ON OSTROWSKI-MERCER INEQUALITIES FOR DIFFERENTIABLE HARMONICALLY CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this work, we prove Ostrowski-Mercer inequalities for differentiable harmonically convex functions. It is also shown that the newly proved inequalities can be converted into some existing inequalities. Furthermore, it is provided that how the newly discovered inequalities can be applied to special means of real numbers.

1. Introduction

The study of different forms of fundamental inequality has been the subject of great interest for well over a century. A variety of mathematicians, interested in both pure and applied mathematics. One of the various ones mathematical basic discoveries of A. M. Ostrowski [25] is the following classical integral inequality:

Theorem 1. *Let $f:[1,\infty)\to\mathbb{R}$ is differentiable functions on $(1,\infty)$ and $f\in L[a,b]$, where $a,b\in[1,\infty)$ with $a<b$. If $|f'(x)|\le M$, then we have following inequality:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{M}{(b-a)}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]. \tag{1.1}$$

Ostrowski inequality has applications in quadrature, theory of probability and optimization, stochastic, statistics, information and the theory of integral operator. A number of scientists have concentrated over the last few years on Ostrowski type inequalities for convex and bounded variation functions. For example, In [13], Dragomir et al. proved some Ostrowski's inequalities for the functions of bounded variations and in [14], he used Lipschitzian mappings and proved Ostrowski's inequalities. After that, in [10], Cerone et al. proved an Ostrowski's inequality for the functions whose second derivatives are bounded. Set [23], proved different Ostrowsk's inequalities for the s-convex functions via Riemann-Liouville fractional integrals. In [22], Sarikaya and Budak use the local fractional integrals and proved Ostrowski's inequalities. In 2018, Budak et al. [8] proved some generalized Ostrowski's inequalities for the twice differentiable functions. Recently, in 2021, Budak et al. [9] proved some Ostrowski's inequalities for quantum differentiable convex functions using the quantum integrals. Until now, a significant number of research papers and books have been published on Ostrowski inequalities and their numerous applications.

In literature, the well-known Jensen inequality [20] states that if f is a convex function on an interval contains in x_n, then

$$f\left(\sum_{j=1}^{n}\lambda_j x_j\right)\le\sum_{j=1}^{n}\lambda_j f(x_j). \tag{1.2}$$

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In convex functions theory, Hermite-Hadamard inequality is very important which was discovered by C. Hermite and J. Hadamard independently (see, also [12], and [24, p.137])

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1.3}$$

where $f: I \rightarrow \mathbb{R}$ is a convex function over I and $a, b \in I$ with $a < b$. In the case of concave mappings, the above inequality satisfies in reverse order.

The following variant of Jensen inequality, known as the Jensen-Mercer, was demonstrated by Mercer [19]:

Theorem 2. *If f is a convex function on $[a,b]$, then the following inequality is true:*

$$f\left(a+b-\sum_{j=1}^{n}\lambda_j x_j\right) \leq f(a)+f(b)-\sum_{j=1}^{n}\lambda_j f(x_j), \tag{1.4}$$

for all $x_j \in [a,b]$ and $\lambda_j \in [0,1]$ with $\sum\limits_{j=1}^{n}\lambda_j = 1$.

In [18], the idea of Jensen-Mercer inequality has been used by Kian and Moslehian, and the following Hermite-Hadamard-Mercer inequality was demonstrated:

$$\begin{aligned} f\left(a+b-\frac{x+y}{2}\right) &\leq \frac{1}{y-x}\int_x^y f(a+b-t)\,dt \\ &\leq \frac{f(a+b-x)+f(a+b-y)}{2} \\ &\leq f(a)+f(b)-\frac{f(x)+f(y)}{2}, \end{aligned} \tag{1.5}$$

where f is convex function on $[a,b]$. Inspired by the work of Kian and Moslehian, in [1], AbdelJawad et al. gave the fractional version of Hermite-Hadamard-Mercer inequalities and in [2], Ali et al. proved the generalized form of Hermite-Hadamard-Mercer inequalities for convex functions. Recently, Chu et al. [11] found the estimates of fractional Hermite-Hadamard-Mercer inequalities for differentiable convex functions.

On the other hand, in 2014, İşcan introduced the following notions of harmonically convex functions and related Hermite-Hadamard type inequalities:

Definition 1. *[16] A mapping $f: I \subset \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}$ is called harmonically convex, if the following inequality holds for all $x, y \in I$ and $t \in [0,1]$:*

$$f\left(\frac{1}{\frac{t}{y}+\frac{1-t}{x}}\right) \leq tf(y)+(1-t)f(x), \tag{1.6}$$

when the inequality (1.6) is reversed, f is described as harmonically concave.

Theorem 3. *[16] For a harmonically convex mapping $f: I \subset \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}$ with $a, b \in I$ and $a < b$, the following inequality exists:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.7}$$

After the work of İşcan, Noor et al. [21] introduced the concepts of log-harmonically convex functions and proved some new Hermite-Hadamard type inequalities. In 2015, İşcan used the s-harmonically convexity and proved Ostrowski's type inequalities in [17]. In [3], Awan et. al. proved Hermite-Hadamard type inequalities for harmonically convex functions via conformable fractional integrals and in [4], they introduced a new class of harmonically convex functions called n-polynomial harmonically convex functions and proved some new inequalities of Hermite-Hadamad type for newly defined class of functions.

