A NEW EXTENSION OF QUANTUM SIMPSON'S AND QUANTUM NEWTON'S TYPE INEQUALITIES FOR QUANTUM DIFFERENTIABLE CONVEX FUNCTIONS

MUHAMMAD AAMIR ALI[1*], HÜSEYIN BUDAK[2], AND ZHIYUE ZHANG[3]

Abstract. In this paper, we prove two identities involving quantum derivatives, quantum integrals, and certain parameters. Using the newly proved identities, we prove new inequalities of Simpson's and Newton's type for quantum differentiable convex functions under certain assumptions. Moreover, we discuss the special cases of our main results and obtain some new and existing Simpson's type inequalities, Newton's type inequalities, midpoint type inequalities and trapezoidal type inequalities.

1. Introduction

Thomas Simpson has evolved essential techniques for the numerical integration and estimation of definite integrals taken into consideration as Simpson's rule during (1710-1761). Nevertheless, a comparable approximation became utilized by J. Kepler nearly earlier than 10 decades, so it's also called Kepler's rule. Simpson's rule consists of the 3-point Newton-Cotes quadrature rule, so estimation primarily based totally on 3 steps quadratic kernel is every so often known as Newton-type results.
1) Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{6}\left[\mathcal{F}(\kappa_1)+4\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\mathcal{F}(\kappa_2)\right].$$

2) Simpson's second formula or Newton-Cotes quadrature formula (Simpson's 3/8 rule).

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)dx \approx \frac{\kappa_2-\kappa_1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right].$$

There are a huge variety of estimations associated with those quadrature rules withinside the literature, certainly considered one among them is the subsequent estimation called Simpson's inequality:

Theorem 1. *Suppose that $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is a four times continuously differentiable mapping on (κ_1,κ_2), and let $\left\|\mathcal{F}^{(4)}\right\|_\infty=\sup_{x\in(\kappa_1,\kappa_2)}\left|\mathcal{F}^{(4)}(x)\right|<\infty$. Then, one has the inequality*

$$\left|\frac{1}{3}\left[\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}+2\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)dx\right|\le\frac{1}{2880}\left\|\mathcal{F}^{(4)}\right\|_\infty(\kappa_2-\kappa_1)^4.$$

In recent years, many writers have focused on Simpson's type inequality in various categories of mappings. Specifically, some mathematicians have worked on the results of Simpson's and Newton's type in obtaining a convex map, because convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics. For example, Dragomir et al. [15] presented the new Simpson's inequalities and their applications in quadrature formulas for numerical integration. In addition, some inequalities of Simpson's type of s-convex functions were determined by Alomari et al. in [6]. Subsequently, Sarikaya et al. note the variance of Simpson's type inequality based on convexity in [34]. For the further studies of this area, one can consult [17, 21, 32].

1991 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. Simpson's inequalities, Newton's inequalities, quantum calculus, convex functions.
[1*]Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, China.
Email: mahr.muhammad.aamir@gmail.com.
[2]Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY.
Email: hsyn.budak@gmail.com.
[3]Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, China.
Email: 05298@njnu.edu.cn.

On the other hand, in the field of q-analysis, many researchers have carried out various studies due to a high demand for mathematics involving quantum calculation modelling. This is the reason why Euler began this stage in the development of what we now know as $q-$calculus and which also serves as a bridge between mathematics and physics. Mathematical areas such as combinatorics, number theory, hypergeometric functions, orthogonal polynomials, and areas of other sciences such as mechanics, quantum theory, and theory of relativity, have received the applications of $q-$calculus [18–20, 22, 24]. Examining the published literature, Euler appears as the first to give some results in this area due to the introduction of the parameter q in the work on infinite series presented by Newton. In the book of Ernst T. [18], about the history of q-calculus, it is read that Jackson was the first to develop and systematize this area; in 1908-1909, Jackson defined the general q-integral and q-difference operator [22]. In 1969, Agarwal described the q-fractional derivative for the first time [1]. During the years 1966 and 1967 the analogue of the fractional integral of Riemman–Liouville appeared in the $q-$calculus setting, this is found in the work published by Al-Salaam [7] . Rajkovic in [33] introduced a Riemman-type definition from a generalization of the definition of $q-$Riemman integral; and Tariboon introduced ${}_{\kappa_1}D_q$-difference operator [8]. Recently, in 2020, Bermudo et al. introduced the notion of ${}^{\kappa_2}D_q$ derivative and integral [10].

In classical analysis, integral inequalities are of particular interest, among them are Simpson's inequality, Ostrowski's inequality, Chebyshev's inequality, and others no less important. All of them have been translated using the q-calculus tools and have been, in some cases using convexity criteria. Many mathematicians have made studies in this area. [2–5, 8, 9, 12–14, 20, 23, 25–31, 35, 38, 39].

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we first present some known definitions and related inequalities in q-calculus. Set the following notation(see, [24]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}, \quad q \in (0,1).$$

Jackson [22] defined the q-Jackson integral of a given function $\mathcal{F}$ from 0 to κ_2 as follows:

$$\int_0^{\kappa_2} \mathcal{F}(x) \ d_qx = (1-q)\kappa_2 \sum_{n=0}^{\infty} q^n \mathcal{F}(\kappa_2 q^n), \text{ where } 0<q<1 \tag{2.1}$$

provided that the sum converges absolutely. Moreover, he defined the q-Jackson integral of a given function over the interval $[\kappa_1, \kappa_2]$ as follows:

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ d_qx = \int_0^{\kappa_2} \mathcal{F}(x) \ d_qx - \int_0^{\kappa_1} \mathcal{F}(x) \ d_qx .$$

Definition 1. *[36] The q_{κ_1}-derivative of mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined as:*

$$ {}_{\kappa_1}D_q\mathcal{F}(x) = \frac{\mathcal{F}(x) - \mathcal{F}(qx + (1-q)\kappa_1)}{(1-q)(x-\kappa_1)}, \ x \neq \kappa_1. \tag{2.2}$$

If $x = \kappa_1$, we define ${}_{\kappa_1}D_q\mathcal{F}(\kappa_1) = \lim_{x\to\kappa_1} {}_{\kappa_1}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 2. *[10] The q^{κ_2}-derivative of mapping $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is defined as:*

$$ {}^{\kappa_2}D_q\mathcal{F}(x) = \frac{\mathcal{F}(qx + (1-q)\kappa_2) - \mathcal{F}(x)}{(1-q)(\kappa_2 - x)}, \ x \neq \kappa_2.$$

If $x = \kappa_2$, we define ${}^{\kappa_2}D_q\mathcal{F}(\kappa_2) = \lim_{x\to\kappa_2} {}^{\kappa_2}D_q\mathcal{F}(x)$ if it exists and it is finite.

Definition 3. *[36] The q_{κ_1}-definite integral of $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ on $[\kappa_1, \kappa_2]$ is defined as:*

$$\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ {}_{\kappa_1}d_qx = (1-q)(\kappa_2-\kappa_1) \sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\kappa_2 + (1-q^n)\kappa_1) = (\kappa_2-\kappa_1) \int_0^1 \mathcal{F}((1-\tau)\kappa_1 + \tau\kappa_2) \ d_q\tau .$$

Alp et al. [8] proved the following q_{κ_1}-Hermite-Hadamard inequalities for convex functions in the setting of quantum calculus:

Theorem 2. *If $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is a convex differentiable function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then we have*

$$\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\ \leq\frac{q\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{[2]_q}. \tag{2.3}$$

In [8] and [28], the authors established some bounds for the left and right hand sides of the inequality (2.3).

