CERTAIN GENERALIZED QUANTUM SIMPSON'S AND QUANTUM NEWTON'S TYPE INEQUALITIES FOR CONVEX FUNCTIONS IN QUANTUM CALCULUS

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Abstract. In this paper first we present some new identities by using the notions of quantum integrals and derivatives which allows us to obtain new quantum Simpson's and quantum Newton's type inequalities for differentiable convex functions by using the q_x-quantum integral and q^y-quantum integral. In particular, this paper generalises and extends previous results obtained by the various authors in the field of quantum and classical integral inequalities.

1. Introduction

Simpson's rules are well-known ways for the numerical integration and numerical estimation of definite integrals. This method is known as developed by Thomas Simpson (1710–1761). However, Johannes Kepler used the same approximation about 100 years ago, so that this method is also known as Kepler's rule. Simpson's rule includes the three-point Newton-Cotes quadrature rule, so estimation based on three steps quadratic kernel is sometimes called as Newton type results.
1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_x^y g(\varkappa)\mathrm{d}\varkappa \approx \frac{y-x}{6}\left[g(x)+4g\left(\frac{x+y}{2}\right)+g(y)\right].$$

2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_x^y g(\varkappa)\mathrm{d}\varkappa \approx \frac{y-x}{8}\left[g(x)+3g\left(\frac{2x+y}{3}\right)+3g\left(\frac{x+2y}{3}\right)+g(y)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimation known as Simpson's inequality:

Theorem 1. *Suppose that $g:[x,y]\to\mathbb{R}$ is a four times continuously differentiable mapping on (x,y), and let $\left\|g^{(4)}\right\|_\infty = \sup_{\varkappa\in(x,y)}\left|g^{(4)}(\varkappa)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{g(x)+g(y)}{2}+2g\left(\frac{x+y}{2}\right)\right]-\frac{1}{y-x}\int_x^y g(\varkappa)d\varkappa\right| \le \frac{1}{2880}\left\|g^{(4)}\right\|_\infty (y-x)^4.$$

In recent years, many authors have focused on Simpson's type inequalities for various classes of functions. Specifically, some mathematicians have worked on Simpson's and Newton's type results for convex mappings, because convexity theory is an effective and powerful method for solving a large number of problems which arise within different branches of pure and applied mathematics. For example, Dragomir et al. [14] presented new Simpson's type results and their applications to quadrature formulas in numerical integration. What is more, some inequalities of Simpson's type for s-convex functions are deduced by Alomari et al. in [6]. Afterwards, Sarikaya et al. observed the variants of Simpson's type inequalities based on convexity in [33]. In [27] and [29], the authors provided some Newton's type inequalities for harmonic convex and p-harmonic convex functions. Additionally, new Newton's type inequalities for functions whose local fractional derivatives are generalized convex are given by Iftikhar et al. in [18].

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On the other hand, quantum calculus (shortly, q-calculus) deals with the calculus without limits, where we can obtain classical mathematical formulas as $q \to 1$. Firstly introduced by Euler (1707–1783) in the track of Newton's infinite series, the study of q-calculus was introduced in the early Twentieth Century after the work of Jackson (1910) on defining an integral later known as the q-Jackson integral (see, [13, 15, 21, 23]). In q-calculus, the classical derivative is replaced by the q-difference operator in order to deal with non-differentiable functions, for more details (see, [5,12]). Applications of q-calculus can be found in various fields of mathematics and physics, and the interested readers are referred to [7, 20, 31, 38].

Many integral inequalities well known in classical analysis such as Hölder inequality, Hermite-Hadamard inequality and Ostrowski inequality, Simpson's inequality, Newton's inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev and other integral inequalities have been proved and applied to q-calculus using classical convexity. Many mathematicians have done studies in q-calculus analysis, the interested reader can check [1–4, 9–11, 16, 19, 22, 24–26, 28, 30, 34, 37, 39, 40].

The organization of this paper is as follows: In Section 2, a brief description of the concepts of q-calculus and some related works in this direction are given. In Section 3, we offer four different identities involving quantum numbers, quantum derivatives, and quantum integrals. The Simpson's and Newton's type inequalities for differentiable convex functions are established in Section 4 and Section 5, respectively. The relationship between the results presented herein and comparable results in the literature is also studied. Section 6 contains some conclusions and further directions for the future research. We believe that the study initiated in this paper may inspire new research in this area.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we first present the definitions and some properties of quantum derivatives and quantum integrals. We also mention some well-known inequalities for quantum integrals.

In [21], Jackson defined the q-Jackson integral from 0 to y for $0 < q < 1$ as follows:

$$\int_0^y g(\varkappa)\, d_q\varkappa = (1-q)\, y \sum_{k=0}^{\infty} q^k g\left(yq^k\right) \tag{2.1}$$

provided the sum converge absolutely [21].
Moreover, he defined the q-Jackson integral on any closed interval $[x, y]$:

$$\int_x^y g(\varkappa)\, d_q\varkappa = \int_0^y g(\varkappa)\, d_q\varkappa - \int_0^x g(\varkappa)\, d_q\varkappa . \tag{2.2}$$

Tariboon and Ntouyas defined the following q_x-derivative and q_x-integral:

Definition 1. *[35] Let $g : [x, y] \to \mathbb{R}$ be a continuous function. The q_x-derivative of g at $\varkappa \in [x, y]$ is characterized by the expression*

$${}_xD_q g(\varkappa) = \frac{g(\varkappa) - g(q\varkappa + (1-q)x)}{(1-q)(\varkappa - x)}, \ \varkappa \neq x. \tag{2.3}$$

Since $g : [x, y] \to \mathbb{R}$ is a continuous function, we can state

$${}_xD_q g(x) = \lim_{\varkappa \to x} {}_xD_q g(\varkappa) .$$

The function g is said to be q_x- differentiable on $[x, y]$ if ${}_xD_q g(\varkappa)$ exists for all $\varkappa \in [x, y]$. If $x = 0$ in (2.3), then ${}_0D_q g(\varkappa) = D_q g(\varkappa)$, where $D_q g(\varkappa)$ is the familiar q-derivative of g at $\varkappa \in [x, y]$ defined by the expression (see, [23])

$$D_q g(\varkappa) = \frac{g(\varkappa) - g(q\varkappa)}{(1-q)\varkappa}, \ \varkappa \neq 0.$$

Definition 2. *[35] Suppose that $g:[x,y]\to\mathbb{R}$ is a continuous function. Then, the q_x-definite integral on $[x,y]$ is defined as*

$$\int_x^y g(\varkappa)\ {}_xd_q\varkappa = (1-q)(y-x)\sum_{k=0}^{\infty} q^k g\left(q^k y + \left(1-q^k\right)x\right).$$

Alp et al. proved quantum Hermite-Hadamard inequalities for q_x-integrals by utilizing convex functions as follows:

Theorem 2. *[3] Suppose that $g:[x,y]\to\mathbb{R}$ is a convex differentiable function on $[x,y]$ and $0<q<1$. Then, we have the quantum Hermite-Hadamard inequalities*

$$g\left(\frac{qx+y}{[2]_q}\right) \le \frac{1}{y-x}\int_x^y g(\varkappa)\ {}_xd_q\varkappa \le \frac{qg(x)+g(y)}{[2]_q}. \tag{2.4}$$

In [3] and [28], Alp et al. and Noor et al. established some bounds for the left and right hand sides of the inequality (2.4), respectively.
On the other hand, Bermudo et al. defined a new quantum derivative and integral which are called q^y-derivative and q^y-integral, which can be expressed as:

Definition 3. *[8] Let $g:[x,y]\to\mathbb{R}$ be a continuous function. The q^y-derivative of g at $\varkappa\in[x,y]$ is characterized by the expression*

$${}^yD_q g(\varkappa) = \frac{g(q\varkappa+(1-q)y)-g(\varkappa)}{(1-q)(y-\varkappa)},\ \ \varkappa\neq y.$$

Definition 4. *[8] Let $g:[x,y]\to\mathbb{R}$ be a continuous function. Then, the q^y-definite integral on $[x,y]$ is defined as*

$$\int_x^y g(\varkappa)\ {}^yd_q\varkappa = (1-q)(y-x)\sum_{k=0}^{\infty} q^k g\left(q^k x + \left(1-q^k\right)y\right).$$

Bermudo et al. also proved the corresponding quantum Hermite-Hadamard inequalities for q^y-integral:

Theorem 3. *[8] If $g:[x,y]\to\mathbb{R}$ is a convex differentiable function on $[x,y]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities are as follows:*

$$g\left(\frac{x+qy}{[2]_q}\right) \le \frac{1}{y-x}\int_x^y g(\varkappa)\ {}^yd_q\varkappa \le \frac{g(x)+qg(y)}{[2]_q}. \tag{2.5}$$

We have to give the following notation which will be used frequently in the next sections (see, [23]):

$$[n]_q = \frac{q^n-1}{q-1}.$$

Additionally, we shall use the following lemma in the main results:

Lemma 1. *[36] We have the equality*

$$\int_x^y (\varkappa-x)^{\alpha}\ {}_xd_q\varkappa = \frac{(y-x)^{\alpha+1}}{[\alpha+1]_q}$$

for $\alpha\in\mathbb{R}\backslash\{-1\}$.

