# LONG TIME BEHAVIOR OF A WAVE EQUATION WITH TIME-VARYING DELAY AND ACOUSTIC BOUNDARY CONDITIONS

JONG YEOUL PARK AND JAE UG JEONG*

Department of Mathematics, Pusan National University, Pusan 609-735, South Korea
Department of Mathematics, Dongeui University, Pusan 47340, South Korea

ABSTRACT. In this paper, we consider the following wave equation with time-varying delay and acoustic boundary conditions

$$u_{tt}(t) - \Delta u(t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau(t)) + f(u) = h(x)$$

in a bounded domain. By virtue of Galerkin method, we prove the the existence and uniqueness of global solution under some general assumptions for the above equation. And the existence of a compact global attractor is proved.

AMS Mathematics Subject Classification: 35B41; 35L05; 37B55

*Key words and phrases*: Wave equation; Time delay; Existence solution; Global attractor

## 1. Introduction

In this paper, we consider the following wave equation with time-varying delay and acoustic boundary conditions:

$$\begin{cases} u_{tt}(t) - \Delta u(t) + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau(t)) \\ \quad +f(u) = h(x), \quad x \in \Omega, \quad t \geq 0, \\ u = 0, \quad x \in \Gamma_0, \quad t \geq 0, \\ \delta_{tt}(t) + k\delta_t(t) + \delta(t) = -u_t(t), \quad x \in \Gamma_1, \quad t \geq 0, \\ \frac{\partial u(t)}{\partial \nu} = \delta_t(t), \quad x \in \Gamma_1, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ \delta(x,0) = \delta_0(x), \quad \delta_t(x,0) = \delta_1(x), \quad x \in \Gamma_1, \\ u_t(x,t) = f_0(x,t), \quad x \in \Omega, \quad t \in [-\tau(0), 0), \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, $\Gamma_0$ and $\Gamma_1$ are closed and disjoint with meas$(\Gamma_0) > 0$, $\nu$ represent the outward normal to $\Gamma$ and $k$ is a positive constant. Moreover, $f$ and $h$ are

* Corresponding author. Tel:+820518901472; Fax:+820518902618.

external forcing terms, $\mu_1$, $\mu_2$ are some constants, $\tau(t)$ represents the time delay and $u_0$, $u_1$, $\delta_0$, $\delta_1$, $f_0$ are given functions belonging to some suitable spaces.

The acoustic boundary conditions are introduced by Morse and Ingard [13] and developed by Beale and Rosenerans [3]. In [3], the authors proved the global existence and regularity of the linear problem. Some authors studied the existence and decay of solutions for wave equations with acoustic boundary conditions.

The time delay aries in many physical, chemical, biological and economical phenomena. Because this phenomena depend not only on the present state but also on the past history of the system in a more complicated way. The differential equations with time delay effects becomes an active area of research (see [6,9,15]).

In [8], Frigeri considered the semilinear damped wave equation with an acoustic boundary condition:

$$\begin{cases} u_{tt} + \omega u_t - \Delta u + u + f(u) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \\ \delta_{tt} + \nu \delta_t + \delta = -u_t \quad \text{on} \quad \Gamma \times (0, \infty), \\ \delta_t = \frac{\partial u}{\partial \mathbf{n}} \quad \text{on} \quad \Gamma \times (0, \infty), \end{cases}$$

where $\mathbf{n}$ is the exterior normal to $\Gamma$ and $\omega$, $\nu$ are an interior and a surface damping parameter, respectively. The author proved the existence of global attractors for semilinear damped wave equations with an acoustic boundary condition. Moreover, Ma and Souza [12] considered a non-autonomous wave equation with acoustic boundary condition of the following form:

$$\begin{cases} u_{tt} - \Delta u + \omega u_t + u + f(u) = h, \quad x \in \Omega, \quad t \geq \tau, \\ \delta_{tt} + \nu \delta_t + \delta = -u_t, \quad x \in \Gamma, \quad t \geq \tau, \\ \delta_t = \partial_{\mathbf{n}} u, \quad x \in \Gamma, \quad t \geq \tau \\ u(x, \tau) = u_\tau^0(x), \quad u_t(x, \tau) = u_\tau^1(x), \quad x \in \Omega, \\ \delta(x, \tau) = \delta_\tau^0(x), \quad \delta_t(x, \tau) = \delta_\tau^1(x), \quad x \in \Gamma, \end{cases}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^3$ with regular boundary $\Gamma$ and $\omega, \nu > 0$ are damping coefficients. They also investigated the existence of a pullback attractor and the upper semicontinuity of pullback attractors.