Recently, for harmonically convex functions, Dragomir proved the following Jensen's inequality:

Theorem 4 (Jensen inequality)**.** *[15] For harmonically convex functions, the following inequality is true:*

$$f\left(\frac{1}{\sum\limits_{i=1}^{n}\frac{\lambda_i}{x_i}}\right)\leq\sum_{i=1}^{n}\lambda_i f\left(x_i\right), \tag{1.8}$$

where $\sum\limits_{i=1}^{n}\lambda_i=1$.

Inspired by the work of Mercer, Baloch et al. gave the Jensen-Mercer inequality for harmonically convex functions that can be stated as:

Theorem 5 (Jensen-Mercer inequality)**.** *[6] For harmonically convex functions on* $[a,b]$*, the following inequality is true:*

$$f\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\sum\limits_{i=1}^{n}\frac{\lambda_i}{x_i}}\right)\leq f\left(a\right)+f\left(b\right)-\sum_{i=1}^{n}\lambda_i f\left(x_i\right), \tag{1.9}$$

where $\sum\limits_{i=1}^{n}\lambda_i=1$*,* $x_i\in[a,b]$ *and* $\lambda_i\in[0,1]$.

After that in [7], Baloch et al. proved the following inequalities of Hermite-Hadamard-Mercer type inequalities:

Theorem 6. *For a harmonically convex mapping* $f:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ *with* $a,b\in I$ *and* $a<b$*, the following inequality holds:*

$$\begin{aligned} f\left(\frac{1}{\frac{a+b}{ab}-\frac{1}{2}\left(\frac{x+y}{xy}\right)}\right) &\leq \frac{xy}{y-x}\int_{\frac{aby}{by+ay-ab}}^{\frac{abx}{bx+ax-ab}}\frac{f\left(t\right)}{t^2}dt \\ &\leq f\left(a\right)+f\left(b\right)-\frac{f\left(x\right)+f\left(y\right)}{2}. \end{aligned} \tag{1.10}$$

Motivated from the discussed literature, we use the Jensen-Mercer inequality for differentiable harmonically convex functions to establish some new Ostrowski's type inequalities with applications.

2. Main Results

New Ostrowski-Mercer inequalities are established for differentiable harmonically convex functions in this section. For this, we first give a new integral identity that will serve as an auxiliary to produce subsequent results for advancement. For brevity, we use the following notations.

$$L=\frac{1}{x}+\frac{1}{a}-\frac{1}{u_1},\ M=\frac{1}{x}+\frac{1}{a}-\frac{1}{v},$$

$$R=\frac{1}{x}+\frac{1}{b}-\frac{1}{u_2},\ P=\frac{1}{x}+\frac{1}{b}-\frac{1}{v},$$

$$\lambda_1(a,x,s,\alpha,\rho)=\frac{\beta(\rho+s+1,1)}{x^{2\alpha}}\cdot{}_2F_1\left(2\alpha,\rho+s+1;\rho+s+2,1-\frac{a}{x}\right),$$

$$\lambda_2(a,x,s,\alpha,\rho)=\frac{\beta(s+1,\rho+1)}{a^{2\alpha}}\cdot{}_2F_1\left(2\alpha,s+1;\rho+s+2,1-\frac{x}{a}\right),$$

$$\lambda_3(b,x,s,\alpha,\rho)=\frac{\beta(\rho+s+1,1)}{x^{2\alpha}}\cdot{}_2F_1\left(2\alpha,\rho+s+1;\rho+s+2,1-\frac{b}{x}\right),$$

and

$$\lambda_4(b,x,s,\alpha,\rho)=\frac{\beta(s+1,\rho+1)}{b^{2\alpha}}\cdot{}_2F_1\left(2\alpha,s+1;\rho+s+2,1-\frac{x}{b}\right),$$

where β is the Euler beta function

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\int_0^1(t)^{x-1}(1-t)^{y-1}dt,\ x,y>0.$$

and ${}_2F_1$ is hyper-geometric function

$${}_2F_1\left(a,b:c,z\right)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1.$$

Lemma 1. *Let $f:I\subset\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable function on I°. If $f\in L[a,b]$, then for all $x,u_1,u_2,v\in I$ and $t\in[0,1]$, the following equality satisfies:*

$$\begin{aligned}(2.1)\qquad & \left(\frac{v-u_1}{vu_1}\right)^2\int_0^1\frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2}f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)dt\\ & -\left(\frac{u_2-v}{vu_2}\right)^2\int_0^1\frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2}f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)dt\\ = & \left(\frac{v-u_1}{vu_1}\right)f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right)f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\\ & -\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}}\frac{f(\omega)}{\omega^2}d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}}\frac{f(\omega)}{\omega^2}d\omega\right].\end{aligned}$$

Proof. It is enough to memories that:

$$\begin{aligned}(2.2)\qquad \mathcal{J} = & \left(\frac{v-u_1}{vu_1}\right)^2\int_0^1\frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2}f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)dt\\ & -\left(\frac{u_2-v}{vu_2}\right)^2\int_0^1\frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2}f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)dt\end{aligned}$$

$$(2.3)\qquad \mathcal{J}=\left(\frac{v-u_1}{vu_1}\right)^2I_1-\left(\frac{u_2-v}{vu_2}\right)^2I_2.$$

Utilizing the integration by parts, we get the equalities:

$$\begin{aligned}(2.4)\qquad I_1 = & \int_0^1\frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2}f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)dt\\ = & \frac{(u_1v)}{(v-u_1)}f\left(\frac{axu_1}{au_1+xu_1-ax}\right)-\frac{(u_1v)^2}{(v-u_1)^2}\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}}\frac{f(\omega)}{\omega^2}d\omega,\end{aligned}$$

and

$$\begin{aligned}(2.5)\qquad I_2 = & \int_0^1\frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2}f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)dt\\ = & \frac{(u_2v)}{(v-u_2)}f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)+\frac{(u_2v)^2}{(v-u_2)^2}\int_{\frac{bxu_1}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}}\frac{f(\omega)}{\omega^2}d\omega.\end{aligned}$$