On the other hand, in [10], Bermudo et al. gave the following definition and obtained the related Hermite-Hadamard type inequalities:

Definition 4. *[10] The q^{κ_2}-definite integral of $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ on $[\kappa_1,\kappa_2]$ is defined as:*

$$\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ =(1-q)(\kappa_2-\kappa_1)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_1+(1-q^n)\kappa_2)=(\kappa_2-\kappa_1)\int_0^1\mathcal{F}(\tau\kappa_1+(1-\tau)\kappa_2)\ \ d_q\tau\ .$$

Theorem 3. *[10] If $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ is a convex differentiable function on $[\kappa_1,\kappa_2]$ and $0<q<1$. Then, q-Hermite-Hadamard inequalities are given as follows:*

$$\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ \leq\frac{\mathcal{F}(\kappa_1)+q\mathcal{F}(\kappa_2)}{[2]_q}. \tag{2.4}$$

From Theorem 2 and Theorem 3, one can obtain the following inequalities:

Corollary 1. *[10] For any convex function $\mathcal{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ and $0<q<1$, we have*

$$\mathcal{F}\left(\frac{q\kappa_1+\kappa_2}{[2]_q}\right)+\mathcal{F}\left(\frac{\kappa_1+q\kappa_2}{[2]_q}\right)\leq\frac{1}{\kappa_2-\kappa_1}\left\{\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\ +\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ \right\}\leq\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2) \tag{2.5}$$

and

$$\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\leq\frac{1}{2(\kappa_2-\kappa_1)}\left\{\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\ +\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}^{\kappa_2}d_qx\ \right\}\leq\frac{\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)}{2}. \tag{2.6}$$

In [11], Budak proved the left and right bounds of the inequality (2.4).

The primary goal of this study is to show a few new generalizations of Simpson's and Newton's inequalities for quantum differentiable convex functions in the setting of quantum calculus. This is the number one motivation of this paper. The thoughts and techniques of the paper may also open new venues for similar studies on this field.

3. Crucial Identities

In this section, we prove three different identities to obtain the main results of this paper.

Let's start with the following useful Lemma.

Lemma 1. *If $\mathcal{F}:[\kappa_1,\kappa_2]\subset\mathbb{R}\to\mathbb{R}$ is a q_{κ_1}-differentiable function on (κ_1,κ_2) such that ${}_{\kappa_1}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1,\kappa_2]$, then we have the following identity:*

$$\begin{aligned}&\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\\&=(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}}(q\tau-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau+\int_{\frac{1}{2}}^1(q\tau-\mu)_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\right]\end{aligned} \tag{3.1}$$

where $q\in(0,1)$.

Proof. From Definition 1, we have

$${}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)=\frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)\tau}. \tag{3.2}$$

Using the fundamental properties of quantum integrals and from equality (3.2), we obtain that

$$\begin{aligned}
&\int_0^{\frac{1}{2}} (q\tau-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau+\int_{\frac{1}{2}}^1 (q\tau-\mu)_{\kappa_1} D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau \qquad (3.3)\\
=\ &\int_0^{\frac{1}{2}} (\mu-\lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau+\int_0^1 (q\tau-\mu)_{\kappa_1} D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\\
=\ &(\mu-\lambda)\int_0^{\frac{1}{2}} \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau\\
&+q\int_0^1 \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)}d_q\tau\\
&-\mu\int_0^1 \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau.
\end{aligned}$$

From Definition 3, we have the following relations

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau \qquad (3.4)\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}\mathcal{F}\left(\frac{q^n}{2}\kappa_2+\left(1-\frac{q^n}{2}\right)\kappa_1\right)-\sum_{n=0}^{\infty}\mathcal{F}\left(\frac{q^{n+1}}{2}\kappa_2+\left(1-\frac{q^{n+1}}{2}\right)\kappa_1\right)\right]\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\mathcal{F}(\kappa_1)\right],
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)\tau}d_q\tau \qquad (3.5)\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\mathcal{F}(\kappa_2)-\mathcal{F}(\kappa_1)\right]
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \frac{\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)-\mathcal{F}(q\tau\kappa_2+(1-q\tau)\kappa_1)}{(1-q)(\kappa_2-\kappa_1)}d_q\tau \qquad (3.6)\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1)-\sum_{n=0}^{\infty}q^n\mathcal{F}\left(q^{n+1}\kappa_2+\left(1-q^{n+1}\right)\kappa_1\right)\right]\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1)-\frac{1}{q}\sum_{n=1}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1)\right]\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1)-\frac{1}{q}\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\kappa_2+(1-q^n)\kappa_1)+\frac{1}{q}\mathcal{F}(\kappa_2)\right]\\
=\ &\frac{1}{\kappa_2-\kappa_1}\left[\frac{1}{q}\mathcal{F}(\kappa_2)-\frac{1}{q(\kappa_2-\kappa_1)}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right].
\end{aligned}$$

By substituting the computed integrals (3.4)-(3.6) in (3.3), we obtain the required identity (3.1) and the proof is completed. □

Remark 1. *In Lemma 1, if we choose $\lambda=\frac{1}{6}$ and $\mu=\frac{5}{6}$, then we obtain [37, Lemma 3].*

Remark 2. *In Lemma 1, if we choose $\lambda=\mu=\frac{q}{[2]_q}$, then we obtain [35, Lemma 3.1].*

Corollary 2. *In Lemma 1, if we choose $\lambda=0$ and $\mu=1$, then we obtain the following new identity*

$$\begin{aligned}
&\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\\
=\ &(\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}} q\tau\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau+\int_{\frac{1}{2}}^1 (q\tau-1)_{\kappa_1} D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\,d_q\tau\right].
\end{aligned}$$

Remark 3. *In Lemma 1, if we take the limit $q \to 1^-$, then we have [16, Lemma 2.1 for $m = 1$].*

Lemma 2. *If $\mathcal{F} : [\kappa_1, \kappa_2] \subset \mathbb{R} \to \mathbb{R}$ is a q_{κ_1}-differentiable function on (κ_1, κ_2) such that ${}_{\kappa_1}D_q\mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$, then we have the following identity:*

$$\lambda\mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{2\kappa_1 + \kappa_2}{3}\right) + (\nu - \mu)\mathcal{F}\left(\frac{\kappa_1 + 2\kappa_2}{3}\right) + (1 - \nu)\mathcal{F}(\kappa_2) - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx \tag{3.7}$$
$$= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{3}}(q\tau - \lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau + \int_{\frac{1}{3}}^{\frac{2}{3}}(q\tau - \mu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau\right.$$
$$\left. + \int_{\frac{2}{3}}^{1}(q\tau - \nu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau\right]$$

where $q \in (0, 1)$.

Proof. From the fundamental properties of quantum integrals, we have

$$\int_0^{\frac{1}{3}}(q\tau - \lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau + \int_{\frac{1}{3}}^{\frac{2}{3}}(q\tau - \mu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau$$
$$+ \int_{\frac{2}{3}}^{1}(q\tau - \nu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau$$
$$= \int_0^{\frac{1}{3}}(\mu - \lambda)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau + \int_0^{\frac{2}{3}}(\nu - \mu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau$$
$$+ \int_0^{1}(q\tau - \nu)\ {}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\,d_q\tau.$$

If the same steps in the proof of Lemma 1 are applied for the rest of this proof, we can obtain the desired identity (3.7). □

Remark 4. *If we take $\lambda = \frac{1}{8}$, $\mu = \frac{1}{2}$, and $\nu = \frac{7}{8}$ in Lemma 2, then we obtain [17, Lemma 2].*

Remark 5. *If we take $\lambda = \mu = \nu = \frac{q}{[2]_q}$, in Lemma 2, then we obtain [35, Lemma 3.1].*

Corollary 3. *If we take the limit $q \to 1^-$ in Lemma 2, then we obtain the following new identity*

$$\lambda\mathcal{F}(\kappa_1) + (\mu - \lambda)\mathcal{F}\left(\frac{2\kappa_1 + \kappa_2}{3}\right) + (\nu - \mu)\mathcal{F}\left(\frac{\kappa_1 + 2\kappa_2}{3}\right) + (1 - \nu)\mathcal{F}(\kappa_2) - \frac{1}{\kappa_2 - \kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx$$
$$= (\kappa_2 - \kappa_1)\left[\int_0^{\frac{1}{3}}(\tau - \lambda)\mathcal{F}'(\tau\kappa_2 + (1-\tau)\kappa_1)\,d\tau + \int_{\frac{1}{3}}^{\frac{2}{3}}(\tau - \mu)\mathcal{F}'(\tau\kappa_2 + (1-\tau)\kappa_1)\,d\tau\right.$$
$$\left. + \int_{\frac{2}{3}}^{1}(\tau - \nu)\mathcal{F}'(\tau\kappa_2 + (1-\tau)\kappa_1)\,d\tau\right]$$

For brevity, let us prove another lemma that will be used frequently in the main results.

