the following result*

$$\frac{1}{2}\left[\Phi\left(\frac{q\alpha + \beta}{[2]_q}\right) + \Phi\left(\frac{\alpha + q\beta}{[2]_q}\right)\right] \supseteq \frac{1}{2(\beta - \alpha)}\left[\int_\alpha^\beta \Phi(u)\ {}_\alpha d_q^I u + \int_\alpha^\beta \Phi(u)\ {}^\beta d_q^I u\right] \supseteq \frac{\Phi(\alpha) + \Phi(\beta)}{2}.$$

Corollary 2. *Let $\Phi = \left[\underline{\Phi},\ \overline{\Phi}\right] : [\alpha, \beta] \to I_\gamma^+$ be an interval-valued convex on $[\alpha, \beta]$. Then we have the following result*

$$\Phi\left(\frac{\alpha + \beta}{2}\right) \supseteq \frac{1}{2(\beta - \alpha)}\left[\int_\alpha^\beta \Phi(u)\ {}_\alpha d_q^I u + \int_\alpha^\beta \Phi(u)\ {}^\beta d_q^I u\right] \supseteq \frac{\Phi(\alpha) + \Phi(\beta)}{2}. \tag{4.6}$$

Proof. It is enough to see that by the interval-valued convexity of Φ,

$$\Phi\left(\frac{\alpha+\beta}{2}\right)=\Phi\left(\frac{1}{2}\frac{q\alpha+\beta}{[2]_q}+\frac{1}{2}\frac{\alpha+q\beta}{[2]_q}\right)\supseteq\frac{1}{2}\left[\Phi\left(\frac{q\alpha+\beta}{[2]_q}\right)+\Phi\left(\frac{\alpha+q\beta}{[2]_q}\right)\right].$$

□

Now, on the basis of the techniques that used to introduce the notions of Definitions 7, 9, 11 and 12, we can give the following new definitions of q-integrals for the functions of two variables.

Definition 13. *Suppose that $\Phi=\left[\underline{\Phi},\overline{\Phi}\right]:\Delta\to I_\gamma$ is a continuous interval-valued function. Then, the following definite $q_{\alpha\gamma}$, q_α^δ, q_γ^β and $q^{\beta\delta}$ integrals on Δ are defined by*

$$\begin{aligned}&\int_\alpha^u\int_\gamma^v\Phi(\tau,\sigma)\ {}_\gamma d_{q_2}^I\sigma\ \ {}_\alpha d_{q_1}^I\tau\\=&\ (1-q_1)(1-q_2)(u-\alpha)(v-\gamma)\\&\times\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^m\Phi\left(q_1^nu+(1-q_1^n)\alpha,q_2^mv+(1-q_2^m)\gamma\right),\end{aligned}$$

$$\begin{aligned}&\int_\alpha^u\int_v^\delta\Phi(\tau,\sigma)\ {}^\delta d_{q_2}^I\sigma\ \ {}_\alpha d_{q_1}^I\tau\\=&\ (1-q_1)(1-q_2)(u-\alpha)(\delta-v)\\&\times\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^m\Phi\left(q_1^nu+(1-q_1^n)\alpha,q_2^mv+(1-q_2^m)\delta\right),\end{aligned}$$

$$\begin{aligned}&\int_u^\beta\int_\gamma^v\Phi(\tau,\sigma)\ {}_\gamma d_{q_2}^I\sigma\ \ {}^\beta d_{q_1}^I\tau\\=&\ (1-q_1)(1-q_2)(\beta-u)(v-\gamma)\\&\times\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^m\Phi\left(q_1^nu+(1-q_1^n)\beta,q_2^mv+(1-q_2^m)\gamma\right)\end{aligned}$$

and

$$\begin{aligned}&\int_u^\beta\int_v^\delta\Phi(\tau,\sigma)\ {}^\delta d_{q_2}^I\sigma\ \ {}^\beta d_{q_1}^I\tau\\=&\ (1-q_1)(1-q_2)(\beta-u)(\delta-v)\\&\times\sum_{n=0}^\infty\sum_{m=0}^\infty q_1^nq_2^m\Phi\left(q_1^nu+(1-q_1^n)\beta,q_2^mv+(1-q_2^m)\delta\right)\end{aligned}$$

respectively, for $(u,v)\in\Delta$.

Remark 2. *It is very easy to observe that*

$$\begin{aligned}\int_\alpha^\beta\int_\gamma^\delta\Phi(\tau,\sigma)\ {}_\gamma d_{q_2}^I\sigma\ \ {}_\alpha d_{q_1}^I\tau&=\int_\alpha^\beta\int_\gamma^\delta\Phi(\tau,\sigma)\ {}^\delta d_{q_2}^I\sigma\ \ {}_\alpha d_{q_1}^I\tau=\int_\alpha^\beta\int_\gamma^\delta\Phi(\tau,\sigma)\ {}_\gamma d_{q_2}^I\sigma\ \ {}^\beta d_{q_1}^I\tau\\&=\int_\alpha^\beta\int_\gamma^\delta\Phi(\tau,\sigma)\ {}^\delta d_{q_2}^I\sigma\ \ {}^\beta d_{q_1}^I\tau=\int_\alpha^\beta\int_\gamma^\delta\Phi(\tau,\sigma)\ d^I\sigma\ \ d^I\tau\end{aligned}$$

by the taking the limits $q_1,q_2\to1^-$ (see, [34]).