In [11], Liu et al. studied the existence, uniqueness and asymptotic behavior of global solution of the following class of a wave equation with time delay:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) \\ \quad + f(x,u) = h(x), \quad x \in \Omega, \quad t > 0, \\ u(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega, \\ u_t(x, t-\tau) = f_0(x, t-\tau), \quad x \in \Omega, \quad 0 < t < \tau, \end{cases}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with a sufficiently smooth boundary $\partial\Omega$, $f$ and $h$ are external forcing terms, $\mu_1$ and $\mu_2$ are some constants, $\tau > 0$ represents the time delay, $u_0$, $u_1$, $f_0$ are given functions.

Recently, Feng [7] studied the long-time dynamics of a plate equation with memory and time delay of the following form:

$$\begin{cases} u_{tt}(x,t) + \alpha\Delta^2 u(x,t) - \int_{-\infty}^{t} g(t-s)\Delta^2 u(x,s)ds + \mu_1 u_t(x,t) \\ \quad +\mu_2 u_t(x,t-\tau(t)) + f(u) = h(x), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u_t(x,t) = f_0(x,t), \quad x \in \Omega, \quad t \in [-\tau(0), 0), \\ u(x,0) = 0, \quad \Delta u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Moreover, Park [16] considered the following von Karman system:

$$\begin{cases} u_{tt} + \Delta^2 u + a_0 u_t(x,t) + a_1 u_t(x,t-\tau) = [u, F(u)] + g, \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ \Delta^2 F(u) = -[u,u] \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ u = \frac{\partial u}{\partial \nu} = 0, \quad F(u) = \frac{\partial F(u)}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \\ u(0) = u_0, \quad u_t(0) = u_1 \quad \text{on} \quad \Omega, \\ u_t(x,t) = f_0(x,t) \quad \text{for} \quad (x,t) \in \Omega \times (-\tau, 0), \end{cases}$$

where $a_0, a_1$ are real numbers, $\tau > 0$ is time delay, $g \in L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (-\tau, 0))$. The author investigated the long-time dynamics of a von Karman equation with time delay.

Motivated and inspired by the works mentioned above results, we prove the existence of a compact global attractor of the wave equation with time-varying delay and acoustic boundary conditions (1.1) under suitable assumptions. To the best of our knowledge, the long time behavior of a wave equation with time-varying delay and acoustic boundary conditions has not yet been considered. It is presently our concern.

The plan of this paper is as follows. In section2, we present some notations and assumptions needed for our work. Moreover, we recall the preliminaries facts which are used throughout this work. In section 3, we get the main results. The proof of main theorem is given in section 4.

## 2. Preliminaries

In this section, we present some notations and assumptions that we shall use in order to prove our results.

Let $H^1(\Omega)$ be the real sobolev space of first order, $\|\cdot\|$ be a $L^2$-norm and $(\cdot,\cdot)$ be the scalar product in $L^2(\Omega)$, *i.e.*, $(u,v) = \int_\Omega u(x,t)v(x,t)dx$. Also, we mean by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \leq q < \infty$. We denote by $V$ the closure in $H^1(\Omega)$ of $\{u \in C^1(\overline{\Omega}) : u = 0 \quad \text{on} \quad \Gamma_0\}$. Since $\Gamma_0$ has nonempty interior and $\Omega$ is a regular domain, $V = \{u \in H^1(\Omega) : \gamma_0(u) = 0 \quad \text{on} \quad \Gamma_0\}$ is a closed subspace of $H^1(\Omega)$, where $\gamma_0 : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ (see [1]). We define in $V$ the inner product and norm by

$$((u,v)) = \sum_{i=1}^{n} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx, \quad \|u\|^2 = \sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial u}{\partial x_i} \right|^2 dx,$$

which are equivalent to the usual inner product and the norm in $H^1(\Omega)$.

The poincare inequality holds on $V$, *i.e.*, there exists a constant $c_* > 0$ such that for $u \in V$,

$$\|u(t)\|_p^2 \le c_* \|\nabla u(t)\|^2, \quad 2 \le p \le \overline{p}, \tag{2.1}$$

where

$$\overline{p} = \begin{cases} \frac{2n-2}{n-2} & \text{if } n \ge 3, \\ +\infty & \text{if } n = 1, 2. \end{cases} \tag{2.2}$$

The trace map $\gamma_0$ is a continuous function. Then there exists a constant $\widetilde{c}_* > 0$ such that

$$\|\gamma_0(u(t))\|_{\Gamma_1}^2 \le \widetilde{c}_* \|\nabla u(t)\|^2, \quad \forall u \in V. \tag{2.3}$$

Now we define the phase space $\mathcal{H} = H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)$ equipped with the norm

$$\begin{aligned} \|(u(t), u_t(t), \delta(t), \delta_t(t))\|_{\mathcal{H}}^2 = \|\nabla u(t)\|^2 + \|u_t(t)\|^2 + \|\delta(t)\|_{L^2(\Gamma_1)}^2 \\ + \|\delta_t(t)\|_{L^2(\Gamma_1)}^2. \end{aligned}$$

Let us state assumptions on the external forcing terms $f(u(t))$, $h(x)$.