3. Identities

In this section, we offer some new identities by using the notions of quantum integrals and quantum derivatives. These identities are necessary to prove the main results of this paper.

Lemma 2. *Let $g : [x, y] \to \mathbb{R}$ be a q^y- differentiable function on (x, y) and $0 < q < 1$. If ${}^yD_q g$ is continuous and integrable on $[x, y]$, then for $h \in (0, 1)$ with $\frac{1}{[n]_q} \leq [h]_q \leq \frac{[n-1]_q}{[n]_q}$, one has the identity:*

$$\frac{1}{y-x}\int_x^y g(t)\ {}^y d_q t \ - \frac{1}{[n]_q}\left[g(x) + q^2[n-2]_q\, g\left(y + [h]_q(x-y)\right) + qg(y)\right] \tag{3.1}$$
$$= q(y-x)\int_0^1 \Lambda(t, [n]_q, [h]_q)\ {}^y D_q g(tx + (1-t)y)\quad d_q t$$

where

$$\Lambda(t, [n]_q, [h]_q) = \begin{cases} t - \frac{1}{[n]_q}, & t \in \left[0, [h]_q\right), \\ t - \frac{[n-1]_q}{[n]_q}, & t \in \left[[h]_q, 1\right]. \end{cases}$$

Proof. Using the formula (2.2), from the definition of the function $\Lambda(t, [n]_q, [h]_q)$, we find that

$$\int_0^1 \Lambda(t, [n]_q, [h]_q)\ {}^y D_q g(tx + (1-t)y)\quad d_q t \tag{3.2}$$
$$= \frac{[n-1]_q - 1}{[n]_q}\int_0^{[h]_q} {}^y D_q g(tx + (1-t)y)\, d_q t$$
$$+ \int_0^1 t\ {}^y D_q g(tx + (1-t)y)\, d_q t$$
$$- \frac{[n-1]_q}{[n]_q}\int_0^1 {}^y D_q g(tx + (1-t)y)\, d_q t.$$

By the Definition 3, one also has

$${}^y D_q g(tx + (1-t)y) = \frac{g(qtx + (1-qt)y) - g(tx + (1-t)y)}{(1-q)(y-x)t}.$$

Now, if we substitute the above equation in (3.2), we obtain

$$\int_0^1 \Lambda(t, [n]_q, [h]_q)\ {}^y D_q g(tx + (1-t)y)\, d_q t \tag{3.3}$$
$$= \frac{[n-1]_q - 1}{[n]_q}\int_0^{[h]_q} \frac{g(qtx + (1-qt)y) - g(tx + (1-t)y)}{(1-q)(y-x)t}\ d_q t$$
$$+ \int_0^1 \frac{g(qtx + (1-qt)y) - g(tx + (1-t)y)}{(1-q)(y-x)}\ d_q t$$
$$- \frac{[n-1]_q}{[n]_q}\int_0^1 \frac{g(qtx + (1-qt)y) - g(tx + (1-t)y)}{(1-q)(y-x)t}\ d_q t\ .$$

Calculating the first integral in the right side of (3.3) by taking into account the case when $x=0$ in Definition 2, it is found that

$$\begin{aligned}
&\int_0^{[h]_q} \frac{g(qtx+(1-qt)y)-g(tx+(1-t)y)}{(1-q)(y-x)t}\ d_qt \qquad (3.4)\\
&= \frac{1}{(y-x)}[h]_q\left\{\sum_{k=0}^{\infty} q^k \frac{g\left(q^{k+1}[h]_q x+\left(1-q^{k+1}[h]_q\right)y\right)}{q^k[h]_q}\right.\\
&\left.-\sum_{k=0}^{\infty} q^k \frac{g\left(q^{k}[h]_q x+\left(1-q^{k}[h]_q\right)y\right)}{q^k[h]_q}\right\}\\
&= \frac{1}{(y-x)}\left\{g(y)-g\left(y+[h]_q(x-y)\right)\right\}.
\end{aligned}$$

If we similarly observe the other integrals in the right side of (3.3), from Definition 4, then we get

$$\begin{aligned}
&\int_0^{1} \frac{g(qtx+(1-qt)y)-g(tx+(1-t)y)}{(1-q)(y-x)}\ d_qt \qquad (3.5)\\
&= \frac{1}{y-x}\left\{\frac{1}{q(y-x)}\int_x^y g(t)\ {}^{y}d_qt\ -\frac{1}{q}g(x)\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{1} \frac{g(qtx+(1-qt)y)-g(tx+(1-t)y)}{(1-q)(y-x)t}\ d_qt \qquad (3.6)\\
&= \frac{1}{y-x}\{g(y)-g(x)\}
\end{aligned}$$

Substituting the expressions (3.4)-(3.6) in (3.3), and later multiplying both sides of the resulting identity by $q(y-x)$, the equality (3.1) can be captured. □

Remark 1. *In Lemma 2, if we consider the limit $q\to 1^-$, then we obtain the following identity*

$$\begin{aligned}
&\frac{1}{y-x}\int_x^y g(t)\,dt-\frac{1}{n}\left[g(x)+(n-2)\,g(y+h(x-y))+g(y)\right]\\
&=(y-x)\int_0^1 \Lambda(t,n,h)\ g'(tx+(1-t)y)\quad dt
\end{aligned}$$

where

$$\Lambda(t,n,h)=\begin{cases} t-\frac{1}{n}, & t\in[0,h),\\ t-\frac{n-1}{n}, & t\in[h,1]\end{cases}$$

which can be viewed as a special case of the identity given in [32, Lemma 1].

Remark 2. *In Lemma 2, if we assume $n=2$ and $[h]_q=\frac{1}{[2]_q}$, then we have the following identity*

$$\frac{g(x)+qg(y)}{[2]_q}-\frac{1}{y-x}\int_x^y g(\varkappa)\ {}^{y}d_q\varkappa=\frac{q(y-x)}{[2]_q}\int_0^1\left(1-[2]_q t\right)\ {}^{y}D_q g(tx+(1-t)y)\,d_qt$$

which is given by Budak in [11, Lemma 1].

Corollary 1. *In Lemma 2, if we take $n=6$ and $[h]_q = \frac{1}{[2]_q}$, then we obtain the following identity*

$$\frac{1}{y-x}\int_x^y g(t)\ {}^y d_q t \ - \frac{1}{[6]_q}\left[g(x) + q^2[4]_q\, g\left(\frac{x+qy}{2}\right) + qg(y)\right]$$

$$= q(y-x)\int_0^1 \Lambda\left(t,[6]_q,\frac{1}{[2]_q}\right)\ {}^y D_q g(tx+(1-t)y)\quad d_q t$$

where

$$\Lambda\left(t,[6]_q,\frac{1}{[2]_q}\right) = \begin{cases} t-\frac{1}{[6]_q}, & t\in\left[0,\frac{1}{[2]_q}\right), \\ t-\frac{[5]_q}{[6]_q}, & t\in\left[\frac{1}{[2]_q},1\right] \end{cases}$$

which is given by Erden et al. in [17].

Lemma 3. *Let $g:[x,y]\to\mathbb{R}$ be a q_x- differentiable function on (x,y) and $0<q<1$. If ${}_xD_q g$ is continuous and integrable on $[x,y]$, then for $h\in(0,1)$ with $\frac{1}{[n]_q}\le [h]_q \le \frac{[n-1]_q}{[n]_q}$, one has the identity:*

$$\frac{1}{[n]_q}\left[qg(x) + q^2[n-2]_q\, g\Big(x+[h]_q(y-x)\Big) + g(y)\right] - \frac{1}{y-x}\int_x^y g(t)\ {}_x d_q t \tag{3.7}$$

$$= q(y-x)\int_0^1 \Lambda(t,[n]_q,[h]_q)\ {}_x D_q g(ty+(1-t)x)\quad d_q t$$

where

$$\Lambda(t,[n]_q,[h]_q) = \begin{cases} t-\frac{1}{[n]_q}, & t\in\left[0,[h]_q\right), \\ t-\frac{[n-1]_q}{[n]_q}, & t\in\left[[h]_q,1\right]. \end{cases}$$

Proof. If the techniques used in the proof of Lemma 2 are applied by taking into account the Definitions of q_x-derivative and q_x-integral, the equality (3.7) can be proved. □

Remark 3. *In Lemma 3, if we take the limit $q\to 1^-$, then we have the following identity:*

$$\frac{1}{n}\left[g(x) + (n-2)\,g(x+h(y-x)) + g(y)\right] - \frac{1}{y-x}\int_x^y g(t)\,dt$$

$$= (y-x)\int_0^1 \Lambda(t,n,h)\ g'(ty+(1-t)x)\quad dt$$

where

$$\Lambda(t,n,h) = \begin{cases} t-\frac{1}{n}, & t\in[0,h), \\ t-\frac{n-1}{n}, & t\in[h,1] \end{cases}$$

which can be viewed as a special case of the identity given in [32, Lemma 1].