5. Some New Iq_1q_2-Hermite-Hadamard inclusions

In this section, we deal with the Hermite-Hadamard type inclusions for co-ordinated convex interval-valued functions using the newly defined Iq_1q_2-integrals in the last section.

Theorem 11. *Let $\Phi = [\underline{\Phi}, \overline{\Phi}] : \Delta \to I_\gamma^+$ be a co-ordinated convex interval-valued function on Δ. The following inclusions of Hermite-Hadamard type hold for $q_{\alpha\gamma}$-integral:*

$$\begin{aligned}
&\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right) \\
\supseteq\ & \frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi\left(u, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\ {}_\alpha d_{q_1}^I u + \frac{1}{\delta-\gamma}\int_\gamma^\delta \Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, v\right)\ {}_\gamma d_{q_2}^I v\right] \\
\supseteq\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v\ {}_\alpha d_{q_1}^I u \\
\supseteq\ & \frac{q_1}{2[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\alpha,v)\ {}_\gamma d_{q_2}^I v + \frac{1}{2[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\beta,v)\ {}_\gamma d_{q_2}^I v \\
& + \frac{q_2}{2[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\gamma)\ {}_\alpha d_{q_1}^I u + \frac{1}{2[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\delta)\ {}_\alpha d_{q_1}^I u \\
\supseteq\ & \frac{q_1q_2\Phi(\alpha,\gamma)+q_1\Phi(\alpha,\delta)+q_2\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}.
\end{aligned} \tag{5.1}$$

Proof. Since Φ is a co-ordinated convex interval-valued function on co-ordinates Δ, therefore, $\Phi_u : [\gamma,\delta] \to I_\gamma^+$, $\Phi_u(v) = \Phi(u,v)$ is a convex interval-valued function on $[\gamma,\delta]$ and for all $u \in [\alpha,\beta]$. From inclusion (4.3), we have

$$\Phi_u\left(\frac{q_2\gamma+\delta}{[2]_{q_2}}\right) \supseteq \frac{1}{\delta-\gamma}\int_\gamma^\delta \Phi_u(v)\ {}_\gamma d_{q_2}^I v \supseteq \frac{q_2\Phi_u(\gamma)+\Phi_u(\delta)}{[2]_{q_2}},$$

which implies that

$$\Phi\left(u, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right) \supseteq \frac{1}{\delta-\gamma}\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v \supseteq \frac{q_2\Phi(u,\gamma)+\Phi(u,\delta)}{[2]_{q_2}}. \tag{5.2}$$

Iq_1-Integrating (5.2) with respect to u over $[\alpha,\beta]$ and dividing both sides by $\beta-\alpha$, we have

$$\begin{aligned}
&\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi\left(u, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\ {}_\alpha d_{q_1}^I u \\
\supseteq\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v\ {}_\alpha d_{q_1}^I u \\
\supseteq\ & \frac{q_2}{[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\gamma)\ {}_\alpha d_{q_1}^I u + \frac{1}{[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\delta)\ {}_\alpha d_{q_1}^I u.
\end{aligned} \tag{5.3}$$

By using the same process, $\Phi_v : [\alpha,\beta] \to I_\gamma^+$, $\Phi_v(u) = \Phi(u,v)$ is a convex interval-valued function on $[\alpha,\beta]$ and $v \in [\gamma,\delta]$, we have

$$\begin{aligned}
&\frac{1}{\delta-\gamma}\int_\gamma^\delta \Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, v\right)\ {}_\gamma d_{q_2}^I v \\
\supseteq\ & \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v\ {}_\alpha d_{q_1}^I u \\
\supseteq\ & \frac{q_1}{[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\alpha,v)\ {}_\gamma d_{q_2}^I v + \frac{1}{[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\beta,v)\ {}_\gamma d_{q_2}^I v.
\end{aligned} \tag{5.4}$$

Thus, we can obtain the second and third inclusions in (5.1) by summing the inclusions (5.3) and (5.4). For the proof of first inclusion in (5.1), again if we use inclusion (4.3), then we have

$$\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right) \supseteq \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi\left(u, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\,{}_{\alpha}d_{q_1}^{I}u, \tag{5.5}$$

$$\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right) \supseteq \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, v\right)\,{}_{\gamma}d_{q_2}^{I}v. \tag{5.6}$$

Hence, after summing the inclusions (5.5) and (5.6), we obtain first inclusion in (5.1). Finally, from the right part of (4.3), we get

$$\frac{q_2}{2\,[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\,{}_{\alpha}d_{q_1}^{I}u \supseteq \frac{q_2}{2\,[2]_{q_2}}\frac{q_1\Phi(\alpha,\gamma)+\Phi(\beta,\gamma)}{[2]_{q_1}}, \tag{5.7}$$

$$\frac{1}{2\,[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\,{}_{\alpha}d_{q_1}^{I}u \supseteq \frac{1}{2\,[2]_{q_2}}\frac{q_1\Phi(\alpha,\delta)+\Phi(\beta,\delta)}{[2]_{q_1}}, \tag{5.8}$$

$$\frac{q_1}{2\,[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\,{}_{\gamma}d_{q_2}^{I}v \supseteq \frac{q_1}{2\,[2]_{q_1}}\frac{q_2\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)}{[2]_{q_2}}, \tag{5.9}$$