(H1) Concerning the forcing term $f \in C^1(\mathbb{R})$, we assume that

$$f(x,0) = 0, \quad |f(u) - f(v)| \le c_f(1 + |u|^p + |v|^p)|u - v|, \quad \forall u, v \in \mathbb{R}, \tag{2.4}$$

where $c_f > 0$ and

$$0 < p \le \frac{2}{n-2} \quad \text{if} \quad n \ge 3 \quad \text{or} \quad p > 0 \quad \text{if} \quad n = 1, 2. \tag{2.5}$$

In addition, we assume that there exist constants $c_f > 0$ and $\beta \in [0, \frac{5}{8c_*})$ such that

$$\widetilde{f}(u(t)) \ge -\frac{\beta}{2} u^2(t) - c_f \quad \text{and} \quad f(u(t))u(t) - \widetilde{f}(u(t)) \ge -\frac{\beta}{2} u^2(t) - c_f, \quad \forall u \in \mathbb{R}, \tag{2.6}$$

where $\widetilde{f}(u(t)) = \int_0^u f(s)ds$.

(H2) With respect to $h(x)$, we assume that

$$h \in L^2(\Omega). \tag{2.7}$$

(H3) With respect to the delay $\tau(t)$, we assume that there exist two positive constants $\tau_0$ and $\tau_1$ such that

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0, \tag{2.8}$$

and further

$$\tau(t) \in W^{2,\infty}(0,T), \quad \tau'(t) \le d < 1, \quad \forall T, t > 0. \tag{2.9}$$

The existence of solutions for problem (1.1) can be proved by the Fadeo-Galerkin method (see [11,12,14]).

**Theorem 2.1.** Assume that (2.1)-(2.9) hold. Then given initial data $(u_0, u_1, \delta_0, \delta_1, f_0) \in \mathcal{H} \times L^2(\Omega, (-\tau(0), 0))$, problem (1.1) has a unique weak solution $z(t) = (u(t), u_t(t), \delta(t), \delta_t(t))$ satisfying

$$z(t) \in C((0,\infty), \mathcal{H}). \tag{2.10}$$

In addition, if $z^i(t) = (u^i(t), u^i_t(t), \delta^i(t), \delta^i_t(t))$, $i = 1, 2$, are two weak solutions of (1.1), then for any $T > 0$

$$\|z^1(t) - z^2(t)\|^2_{\mathcal{H}} \le e^{C_0 T} \|z^1_0 - z^2_0\|^2_{\mathcal{H}},$$

where $C_0$ is a constant depending on the initial data.

**Remark 2.1.** The uniqueness of problem (1.1) defines the operator $S(t) : \mathcal{H} \to \mathcal{H}$ such that

$$S(t)(u_0, u_1, \delta_0, \delta_1) = (u(t), u_t(t), \delta(t), \delta_t(t)), \quad \forall t > 0,$$

where $(u(t), u_t(t), \delta(t), \delta_t(t))$ is the weak solution corresponding to initial data $(u_0, u_1, \delta_0, \delta_1) \in \mathcal{H}$. It turns out that $S(t)$ satisfies the semigroup properties $S(0) = I$ and $S(t+s) = S(t)S(s)$. Moreover, the continuous dependence on the initial data in $\mathcal{H}$ and the condition (2.10) imply that $S(t)$ is strongly continuous on $\mathcal{H}$. Then the long-time dynamic of problem (1.1) can be studied by the continuous dynamical system $(\mathcal{H}, S(t))$.

The energy functional to problem (1.1) is defined by

$$\begin{aligned} E(t) = &\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\delta_t(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|\delta(t)\|^2_{L^2(\Gamma_1)} \\ &+ \int_\Omega \big(\widetilde{f}(u(t)) - h(x)u(t)\big)dx. \end{aligned}$$

Then we can get the following lemma concerning the energy functional $E(t)$.

**Lemma 2.1.** For a weak solution $(u(t), u_t(t), \delta(t), \delta_t(t)) \in \mathcal{H}$, the energy functional $E(t)$ satisfies that there exists a a constant $\beta_0 > 0$ such that

$$E(t) \ge \beta_0 \big(\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \|\delta(t)\|^2_{L^2(\Gamma_1)} + \|\delta_t(t)\|^2_{L^2(\Gamma_1)}\big) - K, \tag{2.11}$$

where $K = c_f|\Omega| + \frac{\|h\|^2}{c_*\rho}$ and $\rho > 0$.