Remark 4. *If we assume $n=2$ and $[h]_q=\frac{1}{[2]_q}$ in Lemma 3, then we obtain [28, Lemma 3.1].*

Corollary 2. *If we consider $n=6$ and $[h]_q=\frac{1}{[2]_q}$ in Lemma 3, then we obtain the following identity*

$$\frac{1}{[6]_q}\left[qg(x) + q^2[4]_q\, g\left(\frac{qx+y}{[2]_q}\right) + g(y)\right] - \frac{1}{y-x}\int_x^y g(t)\ {}_x d_q t$$

$$= q(y-x)\int_0^1 \Lambda\left(t,[6]_q,\frac{1}{[2]_q}\right)\ {}_x D_q g(ty+(1-t)x)\quad d_q t$$

where

$$\Lambda\left(t,[6]_q,\frac{1}{[2]_q}\right)=\begin{cases} t-\frac{1}{[6]_q}, & t\in\left[0,\frac{1}{[2]_q}\right), \\ t-\frac{[5]_q}{[6]_q}, & t\in\left[\frac{1}{[2]_q},1\right]. \end{cases}$$

Lemma 4. *Let $g:[x,y]\to\mathbb{R}$ be a q^y- differentiable function on (x,y) and $0<q<1$. If ${}^yD_qg$ is continuous and integrable on $[x,y]$, then for $h\in(0,1)$ with $\frac{1}{[n]_q}\le[h]_q\le\frac{[n-1]_q}{[n]_q}$, one has the identity:*

$$\begin{aligned}(3.8)\qquad & \frac{1}{y-x}\int_x^y g(\varkappa)\ {}^yd_q\varkappa-\frac{1}{[n]_q}\left[g(x)+\frac{q\left([n]_q-[10-n]_q\right)}{[10-n]_q}g\left(y+[h]_q(x-y)\right)\right.\\ & \left.+\frac{[n-1]_q[10-n]_q-[n]_q}{[n]_q[10-n]_q}g\left(y+[1-h]_q(x-y)\right)+qg(y)\right]\\ = {} & q(y-x)\int_0^1\Delta\left(t,[n]_q,[h]_q\right)\ {}^yD_qg(tx+(1-t)y)\ d_qt\end{aligned}$$

where

$$\Delta(t,[n]_q,[h]_q)=\begin{cases} t-\frac{1}{[n]_q}, & t\in\left[0,[h]_q\right) \\ t-\frac{1}{[10-n]_q}, & t\in\left[[h]_q,[1-h]_q\right) \\ t-\frac{[n-1]_q}{[n]_q}, & t\in\left[[1-h]_q,1\right]. \end{cases}$$

Proof. We can obtain the following equality after applying the fundamental properties of quantum integrals and the definition of $\Delta(t,[n]_q,[h]_q)$:

$$\begin{aligned} & q(y-x)\int_0^1\Delta\left(t,[n]_q,[h]_q\right)\ {}^yD_qg(tx+(1-t)y)\ d_qt\\ = {} & \frac{[n]_q-[10-n]_q}{[n]_q[10-n]_q}\int_0^{[h]_q}{}^yD_qg(tx+(1-t)y)\ d_qt\\ & +\frac{[n-1]_q[10-n]_q-[n]_q}{[n]_q[10-n]_q}\int_0^{[1-h]_q}{}^yD_qg(tx+(1-t)y)\ d_qt\\ & -\frac{[n-1]_q}{[n]_q}\int_0^1{}^yD_qg(tx+(1-t)y)\ d_qt\\ & +\int_0^1 t\ {}^yD_qg(tx+(1-t)y)\ d_qt.\end{aligned}$$

Following arguments similar to those in the proof of Lemma 2, the required identity can be obtained. □

Lemma 5. *Let $g:[x,y]\to\mathbb{R}$ be a q_x- differentiable function on (x,y) and $0<q<1$. If ${}_xD_qg$ is continuous and integrable on $[x,y]$, then for $h\in(0,1)$ with $\frac{1}{[n]_q}\le[h]_q\le\frac{[n-1]_q}{[n]_q}$, one has the identity:*

$$\begin{aligned}(3.9)\qquad & \frac{1}{[n]_q}\left[qg(x)+\frac{[n-1]_q[10-n]_q-[n]_q}{[n]_q[10-n]_q}g\left(x+[1-h]_q(y-x)\right)\right.\\ & \left.+\frac{q\left([n]_q-[10-n]_q\right)}{[10-n]_q}g\left(x+[h]_q(y-x)\right)+g(y)\right]-\frac{1}{y-x}\int_x^y g(\varkappa)\ {}_xd_q\varkappa\\ = {} & q(y-x)\int_0^1\Delta\left(t,[n]_q,[h]_q\right)\ {}_xD_qg(ty+(1-t)x)\ d_qt\end{aligned}$$

where

$$\Delta(t,[n]_q,[h]_q)=\begin{cases} t-\frac{1}{[n]_q}, & t\in\left[0,[h]_q\right) \\ t-\frac{1}{[10-n]_q}, & t\in\left[[h]_q,[1-h]_q\right) \\ t-\frac{[n-1]_q}{[n]_q}, & t\in\left[[1-h]_q,1\right]. \end{cases}$$

Proof. If the techniques used in the proof of Lemmas 2, 3, and 4 are applied by taking into account the Definitions of q_x-derivative and q_x-integral, the equality (3.9) can be proved. □

4. Simpson's Inequalities for Quantum Integrals

In this section, we offer some new quantum boundaries for quantum Simpson's inequalities via q_x-integrals and q^y-integrals. Moreover, we give some calculated integrals that will be used in the new results.

Lemma 6. *The following quantum integrals hold:*

$$\int_0^{[h]_q} t\left|t-\frac{1}{[n]_q}\right| d_qt = A_1(q)=\frac{2q^2+[h]_q^2[n]_q^2\left([h]_q[n]_q[2]_q-[3]_q\right)}{[n]_q^3[3]_q[2]_q}, \tag{4.1}$$

$$\int_{[h]_q}^1 t\left|t-\frac{[n-1]_q}{[n]_q}\right| d_qt = A_2(q)=2\frac{q^2[n-1]_q^3}{[3]_q^3[2]_q[3]_q}+\frac{1+[h]_q^3}{[3]_q}-\frac{[n-1]_q\left(1+[h]_q^2\right)}{[n]_q[2]_q}, \tag{4.2}$$

$$\int_0^{[h]_q} (1-t)\left|t-\frac{1}{[n]_q}\right| d_qt = A_3(q)=\frac{2q+[h]_q[n]_q\left([h]_q[n]_q-[2]_q\right)}{[n]_q^2[2]_q}-A_1(q), \tag{4.3}$$

and

$$\int_{[h]_q}^1 (1-t)\left|t-\frac{[n-1]_q}{[n]_q}\right| d_qt \tag{4.4}$$

$$= A_4(q)=2\frac{q[n-1]_q^2}{[2]_q[n]_q^2}-\frac{[n-1]_q\left(1+[h]_q\right)}{[n]_q}+\frac{1+[h]_q^2}{[2]_q}-A_2(q).$$

Proof. Since $[h]_q\geq\frac{1}{[n]_q}$, we have

$$\begin{aligned}\int_0^{[h]_q} t\left|t-\frac{1}{[n]_q}\right| d_qt &= \int_0^{\frac{1}{[n]_q}} t\left(\frac{1}{[n]_q}-t\right)d_qt+\int_{\frac{1}{[n]_q}}^{[h]_q} t\left(t-\frac{1}{[n]_q}\right)d_qt \\ &= 2\int_0^{\frac{1}{[n]_q}} t\left(\frac{1}{[n]_q}-t\right)d_qt+\int_0^{[h]_q} t\left(t-\frac{1}{[n]_q}\right)d_qt \\ &= \frac{2q^2+[h]_q^2[n]_q^2\left([h]_q[n]_q[2]_q-[3]_q\right)}{[n]_q^3[3]_q[2]_q}.\end{aligned}$$

Similarly, the quantum integrals (4.2)-(4.4) can be computed and the proof is finished. □

4.1. Simpson's Inequalities for q_x-integrals.

In this subsection, we prove quantum simpson's inequalities using the Lemma 2.

Theorem 4. *We assume that the conditions of Lemma 2 hold. If the mapping $|{}^yD_qg|$ is convex on $[x,y]$, then the following inequality holds for q^y-integral:*

$$\left|\frac{1}{y-x}\int_x^y g(t)\ {}^yd_qt\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\,g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right| \tag{4.5}$$

$$\leq\ q(y-x)\left[|{}^yD_qg(x)|\left\{A_1(q)+A_2(q)\right\}+|{}^yD_qg(y)|\left\{A_3(q)+A_4(q)\right\}\right]$$

where $A_1(q)-A_4(q)$ are given in (4.1)-(4.4), respectively.