We obtain our required equality (2.1) by putting equality (2.4) and (2.5) in (2.3). □

Remark 1. *In Lemma 1, if we set $u_1=a$, $u_2=b$ and $v=x$, then Lemma 1 reduces to [17, Lemma 2.1].*

Theorem 7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$. If $|f'|$ is harmonically convex function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}
(2.6)\qquad & \Bigg| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \\
& - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega \right] \Bigg| \\
\leq\ & \left(\frac{v-u_1}{u_1v}\right)^2 \Big[\lambda_1(L,M,0,1,1)(|f'(x)|+|f'(a)|) - \\
& \lambda_1(L,M,1,1,1)(|f'(u_1)|) - \lambda_2(L,M,1,1,1)(|f'(v)|) \Big] - \\
& \left(\frac{u_2-v}{u_2v}\right)^2 \Big[\lambda_3(R,P,0,1,1)(|f'(x)|+|f'(a)|) - \\
& \lambda_3(R,P,1,1,1)(|f'(u_2)|) - \lambda_4(R,P,1,1,1)(|f'(v)|) \Big],
\end{aligned}$$

where $x,u_1,u_2,v \in [a,b]$.

Proof. Taking the modulus in Lemma 1 and using the properties of the modulus, we have

$$\begin{aligned}
(2.7)\qquad & \Bigg| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \\
& - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega \right] \Bigg| \\
\leq\ & \left(\frac{v-u_1}{u_1v}\right)^2 \int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2} \left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)\right| dt \\
& - \left(\frac{u_2-v}{u_2v}\right)^2 \int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2} \left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)\right| dt.
\end{aligned}$$

By using the Jensen-Mercer inequality for (2.7), we obtain

$$\begin{aligned}
(2.8)\qquad & \Bigg| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \\
& - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega \right] \Bigg| \\
\leq\ & \left(\frac{v-u_1}{vu_1}\right)^2 \int_0^1 \frac{t}{(Lt+(1-t)M)^2} \Big(|f'(x)|+|f'(a)|-t|f'(u_1)|-(1-t)|f'(v)| \Big) dt \\
& - \left(\frac{u_2-v}{u_2v}\right)^2 \int_0^1 \frac{t}{(Rt+(1-t)P)^2} \Big(|f'(x)|+|f'(b)|-t|f'(u_2)|-(1-t)|f'(v)| \Big) dt.
\end{aligned}$$

By integrating and simplification, we get

$$
\begin{aligned}
(2.9)\quad & \Bigg|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \\
& -\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right]\Bigg| \\
\leq\ & \left(\frac{v-u_1}{vu_1}\right)^2\Big[\lambda_1(L, M, 0,1,1)(|f'(x)|+|f'(a)|)- \\
& \lambda_1(L, M, 1,1,1)(|f'(u_1)|)-\lambda_2(L, M, 1,1,1)(|f'(v)|)\Big]- \\
& \left(\frac{u_2-v}{u_2v}\right)^2\Big[\lambda_3(R, P, 0,1,1)(|f'(x)|+|f'(a)|)- \\
& \lambda_3(R, P, 1,1,1)(|f'(u_2)|)-\lambda_4(R, P, 1,1,1)(|f'(v)|)\Big].
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
\int_0^1 \frac{t}{(Lt-(1-t)M)^2} dt &= \frac{\beta(2,1)}{M^2}\, {}_2F_1\left(2,2,3;1-\frac{L}{M}\right), \\
\int_0^1 \frac{t}{(Rt-(1-t)P)^2} dt &= \frac{\beta(2,1)}{P^2}\, {}_2F_1\left(2,2,3;1-\frac{R}{P}\right), \\
\int_0^1 \frac{t^2}{(Lt-(1-t)M)^2} dt &= \frac{\beta(3,1)}{M^2}\, {}_2F_1\left(2,3,4;1-\frac{L}{M}\right), \\
\int_0^1 \frac{t^2}{(Rt-(1-t)P)^2} dt &= \frac{\beta(3,1)}{P^2}\, {}_2F_1\left(2,3,4;1-\frac{R}{P}\right), \\
\int_0^1 \frac{t(1-t)}{(Lt-(1-t)M)^2} dt &= \frac{\beta(2,2)}{M^2}\, {}_2F_1\left(2,2,4;1-\frac{L}{M}\right),
\end{aligned}
$$

and

$$
\int_0^1 \frac{t(1-t)}{(Rt-(1-t)P)^2} dt = \frac{\beta(2,2)}{P^2}\, {}_2F_1\left(2,2,4;1-\frac{R}{P}\right)
$$

which completes the proof. □

Remark 2. *In Theorem 7, if we set $u_1=a$, $u_2=b$ and $v=x$, then Theorem 7 becomes [17, Theorem 2.2 for $s=q=1$] and [5, Theorem 26 for $s=m=q=1$].*