**Proof.** To prove (2.11), we define

$$\widetilde{E}(t) = E(t) + K,$$

where $K = c_f|\Omega| + \frac{1}{c_* \rho}\|h\|^2$ and $\rho > 0$ is a constant. By using (2.6) and Young's inequality, we obtain

$$\int_\Omega \widetilde{f}(u(t))dx \geq -\frac{\beta c_*}{2}\|\nabla u(t)\|^2 - c_f|\Omega|$$

and for any $\rho > 0$

$$-\int_\Omega h(x)u(t)dx \geq -\frac{\rho}{4}\|\nabla u(t)\|^2 - \frac{1}{c_*\rho}\|h\|^2.$$

Then we get

$$\widetilde{E}(t) \geq \left(\frac{1}{2} - \frac{\beta c_*}{2} - \frac{\rho}{4}\right)\|\nabla u(t)\|^2 + \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\delta_t(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|\delta(t)\|^2_{L^2(\Gamma_1)}.$$

Noticing $\beta \in [0, \frac{1}{c_*})$, taking $\rho > 0$ so small, we derive

$$E(t) \geq \beta_0\big(\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \|\delta_t(t)\|^2_{L^2(\Gamma_1)} + \|\delta(t)\|^2_{L^2(\Gamma_1)}\big) - K.$$

The proof is complete.

## 3. Long-time dynamics

First we recall some fundamentals of the theory of infinite dimensional dynamical systems. They can be found in, for instant, Bain and Vishik [2], Chueshov and Lasiecka [4,5], Hale [10].

A compact set $\mathcal{A} \subset \mathcal{H}$ is a global attractor for a dynamical system $(\mathcal{H}, S(t))$ if it is fully invariant and uniformly attracting, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$ and for every bounded subset $B \subset \mathcal{H}$,

$$\lim_{t\to\infty} \mathrm{dist}_{\mathcal{H}}(S(t)B, \mathcal{A}) = 0,$$

where $\mathrm{dist}_{\mathcal{H}}$ is the Housdorff semidistance in $\mathcal{H}$.

A bounded set $\mathcal{B} \subset \mathcal{H}$ is an absorbing set for $S(t)$ if any bounded set $B \subset \mathcal{H}$, there exists $t_B > 0$ satisfying

$$S(t)B \subset \mathcal{B}, \quad \forall t \geq t_B,$$

which characterizes $S(t)$ as a dissipative semigroup.

A semigroup $S(t)$ is asymptotically smooth in $\mathcal{H}$ if for any bounded positively invariant set $B \subset \mathcal{H}$, there exists a compact set $K \subset \overline{B}$ such that

$$\lim_{t\to+\infty} \mathrm{dist}_{\mathcal{H}}(S(t)B, K) = 0.$$

The following theorem is well-known result (see [2,5,10]).

**Theorem 3.1.** A dissipative dynamical system $(\mathcal{H}, S(t))$ has a compact global attractor if and only if it is asymptotically smooth.

We present here a more recent method by Chueshov and Lasiecka [5] to verify the asymptotic smoothness property.

**Theorem 3.2.** Suppose that for any positively bounded invariant set $B \subset \mathcal{H}$ and for any $\varepsilon > 0$, there exists $T = T(\varepsilon, B)$ such that

$$\|S(t)x - S(T)y\|_{\mathcal{H}} \leq \varepsilon + \phi_T(x, y), \quad \forall x, y \in B,$$

where $\phi_T : B \times B \to \mathbb{R}$ satisfies

$$\liminf_{n\to\infty} \liminf_{m\to\infty} \phi_T(z_n, z_m) = 0 \tag{3.1}$$

for any sequence $\{z_n\}_{n\in\mathbb{N}}$ in $B$. Then $S(t)$ is asymptotically smooth in $\mathcal{H}$.

When a suitable smallness condition on the time-delay feedback is satisfied (*i.e.*, $0 < |\mu_2| < \mu_1$), our main result in this paper is the following.

**Theorem 3.3.** Assume that the hypotheses of Theorem 2.1 and $0 < |\mu_2| < \sqrt{1-d}\mu_1$ hold. Then the dynamical system $(\mathcal{H}, S(t))$ corresponding to the system (1.1) possesses a compact global attractor $\mathcal{A}$.

## 4. Proof of the main result

In order to prove Theorem 3.3, we will apply the abstract results presented in Section 3. The first step is to show that the dynamical system $(\mathcal{H}, S(t))$ is dissipative. The second step is to verify that it is asymptotically smooth. Then the existence of a compact global attractor is guaranteed by Theorem 3.1.