Proof. By taking the modulus in Lemma 2 and using the properties of the modulus, we obtain that

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y g(t)\ {}^y d_q t\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\, g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right| \\
\le\ & q(y-x)\int_0^{[h]_q}\left|t-\frac{1}{[n]_q}\right|\left|\ {}^y D_q g(tx+(1-t)y)\right| d_q t \\
&+q(y-x)\int_{[h]_q}^1\left|t-\frac{[n-1]_q}{[n]_q}\right|\left|\ {}^y D_q g(tx+(1-t)y)\right| d_q t.
\end{aligned}
\tag{4.6}
$$

Since the mapping $|{}^y D_q g|$ is convex on $[x,y]$, therefore, we have

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y g(t)\ {}^y d_q t\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\, g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right| \\
\le\ & q(y-x)\left|{}^y D_q g(x)\right|\left[\int_0^{[h]_q} t\left|t-\frac{1}{[n]_q}\right| d_q t+\int_{[h]_q}^1 t\left|t-\frac{[n-1]_q}{[n]_q}\right| d_q t\right] \\
&+q(y-x)\left|{}^y D_q g(y)\right|\left[\int_0^{[h]_q}(1-t)\left|t-\frac{1}{[n]_q}\right| d_q t+\int_{[h]_q}^1 (1-t)\left|t-\frac{[n-1]_q}{[n]_q}\right| d_q t\right].
\end{aligned}
$$

From (4.1)-(4.4), we have

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y g(t)\ {}^y d_q t\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\, g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right| \\
\le\ & q(y-x)\left|{}^y D_q g(x)\right|\{A_1(q)+A_2(q)\}+q(y-x)\left|{}^y D_q g(y)\right|\{A_3(q)+A_4(q)\}
\end{aligned}
$$

which ends the proof. □

Remark 5. *If we set the limit $q\to 1^-$ in Theorem 4, then we obtain the following Simpson's type inequality*

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y g(t)\ dt\ -\frac{1}{n}\left[g(x)+(n-2)\,g(y+h(x-y))+g(y)\right]\right| \\
\le\ & (y-x)\left[\left|g'(x)\right|\{A_1(1)+A_2(1)\}+\left|g'(y)\right|\{A_3(1)+A_4(1)\}\right]
\end{aligned}
$$

which can be viewed as a special case of the inequality given in [32, Theorem 2.1].

Remark 6. *In Theorem 4, if we take $n=2$ and $[h]_q=\frac{1}{[2]_q}$, then we have the following trapezoidal type inequality*

$$
\begin{aligned}
&\left|\frac{g(x)+qg(y)}{[2]_q}-\frac{1}{y-x}\int_x^y g(\varkappa)\ {}^y d_q\varkappa\right| \\
\le\ & (y-x)\left[\left|\ {}^y D_q g(x)\right|\frac{q^2\left(1+4q+q^2\right)}{[3]_q[2]_q^4}+\left|\ {}^y D_q g(y)\right|\frac{q^2\left(1+3q^2+2q^3\right)}{[3]_q[2]_q^4}\right]
\end{aligned}
$$

which is given by Budak in [11, Theorem 3].

Corollary 3. *If we use $n=6$ and $[h]_q=\frac{1}{[2]_q}$ in Theorem 4, then we have the following Simpson's type inequality*

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y g(t)\ {}^y d_q t\ -\frac{1}{[6]_q}\left[g(x)+q^2[4]_q\, g\left(\frac{x+qy}{2}\right)+qg(y)\right]\right| \\
\le\ & q(y-x)\left[\left|{}^y D_q g(x)\right|\{\mathcal{A}_1(q)+\mathcal{A}_2(q)\}+\left|{}^y D_q g(y)\right|\{\mathcal{A}_3(q)+\mathcal{A}_4(q)\}\right]
\end{aligned}
$$

where

$$\begin{aligned}
\mathcal{A}_1(q) &= \frac{2q^2[2]_q^2+[6]_q^2\left([6]_q-[3]_q\right)}{[2]_q^3[3]_q[6]_q^3},\\
\mathcal{A}_2(q) &= \frac{2q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{[6]_q\left(1+[2]_q^3\right)-[3]_q[5]_q\left(1+[2]_q^2\right)}{[2]_q^3[3]_q[6]_q},\\
\mathcal{A}_3(q) &= 2\frac{q[3]_q[6]_q-q^2}{[2]_q[3]_q[6]_q^3}+\frac{1}{[2]_q^3}\left(\frac{q+q^2}{[3]_q}-\frac{q^2+2q}{[6]_q}\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_4(q) &= 2\frac{q[5]_q^2[6]_q[3]_q-q^2[5]_q^3}{[2]_q[3]_q[6]_q^3}+\frac{q^2}{[2]_q[3]_q}-\frac{q[5]_q}{[2]_q[6]_q}\\
&\quad-\frac{1}{[2]_q^3}\left[\frac{[5]_q\left(2q+q^2\right)}{[6]_q}-\frac{q+q^2}{[3]_q}\right]
\end{aligned}$$

which is given by Erden et al. in [17].

Remark 7. *In Corollary 3, if we take the limit $q\to 1^-$, then we have the following inequality of Simpson's like for the function whose modulus values of the first derivative are convex (see, [6]):*

$$\begin{aligned}
&\left|\frac{1}{6}\left[g(x)+4g\left(\frac{x+y}{2}\right)+g(y)\right]-\frac{1}{y-x}\int_x^y g(t)\,dt\right| \\
&\leq \frac{5(y-x)}{72}\left[\left|g'(x)\right|+\left|g'(y)\right|\right].
\end{aligned}\tag{4.7}$$

Theorem 5. *We assume that the conditions of Lemma 2 hold. If the mapping $\left|{}^{y}D_q g\right|^{p_1}$, $p_1\geq 1$ is convex on $[x,y]$, then we have the following inequality*

$$\begin{aligned}
&\left|\frac{1}{y-x}\int_x^y g(t)\ {}^{y}d_qt\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\,g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right|\\
&\leq q(y-x)\left[\left(\frac{2q}{[2]_q[n]_q^2}+\frac{[h]_q\left([h]_q[n]_q-[2]_q\right)}{[n]_q[2]_q}\right)^{1-\frac{1}{p_1}}\right.\\
&\quad\times\left(A_1(q)\left|{}^{y}D_qg(x)\right|^{p_1}+A_3(q)\left|{}^{y}D_qg(y)\right|^{p_1}\right)^{\frac{1}{p_1}}\\
&\quad+\left(2\frac{q[n-1]_q}{[2]_q[n]_q^2}+\frac{1}{[2]_q}-\frac{[n-1]_q}{[n]_q}-\frac{[n-1]_q[h]_q[2]_q-[h]_q^2[n]_q}{[n]_q[2]_q}\right)^{1-\frac{1}{p_1}}\\
&\quad\left.\times\left(A_2(q)\left|{}^{y}D_qg(x)\right|^{p_1}+A_4(q)\left|{}^{y}D_qg(y)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}\tag{4.8}$$

where $A_1(q)-A_4(q)$ are given in (4.1)-(4.4), respectively.

Proof. By reconsidering the inequality (4.6), applying the well-known power mean inequality for the quantum integrals and the convexity of $|{}^{y}D_{q}g|^{p_1}$, $p_1 \geq 1$, we find that

$$\begin{aligned}
&\left|\frac{1}{y-x}\int_{x}^{y} g(t)\ {}^{y}d_{q}t\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\, g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right| \\
\leq\ & q(y-x)\left[\left(\int_0^{[h]_q}\left|t-\frac{1}{[n]_q}\right|d_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^{[h]_q}\left|t-\frac{1}{[n]_q}\right|\left|\ {}^{y}D_qg(tx+(1-t)y)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_{[h]_q}^{1}\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt\right)^{1-\frac{1}{p_1}}\left(\int_{[h]_q}^{1}\left|t-\frac{[n-1]_q}{[n]_q}\right|\left|\ {}^{y}D_qg(tx+(1-t)y)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}\right] \\
\leq\ & q(y-x)\left[\left(\int_0^{[h]_q}\left|t-\frac{1}{[n]_q}\right|d_qt\right)^{1-\frac{1}{p_1}}\right. \\
&\times\left(|{}^{y}D_qg(x)|^{p_1}\int_0^{[h]_q}t\left|t-\frac{1}{[n]_q}\right|d_qt+|{}^{y}D_qg(y)|^{p_1}\int_0^{[h]_q}(1-t)\left|t-\frac{1}{[n]_q}\right|d_qt\right)^{\frac{1}{p_1}} \\
&+\left(\int_{[h]_q}^{1}\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt\right)^{1-\frac{1}{p_1}} \\
&\left.\times\left(|{}^{y}D_qg(x)|^{p_1}\int_{[h]_q}^{1}t\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt+|{}^{y}D_qg(y)|^{p_1}\int_{[h]_q}^{1}(1-t)\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