Corollary 1. *(Ostrowski-Mercer Inequality): In Theorem 7, if we choose $|f'(t)| \leq \mathcal{M},\ \forall\ t \in [a,b]$, then we have the following Ostrowski-Mercer inequality:*

$$
\begin{aligned}
(2.10)\quad & \Bigg|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \\
& -\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right]\Bigg| \\
\leq\ & \mathcal{M}\Bigg\{\left(\frac{v-u_1}{vu_1}\right)^2\Big[\lambda_1(L, M, 0,1,1)\Big]-\left(\frac{u_2-v}{u_2v}\right)^2\Big[\lambda_3(R, P, 0,1,1)\Big]\Bigg\}.
\end{aligned}
$$

Proof. The result can be easily obtained by using $\left|f'\left(\frac{1}{\left(\frac{1}{x}+\frac{1}{a}-\left(\frac{t}{u_1}+\frac{1-t}{v}\right)\right)}\right)\right| \leq \mathcal{M}$ and $\left|f'\left(\frac{1}{\left(\frac{1}{x}+\frac{1}{b}-\left(\frac{t}{u_2}+\frac{1-t}{v}\right)\right)}\right)\right| \leq \mathcal{M}$. □

Remark 3. *In Corollary 1, if we set $u_1=a$, $u_2=b$ and $v=x$, then Corollary 1 becomes [17, Corollary 2.1 for $s=q=1$] and [5, Corollary 28 for $s=m=q=1$].*

Theorem 8. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$. If $|f'|^q$, $q > 1$ is harmonically convex function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}
&\left|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2 v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\right.\\
&\left.-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2}d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2}d\omega\right]\right|\\
\leq\ & \left(\frac{v-u_1}{vu_1}\right)^2 \left[\lambda_1(L,M,0,p,p)\right]^{\frac{1}{p}} \left[|f'(x)|^q + |f'(a)|^q - \frac{1}{2}(|f'(u_1)|^q + |f'(v)|^q)\right]^{\frac{1}{q}}\\
&-\left(\frac{u_2-v}{u_2 v}\right)^2 \left[\lambda_3(R,P,0,p,p)\right]^{\frac{1}{p}} \left[|f'(x)|^q + |f'(b)|^q - \frac{1}{2}(|f'(u_2)|^q + |f'(v)|^q)\right]^{\frac{1}{q}},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $x, u_1, u_2, v \in [a,b]$.

Proof. Taking the modulus in Lemma 1 and from properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2 v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\right.\\
&\left.-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2}d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2}d\omega\right]\right|\\
\leq\ & \left(\frac{v-u_1}{vu_1}\right)^2 \int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2} \left|f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)\right| dt\\
&-\left(\frac{u_2-v}{u_2 v}\right)^2 \int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2} \left|f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)\right| dt.
\end{aligned} \tag{2.11}$$

By applying the Hölder inequality, we get

$$\begin{aligned}
&\left|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2 v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\right.\\
&\left.-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2}d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2}d\omega\right]\right|\\
\leq\ & \left(\frac{v-u_1}{vu_1}\right)^2 \left[\int_0^1 \left(\frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2}\right)^p dt\right]^{\frac{1}{p}} \left[\int_0^1 \left|f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)\right|^q dt\right]^{\frac{1}{q}}\\
&-\left(\frac{u_2-v}{u_2 v}\right)^2 \left[\int_0^1 \left(\frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2}\right)^p dt\right]^{\frac{1}{p}} \left[\int_0^1 \left|f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)\right|^q dt\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.12}$$

Applying the Jensen-Mercer inequality, we get

$$
\begin{aligned}
(2.13)\qquad & \left| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2 v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \right. \\
& \left. - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega \right] \right| \\
\leq & \left(\frac{v-u_1}{vu_1}\right)^2 \left[\int_0^1 \left(\frac{t}{(tL+(1-t)M)^2}\right)^p dt \right]^{\frac{1}{p}} \\
& \left[|f'(x)|^q + |f'(a)|^q - t|f'(u_1)|^q - (1-t)|f'(v)|^q \right] dt \Big]^{\frac{1}{q}} \\
& - \left(\frac{u_2-v}{u_2 v}\right)^2 \left[\int_0^1 \left(\frac{t}{(Rt+(1-t)P)^2}\right)^p dt \right]^{\frac{1}{p}} \\
& \left[|f'(x)|^q + |f'(b)|^q - t|f'(u_2)|^q - (1-t)|f'(v)|^q \right] dt \Big]^{\frac{1}{q}}.
\end{aligned}
$$

On integrating and simplification, we get

$$
\begin{aligned}
(2.14)\qquad & \left| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2 v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \right. \\
& \left. - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega \right] \right| \\
\leq & \left(\frac{v-u_1}{vu_1}\right)^2 \left[\lambda_1(L,M,0,p,p) \right]^{\frac{1}{p}} \left[|f'(x)|^q + |f'(a)|^q - \frac{1}{2}(|f'(u_1)|^q + |f'(v)|^q) \right]^{\frac{1}{q}} \\
& - \left(\frac{u_2-v}{u_2 v}\right)^2 \left[\lambda_3(R,P,0,p,p) \right]^{\frac{1}{p}} \left[|f'(x)|^q + |f'(a)|^q - \frac{1}{2}(|f'(u_1)|^q + |f'(v)|^q) \right]^{\frac{1}{q}}
\end{aligned}
$$

where

$$
\int_0^1 \frac{t^p}{(Lt-(1-t)M)^{2p}} dt = \frac{\beta(p+2,1)}{M^{2p}} {}_2F_1\left(2p, p+2, p+3; 1-\frac{L}{M}\right)
$$

and the proof is completed. □

Remark 4. *In Theorem 8, if we set $u_1 = a$, $u_2 = b$ and $v = x$, then Theorem 8 becomes [17, Theorem 2.6 for $s = q = 1$] and [5, Theorem 42 for $s = m = 1$]..*