Lemma 3. *The following quantum integrals holds for $\lambda, \mu, \nu \geq 0$:*

$$\Omega_{11} = \int_0^{\frac{1}{2}}|q\tau - \lambda|\,d_q\tau = \begin{cases} \frac{8\lambda^2 + q}{4[2]_q} - \frac{\lambda}{2}, & q > 2\lambda, \\ \frac{\lambda}{2} - \frac{q}{4[2]_q}, & q \leq 2\lambda, \end{cases} \tag{3.8}$$

$$\Omega_{12} = \int_{\frac{1}{2}}^{1}|q\tau - \mu|\,d_q\tau = \begin{cases} \frac{\mu}{2} - \frac{3q}{4[2]_q}, & q < \mu, \\ \frac{8\mu^2 + 5q}{4[2]_q} - \frac{3\mu}{2}, & \mu \leq q \leq 2\mu, \\ \frac{3q}{4[2]_q} - \frac{\mu}{2}, & q > 2\mu, \end{cases} \tag{3.9}$$

$$\Omega_{13}=\int_0^{\frac{1}{3}}|q\tau-\lambda|\,d_q\tau=\begin{cases}\frac{2\lambda^2}{[2]_q}+\frac{q}{9[2]_q}-\frac{\lambda}{3}, & q>3\lambda,\\ \frac{\lambda}{3}-\frac{q}{9[2]_q}, & q\leq 3\lambda,\end{cases}\tag{3.10}$$

$$\Omega_{14}=\int_{\frac{1}{3}}^{\frac{2}{3}}|q\tau-\mu|\,d_q\tau=\begin{cases}\frac{\mu}{3}-\frac{q}{3[2]_q}, & q<\frac{3\mu}{2},\\ \frac{18\mu^2+5q}{9[2]_q}-\mu, & \frac{3\mu}{2}\leq q\leq 3\mu,\\ \frac{q}{3[2]_q}-\frac{\mu}{3}, & q>3\mu,\end{cases}\tag{3.11}$$

$$\Omega_{15}=\int_{\frac{2}{3}}^{1}|q\tau-\nu|\,d_q\tau=\begin{cases}\frac{\nu}{3}-\frac{5q}{9[2]_q}, & q<\nu,\\ \frac{18\nu^2+13q}{9[2]_q}-\frac{5\nu}{3}, & \nu\leq q\leq \frac{3\nu}{2},\\ \frac{5q}{9[2]_q}-\frac{\nu}{3}, & q>\frac{3\nu}{2},\end{cases}\tag{3.12}$$

$$\Omega_{1}=\int_0^{\frac{1}{2}}\tau\,|q\tau-\lambda|\,d_q\tau=\begin{cases}\frac{2\lambda^3}{[2]_q[3]_q}+\frac{q}{8[3]_q}-\frac{\lambda}{4[2]_q}, & q>2\lambda,\\ \frac{\lambda}{4[2]_q}-\frac{q}{8[3]_q}, & q\leq 2\lambda,\end{cases}\tag{3.13}$$

$$\begin{aligned}\Omega_2 &= \int_0^{\frac{1}{2}}(1-\tau)\,|q\tau-\lambda|\,d_q\tau\\ &= \Omega_{11}-\Omega_1\\ &= \begin{cases}\frac{8\lambda^2+\lambda+q}{4[2]_q}-\frac{\lambda}{2}-\frac{q}{8[3]_q}-\frac{2\lambda^3}{[2]_q[3]_q}, & q>2\lambda,\\ \frac{\lambda}{2}-\frac{\lambda+q}{4[2]_q}+\frac{q}{8[3]_q}, & q\leq 2\lambda,\end{cases}\end{aligned}\tag{3.14}$$

$$\begin{aligned}\Omega_3 &= \int_{\frac{1}{2}}^{1}\tau\,|q\tau-\mu|\,d_q\tau=\\ &= \begin{cases}\frac{3\mu}{4[2]_q}-\frac{7q}{8[3]_q}, & q<\mu,\\ \frac{2\mu^3}{[2]_q[3]_q}-\frac{5\mu}{4[2]_q}+\frac{9q}{8[3]_q}, & \mu\leq q\leq 2\mu,\\ \frac{7q}{8[3]_q}-\frac{3\mu}{4[2]_q}, & q>2\mu,\end{cases}\end{aligned}\tag{3.15}$$

$$\begin{aligned}\Omega_4 &= \int_{\frac{1}{2}}^{1}(1-\tau)\,|q\tau-\mu|\,d_q\tau=\\ &= \Omega_{12}-\Omega_3\\ &= \begin{cases}\frac{\mu}{2}-\frac{3(\mu+q)}{4[2]_q}+\frac{7q}{8[3]_q}, & q<\mu,\\ \frac{8\mu^2+5q+5\mu}{4[2]_q}-\frac{3\mu}{2}-\frac{9q}{8[3]_q}-\frac{2\mu^3}{[2]_q[3]_q}, & \mu\leq q\leq 2\mu,\\ \frac{3(\mu+q)}{4[2]_q}-\frac{\mu}{2}-\frac{7q}{8[3]_q}, & q>2\mu,\end{cases}\end{aligned}\tag{3.16}$$

$$\Omega_5=\int_0^{\frac{1}{3}}\tau\left|q\tau-\lambda\right|d_q\tau=\begin{cases}\frac{2\lambda^3}{[2]_q[3]_q}+\frac{q}{27[3]_q}-\frac{\lambda}{9[2]_q}, & q>3\lambda,\\ \frac{\lambda}{9[2]_q}-\frac{q}{27[3]_q}, & q\leq 3\lambda,\end{cases}\tag{3.17}$$

$$\begin{aligned}\Omega_6 &= \int_0^{\frac{1}{3}}(1-\tau)\left|q\tau-\lambda\right|d_q\tau=\\ &= \Omega_{13}-\Omega_5\\ &= \begin{cases}\frac{18\lambda^2+\lambda+q}{9[2]_q}-\frac{\lambda}{3}-\frac{q}{27[3]_q}-\frac{2\lambda^3}{[2]_q[3]_q}, & q>2\lambda,\\ \frac{\lambda}{3}-\frac{\lambda+q}{9[2]_q}+\frac{q}{27[3]_q}, & q\leq 2\lambda,\end{cases}\end{aligned}\tag{3.18}$$

$$\Omega_7=\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\mu\right|d_q\tau=\begin{cases}\frac{\mu}{3[2]_q}-\frac{7q}{27[3]_q}, & q<\frac{3\mu}{2},\\ \frac{2\mu^3}{[2]_q[3]_q}-\frac{5\mu}{9[2]_q}+\frac{q}{3[3]_q}, & \frac{3\mu}{2}\leq q\leq 3\mu,\\ \frac{7q}{27[3]_q}-\frac{\mu}{3[2]_q}, & q>3\mu,\end{cases}\tag{3.19}$$

$$\begin{aligned}\Omega_8 &= \int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|q\tau-\mu\right|d_q\tau\\ &= \Omega_{14}-\Omega_7\\ &= \begin{cases}\frac{\mu}{3}-\frac{q+\mu}{3[2]_q}+\frac{7q}{27[3]_q}, & q<\frac{3\mu}{2},\\ \frac{18\mu^2+5q+5\mu}{9[2]_q}-\mu-\frac{q}{3[3]_q}-\frac{2\mu^3}{[2]_q[3]_q}, & \frac{3\mu}{2}\leq q\leq 3\mu,\\ \frac{q+\mu}{3[2]_q}-\frac{\mu}{3}-\frac{7q}{27[3]_q}, & q>3\mu,\end{cases}\end{aligned}\tag{3.20}$$