$$\frac{1}{2\,[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\,{}_{\gamma}d_{q_2}^{I}v \supseteq \frac{1}{2\,[2]_{q_1}}\frac{q_2\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{[2]_{q_2}} \tag{5.10}$$

and after summing the inclusions (5.7)-(5.10) the proof is finished. □

Theorem 12. *Let $\Phi = \left[\underline{\Phi}, \overline{\Phi}\right] : \Delta \to I_{\gamma}^{+}$ be a co-ordinated convex interval-valued function on Δ. The following inclusions of Hermite-Hadamard type hold for q_{α}^{δ}-integral:*

$$\begin{aligned}
&\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, \frac{\gamma+q_2\delta}{[2]_{q_2}}\right) \\
&\supseteq \frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\Phi\left(u, \frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\,{}_{\alpha}d_{q_1}^{I}u + \frac{1}{\delta-\gamma}\int_{\gamma}^{\delta}\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}}, v\right)\,{}^{\delta}d_{q_2}^{I}v\right] \\
&\supseteq \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\,{}^{\delta}d_{q_2}^{I}v\,{}_{\alpha}d_{q_1}^{I}u \\
&\supseteq \frac{q_1}{2\,[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\,{}^{\delta}d_{q_2}^{I}v + \frac{1}{2\,[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\,{}^{\delta}d_{q_2}^{I}v \\
&+ \frac{1}{2\,[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\,{}_{\alpha}d_{q_1}^{I}u + \frac{q_2}{2\,[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\,{}_{\alpha}d_{q_1}^{I}u \\
&\supseteq \frac{q_1\Phi(\alpha,\gamma)+q_1q_2\Phi(\alpha,\delta)+\Phi(\beta,\gamma)+q_2\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}
\end{aligned} \tag{5.11}$$

for all $q_1, q_2 \in (0,1)$.

Proof. Following arguments similar to those in the proof of Theorem 11 by taking into account the double inclusion (4.5), the desired inclusion (5.11) can be attained. □

Theorem 13. *Let $\Phi = \left[\underline{\Phi}, \overline{\Phi}\right] : \Delta \to I_\gamma^+$ be a co-ordinated convex interval-valued function on Δ. The following inclusions of Hermite-Hadamard type hold for q_γ^β-integral:*

$$\begin{aligned}
(5.12)\qquad &\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}}, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right) \\
&\supseteq \frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi\left(u, \frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\ {}^\beta d_{q_1}^I u\ + \frac{1}{\delta-\gamma}\int_\gamma^\delta \Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}}, v\right)\ {}_\gamma d_{q_2}^I v\ \right] \\
&\supseteq \frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v\ \ {}^\beta d_{q_1}^I u \\
&\supseteq \frac{1}{2[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\alpha,v)\ {}_\gamma d_{q_2}^I v\ + \frac{q_1}{2[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\beta,v)\ {}_\gamma d_{q_2}^I v \\
&+ \frac{q_2}{2[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\gamma)\ {}^\beta d_{q_1}^I u\ + \frac{1}{2[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\delta)\ {}^\beta d_{q_1}^I u \\
&\supseteq \frac{q_2\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)+q_1q_2\Phi(\beta,\gamma)+q_1\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}
\end{aligned}$$

for all $q_1, q_2 \in (0,1)$.

Proof. Following arguments similar to those in the proof of Theorem 11 by taking into account the double inclusion (4.5), the desired inclusion (5.12) can be attained. □

Theorem 14. *Let $\Phi = \left[\underline{\Phi}, \overline{\Phi}\right] : \Delta \to I_\gamma^+$ be a co-ordinated convex interval-valued function on Δ. The following inclusions of Hermite-Hadamard type hold for $q^{\beta\delta}$-integral:*

$$\begin{aligned}
(5.13)\qquad &\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}}, \frac{\gamma+q_2\delta}{[2]_{q_2}}\right) \\
\supseteq\ \ &\frac{1}{2}\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \Phi\left(u, \frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\ {}^\beta d_{q_1}^I u + \frac{1}{\delta-\gamma}\int_\gamma^\delta \Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}}, v\right)\ {}^\delta d_{q_2}^I v\right] \\
\supseteq\ \ &\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}^\delta d_{q_2}^I v\ {}^\beta d_{q_1}^I u \\
\supseteq\ \ &\frac{1}{2[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\alpha,v)\ {}^\delta d_{q_2}^I v + \frac{q_1}{2[2]_{q_1}(\delta-\gamma)}\int_\gamma^\delta \Phi(\beta,v)\ {}^\delta d_{q_2}^I v \\
&+\frac{1}{2[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\gamma)\ {}^\beta d_{q_1}^I u + \frac{q_2}{2[2]_{q_2}(\beta-\alpha)}\int_\alpha^\beta \Phi(u,\delta)\ {}^\beta d_{q_1}^I u \\
\supseteq\ \ &\frac{\Phi(\alpha,\gamma)+q_2\Phi(\alpha,\delta)+q_1\Phi(\beta,\gamma)+q_1q_2\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}.
\end{aligned}$$

Proof. Following arguments similar to those in the proof of Theorem 11 by taking into account the double inclusion (4.5), the desired inclusion (5.13) can be attained. □