Inspired by [14], the modified energy functional corresponding to the problem (1.1) is given by

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\delta_t(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|\delta(t)\|^2_{L^2(\Gamma_1)} \\ & + \frac{\xi}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{\sigma(s-t)}u_t^2(x,s)dxds + \int_\Omega \big(\widetilde{f}(u(t)) - h(x)u(t)\big)dx, \end{aligned} \tag{4.1}$$

where $\xi > 0$ will be determined later and the constant $\sigma > 0$, as below, has been introduced in [14]:

$$\sigma < \frac{1}{\tau_1}\left|\log\frac{|\mu_2|}{\sqrt{1-d}}\right|.$$

To prove our result, we obtain the following lemmas.

**Lemma 4.1.** Let $(u(t), u_t(t), \delta(t), \delta_t(t))$ be the weak solution of (1.1). Then the energy functional defined by (4.1) satisfies

$$\begin{aligned} E'(t) \leq {} & \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2}\right)\|u_t(t)\|^2 + \left(\frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2e^{\sigma\tau_1}}\right)\|u_t(t-\tau(t))\|^2 \\ & - \frac{\sigma\xi}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{-\sigma(t-s)}u_t^2(x,s)dxds - k\|\delta_t(t)\|^2_{L^2(\Gamma_1)} \\ \leq {} & 0. \end{aligned} \tag{4.2}$$

**Proof.** Differentiating (4.1) and using (1.1), we obtain

$$\begin{aligned}E'(t) = &-\mu_1\|u_t(t)\|^2 - \mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx + \frac{\xi}{2}\|u_t(t)\|^2\\ &- \frac{\xi}{2}e^{-\sigma\tau(t)}\int_\Omega u_t^2(t-\tau(t))(1-\tau'(t))dx\\ &- \frac{\sigma\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\sigma(t-s)}u_t^2(x,s)dxds\\ &- k\|\delta_t(t)\|^2_{L^2(\Gamma_1)}.\end{aligned}$$

By using (2.8) and (2.9), we get

$$\begin{aligned}E'(t) \le &-\mu_1\|u_t(t)\|^2 - \mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx + \frac{\xi}{2}\|u_t(t)\|^2\\ &- \frac{\xi}{2}(1-d)e^{-\sigma\tau_1}\int_\Omega u_t^2(t-\tau(t))dx - \frac{\sigma\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\sigma(t-s)}u_t^2(x,s)dxds\\ &- k\|\delta_t(t)\|^2_{L^2(\Gamma_1)}. \end{aligned} \tag{4.3}$$

It follows from Young's inequality that

$$-\mu_2\int_\Omega u_t(t)u_t(t-\tau(t))dx \le \frac{|\mu_2|}{2\sqrt{1-d}}\|u_t(t)\|^2 + \frac{|\mu_2|\sqrt{1-d}}{2}\int_\Omega u_t^2(t-\tau(t))dx. \tag{4.4}$$

Thus from (4.3) and (4.4), we deduce that

$$\begin{aligned}E'(t) \le &\left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2}\right)\|u_t(t)\|^2 + \left\{\frac{|\mu_2|\sqrt{1-d}}{2}\right.\\ &\left.- \frac{\xi}{2}e^{-\sigma\tau_1}(1-d)\right\}\|u_t^2(t-\tau(t))\|^2 - \frac{\sigma\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{-\sigma(t-s)}u_t^2(x,s)dxds\\ &- k\|\delta_t(t)\|^2_{L^2(\Gamma_1)}.\end{aligned}$$

Notice that $e^{\sigma\tau_1} \to 1^+$ as $\sigma \to 0^+$. Using the continuity of set of real numbers, we can choose $\sigma > 0$ so small that there exists a positive constant $\xi > 0$ such that

$$\frac{e^{\sigma\tau_1}|\mu_2|}{\sqrt{1-d}} < \xi < \mu_1. \tag{4.5}$$

It follows from (4.5) that

$$\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2} < 0, \tag{4.6}$$

and

$$\frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi}{2}e^{-\sigma\tau_1}(1-d) < 0. \tag{4.7}$$

Therefore

$$E'(t) \le 0.$$

The proof is complete.

Now, we define a Lyapunov functional

$$E_\varepsilon(t) = E(t) + \varepsilon\phi(t), \quad \varepsilon > 0, \tag{4.8}$$

where

$$\phi(t) = \int_\Omega u(t)u_t(t)dx + \int_{\Gamma_1} \delta(t)\delta_t(t)d\Gamma + \int_{\Gamma_1} u(t)\delta(t)d\Gamma. \tag{4.9}$$

Then, first we prove there exists $\varepsilon_0 > 0$ such that

$$\frac{1}{2}E(t) - \frac{1}{2}K \le E_\varepsilon(t) \le \frac{3}{2}E(t) + \frac{1}{2}K, \quad \forall\varepsilon \in [0, \varepsilon_0], \tag{4.10}$$

where $K = c_f|\Omega| + \frac{1}{c_*\rho}\|h\|^2$ and $\rho > 0$.