From (4.1)-(4.4), we obtain that

$$\begin{aligned}
&\left|\frac{1}{y-x}\int_{x}^{y} g(t)\ {}^{y}d_{q}t\ -\frac{1}{[n]_q}\left[g(x)+q^2[n-2]_q\, g\left(y+[h]_q(x-y)\right)+qg(y)\right]\right| \\
\leq\ & q(y-x)\left[\left(\int_0^{[h]_q}\left|t-\frac{1}{[n]_q}\right|d_qt\right)^{1-\frac{1}{p_1}}\left(A_1(q)\left|\ {}^{y}D_qg(x)\right|^{p_1}+A_3(q)\left|\ {}^{y}D_qg(y)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_{[h]_q}^{1}\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt\right)^{1-\frac{1}{p_1}}\left(A_2(q)\left|\ {}^{y}D_qg(x)\right|^{p_1}+A_4(q)\left|\ {}^{y}D_qg(y)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned}\tag{4.9}$$

We also observe that

$$\begin{aligned}
\int_0^{[h]_q}\left|t-\frac{1}{[n]_q}\right|d_qt &= \int_0^{\frac{1}{[n]_q}}\left(\frac{1}{[n]_q}-t\right)d_qt+\int_{\frac{1}{[n]_q}}^{1}\left(t-\frac{1}{[n]_q}\right)d_qt \\
&= \frac{2q}{[2]_q[n]_q^2}+\frac{[h]_q\left([h]_q[n]_q-[2]_q\right)}{[n]_q[2]_q}
\end{aligned}\tag{4.10}$$

and

$$\begin{aligned}
&\int_{[h]_q}^{1}\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt \\
&= \int_{[h]_q}^{\frac{[n-1]_q}{[n]_q}}\left(\frac{[n-1]_q}{[n]_q}-t\right)d_qt+\int_{\frac{[n-1]_q}{[n]_q}}^{1}\left(t-\frac{[n-1]_q}{[n]_q}\right)d_qt \\
&= 2\frac{q[n-1]_q}{[2]_q[n]_q^2}+\frac{1}{[2]_q}-\frac{[n-1]_q}{[n]_q}-\frac{[n-1]_q[h]_q[2]_q-[h]_q^2[n]_q}{[n]_q[2]_q}.
\end{aligned}\tag{4.11}$$

Substituting the computed integrals (4.10) and (4.11) in (4.9), we obtain the inequality (4.8) and the proof is ended. □

Corollary 4. *In Theorem 5, if we take the limit $q \to 1^-$, then we obtain the following Simpson's type inequality*

$$\begin{aligned}
&\left| \frac{1}{y-x} \int_x^y g(t) \quad dt \quad - \frac{1}{n} \left[g(x) + (n-2)\, g(y + h(x-y)) + g(y) \right] \right| \\
\leq \quad & (y-x) \left[\left(\frac{1}{n^2} + \frac{h(hn-2)}{2n} \right)^{1-\frac{1}{p_1}} \right. \\
& \times \left(A_1(1) \left| g'(x) \right|^{p_1} + A_3(1) \left| g'(y) \right|^{p_1} \right)^{\frac{1}{p_1}} \\
& + \left(\frac{n-1}{n^2} + \frac{1}{2} - \frac{n-1}{n} - \frac{2(n-1)h - h^2 n}{2n} \right)^{1-\frac{1}{p_1}} \\
& \left. \times \left(A_2(1) \left| g'(x) \right|^{p_1} + A_4(1) \left| \, g'(y) \right|^{p_1} \right)^{\frac{1}{p_1}} \right].
\end{aligned}$$

Corollary 5. *In Theorem 5, if we set $n = 6$ and $[h]_q = \frac{1}{[2]_q}$, then we obtain the following Simpson's type inequality*

$$\begin{aligned}
&\left| \frac{1}{y-x} \int_x^y g(t) \quad {}^y d_q t \quad - \frac{1}{[6]_q} \left[g(x) + q^2 [4]_q\, g\left(y + [h]_q (x-y) \right) + q g(y) \right] \right| \\
\leq \quad & q(y-x) \left[\left(\frac{2q}{[2]_q [6]_q^2} + \frac{q^3 [3]_q - q}{[6]_q [2]_q^3} \right)^{1-\frac{1}{p_1}} \right. \\
& \times \left(\mathcal{A}_1(q) \left| {}^y D_q g(x) \right|^{p_1} + \mathcal{A}_3(q) \left| {}^y D_q g(y) \right|^{p_1} \right)^{\frac{1}{p_1}} \\
& + \left(2q \frac{[5]_q^2}{[2]_q [6]_q^2} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q [2]_q^2 - [6]_q}{[6]_q [2]_q^3} \right)^{1-\frac{1}{p_1}} \\
& \left. \times \left(\mathcal{A}_2(q) \left| {}^y D_q g(x) \right|^{p_1} + \mathcal{A}_4(q) \left| {}^y D_q g(y) \right|^{p_1} \right)^{\frac{1}{p_1}} \right]
\end{aligned} \tag{4.12}$$

where $A_1(q) - A_4(q)$ are given in Corollary 3. The above inequality of Simpson's type for differentiable convex functions was given by Erden et al. in [17].

Remark 8. *In Corollary 5, if we use the limit $q \to 1^-$, then we have the following inequality given by Alomari et al. (see, [6]):*

$$\begin{aligned}
&\left| \frac{1}{6} \left[g(x) + 4g\left(\frac{x+y}{2} \right) + g(y) \right] - \frac{1}{y-x} \int_x^y g(t)\, dt \right| \\
\leq \quad & \frac{1}{(1296)^{\frac{1}{p_1}}} \left(\frac{5}{72} \right)^{1-\frac{1}{p_1}} (y-x) \\
& \times \left(\left[61 \left| g'(x) \right|^{p_1} + 29 \left| g'(y) \right|^{p_1} \right]^{\frac{1}{p_1}} + \left[29 \left| g'(x) \right|^{p_1} + 61 \left| g'(y) \right|^{p_1} \right]^{\frac{1}{p_1}} \right).
\end{aligned} \tag{4.13}$$

Remark 9. *In Theorem 5, if we assume $n = 2$ and $[h]_q = \frac{1}{[2]_q}$, then we obtain the following trapezoidal type inequality*

$$\begin{aligned}
&\left| \frac{g(x) + qg(y)}{[2]_q} - \frac{1}{y-x} \int_x^y g(\varkappa)\ {}^y d_q \varkappa \right| \\
\leq\ & \frac{q(y-x)}{[2]_q} \left[\left(\frac{q\left(2+q+q^3\right)}{[2]_q^3} \right)^{1-\frac{1}{p_1}} \right. \\
&\times \left. \left(|\ {}^y D_q g(x)|^{p_1} \frac{q\left(1+4q+q^2\right)}{[3]_q [2]_q^3} + |\ {}^y D_q g(y)|^{p_1} \frac{q\left(1+3q^2+2q^3\right)}{[3]_q [2]_q^3} \right)^{\frac{1}{p_1}} \right]
\end{aligned}$$

which is given by Budak in [11, Theorem 4].

4.2. **Simpson's Inequalities for q_x-integrals.** In this subsection, we establish quantum Simpson's inequalities by using the Lemma 3.

Theorem 6. *We assume that the conditions of Lemma 3 hold. If the mapping $|{}_x D_q g|$ is convex on $[x, y]$, then the following inequality holds for q_x-integral:*

$$\begin{aligned}
&\left| \frac{1}{[n]_q} \left[qg(x) + q^2 [n-2]_q\, g\left(x + [h]_q (y-x)\right) + g(y) \right] - \frac{1}{y-x} \int_x^y g(t)\ {}_x d_q t \right| \qquad (4.14) \\
\leq\ & q(y-x) \left[|{}_x D_q g(y)| \left\{ A_1(q) + A_2(q) \right\} + |{}_x D_q g(x)| \left\{ A_3(q) + A_4(q) \right\} \right]
\end{aligned}$$

where $A_1(q) - A_4(q)$ are given in (4.1)-(4.4), respectively.

Proof. If we apply the techniques used in the proof of Theorem 4 and consider the Lemma 3, then we can prove the inequality (4.14). □

Remark 10. *If we set $q \to 1^-$ in Theorem 6, then we obtain the following Simpson's type inequality*

$$\begin{aligned}
&\left| \frac{1}{y-x} \int_x^y g(t)\ dt\ - \frac{1}{n} \left[g(x) + (n-2)\, g(y + h(x-y)) + g(y) \right] \right| \\
\leq\ & (y-x) \left[|g'(y)| \left\{ A_1(1) + A_2(1) \right\} + |g'(x)| \left\{ A_3(1) + A_4(1) \right\} \right]
\end{aligned}$$

which can be viewed as a special case of the inequality given in [32, Theorem 2.1].