Remark 3. *If we take the limits $q_1 \to 1^-$ and $q_2 \to 1^-$ in Theorems 11, 12, 13, and 14, then each Theorem reduces to Theorem 2.*

Remark 4. *If $\underline{\Phi} = \overline{\Phi}$ in Theorems 11, 12, 13, and 14, then Theorems 11, 12, 13, and 14 reduces to Theorem 2 in [5], Theorems 3.4, 3.5, and 3.6 in [10], respectively.*

On the other hand, summing up the results in Theorems 11, 12, 13, and 14 yields the next corollary:

Corollary 3. *Let $\Phi=\left[\underline{\Phi},\overline{\Phi}\right]:\Delta\rightarrow I_{\gamma}^{+}$ be a co-ordinated convex interval-valued function on Δ. Then we have,*

$$
\begin{aligned}
&\frac{1}{4}\left[\Phi\left(\frac{\alpha q_1+\beta}{[2]_{q_1}},\frac{\gamma q_2+\delta}{[2]_{q_2}}\right)+\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\right.\\
&\left.+\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)+\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\right]\\
&\supseteq\frac{1}{8(\beta-\alpha)}\int_{\alpha}^{\beta}\left[\Phi\left(u,\frac{\gamma q_2+\delta}{[2]_{q_2}}\right)+\Phi\left(u,\frac{\gamma+q_2\delta}{[2]_{q_2}}\right)\right]\ {}_{\alpha}d_{q_1}^{I}u\\
&+\frac{1}{8(\beta-\alpha)}\int_{\alpha}^{\beta}\left[\Phi\left(u,\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)+\Phi\left(u,\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\right]\ {}^{\beta}d_{q_1}^{I}u\\
&+\frac{1}{8(\delta-\gamma)}\int_{\gamma}^{\delta}\left[\Phi\left(\frac{\alpha q_1+\beta}{[2]_{q_1}},v\right)+\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},v\right)\right]\ {}_{\gamma}d_{q_2}^{I}v\\
&+\frac{1}{8(\delta-\gamma)}\int_{\gamma}^{\delta}\left[\Phi\left(\frac{q_1\alpha+\beta}{[2]_{q_1}},v\right)+\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},v\right)\right]\ {}^{\delta}d_{q_2}^{I}v\\
&\supseteq\frac{1}{4(\beta-\alpha)(\delta-\gamma)}\left[\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}_{\gamma}d_{q_2}^{I}v\ \ {}_{\alpha}d_{q_1}^{I}u+\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}^{\delta}d_{q_2}^{I}v\ \ {}_{\alpha}d_{q_1}^{I}u\right.\\
&\left.+\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}_{\gamma}d_{q_2}^{I}v\ \ {}^{\beta}d_{q_1}^{I}u+\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}^{\delta}d_{q_2}^{I}v\ \ {}^{\beta}d_{q_1}^{I}u\right]\\
&\supseteq\frac{1}{8(\delta-\gamma)}\left[\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}_{\gamma}d_{q_2}^{I}v+\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}_{\gamma}d_{q_2}^{I}v\right.\\
&\left.+\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}^{\delta}d_{q_2}^{I}v+\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}^{\delta}d_{q_2}^{I}v\right]\\
&+\frac{1}{8(\beta-\alpha)}\left[\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}_{\alpha}d_{q_1}^{I}u+\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}_{\alpha}d_{q_1}^{I}u\right.\\
&\left.+\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}^{\beta}d_{q_1}^{I}u+\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}^{\beta}d_{q_1}^{I}u\right]\\
&\supseteq\frac{\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)+\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{4}
\end{aligned}
$$

for $q_1,q_2\in(0,1)$.

Remark 5. *By taking the limits $q_1\rightarrow 1^{-}$ and $q_2\rightarrow 1^{-}$ in Corollary 3, then Corollary 3 reduces to Theorem 2.*

Remark 6. *If $\underline{\Phi}=\overline{\Phi}$ in Corollary 3, then Corollary 3 reduces to Corollary 3.1 in [10].*

Corollary 4. *Let $\Phi = \left[\underline{\Phi}, \overline{\Phi}\right] : \Delta \to I_\gamma^+$ be a co-ordinated convex interval-valued function on Δ. Then we have,*