$$\Omega_9=\int_{\frac{2}{3}}^{1}\tau\left|q\tau-\nu\right|d_q\tau=\begin{cases}\frac{5\nu}{9[2]_q}-\frac{19q}{27[3]_q}, & q<\nu,\\ \frac{2\nu^3}{[2]_q[3]_q}-\frac{13\nu}{9[2]_q}+\frac{35q}{27[3]_q}, & \nu\leq q\leq\frac{3\nu}{2},\\ \frac{19q}{27[3]_q}-\frac{5\nu}{9[2]_q}, & q>\frac{3\nu}{2},\end{cases}\tag{3.21}$$

$$\begin{aligned}\Omega_{10} &= \int_{\frac{2}{3}}^{1}(1-\tau)\left|q\tau-\nu\right|d_q\tau\\ &= \Omega_{15}-\Omega_9\\ &= \begin{cases}\frac{\nu}{3}-\frac{5(q+\nu)}{9[2]_q}+\frac{19q}{27[3]_q}, & q<\nu,\\ \frac{18\nu^2+13q+13\nu}{9[2]_q}-\frac{5\nu}{3}-\frac{35q}{27[3]_q}-\frac{2\nu^3}{[2]_q[3]_q}, & \nu\leq q\leq\frac{3\nu}{2},\\ \frac{5(q+\nu)}{9[2]_q}-\frac{\nu}{3}-\frac{19q}{27[3]_q}, & q>\frac{3\nu}{2}.\end{cases}\end{aligned}\tag{3.22}$$

Proof. Case I: Let $q > 2\lambda$.

By the definition q-integral, we have

$$\begin{aligned}\Omega_1 &= \int_0^{\frac{1}{2}} \tau \left|q\tau - \lambda\right| d_q\tau \\ &= \int_0^{\frac{\lambda}{q}} \tau\left(\lambda - q\tau\right) d_q\tau + \int_{\frac{\lambda}{q}}^{\frac{1}{2}} \tau\left(\lambda - q\tau\right) d_q\tau \\ &= 2\int_0^{\frac{\lambda}{q}} \tau\left(\lambda - q\tau\right) d_q\tau + \int_0^{\frac{1}{2}} \tau\left(\lambda - q\tau\right) d_q\tau \\ &= \frac{2\lambda^3}{[2]_q [3]_q} + \frac{q}{8[3]_q} - \frac{\lambda}{4[2]_q}.\end{aligned}$$

Case I: Let $q \leq 2\lambda$.

From definition quantum integral, we get

$$\Omega_1 = \int_0^{\frac{1}{2}} \tau\left|q\tau - \lambda\right| d_q\tau = \int_0^{\frac{1}{2}} \tau\left(\lambda - q\tau\right) d_q\tau = \frac{\lambda}{4[2]_q} - \frac{q}{8[3]_q}.$$

This gives the proof of the equality (3.13). The others can be calculated in similar way. □

4. Simpson's type inequalities for quantum integrals

In this section, we prove a new generalization of quantum Simpson's inequalities for quantum differentiable convex functions via quantum integrals.

Theorem 4. *We assume that the given conditions of Lemma 1 hold. If the mapping* $\left|{}_{\kappa_1}D_q\mathcal{F}\right|$ *is convex on* $[\kappa_1, \kappa_2]$*, then the following Simpson's type inequality holds:*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\ &\leq (\kappa_2-\kappa_1)\left[(\Omega_1+\Omega_3)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right| + (\Omega_2+\Omega_4)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned} \tag{4.1}$$

where Ω_1-Ω_4 *are given in (3.13)-(3.16), respectively.*

Proof. Taking the modulus in Lemma 1 and using the convexity of $\left|{}_{\kappa_1}D_q\mathcal{F}\right|$, we obtain

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1) + (1-\mu)\mathcal{F}(\kappa_2) + (\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\ \leq\ & (\kappa_2-\kappa_1)\left[\int_0^{\frac{1}{2}} \left|q\tau-\lambda\right| \left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\right| d_q\tau + \int_{\frac{1}{2}}^{1} \left|q\tau-\mu\right| \left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2 + (1-\tau)\kappa_1)\right| d_q\tau\right] \\ \leq\ & (\kappa_2-\kappa_1)\left[\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|\left\{\int_0^{\frac{1}{2}} \tau\left|q\tau-\lambda\right| d_q\tau + \int_{\frac{1}{2}}^{1} \tau\left|q\tau-\mu\right| d_q\tau\right\}\right. \\ &\left. + \left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\left\{\int_0^{\frac{1}{2}} (1-\tau)\left|q\tau-\lambda\right| d_q\tau + \int_{\frac{1}{2}}^{1} (1-\tau)\left|q\tau-\mu\right| d_q\tau\right\}\right] \\ =\ & (\kappa_2-\kappa_1)\left[(\Omega_1+\Omega_3)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right| + (\Omega_2+\Omega_4)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned}$$

which completes the proof. □

Remark 6. *If we take the limit* $q \to 1^-$ *in Theorem 4, then we have [16, Theorem 2.1 for* $s = m = 1$*].*

Remark 7. *If we assume* $\lambda = \mu = \frac{q}{[2]_q}$ *in Theorem 4, then we obtain [35, Theorem 4.1].*

Corollary 4. *In Theorem 4, if we choose $\lambda = 0$ and $\mu = 1$, then we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\frac{3}{4[2]_q[3]_q}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+\frac{2q^2+2q-1}{4[2]_q[3]_q}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right].
\end{aligned}$$

Corollary 5. *If we assume $\lambda = \frac{1}{6}$ and $\mu = \frac{5}{6}$ in Theorem 4, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[(\Omega_1^*+\Omega_3^*)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+(\Omega_2^*+\Omega_4^*)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]
\end{aligned}$$

where

$$\Omega_1^* = \int_0^{\frac{1}{2}}\tau\left|q\tau-\frac{1}{6}\right|d_q\tau = \begin{cases}\frac{2}{216.[2]_q[3]_q}+\frac{q}{8[3]_q}-\frac{1}{24.[2]_q}, & \frac{1}{3}<q<1, \\ \frac{1}{24.[2]_q}-\frac{q}{8[3]_q}, & 0<q\leq\frac{1}{3},\end{cases}$$

$$\begin{aligned}
\Omega_2^* &= \int_0^{\frac{1}{2}}(1-\tau)\left|q\tau-\frac{1}{6}\right|d_q\tau \\
&= \begin{cases}\frac{7}{72.[2]_q}+\frac{q}{4[2]_q}-\frac{1}{12}-\frac{q}{8[3]_q}-\frac{2}{216.[2]_q[3]_q}, & \frac{1}{3}<q<1, \\ \frac{1}{12}-\frac{1}{24[2]_q}-\frac{q}{4[2]_q}+\frac{q}{8[3]_q}, & 0<q\leq\frac{1}{3},\end{cases}
\end{aligned}$$

$$\begin{aligned}
\Omega_3^* &= \int_{\frac{1}{2}}^1\tau\left|q\tau-\frac{5}{6}\right|d_q\tau = \\
&= \begin{cases}\frac{15}{24.[2]_q}-\frac{7q}{8[3]_q}, & 0<q<\frac{5}{6}, \\ \frac{250}{216.[2]_q[3]_q}-\frac{25}{24.[2]_q}+\frac{9q}{8[3]_q}, & \frac{5}{6}\leq q<1,\end{cases}
\end{aligned}$$

$$\begin{aligned}
\Omega_4^* &= \int_{\frac{1}{2}}^1(1-\tau)\left|q\tau-\frac{5}{6}\right|d_q\tau = \\
&= \begin{cases}\frac{5}{12}-\frac{15}{24.[2]_q}-\frac{3q}{4[2]_q}+\frac{7q}{8[3]_q}, & 0<q<\frac{5}{6}, \\ \frac{50}{216.[2]_q}+\frac{5q}{4[2]_q}-\frac{25}{24.[2]_q}-\frac{15}{12}-\frac{9q}{8[3]_q}-\frac{250}{216.[2]_q[3]_q}, & \frac{5}{6}\leq q<1\end{cases}
\end{aligned}$$

which is given by Tunç et al. in [37, Theorem 1], the coefficients of $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|$ and $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|$ in this inequality are more modified than the inequality of Tunç et al..