Remark 11. *In Theorem 6, if we take $n = 2$ and $[h]_q = \frac{1}{[2]_q}$, then we have the following trapezoidal type inequality*

$$\begin{aligned}
&\left| \frac{qg(x) + g(y)}{[2]_q} - \frac{1}{y-x} \int_x^y g(\varkappa)\ {}_x d_q \varkappa \right| \\
\leq\ & (y-x) \left[|\ {}_x D_q g(y)| \frac{q^2\left(1+4q+q^2\right)}{[3]_q [2]_q^4} + |\ {}_x D_q g(x)| \frac{q^2\left(1+3q^2+2q^3\right)}{[3]_q [2]_q^4} \right]
\end{aligned}$$

which is given by Noor et al. in [28, Theorem 3.2 for $r = 1$].

Corollary 6. *If we use $n = 6$ and $[h]_q = \frac{1}{[2]_q}$ in Theorem 6, then we have the following Simpson's type inequality*

$$\begin{aligned}
&\left| \frac{1}{[6]_q} \left[qg(x) + q^2 [4]_q\, g\left(\frac{qx+y}{[2]_q} \right) + g(y) \right] - \frac{1}{y-x} \int_x^y g(t)\ {}_x d_q t \right| \\
\leq\ & q(y-x) \left[|{}_x D_q g(y)| \left\{ \mathcal{A}_1(q) + \mathcal{A}_2(q) \right\} + |{}_x D_q g(x)| \left\{ \mathcal{A}_3(q) + \mathcal{A}_4(q) \right\} \right]
\end{aligned}$$

where $\mathcal{A}_1(q) - \mathcal{A}_4(q)$ are given in Corollary 3.

Remark 12. *In Corollary 6, if we take the limit $q \to 1^-$, then we obtain the inequality (4.7).*

Theorem 7. *We assume that the conditions of Lemma 3 hold. If the mapping* $|_x D_q g|^{p_1}$, $p_1 \geq 1$ *is convex on* $[x, y]$, *then we have the following inequality*

$$\left| \frac{1}{[n]_q} \left[qg(x) + q^2 [n-2]_q g\left(x + [h]_q (y-x)\right) + g(y) \right] - \frac{1}{y-x} \int_x^y g(t) \ {}_x d_q t \right| \tag{4.15}$$

$$\leq q(y-x) \Bigg[\left(\frac{2q}{[2]_q [n]_q^2} + \frac{[h]_q \left([h]_q [n]_q - [2]_q\right)}{[n]_q [2]_q} \right)^{1-\frac{1}{p_1}}$$

$$\times \left(A_1(q) \left| {}_x D_q g(y)\right|^{p_1} + A_3(q) \left| {}_x D_q g(x)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

$$+ \left(2\frac{q[n-1]_q}{[2]_q [n]_q^2} + \frac{1}{[2]_q} - \frac{[n-1]_q}{[n]_q} - \frac{[n-1]_q [h]_q [2]_q - [h]_q^2 [n]_q}{[n]_q [2]_q} \right)^{1-\frac{1}{p_1}}$$

$$\times \left(A_2(q) \left| {}_x D_q g(y)\right|^{p_1} + A_4(q) \left| {}_x D_q g(x)\right|^{p_1}\right)^{\frac{1}{p_1}} \Bigg]$$

where $A_1(q) - A_4(q)$ *are given in (4.1)-(4.4), respectively.*

Proof. If we apply the techniques used in the proof of Theorem 5 and consider the Lemma 3, then we can prove the inequality (4.15). □

Corollary 7. *In Theorem 7, if we take the limit* $q \to 1^-$, *then we obtain the following Simpson's type inequality*

$$\left| \frac{1}{n} \left[g(x) + (n-2) g(y + h(x-y)) + g(y) \right] - \frac{1}{y-x} \int_x^y g(t) \ dt \right|$$

$$\leq (y-x) \Bigg[\left(\frac{1}{n^2} + \frac{h(hn-2)}{2n} \right)^{1-\frac{1}{p_1}}$$

$$\times \left(A_1(1) \left|g'(y)\right|^{p_1} + A_3(1) \left|g'(x)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

$$+ \left(\frac{n-1}{n^2} + \frac{1}{2} - \frac{n-1}{n} - \frac{2(n-1)h - h^2 n}{2n} \right)^{1-\frac{1}{p_1}}$$

$$\times \left(A_2(1) \left|g'(y)\right|^{p_1} + A_4(1) \left| g'(x)\right|^{p_1}\right)^{\frac{1}{p_1}} \Bigg].$$

Corollary 8. *In Theorem 7, if we set* $n = 6$ *and* $[h]_q = \frac{1}{[2]_q}$, *then we obtain the following Simpson's type inequality*

$$\left| \frac{1}{[6]_q} \left[qg(x) + q^2 [4]_q g\left(\frac{qx+y}{[2]_q}\right) + g(y) \right] - \frac{1}{y-x} \int_x^y g(t) \ {}_x d_q t \right|$$

$$\leq q(y-x) \Bigg[\left(\frac{2q}{[2]_q [6]_q^2} + \frac{q^3 [3]_q - q}{[6]_q [2]_q^3} \right)^{1-\frac{1}{p_1}}$$

$$\times \left(\mathcal{A}_1(q) \left| {}_x D_q g(y)\right|^{p_1} + \mathcal{A}_3(q) \left| {}_x D_q g(x)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

$$+ \left(2q\frac{[5]_q^2}{[2]_q [6]_q^2} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q [2]_q^2 - [6]_q}{[6]_q [2]_q^3} \right)^{1-\frac{1}{p_1}}$$

$$\times \left(\mathcal{A}_2(q) \left| {}_x D_q g(y)\right|^{p_1} + \mathcal{A}_4(q) \left| {}_x D_q g(x)\right|^{p_1}\right)^{\frac{1}{p_1}} \Bigg]$$

where $\mathcal{A}_1(q) - \mathcal{A}_4(q)$ *are given in Corollary 3.*

Remark 13. *If we take the limit* $q \to 1^-$ *in Corollary 8, then we obtain the inequality (4.13).*

Remark 14. *In Theorem 7, if we assume $n=2$ and $[h]_q=\frac{1}{[2]_q}$, then we obtain the following trapezoidal type inequality*

$$\begin{aligned}&\left|\frac{qg(x)+g(y)}{[2]_q}-\frac{1}{y-x}\int_x^y g(\varkappa)\ {}_xd_q\varkappa\right|\\ \leq\ & \frac{q(y-x)}{[2]_q}\left[\left(\frac{q\left(2+q+q^3\right)}{[2]_q^3}\right)^{1-\frac{1}{p_1}}\right.\\ &\left.\times\left(\left|\ {}^yD_qg(y)\right|^{p_1}\frac{q\left(1+4q+q^2\right)}{[3]_q[2]_q^3}+\left|\ {}^yD_qg(x)\right|^{p_1}\frac{q\left(1+3q^2+2q^3\right)}{[3]_q[2]_q^3}\right)^{\frac{1}{p_1}}\right]\end{aligned}$$

which is given by Noor et al. in [28, Theorem 3.2].

5. Newton's inequalities for quantum integrals

In this section, we offer some new quantum boundaries for quantum Newton's inequalities via q_x-integral and q^y-integral. Moreover, we give some calculated integrals that will be used in the new results.

Lemma 7. *The following equalities hold for the quantum integrals:*

$$\begin{aligned}&\int_0^{[h]_q} t\left|t-\frac{1}{[n]_q}\right|d_qt \qquad (5.1)\\ =\ & B_1(q)=\frac{2q^2}{[n]_q^3[2]_q[3]_q}+\frac{[h]_q^2\left([h]_q[2]_q[n]_q-[3]_q\right)}{[2]_q[3]_q[n]_q},\end{aligned}$$

$$\begin{aligned}&\int_0^{[h]_q}(1-t)\left|t-\frac{1}{[n]_q}\right|d_qt \qquad (5.2)\\ =\ & B_2(q)=\frac{2q}{[n]_q^2[2]_q}+\frac{[h]_q\left([h]_q[n]_q-[2]_q\right)}{[2]_q[n]_q}-B_1(q),\end{aligned}$$

$$\begin{aligned}&\int_{[h]_q}^{[1-h]_q} t\left|t-\frac{1}{[10-n]_q}\right|d_qt \qquad (5.3)\\ =\ & B_3(q)=\frac{2q^2}{[10-n]_q^3[2]_q[3]_q}+\frac{[h]_q^3+[1-h]_q^3}{[3]_q}-\frac{[1-h]_q^2+[h]_q^2}{[10-n]_q[2]_q},\end{aligned}$$

$$\begin{aligned}&\int_{[h]_q}^{[1-h]_q}(1-t)\left|t-\frac{1}{[10-n]_q}\right|d_qt \qquad (5.4)\\ =\ & B_4(q)=\frac{2q}{[10-n]_q[2]_q}+\frac{[h]_q^2+[1-h]_q^2}{[2]_q}-\frac{[h]_q+[1-h]_q}{[10-n]_q}-B_3(q),\end{aligned}$$

$$\int_{[1-h]_q}^{1} t\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt \qquad (5.5)$$

$$= B_5(q) = \frac{2q^2[n-1]_q^3}{[n]_q^3[2]_q[3]_q} + \frac{1+[1-h]_q^3}{[3]_q} - \frac{[n-1]_q\left(1+[1-h]_q^2\right)}{[n]_q[2]_q},$$

$$\int_{[1-h]_q}^{1}(1-t)\left|t-\frac{[n-1]_q}{[n]_q}\right|d_qt \tag{5.6}$$

$$= B_6(q) = \frac{2q[n-1]_q^2}{[n]_q^2[2]_q} + \frac{[1-h]_q^2+1}{[2]_q} - \frac{[n-1]_q\left(1+[1-h]_q\right)}{[n]_q} - B_5(q).$$

5.1. **Newton's inequalities for q^y-quantum integrals.** In this subsection, we offer some new results about the Newton's inequalities for q^y-quantum differentiable convex functions via the q^y-quantum integral.