$$
\begin{aligned}
(5.14)\qquad & \Phi\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}\right) \\
& \supseteq \frac{1}{4(\beta-\alpha)}\left[\int_\alpha^\beta \Phi\left(u, \frac{\gamma+\delta}{2}\right)\ {}_\alpha d_{q_1}^I u + \int_\alpha^\beta \Phi\left(u, \frac{\gamma+\delta}{2}\right)\ {}^\beta d_{q_1}^I u\right] \\
& + \frac{1}{4(\delta-\gamma)}\left[\int_\gamma^\delta \Phi\left(\frac{\alpha+\beta}{2}, v\right)\ {}_\gamma d_{q_2}^I v + \int_\gamma^\delta \Phi\left(\frac{\alpha+\beta}{2}, v\right)\ {}^\delta d_{q_2}^I v\right] \\
& \supseteq \frac{1}{4(\beta-\alpha)(\delta-\gamma)}\left[\int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v\ {}_\alpha d_{q_1}^I u + \int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}^\delta d_{q_2}^I v\ {}_\alpha d_{q_1}^I u\right. \\
& \left. + \int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}_\gamma d_{q_2}^I v\ {}^\beta d_{q_1}^I u + \int_\alpha^\beta\int_\gamma^\delta \Phi(u,v)\ {}^\delta d_{q_2}^I v\ {}^\beta d_{q_1}^I u\right] \\
& \supseteq \frac{1}{8(\delta-\gamma)}\left[\int_\gamma^\delta \Phi(\alpha,v)\ {}_\gamma d_{q_2}^I v + \int_\gamma^\delta \Phi(\beta,v)\ {}_\gamma d_{q_2}^I v\right. \\
& \left. + \int_\gamma^\delta \Phi(\alpha,v)\ {}^\delta d_{q_2}^I v + \int_\gamma^\delta \Phi(\beta,v)\ {}^\delta d_{q_2}^I v\right] \\
& + \frac{1}{8(\beta-\alpha)}\left[\int_\alpha^\beta \Phi(u,\gamma)\ {}_\alpha d_{q_1}^I u + \int_\alpha^\beta \Phi(u,\delta)\ {}_\alpha d_{q_1}^I u\right. \\
& \left. + \int_\alpha^\beta \Phi(u,\gamma)\ {}^\beta d_{q_1}^I u + \int_\alpha^\beta \Phi(u,\delta)\ {}^\beta d_{q_1}^I u\right] \\
& \supseteq \frac{\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)+\Phi(\beta,\gamma)+\Phi(\beta,\delta)}{4}
\end{aligned}
$$

for $q_1, q_2 \in (0,1)$.

Proof. By the first inclusion in (4.6), we have

$$
(5.15)\qquad \Phi\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}\right) \supseteq \frac{1}{2(\beta-\alpha)}\left\{\int_\alpha^\beta \Phi\left(u, \frac{\gamma+\delta}{2}\right)\ {}_\alpha d_{q_1}^I u + \int_\alpha^\beta \Phi\left(u, \frac{\gamma+\delta}{2}\right)\ {}^\beta d_{q_1}^I u\right\}
$$

and

$$
(5.16)\qquad \Phi\left(\frac{\alpha+\beta}{2}, \frac{\gamma+\delta}{2}\right) \supseteq \frac{1}{2(\delta-\gamma)}\left\{\int_\gamma^\delta \Phi\left(\frac{\alpha+\beta}{2}, v\right)\ {}_\gamma d_{q_2}^I v + \int_\gamma^\delta \Phi\left(\frac{\alpha+\beta}{2}, v\right)\ {}^\delta d_{q_2}^I v\right\}.
$$

By the inclusion (5.15) and (5.16), we have the first inclusion in (5.14).

Since Φ is a co-ordinated convex interval-valued function, we have

$$\begin{aligned} &\frac{1}{2(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi\left(u,\frac{\gamma+\delta}{2}\right)\ {}_{\alpha}d_{q_1}^{I}u \\ =\ &\frac{1}{2(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi\left(u,\frac{1}{2}\frac{q_2\gamma+\delta}{[2]_{q_2}}+\frac{1}{2}\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\ {}_{\alpha}d_{q_1}^{I}u \\ \supseteq\ &\frac{1}{4(\beta-\alpha)}\int_{\alpha}^{\beta}\left[\Phi\left(u,\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)+\Phi\left(u,\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\right]\ {}_{\alpha}d_{q_1}^{I}u\ , \end{aligned} \tag{5.17}$$

$$\begin{aligned} &\frac{1}{2(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi\left(u,\frac{\gamma+\delta}{2}\right)\ {}^{\beta}d_{q_1}^{I}\tau \\ =\ &\frac{1}{2(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi\left(u,\frac{1}{2}\frac{q_2\gamma+\delta}{[2]_{q_2}}+\frac{1}{2}\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\ {}^{\beta}d_{q_1}^{I}\tau \\ \supseteq\ &\frac{1}{4(\beta-\alpha)}\int_{\alpha}^{\beta}\left[\Phi\left(u,\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)+\Phi\left(u,\frac{q_2\gamma+\delta}{[2]_{q_2}}\right)\right]\ {}^{\beta}d_{q_1}^{I}u\ , \end{aligned} \tag{5.18}$$

$$\begin{aligned} &\frac{1}{2(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi\left(\frac{\alpha+\beta}{2},v\right)\ {}_{\gamma}d_{q_2}^{I}v \\ =\ &\frac{1}{2(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi\left(\frac{1}{2}\frac{\alpha q_1+\beta}{[2]_{q_1}}+\frac{1}{2}\frac{\alpha+q_1\beta}{[2]_{q_1}},v\right)\ {}_{\gamma}d_{q_2}^{I}v \\ \supseteq\ &\frac{1}{4(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi\left(\frac{\alpha q_1+\beta}{[2]_{q_1}},v\right)+\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},v\right)\ {}_{\gamma}d_{q_2}^{I}v \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} &\frac{1}{2(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi\left(\frac{\alpha+\beta}{2},v\right)\ {}^{\delta}d_{q_2}^{I}v \\ =\ &\frac{1}{2(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi\left(\frac{1}{2}\frac{\alpha q_1+\beta}{[2]_{q_1}}+\frac{1}{2}\frac{\alpha+q_1\beta}{[2]_{q_1}},v\right)\ {}^{\delta}d_{q_2}^{I}v \\ \supseteq\ &\frac{1}{4(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi\left(\frac{\alpha q_1+\beta}{[2]_{q_1}},v\right)+\Phi\left(\frac{\alpha+q_1\beta}{[2]_{q_1}},v\right)\ {}^{\delta}d_{q_2}^{I}v\ . \end{aligned} \tag{5.20}$$