Theorem 5. *We assume that the given conditions of Lemma 1 hold. If the mapping $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1 \geq 1$ is convex on $[\kappa_1, \kappa_2]$, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\Big[\Omega_{11}^{1-\frac{1}{p_1}}\left(\Omega_1\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_2\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}} \\
&+\Omega_{12}^{1-\frac{1}{p_1}}\left(\Omega_3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_4\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\Big]
\end{aligned} \tag{4.2}$$

where Ω_{11}, Ω_{12} and Ω_1-Ω_4 are given in (3.8), (3.9), and (3.13)-(3.16), respectively.

Proof. Taking the modulus in Lemma 1 and using the power mean inequality, we have

$$\begin{aligned} &\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\ \leq\ &(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|\,d_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|\,|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)|^{p_1}\,d_q\tau\right)^{\frac{1}{p_1}}\right. \\ &\left.+\left(\int_{\frac{1}{2}}^1|q\tau-\mu|\,d_q\tau\right)^{1-\frac{1}{p_1}}\left(\int_{\frac{1}{2}}^1|q\tau-\mu|\,|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)|^{p_1}\,d_q\tau\right)^{\frac{1}{p_1}}\right]. \end{aligned}$$

By using the convexity of $|{}_{\kappa_1}D_q\mathcal{F}|^{p_1}$, we have

$$\begin{aligned} &\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\ \leq\ &(\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|\,d_q\tau\right)^{1-\frac{1}{p_1}}\right. \\ &\times\left(|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}\int_0^{\frac{1}{2}}\tau|q\tau-\lambda|\,d_q\tau+|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\int_0^{\frac{1}{2}}(1-\tau)|q\tau-\lambda|\,d_q\tau\right)^{\frac{1}{p_1}} \\ &+\left(\int_{\frac{1}{2}}^1|q\tau-\mu|\,d_q\tau\right)^{1-\frac{1}{p_1}} \\ &\left.\times\left(|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}\int_{\frac{1}{2}}^1\tau|q\tau-\mu|\,d_q\tau+|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\int_{\frac{1}{2}}^1(1-\tau)|q\tau-\mu|\,d_q\tau\right)^{\frac{1}{p_1}}\right] \\ =\ &\left[\Omega_{11}^{1-\frac{1}{p_1}}\left(\Omega_1|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}+\Omega_2|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\right)^{\frac{1}{p_1}}\right. \\ &\left.+\Omega_{12}^{1-\frac{1}{p_1}}\left(\Omega_3|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}+\Omega_4|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\right)^{\frac{1}{p_1}}\right] \end{aligned}$$

and the proof is completed. □

Remark 8. *If we take the limit $q\to1^-$ in Theorem 5, then we have [16, Theorem 2.3 for $s=m=1$].*

Remark 9. *If we assume $\lambda=\mu=\frac{q}{[2]_q}$ in Theorem 5, then we obtain [35, Theorem 4.2].*

Corollary 6. *If we assume $\lambda=\frac{1}{6}$ and $\mu=\frac{5}{6}$ in Theorem 5, then we obtain the following inequality*

$$\begin{aligned} &\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\ \leq\ &(\kappa_2-\kappa_1)\left[\Theta_1^{1-\frac{1}{p_1}}\left(\Omega_1^*|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}+\Omega_2^*|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\right)^{\frac{1}{p_1}}\right. \\ &\left.+\Theta_2^{1-\frac{1}{p_1}}\left(\Omega_3^*|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}+\Omega_4^*|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\right)^{\frac{1}{p_1}}\right] \end{aligned}$$

where $\Omega_1^*-\Omega_4^*$ are given in Corollary 5 and

$$\Theta_1=\int_0^{\frac{1}{2}}\left|q\tau-\frac{1}{6}\right|d_q\tau=\begin{cases}\frac{2}{36[2]_q}+\frac{q}{4[2]_q}-\frac{1}{12}, & \frac{1}{3}<q<1,\\ \frac{1}{12}-\frac{q}{4[2]_q}, & 0<q\leq\frac{1}{3},\end{cases}$$

$$\Theta_2=\int_{\frac{1}{2}}^1\left|q\tau-\frac{5}{6}\right|d_q\tau=\begin{cases}\frac{5}{12}-\frac{3q}{4[2]_q}, & 0<q<\frac{5}{6},\\ \frac{50}{36[2]_q}+\frac{5q}{4[2]_q}-\frac{15}{12}, & \frac{5}{6}\leq q<1,\end{cases}$$

which is given by Tunç et al. in [37, Theorem 3], the values of Θ_1, Θ_2 *and the coefficients of* $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}$ *and* $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}$ *in this inequality are more modified than the inequality of Tunç et al..*

Corollary 7. *In Theorem 5, if we choose* $\lambda = 0$ *and* $\mu = 1$*, then we obtain the following midpoint type inequality*

$$\begin{aligned}
&\left|\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left(\frac{q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{q}{8[3]_q} + \left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\frac{q\left([3]_q+q^2\right)}{8[2]_q[3]_q}\right)^{\frac{1}{p_1}} \\
&+\left(\frac{2-q}{4[2]_q}\right)^{1-\frac{1}{p_1}}\left(\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q} + \left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\left(\frac{1}{2}-\frac{3q}{4[2]_q}-\frac{6[3]_q-7q[2]_q}{8[2]_q[3]_q}\right)\right)^{\frac{1}{p_1}}.
\end{aligned}$$

Theorem 6. *We assume that the given conditions of Lemma 1 hold. If the mapping* $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1>1$ *is convex on* $[\kappa_1,\kappa_2]$*, then the following Simpson's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Omega_{16}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q} + \frac{(2q+1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\Omega_{17}^{\frac{1}{r_1}}\left(\frac{3\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{4[2]_q} + \frac{(2q-1)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned} \tag{4.3}$$

where $p_1^{-1}+r_1^{-1}=1$ *and*

$$\Omega_{16} = \int_0^{\frac{1}{2}}|q\tau-\lambda|^{r_1}\,d_q\tau,\ \ \Omega_{17} = \int_{\frac{1}{2}}^1|q\tau-\mu|^{r_1}\,d_q\tau.$$

Proof. Taking the modulus in Lemma 1 and using the Hölder inequality, we have

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|^{r_1}\,d_q\tau\right)^{\frac{1}{r_1}}\left(\int_0^{\frac{1}{2}}\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^1|q\tau-\mu|^{r_1}\,d_q\tau\right)^{\frac{1}{r_1}}\left(\int_{\frac{1}{2}}^1\left|{}_{\kappa_1}D_q\mathcal{F}(\tau\kappa_2+(1-\tau)\kappa_1)\right|^{p_1}d_q\tau\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Applying the convexity of $|{}_{\kappa_1}D_q\mathcal{F}|^{p_1}$, we have

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(1-\mu)\mathcal{F}(\kappa_2)+(\mu-\lambda)\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\left(\int_0^{\frac{1}{2}}|q\tau-\lambda|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}\int_0^{\frac{1}{2}}\tau d_q\tau+|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\int_0^{\frac{1}{2}}(1-\tau)d_q\tau\right)^{\frac{1}{p_1}}\right. \\
&\left.+\left(\int_{\frac{1}{2}}^{1}|q\tau-\mu|^{r_1}d_q\tau\right)^{\frac{1}{r_1}}\left(|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}\int_{\frac{1}{2}}^{1}\tau d_q\tau+|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}\int_{\frac{1}{2}}^{1}(1-\tau)d_q\tau\right)^{\frac{1}{p_1}}\right] \\
=\ & (\kappa_2-\kappa_1)\left[\Omega_{16}^{\frac{1}{r_1}}\left(\frac{|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}}{4[2]_q}+\frac{(2q+1)|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\Omega_{17}^{\frac{1}{r_1}}\left(\frac{3|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}}{4[2]_q}+\frac{(2q-1)|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

and the proof is finished. □

Remark 10. *If we take the limit $q\to1^-$ in Theorem 6, then Theorem 6 becomes [16, Theorem 2.2 for $s=m=1$].*