Corollary 2. *(Ostrowski-Mercer Inequality): In Theorem 8, if we choose $|f'(t)| \leq \mathcal{M}$, $\forall\ t \in [a,b]$, then we have the following Ostrowski-Mercer inequality:*

$$
\begin{aligned}
(2.15)\qquad & \left| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2 v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \right. \\
& \left. - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega \right] \right| \\
\leq & \mathcal{M} \left\{ \left(\frac{v-u_1}{vu_1}\right)^2 \left[\lambda_1(L,M,0,p,p) \right]^{\frac{1}{p}} - \left(\frac{u_2-v}{u_2 v}\right)^2 \left[\lambda_3(R,P,0,p,p) \right]^{\frac{1}{p}} \right\}.
\end{aligned}
$$

Proof. The result can be easily obtained by using $\left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right) \right| \leq \mathcal{M}$ and $\left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right) \right| \leq \mathcal{M}$. □

Remark 5. *In Corollary 2, if we set $u_1 = a$, $u_2 = b$ and $v = x$, then Corollary 2 becomes [17, Corollary 2.5 for $s = q = 1$] and [5, Corollary 44 for $s = m = 1$].*

Theorem 9. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$. If $|f'|^q$, $q \geq 1$ is harmonically convex function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}
&\left| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \right. \\
&\left. - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right] \right| \\
\leq &\left(\frac{v-u_1}{vu_1}\right)^2 \left[\lambda_1(L,M,0,1,1)\right]^{1-\frac{1}{q}} \Big[\lambda_1(L,M,0,1,1)(|f'(x)|^q + |f'(a)|^q) - \\
&\lambda_1(L,M,1,1,1)(|f'(u_1)|^q) - \lambda_2(L,M,1,1,1)(|f'(v)|^q)\Big] \\
&- \left(\frac{u_2-v}{u_2v}\right)^2 \left[\lambda_3(R,P,0,1,1)\right]^{1-\frac{1}{q}} \Big[\lambda_3(R,P,0,1,1)(|f'(x)|^q + |f'(b)|^q) - \\
&\lambda_3(R,P,1,1,1)(|f'(u_2)|^q) - \lambda_4(R,P,1,1,1)(|f'(v)|^q)\Big],
\end{aligned} \tag{2.16}$$

where $x,u_1,u_2,v \in [a,b]$.

Proof. Taking the modulus in Lemma 1 and properties of the modulus, we have

$$\begin{aligned}
&\left| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \right. \\
&\left. - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right] \right| \\
\leq &\left(\frac{v-u_1}{vu_1}\right)^2 \int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2} \left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)\right| dt \\
&- \left(\frac{u_2-v}{u_2v}\right)^2 \int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2} \left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)\right| dt.
\end{aligned} \tag{2.17}$$

By applying the power mean inequality, we have

$$\begin{aligned}
&\left| \left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right) + \left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right) \right. \\
&\left. - \left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega + \int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right] \right| \\
\leq &\left(\frac{v-u_1}{vu_1}\right)^2 \left(\int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2} dt\right)^{1-\frac{1}{q}} \\
&\left[\int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))^2} \left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{a}-(\frac{t}{u_1}+\frac{1-t}{v}))}\right)\right|^q dt\right]^{\frac{1}{q}} \\
&- \left(\frac{u_2-v}{u_2v}\right)^2 \left(\int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2} dt\right)^{1-\frac{1}{q}} \\
&\left[\int_0^1 \frac{t}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))^2} \left| f'\left(\frac{1}{(\frac{1}{x}+\frac{1}{b}-(\frac{t}{u_2}+\frac{1-t}{v}))}\right)\right|^q dt\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.18}$$

By using the Jensen-Mercer inequality, we have

$$\begin{aligned}(2.19)\quad &\Bigg|\left(\frac{v-u_1}{vu_1}\right)f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right)f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\\ &-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}}\frac{f(\omega)}{\omega^2}d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}}\frac{f(\omega)}{\omega^2}d\omega\right]\Bigg|\\ \leq\ &\left(\frac{v-u_1}{vu_1}\right)^2\left(\int_0^1\frac{t}{(tL+(1-t)M)^2}dt\right)^{1-\frac{1}{q}}\\ &\left[\int_0^1\frac{t}{(tL+(1-t)M)^2}\Big(|f'(x)|^q+|f'(a)|^q-t|f'(u_1)|^q-(1-t)|f'(v)|^q\Big)dt\right]^{\frac{1}{q}}\\ &-\left(\frac{u_2-v}{u_2v}\right)^2\left(\int_0^1\frac{t}{(tR+(1-t)M)^2}dt\right)^{1-\frac{1}{q}}\\ &\left[\int_0^1\frac{t}{(tR+(1-t)P)^2}\Big(|f'(x)|^q+|f'(b)|^q-t|f'(u_2)|^q-(1-t)|f'(v)|^q\Big)dt\right]^{\frac{1}{q}}.\end{aligned}$$