By adding the inclusions (5.17)-(5.20) and by using second inclusion in Corollary 3, we obtain the second inclusion in (5.14). □

Remark 7. *By taking the limits $q_1 \to 1^-$ and $q_2 \to 1^-$ in Corollary 4, then Corollary 4 reduces to Theorem 2.*

Remark 8. *If $\underline{\Phi} = \overline{\Phi}$ in Corollary 4, then Corollary 4 reduces to Corollary 3.2 in [10].*

6. Iq_1q_2-TRAPEZOIDAL TYPE INEQUALITIES

In this section, we offer some new inequalities of trapezoidal type for the q_1q_2-differentiable co-ordinated convex functions.

Theorem 15. *Let $\Phi = \left[\underline{\Phi}, \overline{\Phi}\right] : \Delta \to I_\gamma^+$ be a function such that $\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\underline{\Phi}}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ and $\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\overline{\Phi}}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ both are co-ordinated convex functions on Δ. Then, the following q_1q_2-trapezoidal type inequality holds for the interval-valued functions*

$$\begin{aligned} &d_H\left(\Pi_1, \Pi_2\right) \\ \leq\ & \frac{q_1q_2\left(\beta-\alpha\right)\left(\delta-\gamma\right)}{\left[2\right]_{q_1}\left[2\right]_{q_2}}\left[A\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi\left(\alpha,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right| + B\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi\left(\alpha,\delta\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right. \\ & \left. + B\left(q_2\right)A\left(q_1\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi\left(\beta,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right| + B\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\Phi\left(\beta,\delta\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right] \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} \Pi_1 &= \frac{q_1q_2\Phi\left(\alpha,\gamma\right)+q_1\Phi\left(\alpha,\delta\right)+q_2\Phi\left(\beta,\gamma\right)+\Phi\left(\beta,\delta\right)}{\left[2\right]_{q_1}\left[2\right]_{q_2}} + \frac{1}{\left(\beta-\alpha\right)\left(\delta-\gamma\right)}\int_\alpha^\beta\int_\gamma^\delta \Phi\left(u,v\right)\ {}_{\gamma}d^I_{q_2}v\ {}_{\alpha}d^I_{q_1}u, \\ \Pi_2 &= \left[\frac{q_2}{\left[2\right]_{q_2}\left(\beta-\alpha\right)}\int_\alpha^\beta \Phi\left(u,\gamma\right)\ {}_{\alpha}d^I_{q_1}u + \frac{1}{\left[2\right]_{q_2}\left(\beta-\alpha\right)}\int_\alpha^\beta \Phi\left(u,\delta\right)\ {}_{\alpha}d^I_{q_1}u\right. \\ &\quad \left. + \frac{q_1}{\left[2\right]_{q_1}\left(\delta-\gamma\right)}\int_\gamma^\delta \Phi\left(\alpha,v\right)\ {}_{\gamma}d^I_{q_2}v + \frac{1}{\left[2\right]_{q_1}\left(\delta-\gamma\right)}\int_\gamma^\delta \Phi\left(\beta,v\right)\ {}_{\gamma}d^I_{q_1}v\right], \\ A\left(q\right) &= \frac{q\left(1+3q^2+2q^3\right)}{\left[3\right]_q\left[2\right]_q^3},\ B\left(q\right) = \frac{q\left(1+4q+q^2\right)}{\left[3\right]_q\left[2\right]_q^3} \end{aligned}$$

and d_H is the Pompeiu–Hausdorff distance between the intervals.

Proof. Applying the definition of the Pompeiu–Hausdorff distance between the intervals, we have

$$\begin{aligned} & d_H\left(\Pi_1, \Pi_2\right) \\ =\ & \max\left\{\left|\underline{\Pi}_1 - \underline{\Pi}_2\right|, \left|\overline{\Pi}_1 - \overline{\Pi}_2\right|\right\}. \end{aligned} \tag{6.2}$$

Considering the co-ordinated convexity of $\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\underline{\Phi}}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ and from inequality (3.6), we obtain that

$$\begin{aligned} &\left|\underline{\Pi}_1 - \underline{\Pi}_2\right| \\ \leq\ & \frac{q_1q_2\left(\beta-\alpha\right)\left(\delta-\gamma\right)}{\left[2\right]_{q_1}\left[2\right]_{q_2}}\left[A\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\underline{\Phi}\left(\alpha,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right| + B\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\underline{\Phi}\left(\alpha,\delta\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right. \\ & \left. + B\left(q_2\right)A\left(q_1\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\underline{\Phi}\left(\alpha,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right| + B\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\underline{\Phi}\left(\alpha,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right]. \end{aligned} \tag{6.3}$$