Indeed, from (2.1) we deduce

$$\int_{\Gamma_1} u(t)\delta(t)d\Gamma \le \frac{c_*}{2}\|\nabla u(t)\|^2 + \frac{1}{2}\|\delta(t)\|^2_{L^2(\Gamma_1)}$$

and using (2.11) and(4.9), we can write

$$\begin{aligned}\|\phi(t)\| &\le (1 + c_*)\|(u(t), u_t(t), \delta(t), \delta_t(t))\|^2_{\mathcal{H}} \\ &\le (1 + c_*)\beta_0^{-1}(E(t) + K).\end{aligned}$$

Then, taking $\varepsilon_0 = \frac{1}{2}\beta_0(1 + c_*)^{-1}$, we obtain

$$\varepsilon_0\|\phi(t)\| \le \frac{1}{2}(E(t) + K)$$

for $\varepsilon < \varepsilon_0$ and (4.10) follows.

Next, we shall estimate $\phi'(t)$ as

$$\begin{aligned}\phi'(t) \le &-E(t) - \frac{5 - 8\beta c_*}{16}\|\nabla u(t)\|^2 + \left(\frac{3}{2} + 4\mu_1^2 c_*\right)\|u_t(t)\|^2 \\ &+ \left(16c_* + \frac{3}{2} + \frac{k^2}{2}\right)\|\delta_t(t)\|^2_{L^2(\Gamma_1)} \\ &+ 4|\mu_2|^2 c_* \int_\Omega u_t^2(t - \tau(t))dx + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\sigma(s-t)}u_t^2(x, s)dxds \\ &+ C_f, \quad \forall t \ge 0,\end{aligned} \tag{4.11}$$

where $C_f = c_f|\Omega|$. For this purpose, using (4.9) and (1.1), we have

$$\begin{aligned}
\phi'(t) &= \int_\Omega u_t^2(t)dx + \int_\Omega u(t)u_{tt}(t)dx + \int_{\Gamma_1} \delta_t^2(t)d\Gamma \\
&\quad + \int_{\Gamma_1} \delta(t)\delta_{tt}(t)d\Gamma + \int_{\Gamma_1} u_t(t)\delta(t)d\Gamma + \int_{\Gamma_1} u(t)\delta_t(t)d\Gamma \\
&= \int_\Omega u_t^2(t)dx + \int_\Omega \Big\{ \Delta u(t) - \mu_1 u_t(t) - \mu_2 u_t(t-\tau(t)) - f(u(t)) \\
&\quad + h(x) \Big\} u(t)dx + \int_{\Gamma_1} \delta_t^2(t)d\Gamma + \int_{\Gamma_1} \Big\{ -k\delta_t(t) - \delta(t) - u_t(t) \Big\} \delta(t)d\Gamma \\
&\quad + \int_{\Gamma_1} u_t(t)\delta(t)d\Gamma + \int_{\Gamma_1} u(t)\delta_t(t)d\Gamma \\
&= \int_\Omega u_t^2(t)dx + \int_{\Gamma_1} \frac{\partial u(t)}{\partial \nu} u(t)d\Gamma - \int_\Omega |\nabla u(t)|^2 dx \\
&\quad - \mu_1 \int_\Omega u_t(t)u(t)dx - \mu_2 \int_\Omega u_t(t-\tau(t))u(t)dx \\
&\quad - \int_\Omega f(u(t))u(t)dx + \int_\Omega h(x)u(t)dx + \int_{\Gamma_1} \delta_t^2(t)d\Gamma \\
&\quad - k\int_{\Gamma_1} \delta_t(t)\delta(t)d\Gamma - \int_{\Gamma_1} \delta^2(t)d\Gamma + \int_{\Gamma_1} u(t)\delta_t(t)d\Gamma, \quad \forall t \ge 0.
\end{aligned}$$

Inserting $E(t)$ in (4.1) and using (2.6), it becomes

$$\begin{aligned}
\phi'(t) &\le -E(t) - \frac{1-\beta c_*}{2}\|\nabla u(t)\|^2 + \frac{3}{2}\|u_t(t)\|^2 - \frac{1}{2}\|\delta(t)\|^2_{L^2(\Gamma_1)} \\
&\quad + \frac{3}{2}\|\delta_t(t)\|^2_{L^2(\Gamma_1)} + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\sigma(s-t)} u_t(x,s)dxds \\
&\quad - \mu_1 \int_\Omega u_t(t)u(t)dx - \mu_2 \int_\Omega u_t(t-\tau(t))u(t)dx \\
&\quad - k\int_{\Gamma_1} \delta_t(t)\delta(t)d\Gamma + 2\int_{\Gamma_1} u(t)\delta_t(t)d\Gamma + C_f, \quad \forall t \ge 0.
\end{aligned} \tag{4.12}$$