On integrating and simplification, we obtain

$$\begin{aligned}(2.20)\quad &\Bigg|\left(\frac{v-u_1}{vu_1}\right)f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right)f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\\ &-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}}\frac{f(\omega)}{\omega^2}d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}}\frac{f(\omega)}{\omega^2}d\omega\right]\Bigg|\\ \leq\ &\left(\frac{v-u_1}{vu_1}\right)^2\Big[\lambda_1(L,M,0,1,1)\Big]^{1-\frac{1}{q}}\Big[\lambda_1(L,M,0,1,1)(|f'(x)|^q+|f'(a)|^q)-\\ &\lambda_1(L,M,1,1,1)(|f'(u_1)|^q)-\lambda_2(L,M,1,1,1)(|f'(v)|^q)\Big]\\ &-\left(\frac{u_2-v}{u_2v}\right)^2\Big[\lambda_3(R,P,0,1,1)\Big]^{1-\frac{1}{q}}\Big[\lambda_3(R,P,0,1,1)(|f'(x)|^q+|f'(b)|^q)-\\ &\lambda_3(R,P,1,1,1)(|f'(u_2)|^q)-\lambda_4(R,P,1,1,1)(|f'(v)|^q)\Big]\end{aligned}$$

which completes the proof. □

Remark 6. *In Theorem 9, if we set $u_1=a$, $u_2=b$ and $v=x$, then Theorem 9 becomes [17, Theorem 2.2 for $s=1$] and [5, Theorem 34 for $s=m=1$].*

Theorem 10. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$. If $|f'|^q$, $q>1$ is harmonically concave function on $[a,b]$, then we have the following inequality:*

$$\begin{aligned}(2.21)\quad &\Bigg|\left(\frac{v-u_1}{vu_1}\right)f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right)f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\\ &-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}}\frac{f(\omega)}{\omega^2}d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}}\frac{f(\omega)}{\omega^2}d\omega\right]\Bigg|\\ \leq\ &\left(\frac{v-u_1}{vu_1}\right)^2\Big[\lambda_1(L,M,0,p,q)\Big]^{\frac{1}{p}}\left(\left|f'\left(\frac{1}{\frac{1}{x}+\frac{1}{a}-\frac{u_1+v}{2u_1v}}\right)\right|\right)\\ &-\left(\frac{u_2-v}{u_2v}\right)^2\Big[\lambda_3(R,P,0,p,q)\Big]^{\frac{1}{p}}\left(\left|f'\left(\frac{1}{\frac{1}{x}+\frac{1}{b}-\frac{u_2+v}{2u_2v}}\right)\right|\right),\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$ and where $x,u_1,u_2,v\in[a,b]$.

Proof. Taking the modulus in Lemma 1 and using the properties of the modulus, we have

$$\begin{aligned}
&\left|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\right. \\
&\left.-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right]\right| \\
\leq\ &\left(\frac{v-u_1}{vu_1}\right)^2 \int_0^1 \frac{t}{\left(\frac{1}{x}+\frac{1}{a}-\left(\frac{t}{u_1}+\frac{1-t}{v}\right)\right)^2}\left|f'\left(\frac{1}{\left(\frac{1}{x}+\frac{1}{a}-\left(\frac{t}{u_1}+\frac{1-t}{v}\right)\right)}\right)\right| dt \\
&-\left(\frac{u_2-v}{u_2v}\right)^2 \int_0^1 \frac{t}{\left(\frac{1}{x}+\frac{1}{b}-\left(\frac{t}{u_2}+\frac{1-t}{v}\right)\right)^2}\left|f'\left(\frac{1}{\left(\frac{1}{x}+\frac{1}{b}-\left(\frac{t}{u_b}+\frac{1-t}{v}\right)\right)}\right)\right| dt.
\end{aligned} \tag{2.22}$$

By applying the Hölder inequality, we get

$$\begin{aligned}
&\left|\left(\frac{v-u_1}{vu_1}\right) f\left(\frac{axu_1}{au_1+xu_1-ax}\right)+\left(\frac{u_2-v}{u_2v}\right) f\left(\frac{bxu_2}{bu_2+xu_2-bx}\right)\right. \\
&\left.-\left[\int_{\frac{bxu_2}{bu_2+xu_2-bx}}^{\frac{bxv}{bv+xv-bx}} \frac{f(\omega)}{\omega^2} d\omega+\int_{\frac{axv}{av+xv-ax}}^{\frac{axu_1}{au_1+xu_1-ax}} \frac{f(\omega)}{\omega^2} d\omega\right]\right| \\
\leq\ &\left(\frac{v-u_1}{vu_1}\right)^2\left[\int_0^1\left(\frac{t}{\left(\frac{1}{x}+\frac{1}{a}-\left(\frac{t}{u_1}+\frac{1-t}{v}\right)\right)^2}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{1}{\left(\frac{1}{x}+\frac{1}{a}-\left(\frac{t}{u_1}+\frac{1-t}{v}\right)\right)}\right)\right|^q dt\right]^{\frac{1}{q}} \\
&-\left(\frac{u_2-v}{u_2v}\right)^2\left[\int_0^1\left(\frac{t}{\left(\frac{1}{x}+\frac{1}{b}-\left(\frac{t}{u_2}+\frac{1-t}{v}\right)\right)^2}\right)^p dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f'\left(\frac{1}{\left(\frac{1}{x}+\frac{1}{b}-\left(\frac{t}{u_2}+\frac{1-t}{v}\right)\right)}\right)\right|^q dt\right]^{\frac{1}{q}}.
\end{aligned} \tag{2.23}$$