Theorem 8. *We assume that the assumptions of Lemma 4 hold. If the mapping $|{}^yD_qg|$ is convex on $[x,y]$, then the following inequality holds for q^y-integral:*

$$\begin{aligned} &\left|\frac{1}{y-x}\int_x^y g(\varkappa)\ {}^yd_q\varkappa - \frac{1}{[n]_q}\left[g(x)+\frac{q\left([n]_q-[10-n]_q\right)}{[10-n]_q}g\left(y+[h]_q(x-y)\right)\right.\right. \\ &\left.\left.+\frac{[n-1]_q[10-n]_q-[n]_q}{[n]_q[10-n]_q}g\left(y+[1-h]_q(x-y)\right)+qg(y)\right]\right| \\ \leq\ & q(y-x)\left[|{}^yD_qg(x)|\{B_1(q)+B_3(q)+B_5(q)\}+|{}^yD_qg(y)|\{B_2(q)+B_4(q)+B_6(q)\}\right] \end{aligned} \tag{5.7}$$

where $B_1(q)-B_6(q)$ are defined in (5.1)-(5.6), respectively.

Proof. If we apply the techniques used in the proof of Theorem 4 and consider the Lemma 4, then we can prove the inequality (5.7). □

Remark 15. *In Theorem 8, if we take the limit $q\to 1^-$ first, later we consider $n=8$ and $h=\frac{1}{3}$, then we obtain the following inequality:*

$$\begin{aligned} &\left|\frac{1}{y-x}\int_x^y g(t)\,dt - \frac{1}{8}\left[g(x)+3g\left(\frac{2x+y}{3}\right)+3g\left(\frac{x+2y}{3}\right)+g(y)\right]\right| \\ \leq\ & \frac{25}{576}(y-x)\left[|g'(x)|+|g'(y)|\right], \end{aligned} \tag{5.8}$$

which is given by Noor et al. in [29].

Theorem 9. *We assume that the assumptions of Lemma 4 hold. If the mapping $|{}^{y}D_{q}g|^{p_1}$, $p_1 \geq 1$ is convex on $[x,y]$, then the following inequality holds for q^y-integral:*

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_{x}^{y} g(\varkappa)\ {}^{y}d_{q}\varkappa - \frac{1}{[n]_q}\left[g(x) + \frac{q\left([n]_q - [10-n]_q\right)}{[10-n]_q} g\left(y + [h]_q (x-y)\right)\right.\right. \\
&\left.\left. + \frac{[n-1]_q [10-n]_q - [n]_q}{[n]_q [10-n]_q} g\left(y + [1-h]_q (x-y)\right) + qg(y)\right]\right| \\
\leq\ & q(y-x)\left[\left(\frac{2q}{[n]_q^2 [2]_q} + \frac{[h]_q \left([h]_q [n]_q - [2]_q\right)}{[2]_q [n]_q}\right)^{1-\frac{1}{p_1}}\right. \\
&\times \left(B_1(q)\,|\ {}^{y}D_{q}g(x)|^{p_1} + B_2(q)\,|\ {}^{y}D_{q}g(y)|^{p_1}\right)^{\frac{1}{p_1}} \\
&+ \left(\frac{2q}{[10-n]_q [2]_q} + \frac{[h]_q^2 + [1-h]_q^2}{[2]_q} - \frac{[h]_q + [1-h]_q}{[10-n]_q}\right)^{1-\frac{1}{p_1}} \\
&\times \left(B_3(q)\,|\ {}^{y}D_{q}g(x)|^{p_1} + B_4(q)\,|\ {}^{y}D_{q}g(y)|^{p_1}\right)^{\frac{1}{p_1}} \\
&+ \left(\frac{2q[n-1]_q^2}{[n]_q^2 [2]_q} + \frac{[1-h]_q^2 + 1}{[2]_q} - \frac{[n-1]_q \left(1 + [1-h]_q\right)}{[n]_q}\right)^{1-\frac{1}{p_1}} \\
&\left.\times \left(B_5(q)\,|\ {}^{y}D_{q}g(x)|^{p_1} + B_6(q)\,|\ {}^{y}D_{q}g(y)|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}
\tag{5.9}
$$

where $B_1(q) - B_6(q)$ are defined in (5.1)-(5.6), respectively.

Proof. If we apply the techniques used in the proof of Theorem 5 and consider the Lemma 4, then we can prove the inequality (5.9). ☐

Remark 16. *If we take the limit $q \to 1^-$ first, later consider $n = 8$ and $h = \frac{1}{3}$ in Theorem 9, then we have the following inequality*

$$
\begin{aligned}
&\left|\frac{1}{y-x}\int_{x}^{y} g(t)\,dt - \frac{1}{8}\left[g(x) + 3g\left(\frac{2x+y}{3}\right) + 3g\left(\frac{x+2y}{3}\right) + g(y)\right]\right| \\
\leq\ & \frac{y-x}{36}\left\{\left(\frac{17}{16}\right)^{1-\frac{1}{p_1}}\left(\frac{251}{1152}|g'(x)|^{p_1} + \frac{973}{1152}|g'(y)|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&+ \left(\frac{1}{2}|g'(x)|^{p_1} + \frac{1}{2}|g'(y)|^{p_1}\right)^{\frac{1}{p_1}} \\
&\left.+ \left(\frac{17}{16}\right)^{1-\frac{1}{p_1}}\left(\frac{973}{1152}|g'(x)|^{p_1} + \frac{251}{1152}|g'(y)|^{p_1}\right)^{\frac{1}{p_1}}\right\},
\end{aligned}
\tag{5.10}
$$

which can be found in [29].

5.2. **Newton's inequalities for q_x-quantum integrals.** In this subsection, we offer some new results about the Newton's inequalities for q_x-quantum differentiable convex functions via the q_x-quantum integral.

Theorem 10. *We assume that the assumptions of Lemma 5 hold. If the mapping $\left|{}_{x}D_{q}g\right|$ is convex on $[x,y]$, then the following inequality holds for q_x-integral:*

$$\begin{aligned}(5.11)\quad &\left|\frac{1}{[n]_q}\left[qg(x)+\frac{[n-1]_q[10-n]_q-[n]_q}{[n]_q[10-n]_q}g\left(x+[1-h]_q(y-x)\right)\right.\right.\\ &\left.\left.+\frac{q\left([n]_q-[10-n]_q\right)}{[10-n]_q}g\left(x+[h]_q(y-x)\right)+g(y)\right]-\frac{1}{y-x}\int_x^y g(\varkappa)\ {}_{x}d_q\varkappa\right|\\ \leq\ & q(y-x)\left[\left|{}_{x}D_qg(y)\right|\left\{B_1(q)+B_3(q)+B_5(q)\right\}+\left|{}_{x}D_qg(x)\right|\left\{B_2(q)+B_4(q)+B_6(q)\right\}\right]\end{aligned}$$

where $B_1(q)-B_6(q)$ are defined in (5.1)-(5.6), respectively.