Corollary 8. *If we assume $\lambda=\frac{1}{6}$ and $\mu=\frac{5}{6}$ in Theorem 6, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathcal{F}(\kappa_1)+\mathcal{F}(\kappa_2)+\mathcal{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Theta_3^{\frac{1}{r_1}}\left(\frac{|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}}{4[2]_q}+\frac{(2q+1)|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&\left.+\Theta_4^{\frac{1}{r_1}}\left(\frac{3|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|^{p_1}}{4[2]_q}+\frac{(2q-1)|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|^{p_1}}{4[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where

$$\Theta_3=\int_0^{\frac{1}{2}}\left|q\tau-\frac{1}{6}\right|^{r_1}d_q\tau,\ \ \Theta_4=\int_{\frac{1}{2}}^{1}\left|q\tau-\frac{5}{6}\right|^{r_1}d_q\tau.$$

5. Newton's type inequalities for quantum integrals

Some new generalized versions of quantum Newton's inequalities for quantum differentiable convex functions are offered in this section.

Theorem 7. *We assume that the given conditions of Lemma 2 hold. If the mapping $|{}_{\kappa_1}D_q\mathcal{F}|$ is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[(\Omega_5+\Omega_7+\Omega_9)|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)|+(\Omega_6+\Omega_8+\Omega_{10})|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)|\right]
\end{aligned} \tag{5.1}$$

where Ω_5-Ω_{10} are given in (3.17)-(3.22), respectively.

Proof. If we consider Lemma 2 and apply the same method that used in the proof of Theorem 4, then we can obtain the desired inequality (5.1). □

Remark 11. *If we assume $\lambda=\mu=\nu=\frac{q}{[2]_q}$ in Theorem 7, then we obtain [35, Theorem 4.1].*

Corollary 9. *If we take the limit $q \to 1^-$ in Theorem 7, then we obtain the following Newton's type inequality*

$$\begin{aligned}&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right|\\ \leq\ &(\kappa_2-\kappa_1)\left[(\Omega_5^*+\Omega_7^*+\Omega_9^*)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+(\Omega_6^*+\Omega_8^*+\Omega_{10}^*)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned}$$

where

$$\Omega_5^*=\int_0^{\frac{1}{3}}\tau\left|\tau-\lambda\right|d\tau=\frac{\lambda^3}{3}+\frac{1}{81}-\frac{\lambda}{18},$$

$$\Omega_6^*=\int_0^{\frac{1}{3}}(1-\tau)\left|\tau-\lambda\right|d\tau=\frac{18\lambda^2+\lambda+1}{18}-\frac{28}{81}-\frac{\lambda^3}{3},$$

$$\Omega_7^*=\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\mu\right|d\tau=\frac{\mu^3}{3}-\frac{5\mu}{18}+\frac{1}{9}$$

$$\Omega_8^*=\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|\tau-\mu\right|d\tau=\frac{18\mu^2+5+5\mu}{18}-\mu-\frac{1}{9}-\frac{\mu^3}{3},$$

$$\Omega_9^*=\int_{\frac{2}{3}}^{1}\tau\left|\tau-\nu\right|d\tau=\frac{\nu^3}{3}-\frac{13\nu}{18}+\frac{35}{81},$$

$$\Omega_{10}^*=\int_{\frac{2}{3}}^{1}(1-\tau)\left|\tau-\nu\right|d\tau=\frac{18\nu^2+13+13\nu}{18}-\frac{5\nu}{3}-\frac{35}{81}-\frac{\nu^3}{3}.$$

Remark 12. *If we take $\lambda=\frac{1}{8}$, $\mu=\frac{1}{2}$, and $\nu=\frac{7}{8}$ in Theorem 7, then we obtain the following inequality*

$$\begin{aligned}&\left|\frac{1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right|\\ \leq\ &(\kappa_2-\kappa_1)\left[(\Theta_5+\Theta_7+\Theta_9)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|+(\Theta_6+\Theta_8+\Theta_{10})\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|\right]\end{aligned}$$

where

$$\Theta_5=\int_0^{\frac{1}{3}}\tau\left|q\tau-\frac{1}{8}\right|d_q\tau=\begin{cases}\frac{1}{256.[2]_q[3]_q}+\frac{q}{27[3]_q}-\frac{1}{72[2]_q}, & \frac{3}{8}<q<1,\\ \frac{1}{72[2]_q}-\frac{q}{27[3]_q}, & 0<q\leq\frac{3}{8},\end{cases}$$

$$\begin{aligned}\Theta_6&=\int_0^{\frac{1}{3}}(1-\tau)\left|q\tau-\frac{1}{8}\right|d_q\tau=\\ &=\begin{cases}\frac{1}{32[2]_q}+\frac{1}{72[2]_q}+\frac{q}{9[2]_q}-\frac{1}{24}-\frac{q}{27[3]_q}-\frac{1}{256[2]_q[3]_q}, & \frac{1}{4}<q<1,\\ \frac{1}{24}-\frac{1}{72[2]_q}-\frac{q}{9[2]_q}+\frac{q}{27[3]_q}, & 0<q\leq\frac{1}{4},\end{cases}\end{aligned}$$

$$\Theta_7=\int_{\frac{1}{3}}^{\frac{2}{3}}\tau\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{1}{6[2]_q}-\frac{7q}{27[3]_q}, & 0<q<\frac{3}{4},\\ \frac{1}{4[2]_q[3]_q}-\frac{5}{18[2]_q}+\frac{q}{3[3]_q}, & \frac{3}{4}\leq q<1,\end{cases}$$

$$\begin{aligned}\Theta_8&=\int_{\frac{1}{3}}^{\frac{2}{3}}(1-\tau)\left|q\tau-\frac{1}{2}\right|d_q\tau\\ &=\begin{cases}\frac{1}{6}-\frac{q}{3[2]_q}-\frac{1}{6[2]_q}+\frac{7q}{27[3]_q}, & 0<q<\frac{3}{4},\\ \frac{1}{2[2]_q}+\frac{5q}{9[2]_q}+\frac{5}{18[2]_q}-\frac{1}{2}-\frac{q}{3[3]_q}-\frac{1}{4[2]_q[3]_q}, & \frac{3}{5}\leq q<1,\end{cases}\end{aligned}$$

$$\Theta_9=\int_{\frac{2}{3}}^{1}\tau\left|q\tau-\frac{7}{8}\right|d_q\tau=\begin{cases}\frac{35}{72[2]_q}-\frac{19q}{27[3]_q}, & 0<q<\frac{7}{8},\\ \frac{343}{256[2]_q[3]_q}-\frac{91}{72[2]_q}+\frac{35q}{27[3]_q}, & \frac{7}{8}\leq q<1,\end{cases}$$

$$\begin{aligned}
\Theta_{10} &= \int_{\frac{2}{3}}^{1}(1-\tau)\left|q\tau-\frac{7}{8}\right|d_q\tau \\
&= \begin{cases} \frac{7}{24}-\frac{5q}{9[2]_q}-\frac{7}{72[2]_q}+\frac{19q}{27[3]_q}, & 0<q<\frac{7}{8}, \\ \frac{49}{32[2]_q}+\frac{13q}{9[2]_q}+\frac{91}{72[2]_q}-\frac{35}{24}-\frac{35q}{27[3]_q}-\frac{343}{256[2]_q[3]_q}, & \frac{7}{8}\leq q<1 \end{cases}
\end{aligned}$$

which is given by Erden et al. in [17, Theorem 1] but in our inequality the coefficients of $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|$ *and* $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|$ *are in modified form.*