Using Young's inequalities, we obtain

$$-\mu_1 \int_\Omega u(t)u_t(t)dx \le \frac{1}{16}\int_\Omega |\nabla u(t)|^2 dx + 4\mu_1^2 c_* \int_\Omega |u_t(t)|^2 dx, \tag{4.13}$$

$$-\mu_2 \int_\Omega u_t(t-\tau(t))u(t)dx \le \frac{1}{16}\int_\Omega |\nabla u(t)|^2 dx + 4|\mu_2|^2 c_* \int_\Omega u_t^2(t-\tau(t))dx, \tag{4.14}$$

$$-k\int_{\Gamma_1} \delta(t)\delta_t(t)d\Gamma \le \frac{1}{2}\|\delta(t)\|^2_{L^2(\Gamma_1)} + \frac{k^2}{2}\|\delta_t(t)\|^2_{L^2(\Gamma_1)} \tag{4.15}$$

and

$$2\int_{\Gamma_1} u(t)\delta_t(t)d\Gamma \le \frac{1}{16}\int_\Omega |\nabla u(t)|^2dx + 16c_*\|\delta_t(t)\|^2_{L^2(\Gamma_1)}. \tag{4.16}$$

Combining with (4.12)-(4.16), we get

$$\begin{aligned}\phi'(t) \le & -E(t) - \frac{5-8\beta c_*}{16}\|\nabla u(t)\|^2 + \left(\frac{3}{2}+4\mu_1^2c_*\right)\|u_t(t)\|^2 \\ & + \left(16c_* + \frac{3}{2}+\frac{k^2}{2}\right)\|\delta_t(t)\|^2_{L^2(\Gamma_1)} + 4|\mu_2|^2c_*\int_\Omega u_t^2(t-\tau(t))dx \\ & + \frac{\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\sigma(s-t)}u_t^2(x,s)dxds + C_f, \quad \forall t\ge 0.\end{aligned}$$

The proof is complete.

Furthermore, from (4.2), (4.8) and (4.11), we derive

$$\begin{aligned}E'_\varepsilon(t) \le & -\varepsilon E(t) - \left\{\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2} - \varepsilon\left(\frac{3}{2}+4\mu_1^2c_*\right)\right\}\|u_t(t)\|^2 \\ & - \frac{(5-8\beta c_*)\varepsilon}{16}\|\nabla u(t)\|^2 \\ & - \left(\frac{\xi(1-d)}{2e^{\sigma\tau_1}} - \frac{|\mu_2|\sqrt{1-d}}{2} - 4\varepsilon|\mu_2|^2c_*\right)\int_\Omega u_t^2(t-\tau(t))dx \\ & - \frac{\xi}{2}(\sigma-\varepsilon)\int_{t-\tau(t)}^t\int_\Omega e^{-\sigma(t-s)}u_t^2(x,s)dxds \\ & - \left\{k - \varepsilon\left(16c_*+\frac{3}{2}+\frac{k^2}{2}\right)\right\}\|\delta_t(t)\|^2_{L^2(\Gamma_1)} + \varepsilon C_f, \quad \forall t\ge 0. \end{aligned} \tag{4.17}$$

Here, using (4.6), (4.7) and choosing $\varepsilon > 0$ sufficiently small such that

$$\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2} - \varepsilon\left(\frac{3}{2}+4\mu_1^2c_*\right) > 0,$$

$$\frac{\xi(1-d)}{2e^{\sigma\tau_1}} - \frac{|\mu_2|\sqrt{1-d}}{2} - 4\varepsilon|\mu_2|^2c_* > 0,$$

$$\sigma - \varepsilon > 0,$$

$$k - \varepsilon\left(16c_*+\frac{3}{2}+\frac{k^2}{2}\right) > 0.$$

Since $\beta < \frac{5}{8c_*}$, (4.17) becomes

$$E'_\varepsilon(t) \le -\varepsilon E(t) + \varepsilon C_f.$$

Using (4.10), we have

$$E'_\varepsilon(t) \le -\frac{2\varepsilon}{3}E_\varepsilon(t) + \varepsilon\left(\frac{K}{3}+C_f\right), \quad \forall t\ge 0,$$

where $K = c_f|\Omega| + \frac{\|h\|^2}{c_*\rho}$, $\rho > 0$ and by using Gronwall's inequality,

$$E_\varepsilon(t) \le E_\varepsilon(0)e^{-\frac{2\varepsilon}{3}t} + \frac{3}{2}\left(C_f + \frac{K}{3}\right).$$