Since $|f'|^q$ is concave function, therefore from inequality (1.10), we have

$$\int_0^1\left|f'\left(\frac{t}{\left(\frac{1}{x}+\frac{1}{a}-\left(\frac{t}{u_1}+\frac{1-t}{v}\right)\right)}\right)\right|^q dt \leq\left|f'\left(\frac{1}{\frac{1}{x}+\frac{1}{a}-\frac{u_1+v}{2u_1v}}\right)\right|^q, \tag{2.24}$$

and

$$\int_0^1\left|f'\left(\frac{t}{\left(\frac{1}{x}+\frac{1}{b}-\left(\frac{t}{u_2}+\frac{1-t}{v}\right)\right)}\right)\right|^q dt \leq\left|f'\left(\frac{1}{\frac{1}{x}+\frac{1}{b}-\frac{u_2+v}{2u_2v}}\right)\right|^q. \tag{2.25}$$

We obtain our required inequality (2.21) by plugging inequality (2.24) and (2.25) in (2.23). □

3. Application to Special Means

For arbitrary positive numbers κ_1, κ_2 ($\kappa_1 \neq \kappa_2$), we consider the means as follows:

(1) The arithmetic mean
$$\mathcal{A}=\mathcal{A}(\kappa_1, \kappa_2)=\frac{\kappa_1+\kappa_2}{2}.$$
(2) The geometric mean
$$\mathcal{G}=\mathcal{G}(\kappa_1, \kappa_2)=\sqrt{\kappa_1 \kappa_2}.$$
(3) The harmonic means
$$\mathcal{H}=\mathcal{H}(\kappa_1, \kappa_2)=\frac{2\kappa_1 \kappa_2}{\kappa_1+\kappa_2}.$$
(4) The logarithmic mean
$$\mathcal{L}=\mathcal{L}(\kappa_1, \kappa_2)=\frac{\kappa_2-\kappa_1}{\ln \kappa_2-\ln \kappa_1}.$$

These means are often employed in numerical approximations and other fields. However, the following straightforward relationship has been stated in the literature.

$$\mathcal{H} \leq \mathcal{G} \leq \mathcal{L} \leq \mathcal{A}.$$

Proposition 1. *For $a, b \in (0, \infty)$, the following inequality is true:*

$$\begin{aligned}
&\Bigg| \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right) \left(2\mathcal{H}^{-1}(a,x) - u_1^{-1} \right)^{-1} + \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right) \left(2\mathcal{H}^{-1}(x,b) - u_2^{-1} \right)^{-1} \\
&- \Bigg[\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \mathcal{L}\left(\left(\frac{1}{x} + \frac{1}{a} - \frac{1}{u_1} \right)^{-1}, \left(\frac{1}{x} + \frac{1}{a} - \frac{1}{v} \right)^{-1} \right) \\
&+ \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right) \mathcal{L}\left(\left(\frac{1}{x} + \frac{1}{b} - \frac{1}{u_2} \right)^{-1}, \left(\frac{1}{x} + \frac{1}{b} - \frac{1}{v} \right)^{-1} \right) \Bigg] \Bigg| \\
\leq\ & \mathcal{M} \Bigg\{ \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right)^2 \Big[\lambda_1(L,M,0,1,1) \Big] - \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right)^2 \Big[\lambda_3(R,P,0,1,1) \Big] \Bigg\}.
\end{aligned}$$

Proof. The inequality (2.10) in Corollary 1 for mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$ leads to this conclusion. □

Proposition 2. *For $a, b \in (0, \infty)$, the following inequality is true:*

$$\begin{aligned}
&\left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right) \left(2\mathcal{H}^{-1}(a,x) - u_1^{-1} \right)^{-1} + \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right) \left(2\mathcal{H}^{-1}(x,b) - u_2^{-1} \right)^{-1} \\
&- \Bigg[\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \mathcal{L}\left(\left(\frac{1}{x} + \frac{1}{a} - \frac{1}{u_1} \right)^{-1}, \left(\frac{1}{x} + \frac{1}{a} - \frac{1}{v} \right)^{-1} \right) \\
&+ \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right) \mathcal{L}\left(\left(\frac{1}{x} + \frac{1}{b} - \frac{1}{u_2} \right)^{-1}, \left(\frac{1}{x} + \frac{1}{b} - \frac{1}{v} \right)^{-1} \right) \Bigg] \Bigg| \\
\leq\ & \mathcal{M} \Bigg\{ \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right)^2 \Big[\lambda_1(L,M,0,p,p) \Big]^{\frac{1}{p}} - \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right)^2 \Big[\lambda_3(R,P,0,p,p) \Big]^{\frac{1}{p}} \Bigg\}.
\end{aligned}$$

Proof. The inequality (2.15) in Corollary 2 for mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x$ leads to this conclusion. □

Proposition 3. *For $a, b \in (0, \infty)$, the following inequality is true:*

$$\begin{aligned}
&\Bigg| \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right) \left(2\mathcal{H}^{-1}(a,x) - u_1^{-1} \right)^{-2} + \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right) \left(2\mathcal{H}^{-1}(x,b) - u_2^{-1} \right)^{-2} \\
&- \left[\frac{1}{v-u_1} \mathcal{G}^2(u_1,v) + \frac{1}{u_2-v} \mathcal{G}^2(v,u_2) \right] \Bigg| \\
\leq\ & \mathcal{M} \Bigg\{ \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right)^2 \Big[\lambda_1(L,M,0,1,1) \Big] - \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right)^2 \Big[\lambda_3(R,P,0,1,1) \Big] \Bigg\}.
\end{aligned}$$

Proof. The inequality (2.10) in Corollary 1 for mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^2$ leads to this conclusion. □