Theorem 8. *We assume that the given conditions of Lemma 2 hold. If the mapping* $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1\geq 1$ *is convex on* $[\kappa_1,\kappa_2]$, *then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \quad (5.2) \\
\leq\ & (\kappa_2-\kappa_1)\left[\Omega_{13}^{1-\frac{1}{p_1}}\left(\Omega_5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_6\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&+\Omega_{14}^{1-\frac{1}{p_1}}\left(\left(\Omega_7\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_8\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right) \\
&\left.+\Omega_{15}^{1-\frac{1}{p_1}}\left(\Omega_9\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_{10}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where Ω_5-Ω_{10} *and* Ω_{13}-Ω_{15} *are given in (3.17)-(3.22) and (3.10)-(3.12), respectively.*

Proof. If we apply the steps used in the proof of Theorem 5 and taking into account Lemma 2, we can obtain the required inequality (5.2). □

Corollary 10. *If we take the limit* $q\to 1^-$ *in Theorem 8, then we obtain the following Newton's type inequality*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\,dx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Theta_{11}^{1-\frac{1}{p_1}}\left(\Omega_5^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_6^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&+\Theta_{12}^{1-\frac{1}{p_1}}\left(\left(\Omega_7^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_8^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right) \\
&\left.+\Theta_{13}^{1-\frac{1}{p_1}}\left(\Omega_9^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Omega_{10}^*\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where Ω_5^*-Ω_{10}^* *are defined in Corollary 9 and*

$$\Theta_{11}=\int_0^{\frac{1}{3}}|\tau-\lambda|\,d\tau=\lambda^2+\frac{1}{9\,[2]_q}-\frac{\lambda}{3},$$

$$\Theta_{12}=\int_{\frac{1}{3}}^{\frac{2}{3}}|\tau-\mu|\,d\tau=\frac{18\mu^2+5}{18}-\mu,$$

$$\Theta_{13}=\int_{\frac{2}{3}}^{1}|\tau-\nu|\,d\tau=\frac{18\nu^2+13}{18}-\frac{5\nu}{3}.$$

Remark 13. *If we take $\lambda=\frac{1}{8}$, $\mu=\frac{1}{2}$, and $\nu=\frac{7}{8}$ in Theorem 8, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathcal{F}(\kappa_1)+3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+\mathcal{F}(\kappa_2)\right]-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Theta_{14}^{1-\frac{1}{p_1}}\left(\Theta_5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Theta_6\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&+\Theta_{15}^{1-\frac{1}{p_1}}\left(\left(\Theta_7\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Theta_8\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right) \\
&\left.+\Theta_{16}^{1-\frac{1}{p_1}}\left(\Theta_9\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}+\Theta_{10}\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where Θ_5-Θ_{10} are given in Remark 12 and

$$\Omega_{14}=\int_0^{\frac{1}{3}}\left|q\tau-\frac{1}{8}\right|d_q\tau=\begin{cases}\frac{1}{32[2]_q}+\frac{q}{9[2]_q}-\frac{1}{24}, & \frac{3}{8}<q<1,\\ \frac{1}{24}-\frac{q}{9[2]_q}, & 0<q\leq\frac{3}{8},\end{cases}$$

$$\Omega_{15}=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau-\frac{1}{2}\right|d_q\tau=\begin{cases}\frac{1}{6}-\frac{q}{3[2]_q}, & 0<q<\frac{3}{4},\\ \frac{1}{2[2]_q}+\frac{5q}{9[2]_q}-\frac{1}{2}, & \frac{3}{4}\leq q<1,\end{cases}$$

$$\Omega_{16}=\int_{\frac{2}{3}}^{1}\left|q\tau-\frac{7}{8}\right|d_q\tau=\begin{cases}\frac{7}{24}-\frac{5q}{9[2]_q}, & 0<q<\frac{7}{8},\\ \frac{49}{4[2]_q}+\frac{13q}{9[2]_q}-\frac{35}{24}, & \frac{7}{8}\leq q<1\end{cases}$$

which is given by Erden et al. in [17, Theorem 4] but the values of Θ_{14}-Θ_{15} and the coefficients of $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}$, $\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}$ are in more modified form.

Remark 14. *If we assume $\lambda=\mu=\nu=\frac{q}{[2]_q}$ in Theorem 8, then we obtain [35, Theorem 4.2].*

Theorem 9. *We assume that the given conditions of Lemma 2 hold. If the mapping $\left|{}_{\kappa_1}D_q\mathcal{F}\right|^{p_1}$, $p_1>1$ is convex on $[\kappa_1,\kappa_2]$, then the following Newton's type inequality holds:*

$$\begin{aligned}
&\left|\lambda\mathcal{F}(\kappa_1)+(\mu-\lambda)\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right)+(\nu-\mu)\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)+(1-\nu)\mathcal{F}(\kappa_2)-\frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Omega_{18}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q}+\frac{(3q+2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&+\Omega_{19}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{3[2]_q}+\frac{q\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{3[2]_q}\right)^{\frac{1}{p_1}} \\
&\left.+\Omega_{20}^{\frac{1}{r_1}}\left(\frac{5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q}+\frac{(3q-2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}\tag{5.3}$$

where $p_1^{-1}+r_1^{-1}=1$ and

$$\Omega_{18}=\int_0^{\frac{1}{3}}|q\tau-\lambda|^{r_1}d_q\tau,\ \Omega_{19}=\int_{\frac{1}{3}}^{\frac{2}{3}}|q\tau-\mu|^{r_1}d_q\tau,\ \Omega_{20}=\int_{\frac{2}{3}}^{1}|q\tau-\nu|^{r_1}d_q\tau.$$

Proof. If we apply the steps used in the proof of Theorem 6 and taking into account Lemma 2, we can obtain the required inequality (5.3). $\square$

Remark 15. *If we take $\lambda = \frac{1}{8}$, $\mu = \frac{1}{2}$, and $\nu = \frac{7}{8}$ in Theorem 9, then we obtain the following inequality*

$$\begin{aligned}
&\left|\frac{1}{8}\left[\mathcal{F}(\kappa_1) + 3\mathcal{F}\left(\frac{2\kappa_1+\kappa_2}{3}\right) + 3\mathcal{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right) + \mathcal{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2}\mathcal{F}(x)\ {}_{\kappa_1}d_qx\right| \\
\leq\ & (\kappa_2-\kappa_1)\left[\Theta_{17}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q} + \frac{(3q+2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right. \\
&+\Theta_{18}^{\frac{1}{r_1}}\left(\frac{\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{3[2]_q} + \frac{q\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{3[2]_q}\right)^{\frac{1}{p_1}} \\
&\left.+\Theta_{19}^{\frac{1}{r_1}}\left(\frac{5\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_2)\right|^{p_1}}{9[2]_q} + \frac{(3q-2)\left|{}_{\kappa_1}D_q\mathcal{F}(\kappa_1)\right|^{p_1}}{9[2]_q}\right)^{\frac{1}{p_1}}\right]
\end{aligned}$$

where

$$\Theta_{17} = \int_0^{\frac{1}{3}}\left|q\tau - \frac{1}{8}\right|^{r_1} d_q\tau,\ \Theta_{18} = \int_{\frac{1}{3}}^{\frac{2}{3}}\left|q\tau - \frac{1}{2}\right|^{r_1} d_q\tau,\ \Theta_{19} = \int_{\frac{2}{3}}^{1}\left|q\tau - \frac{7}{8}\right|^{r_1} d_q\tau$$

which is given by Erden et al. in [17, Theorem 2] but the values of Θ_{17}-Θ_{19} *are in more modified form.*

6. Conclusions

We conclude our work by mentioning that here, we gave the extension of quantum Simpson's and quantum Newton's inequalities for quantum differentiable convex functions under certain parameters in the setting of quantum calculus. It is important to mention that our results transformed into some new and known results by considering the limit $q \to 1^-$ and by different variations of the involved parameters in our main results. We strongly believe that it is an interesting and new problem for the upcoming researchers who can obtain similar inequalities for other kinds of convexity and quantum integrals.