Similarly, using the co-ordinated convexity of $\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\overline{\Phi}}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ and inequality (3.6), we find that

$$\begin{aligned} &\left|\overline{\Pi}_1 - \overline{\Pi}_2\right| \\ \leq\ & \frac{q_1q_2\left(\beta-\alpha\right)\left(\delta-\gamma\right)}{\left[2\right]_{q_1}\left[2\right]_{q_2}}\left[A\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\overline{\Phi}\left(\alpha,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right| + B\left(q_1\right)A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\overline{\Phi}\left(\alpha,\delta\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right. \\ & \left. + B\left(q_2\right)A\left(q_1\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\overline{\Phi}\left(\beta,\gamma\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right| + B\left(q_1\right)B\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1q_2}\overline{\Phi}\left(\beta,\delta\right)}{{}_{\alpha}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right]. \end{aligned} \tag{6.4}$$

From (6.2)-(6.4), we get that

$$\begin{aligned}
& d_H\left(\Pi_1, \Pi_2\right) \\
= & \max\left\{\left|\underline{\Pi}_1-\underline{\Pi}_2\right|,\left|\overline{\Pi}_1-\overline{\Pi}_2\right|\right\} \\
\leq & \max\left\{\frac{q_1 q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[A\left(q_1\right) A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\alpha, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|+B\left(q_1\right) A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\alpha, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right.\right. \\
& \left.+B\left(q_2\right) A\left(q_1\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\beta, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|+B\left(q_1\right) B\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\beta, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right], \\
& \frac{q_1 q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[A\left(q_1\right) A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\alpha, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|+B\left(q_1\right) A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\alpha, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right. \\
& \left.\left.+B\left(q_2\right) A\left(q_1\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\beta, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|+B\left(q_1\right) B\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\beta, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right]\right\} \\
= & \frac{q_1 q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[A\left(q_1\right) A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\alpha, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|+B\left(q_1\right) A\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\alpha, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right. \\
& \left.+B\left(q_2\right) A\left(q_1\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\beta, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|+B\left(q_1\right) B\left(q_2\right)\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\beta, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right]
\end{aligned}$$

because

$$\begin{aligned}
\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\alpha, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right| &= \max\left\{\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\alpha, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|,\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\alpha, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right\}, \\
\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\alpha, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right| &= \max\left\{\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\alpha, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|,\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\alpha, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right\}, \\
\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\beta, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right| &= \max\left\{\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\beta, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|,\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\beta, \gamma)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right\}
\end{aligned}$$

and

$$\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \Phi(\beta, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right| = \max\left\{\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \underline{\Phi}(\beta, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|,\left|\frac{{}_{\alpha,\gamma}\partial^2_{q_1 q_2} \overline{\Phi}(\beta, \delta)}{{}_{\alpha}\partial_{q_1} \tau\ {}_{\gamma}\partial_{q_2} \sigma}\right|\right\}.$$

So, the proof is completed. □

Similarly, we can write the following Theorems by the help of Theorem 6, Theorem 7, and Theorem 8.

Theorem 16. *Let* $\Phi=\left[\underline{\Phi}, \overline{\Phi}\right]: \Delta \rightarrow I_\gamma^{+}$ *be a function such that* $\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2} \underline{\Phi}}{{}^{\beta}\partial_{q_1} \tau\ {}^{\delta}\partial_{q_2} \sigma}\right|$ *and* $\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2} \overline{\Phi}}{{}^{\beta}\partial_{q_1} \tau\ {}^{\delta}\partial_{q_2} \sigma}\right|$ *both are co-ordinated convex functions on* Δ*. Then, the following* q_1q_2*-trapezoidal type inequality holds for the interval-valued functions*

$$\begin{aligned}
& d_H\left(\Pi_3, \Pi_4\right) \\
\leq & \frac{q_1 q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[B\left(q_1\right) B\left(q_2\right)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2} \Phi(\alpha, \gamma)}{{}^{\beta}\partial_{q_1} \tau\ {}^{\delta}\partial_{q_2} \sigma}\right|+B\left(q_1\right) A\left(q_2\right)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2} \Phi(\alpha, \delta)}{{}^{\beta}\partial_{q_1} \tau\ {}^{\delta}\partial_{q_2} \sigma}\right|\right. \\
& \left.+A\left(q_1\right) B\left(q_2\right)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2} \Phi(\beta, \gamma)}{{}^{\beta}\partial_{q_1} \tau\ {}^{\delta}\partial_{q_2} \sigma}\right|+A\left(q_1\right) A\left(q_2\right)\left|\frac{{}^{\beta,\ \delta}\partial^2_{q_1,q_2} \Phi(\beta, \delta)}{{}^{\beta}\partial_{q_1} \tau\ {}^{\delta}\partial_{q_2} \sigma}\right|\right]
\end{aligned}$$

where

$$\Pi_3 = \frac{\Phi(\alpha,\gamma)+q_1\Phi(\beta,\gamma)+q_2\Phi(\alpha,\delta)+q_1q_2\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}+\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}^{\beta}d_{q_1}^{I}u\ {}^{\delta}d_{q_2}^{I},$$

$$\Pi_4 = \left[\frac{1}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}^{\beta}d_{q_1}^{I}u+\frac{q_2}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}^{\beta}d_{q_1}^{I}u\right.$$
$$\left.+\frac{1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}^{\delta}d_{q_2}^{I}v+\frac{q_1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}^{\delta}d_{q_2}^{I}v\right]$$

and d_H is the Pompeiu–Hausdorff distance between the intervals.