Proposition 4. *For $a, b \in (0, \infty)$, the following inequality is true:*

$$\begin{aligned}
&\Bigg| \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right) \left(2\mathcal{H}^{-1}(a,x) - u_1^{-1} \right)^{-2} + \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right) \left(2\mathcal{H}^{-1}(x,b) - u_2^{-1} \right)^{-2} \\
&- \left[\frac{1}{v-u_1} \mathcal{G}^2(u_1,v) + \frac{1}{u_2-v} \mathcal{G}^2(v,u_2) \right] \Bigg| \\
\leq\ & \mathcal{M} \Bigg\{ \left(\frac{v-u_1}{\mathcal{G}^2(u_1,v)} \right)^2 \Big[\lambda_1(L,M,0,p,p) \Big]^{\frac{1}{p}} - \left(\frac{u_2-v}{\mathcal{G}^2(v,u_2)} \right)^2 \Big[\lambda_3(R,P,0,p,p) \Big]^{\frac{1}{p}} \Bigg\}.
\end{aligned}$$

Proof. The inequality (2.15) in Corollary 2 for mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^2$ leads to this conclusion. □

Conclusion

The Jensen-Mercer inequality has been utilized to prove some new Ostrowski-Mercer type inequalities involving the differentiable harmonically convex function. Moreover, we discussed the special cases of newly proved results and obtained some new and existing inequalities of Ostrowski and Ostrowski-Mercer type. It is an interesting and new problem that the upcoming researcher can prove the similar inequalities for different fractional operators in their future work.

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Author's contributions

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References

[1] T. Abdeljawad, M. A. Ali, P. O. Mohammad, and A. Kashuri, On inequalities of Hermite-Hadamard-Mercer type involving Riemann-Liouville fractional integrals. *AIMS Mathematics* **5** (2020), 7316-7331.

[2] M. M. Ali and A. R. Khan, Generalized integral Mercer's inequality and integral means. *J. Inequal. Spec. Funct.* **10** (2019), 60-76.

[3] M. U. Awan, M. A. Noor, M. V. Mihai, and K. I. Noor, Inequalities via harmonic convex functions: conformable fractional calculus approach. *J. Math. Inequal.* **12** (2018), 143-153.

[4] M. U. Awan, N. Akhtar, and S. Iftikhar, New Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions. *J. Inequal. Appl.* **2020** (2020), 1-12.

[5] I. A. Baloch and İ. İşcan, Some Ostrowski Type Inequalities for Harmonically (s, m)-Convex Functions in Second Sense. *Int. J. Anal.* **2015** Article ID 672675, 9 pages, 2015. https://doi.org/10.1155/2015/672675.

[6] I. A. Baloch, A. A. Mughal, Y. M. Chu, A. Ul Haq, and M. D. L. Sen, A variant of Jensen-type inequality and related results for harmonic convex. *AIMS Mathematics* **5** (2020), 6404-6418.

[7] I. A. Baloch, A. A. Mughal, Y-M. Chu, A. U. Haq and M. D. L. Sen, Improvements and generalization of some results related to the class of harmonically convex functions and applications. *J. Math. Comput. Sci.* **22** (2021), 282-294.

[8] H. Budak, M. Z. Sarikaya and S. S. Dragomir, *Some perturbed Ostrowski type inequalities for twice differentiable functions*. In Advances in Mathematical Inequalities and Applications (pp. 279-294). Birkhäuser, Singapore, 2018.

[9] H. Budak, M. A. Ali, N. Alp, and Y.-M. Chu, Quantum Ostrowski Type Integral Inequalities. *J. Math. Inequal.* In press, 2021.

[10] P. Cerone, S. S. Dragomir and J. Roumeliotis, An inequality of Ostrowski type for mappings whose second derivatives are bounded and applications. *East Asian Math. J.* **15** (1999), 1-9.

[11] H. H. Chu, S. Rashid, Z. Hammouch and Y. M. Chu, New fractional estimates for Hermite-Hadamard-Mercer's type inequalities. *Alexadria Engineering Journal* **59** (2020), 3079-3089.

[12] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*. RGMIA Monographs, Victoria University, 2000.

[13] S. S. Dragomir, On the Ostrowski integral inequality for mappings with bounded variation and applications. *Math. Ineq. Appl.* **2** (1998).

[14] S. S. Dragomir, The Ostrowski integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), 33-37.

[15] S. S. Dragomir, Inequalities of Jensen type for HA-convex functions. *An. Univ. Oradea Fasc. Mat.* **27** (2020), 103-124.

[16] İ. İşcan, Hermite-Hadamard type inequaities for harmonically functions. *Hacet. J. Math. Stat.* **43** (2014), 935-942.

[17] İ. İşcan, Ostrowski type inequalities for harmonically s-convex functions. *Konuralp J. Math.* **3** (2015), 63-74.

[18] M. Kian and M. S. Moslehian, Refinements of the operator Jensen-Mercer inequality. *Electron. J. Linear Algebra* **26** (2013), 742-753.

[19] A. McD. Mercer, A Variant of Jensenis Inequality. *J. Ineq. Pure and Appl. Math* **4** (2003).

[20] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and new inequalities in analysis.* Kluwer Academic Publishers Group, Dordrecht, 1993.

[21] M. A. Noor, K. I. Noor, and M. U. Awan, Some characterizations of harmonically log-convex functions. *Proc. Jangjeon Math. Soc.* **17** (2014), 51-61.

[22] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), 1527-1538.

[23] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **63** (2012), 1147-1154.

[24] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Orderings and Statistical Applications.* Academic Press, Boston, 1992.

[25] A. M. Ostrowski, Über die absolutabweichung einer differentiebaren funktion von ihrem integralmitelwert. *Comment. Math. Helv.* **10** (1938), 226-227.