Author Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Funding

The work was partially supported by the National Natural Science Foundation of China (Grant No. 11971241).

Availability of Data Materials

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Acknowledgments

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Conflicts of Interest

The authors declare that they have no competing interests.

References

[1] R. Agarwal, "A propos d'une note de m. pierre humbert," Comptes rendus de l'Academie des Sciences, vol. 236, no. 21, pp. 2031–2032, 1953.

[2] M. A. Ali, H. Budak, Z. Zhang, and H. Yildrim, *Some new Simpson's type inequalities for co-ordinated convex functions in quantum calculus*, Mathematical Methods in the Applied Sciences, https://doi.org/10.1002/mma.7048.

[3] M. A. Ali, H. Budak, M. Abbas, and Y.-M. Chu, *Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second q^{κ_2}-derivatives*, Adv Differ Equ 2021, 7 (2021). https://doi.org/10.1186/s13662-020-03163-1.

[4] M. A. Ali, Y.-M. Chu, H. Budak, A. Akkurt, and H.Yildrim, *New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions*, Adv Differ Equ 2021, 64 (2021). https://doi.org/10.1186/s13662-021-03226-x

[5] M. A. Ali, M. Abbas, H. Budak, P. Agarwal, G. Murtaza and Yu-Ming Chu, *Quantum variant of Montgomery identity and Ostrowski-type inequalities for the mappings of two variables*, Adv Differ Equ 2021, 25 (2021). https://doi.org/10.1186/s13662-020-03195-7.

[6] M. Alomari, M. Darus, and S. S. Dragomir, *New inequalities of Simpson's type for σ-convex functions with applications*, RGMIA Res. Rep. Coll., vol. 12, no. 4, 2009.

[7] W. Al-Salam, *Some fractional q-integrals and q-derivatives*, Proceedings of the Edinburgh Mathematical Society, vol. 15, no. 2, pp. 135–140, 1966/1967.

[8] N. Alp, M. Z. Sarikaya, M. Kunt and I. Iscan, *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University–Science (2018) 30, 193–203.

[9] N. Alp and M. Z. Sarikaya, *Quantum Hermite-Hadamard's type inequalities for co-ordinated convex functions*, Applied Mathematics E-Notes, 20(2020), 341-356.

[10] S. Bermudo, P. Kórus and J. N. Valdés, *On q-Hermite–Hadamard inequalities for general convex functions*, Acta Mathematica Hungarica, 1-11, 2020.

[11] H. Budak, *Some trapezoid and midpoint type inequalities for newly defined quantum integrals*, Proyecciones Journal of Mathematics, in press.

[12] H. Budak, S. Erden, and M. A. Ali, *Simpson and Newton type inequalities for convex functions via newly defined quantum integrals*, Mathematical Methods in the Applied Sciences (2020).

[13] H. Budak, M. A. Ali, and M. Tarhanaci, *Some New Quantum Hermite–Hadamard-Like Inequalities for Coordinated Convex Functions*, Journal of Optimization Theory and Applications (2020): 1-12.

[14] H. Budak, M. A. Ali, and T. Tunç, *Quantum Ostrowski type integral inequalities for functions of two variables*, Mathematical Methods in the Applied Sciences, In press, 2020.

[15] S. S. Dragomir, R. P. Agarwal, P. Cerone, *On Simpson's inequality and applications*, J. Inequal. Appl. 5 (2000) 533–579.

[16] T. Du, Y. Li, and Z. Yang, *A generalization of Simpson's inequality via differentiable mapping using extended (s,m)-convex functions*, Applied Mathematics and Computations, 293 (2017) 358–369.

[17] S. Erden, S. Iftikhar, M. R. Delavar, P. Kumam, P. Thounthong, and W. Kumam, *On generalizations of some inequalities for convex functions via quantum integrals*, RACSAM (2020) 114:110 https://doi.org/10.1007/s13398-020-00841-3.

[18] T. Ernst, *The History of Q-Calculus And New Method.* Sweden: Department of Mathematics, Uppsala University, 2000.

[19] T. Ernst, A *Comprehensive Treatment of q-Calculus*, Springer Basel (2012).

[20] H. Gauchman, *Integral inequalities in q-calculus*, Comput. Math. Appl. 47 (2004) 281-300.

[21] S. Iftikhar, S. Erden, P. Kumam, and M. U. Awan. *Local fractional Newton's inequalities involving generalized harmonic convex functions*, Advances in Difference Equations 2020 (2020): 1-14.

[22] F. H. Jackson, *On a q-definite integrals*, Quarterly J. Pure Appl. Math. 41 (1910) 193-203.

[23] S. Jhanthanam, T. Jessada , N. Sotiris K., and N. Kamsing, *On q-Hermite-Hadamard Inequalities for Differentiable Convex Functions*, Mathematics 7, no. 7 (2019): 632.

[24] V. Kac and P. Cheung *Quantum calculus*, Springer (2001).

[25] M. A. Khan, M. Noor, E. R. Nwaeze, and Y-M. Chu , *Quantum Hermite–Hadamard inequality by means of a Green function*, Advances in Difference Equations 2020, no. 1 (2020): 1-20.

[26] M. Kunt, İ. İmdat, N. Alp, and M. Z. Sarıkaya, *(p,q)-Hermite–Hadamard inequalities and (p,q)-estimates for midpoint type inequalities via convex and quasi-convex functions*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas 112, no. 4 (2018): 969-992.

[27] W. Liu and Z. Hefeng, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, Journal of Applied Analysis and Computation, 7(2), 501-522.

[28] M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput. 251 (2015) 675–679.

[29] M. A. Noor, K. I. Noor, M. U. Awan, *Some quantum integral inequalities via preinvex functions*, Appl. Math. Comput. 269 (2015) 242–251.

[30] M. Noor, K. Noor, and M. Awan, *Quantum Ostrowski inequalities for q-differentiable convex functions*, J. Math. Inequal, 2016, 10(4), 1013-1018.

[31] E. R. Nwaeze, and A. M. Tameru, *New parameterized quantum integral inequalities via η-quasiconvexity*, Advances in Difference Equations 2019, no. 1 (2019): 425.

[32] M. E. Özdemir, A. O. Akdemir, H. Kavurmaci and M. Avci, *On the Simpson's inequality for co-ordinated convex functions*, Turkish Journal of Analysis and Number Theory, 2014, Vol. 2, No. 5, 165-169.

[33] Stankovic M.S. Marinkovic S.D. Rajković, P.M., *The zeros of polynomials orthogonal with respect to q-integral on several intervals in the complex plane*, 2003(1):178–188, 2003.

[34] M. Z. Sarikaya, E. Set and M. E. Özdemir, *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll. 13(2) (2010), Article2.

[35] W. Sudsutad, S. K. Ntouyas, J. Tariboon, *Quantum integral inequalities for convex functions*, J. Math. Inequal. 9 (3) (2015) 781–793.

[36] J. Tariboon, S. K. Ntouyas, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ. 282 (2013) 1-19.

[37] M. Tunç, E. Göv, and S. Balgeçti, *Simposn type quantum integral inequalities for convex functions,* Miskolc Mathematical Notes, Vol. 19 (2018), No. 1, pp. 649–664, DOI: 10.18514/MMN.2018.1661.

[38] M. Vivas-Cortez, M. A. Ali, A. Kashuri, I. B. Sial, and Z. Zhang, *Some New Newton's Type Integral Inequalities for Co-Ordinated Convex Functions in Quantum Calculus*, Symmetry 12, no. 9 (2020): 1476.

[39] H. Zhuang, W. Liu, J. Park, *Some quantum estimates of Hermite-Hadmard inequalities for quasi-convex functions*, Miskolc Math. Notes, 2016, 17(2).