Theorem 17. *Let $\Phi=\left[\underline{\Phi},\overline{\Phi}\right]:\Delta\to I_{\gamma}^{+}$ be a function such that $\left|\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\underline{\Phi}}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|$ and $\left|\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\overline{\Phi}}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|$ both are co-ordinated convex functions on Δ. Then, the following q_1q_2-trapezoidal type inequality holds for the interval-valued functions*

$$d_H(\Pi_5,\Pi_6)$$
$$\leq \frac{q_1q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[B(q_1)B(q_2)\left|\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\Phi(\beta,\gamma)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|+B(q_1)A(q_2)\left|\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\Phi(\beta,\delta)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|\right.$$
$$\left.+A(q_1)B(q_2)\left|\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\Phi(\alpha,\gamma)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|+A(q_1)A(q_2)\left|\frac{{}_{\alpha}^{\delta}\partial^2_{q_1,q_2}\Phi(\alpha,\delta)}{{}_{\alpha}\partial_{q_1}\tau\ {}^{\delta}\partial_{q_2}\sigma}\right|\right],$$

where

$$\Pi_5 = \frac{q_1\Phi(\alpha,\gamma)+q_1q_2\Phi(\alpha,\delta)+\Phi(\beta,\gamma)+q_2\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}+\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}_{\alpha}d_{q_1}^{I}u\ {}^{\delta}d_{q_2}^{I},$$

$$\Pi_6 = \left[\frac{1}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}_{\alpha}d_{q_1}^{I}u-\frac{q_2}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}_{\alpha}d_{q_1}^{I}u\right.$$
$$\left.+\frac{q_1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}^{\delta}d_{q_2}^{I}v+\frac{1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}^{\delta}d_{q_2}^{I}v\right]$$

and d_H is the Pompeiu–Hausdorff distance between the intervals.

Theorem 18. *Let $\Phi=\left[\underline{\Phi},\overline{\Phi}\right]:\Delta\to I_{\gamma}^{+}$ be a function such that $\left|\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\underline{\Phi}}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ and $\left|\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\overline{\Phi}}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|$ both are co-ordinated convex functions on Δ. Then, the following q_1q_2-trapezoidal type inequality holds for the interval-valued functions*

$$d_H(\Pi_7,\Pi_8)$$
$$\leq \frac{q_1q_2(\beta-\alpha)(\delta-\gamma)}{[2]_{q_1}[2]_{q_2}}\left[B(q_1)B(q_2)\left|\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\Phi(\alpha,\delta)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|+B(q_1)A(q_2)\left|\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\Phi(\alpha,\gamma)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right.$$
$$\left.+A(q_1)B(q_2)\left|\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\Phi(\beta,\delta)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|+A(q_1)A(q_2)\left|\frac{{}_{\gamma}^{\beta}\partial^2_{q_1,q_2}\Phi(\beta,\gamma)}{{}^{\beta}\partial_{q_1}\tau\ {}_{\gamma}\partial_{q_2}\sigma}\right|\right],$$

where

$$\Pi_7 = \frac{q_2\Phi(\alpha,\gamma)+\Phi(\alpha,\delta)+q_1q_2\Phi(\beta,\gamma)+q_1\Phi(\beta,\delta)}{[2]_{q_1}[2]_{q_2}}+\frac{1}{(\beta-\alpha)(\delta-\gamma)}\int_{\alpha}^{\beta}\int_{\gamma}^{\delta}\Phi(u,v)\ {}^{\beta}d_{q_1}^{I}u\ {}_{\gamma}d_{q_2}^{I}v,$$

$$\Pi_8 = \left[\frac{q_2}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\gamma)\ {}^{\beta}d_{q_1}u+\frac{1}{[2]_{q_2}(\beta-\alpha)}\int_{\alpha}^{\beta}\Phi(u,\delta)\ {}^{\beta}d_{q_1}^{I}u\right.$$
$$\left.+\frac{1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\alpha,v)\ {}_{\gamma}d_{q_2}^{I}v+\frac{q_1}{[2]_{q_1}(\delta-\gamma)}\int_{\gamma}^{\delta}\Phi(\beta,v)\ {}_{\gamma}d_{q_2}^{I}v\right]$$

and d_H is the Pompeiu–Hausdorff distance between the intervals.

Remark 9. *If $\underline{\Phi} = \overline{\Phi}$ in Theorems 15, 16, 17, and 18, then Theorems 15, 16, 17, and 18 reduces to Theorems 5, 6, 7, and 8, respectively.*

Remark 10. *If $\underline{\Phi} = \overline{\Phi}$ and $q_1, q_2 \to 1^-$ in Theorems 15, 16, 17, and 18, then all results reduces to [32, Theorem 2].*

7. Conclusion

In this research, we have defined the double quantum integrals for the interval-valued functions of two variables. We have proved some new Hermite-Hadamard type inclusions for co-ordinated convex interval-valued functions using the newly defined double quantum integrals. Furthermore, we have derived new inequalities of trapezoidal type for the interval-valued functions of two variables. It is also proved that the results given in this paper are transformed into some existing results by considering the interval-valued function Φ such that $\underline{\Phi} = \overline{\Phi}$ and $q_1, q_2 \to 1^-$ in the main results of this paper. It is an interesting and new problem that the upcoming researchers can prove the midpoint inequalities, Ostrowski's inequalities, Newton's inequalities, and Simpson's inequalities for different kinds of interval-valued convexities using the Pompeiu–Hausdorff distance between the intervals in their future research.

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