Proof. If we apply the techniques used in the proof of Theorem 4 and consider the Lemma 5, then we can prove the inequality (5.11). □

Remark 17. *In Theorem 10, if we consider the limit $q\to 1^-$ first, later we consider $n=8$ and $h=\frac{1}{3}$, then we recapture the inequality (5.8).*

Theorem 11. *We assume that the assumptions of Lemma 5 hold. If the mapping $\left|{}_{x}D_{q}g\right|^{p_1}$, $p_1\geq 1$ is convex on $[x,y]$, then the following inequality holds for q_x-integral:*

$$\begin{aligned}(5.12)\quad &\left|\frac{1}{[n]_q}\left[qg(x)+\frac{[n-1]_q[10-n]_q-[n]_q}{[n]_q[10-n]_q}g\left(x+[1-h]_q(y-x)\right)\right.\right.\\ &\left.\left.+\frac{q\left([n]_q-[10-n]_q\right)}{[10-n]_q}g\left(x+[h]_q(y-x)\right)+g(y)\right]-\frac{1}{y-x}\int_x^y g(\varkappa)\ {}_{x}d_q\varkappa\right|\\ \leq\ & q(y-x)\left[\left(\frac{2q}{[n]_q^2[2]_q}+\frac{[h]_q\left([h]_q[n]_q-[2]_q\right)}{[2]_q[n]_q}\right)^{1-\frac{1}{p_1}}\right.\\ &\times\left(B_1(q)\left|{}_{x}D_qg(y)\right|^{p_1}+B_2(q)\left|{}_{x}D_qg(x)\right|^{p_1}\right)^{\frac{1}{p_1}}\\ &+\left(\frac{2q}{[10-n]_q[2]_q}+\frac{[h]_q^2+[1-h]_q^2}{[2]_q}-\frac{[h]_q+[1-h]_q}{[10-n]_q}\right)^{1-\frac{1}{p_1}}\\ &\times\left(B_3(q)\left|{}_{x}D_qg(y)\right|^{p_1}+B_4(q)\left|{}_{x}D_qg(x)\right|^{p_1}\right)^{\frac{1}{p_1}}\\ &+\left(\frac{2q[n-1]_q^2}{[n]_q^2[2]_q}+\frac{[1-h]_q^2+1}{[2]_q}-\frac{[n-1]_q\left(1+[1-h]_q\right)}{[n]_q}\right)^{1-\frac{1}{p_1}}\\ &\left.\times\left(B_5(q)\left|{}_{x}D_qg(y)\right|^{p_1}+B_6(q)\left|{}_{x}D_qg(x)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]\end{aligned}$$

where $B_1(q)-B_6(q)$ are defined in (5.1)-(5.6), respectively.

Proof. If we apply the techniques used in the proof of Theorem 5 and consider the Lemma 5, then we can prove the inequality (5.12). □

Remark 18. *In Theorem 11, if we take the limit $q\to 1^-$ first, later we consider $n=8$ and $h=\frac{1}{3}$, then we recapture the inequality (5.10).*

6. Conclusions

We conclude our work by mentioning that here, we proved some new quantum integral inequalities of Simpson's and Newton's type for differentiable convex functions by using the notions of quantum derivatives and quantum integrals. It is important to mention that our results transformed into the some new and known results by considering different choices of the general quantum number and considering the limit $q\to 1^-$ in our main results. We strongly believe that it is an interesting and new problem for the upcoming researchers those can obtain similar inequalities for co-ordinated convex functions in their future work.

References

[1] M. A. Ali, H. Budak, Z. Zhang, and H. Yildrim, *Some New Simpson's type inequalities for co-ordinated convex functions in quantum calculus,* Mathematical Methods in the Applied Sciences, In press, 2020.

[2] M. A. Ali, H. Budak, M. Abbas, and Y.-M. Chu, *Quantum Hermite-Hadamard type inequalities for functions whose second q^y-derivatives absolute value are convex*, Advances in Difference Equations, In press, 2020.

[3] N. Alp, M. Z. Sarikaya, M. Kunt and I. Iscan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University–Science (2018) 30, 193–203.

[4] N. Alp and M. Z. Sarikaya, *Hermite Hadamard's type inequalities for co-ordinated convex functions on quantum integral,* Applied Mathematics E-Notes, 20(2020), 341–356.

[5] Almeida, R.; Torres, D.F.M. *Non-differentiable variational principles in terms of a quantum operator*, Math. Methods Appl. Sci. 2011, 34, 2231–2241.

[6] M. Alomari, M. Darus, and S. S. Dragomir, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll., vol. 12, no. 4, 2009.

[7] R. Askey, J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize the Jacobi polynomails*, Mem. Am. Math. Soc. 1985, 54, 1–55.

[8] S. Bermudo, P. Kórus and J. N. Valdés, *On q-Hermite–Hadamard inequalities for general convex functions*, Acta Mathematica Hungarica, 1-11, 2020.

[9] H. Budak, M. A. Ali, and M. Tarhanaci, *Some New Quantum Hermite–Hadamard-Like Inequalities for Coordinated Convex Functions,* J Optim Theory Appl 186, 899–910 (2020). https://doi.org/10.1007/s10957-020-01726-6.

[10] H. Budak, S. Erden, and M. A. Ali, *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals,* Mathematical Methods in the Applied Sciences, https://doi.org/10.1002/mma.6742.

[11] H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, (submitted), 2020.

[12] J, Cresson, G. S. F. Frederico, D. F. M. Torres, *Constants of motion for Non-differentiable quantum variational problems*, Topol. Methods Nonlinear Anal. 2009, 33, 217–231.

[13] A. M. C. Brito, D. A. Cruz, Symmetric Quantum Calculus. Ph.D. Thesis, Aveiro University, Aveiro, Portugal, 2012.

[14] S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl. 5 (2000) 533–579.

[15] T. Ernst, The History Of Q-Calculus And New Method. Sweden: Department of Mathematics, Uppsala University, 2000.

[16] S. Erden, S.Iftikhar, R. M. Delavar, P. Kumam, P. Thounthong and W. Kumam, *On generalizations of some inequalities for convex functions via quantum integrals*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, (2020) 114(3), 1-15. Doi: 10.1007/s13398-020-00841-3.

[17] S. Eerden, M. A. Ali, and H. Budak, *Simpson type estimations for convex functions via quantum calculus, Submitted (2020).*

[18] S. Iftikhar, P. Komam and S. Erden, *Newton's type Integral Inequalities via Local Fractional Integrals*, Fractals, 28 (3), (2020), 2050037, 13 pages. Doi: 10.1142/S0218348X20500371.

[19] H. Gauchman, *Integral inequalities in q-calculus*, Comput. Math. Appl. 47 (2004) 281-300.

[20] M. E. H. Ismail, P. Simeonov, *q-Difference operators for orthogonal polynomials*, J. Comput. Appl. Math. 2009, 233, 749–761.

[21] F. H. Jackson, *On a q-definite integrals*, Quarterly J. Pure Appl. Math. 41 (1910) 193-203.

[22] S. Jhanthanam, T. Jessada , N. Sotiris K., and N. Kamsing, *On q-Hermite-Hadamard Inequalities for Differentiable Convex Functions*, Mathematics 7, no. 7 (2019): 632.

[23] V. Kac and P. Cheung *Quantum calculus*, Springer (2001).

[24] M. A. Khan, M. Noor, E. R. Nwaeze, and Y.-M. Chu , *Quantum Hermite–Hadamard inequality by means of a Green function*, Advances in Difference Equations 2020, no. 1 (2020): 1-20.

[25] W. Liu and Z. Hefeng, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, Journal of Applied Analysis and Computation, 7(2), 501-522.

[26] M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Appl. Math. Comput. 269 (2015) 242–251.

[27] M. A. Noor, K. I. Noor and S. Iftikhar, *Some Newton's type inequalities for harmonic convex functions.* Journal of Advance Mathematical Studies, 9(1)(2016),07-16.

[28] M. A. Noor, K. I. Noor and M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput. 251 (2015), 675-679.

[29] M. A. Noor, K. I. Noor and S. Iftikhar, *Newton inequalities for p-harmonic convex functions*, Honam Mathematical Journal, 40(2)(2018), 239-250.

[30] E. R. Nwaeze, and A. M. Tameru, *New parameterized quantum integral inequalities via η-quasiconvexity*, Advances in Difference Equations 2019, no. 1 (2019): 425.

[31] D. N. Page, Information in black hole radiation. Phys. Rev. Lett. 1993, 71, 3743–3746.

[32] J. Park, *On Simpson-like type integral inequalities for differentiable preinvex functions,* Applied Mathematical Sciences, Vol. 7, 2013, no. 121, 6009-6021.

[33] M. Z. Sarikaya, E. Set and M. E. Özdemir, *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll. 13(2) (2010), Article2.
[34] W. Sudsutad, S. K. Ntouyas, J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal. 9 (3) (2015) 781–793.
[35] J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ. 282 (2013) 1-19.
[36] J. Tariboon and S. K. Ntouyas, *Quantum integral inequalities on finite intervals,* J. Inequal. Appl. 2014, 2014:121, 13pp.
[37] M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, and Z. Zhang, *Some New Newton's Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus,* Symmetry 2020, 12(9), 1476; https://doi.org/10.3390/sym12091476.
[38] D. Youm, *q-deformed conformal quantum mechanics*, Phys. Rev. D 2000, 62, 095009.
[39] Y. Zhang, T. S. Du, H. Wang and Y. J. Shen, *Different types of quantum integral inequalities via* (α, m)*-convexity*, Journal of inequalities and applications, 2018(1), p.264 (2018).
[40] H. Zhuang, W. Liu, J. Park, *Some quantum estimates of Hermite-Hadmard inequalities for quasi-convex functions*, Mathematics 2019, 7, 152.

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