Using again (4.10) and taking $\delta_1 = \frac{2\varepsilon}{3}$, we obtain

$$E(t) \le (3E(0) + K)e^{-\delta_1 t} + 3C_f + 2K, \quad \forall t \ge 0.$$

In view of (2.11), we conclude that

$$\|(u(t), u_t(t), \delta(t), \delta_t(t)\|^2_{\mathcal{H}} \le \frac{1}{\beta_0}\left\{(3E(0) + K)e^{-\delta_1 t} + \delta_2\right\},$$

where $\delta_2 = 3(C_f + K)$. This shows that any closed ball $\mathcal{B} = \overline{B}(0, r)$ with $r = \sqrt{\frac{\delta_2}{\beta_0}}$ is a bounded absorbing set of $(\mathcal{H}, S(t))$. The existence of a bounded absorbing set implies that for initial data lying in bounded sets $B \subset \mathcal{H}$, the solutions of problem (1.1) are globally bounded. More precisely, let $(u(t), u_t(t), \delta(t), \delta_t(t))$ be a solution of (1.1) with initial data $(u_0, u_1, \delta_0, \delta_1)$ in a bounded set $B$. Then one has

$$\|(u(t), u_t(t), \delta(t), \delta_t(t))\|_{\mathcal{H}} \le C_B, \quad \forall t \ge 0,$$

where $C_B > 0$ is a constant depending on $B$.

**Lemma 4.2.** Under the hypotheses of Theorem 3.3, given a bounded set $B \subset \mathcal{H}$, let $z^1(t) = (u^1(t), u^1_t(t), \delta^1(t), \delta^1_t(t))$, $z^2(t) = (u^2(t), u^2_t(t), \delta^2(t), \delta^2_t(t))$ be two weak solutions of problem (1.1) such that $z^1(0) = (u^1_0, u^1_1, \delta^1_0, \delta^1_1)$ and $z^2(0) = (u^2_0, u^2_1, \delta^2_0, \delta^2_1)$ are in $B$. Then we have

$$\|z^1(t) - z^2(t)\|^2_{\mathcal{H}} \le C_B e^{-\gamma t} + C_B \int_0^t e^{-\gamma(t-s)}\|w(s)\|^2_{2(p+1)} ds, \tag{4.18}$$

where $w = u^1 - u^2$, $\gamma > 0$ is a small positive constant and $C_B$ is a constant depending on a bounded set $B$.

**Proof.** The proof is also divided into three parts.

First, we denote $w(t) = u^1(t) - u^2(t)$ and $\zeta(t) = \delta^1(t) - \delta^2(t)$. Then $(w(t), \zeta(t))$ is a solution of

$$w_{tt}(t) - \Delta w(t) + \mu_1 w_t(t) + \mu_2 w_t(t - \tau(t)) = f(u^1(t)) - f(u^2(t)), \quad x \in \Omega, t \ge 0, \tag{4.19}$$

$$\zeta_{tt}(t) + k\zeta_t(t) + \zeta(t) = -w_t(t), \quad x \in \Gamma_1, \quad t \ge 0, \tag{4.20}$$

$$\frac{\partial w(t)}{\partial \nu} = \zeta_t(t), \quad x \in \Gamma_1, \quad t \ge 0, \tag{4.21}$$

with initial conditions

$$w(0) = u^1_0 - u^2_0, \quad w_1(0) = u^1_1 - u^2_1, \quad x \in \Omega, \tag{4.22}$$

$$\zeta(0) = \delta^1_0 - \delta^2_0, \quad \zeta_1(0) = \delta^1_1 - \delta^2_1, \quad x \in \Gamma_1. \tag{4.23}$$

We define energy functional of problem (4.19)-(4.23) by

$$
\begin{aligned}
G(t) = {} & \frac{1}{2}\|w_t(t)\|^2 + \frac{1}{2}\|\nabla w(t)\|^2 + \frac{1}{2}\|\zeta_t(t)\|^2_{L^2(\Gamma_1)} + \frac{1}{2}\|\zeta(t)\|^2_{L^2(\Gamma_1)} \\
& + \frac{\xi}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{\sigma(s-t)} w_t^2(x,s)\,dxds. \qquad (4.24)
\end{aligned}
$$

Then, differentiating $G(t)$ and using (4.19)-(4.21), we obtain

$$
\begin{aligned}
G'(t) = {} & -\mu_1\|w_t(t)\|^2 - \mu_2\int_\Omega w_t(t-\tau(t))w_t(t)dx \\
& + \int_\Omega \big(f(u^1(t)) - f(u^2(t))\big)w_t(t)dx - k\|\zeta_t(t)\|^2_{L^2(\Gamma_1)} + \frac{\xi}{2}\|w_t(t)\|^2 \\
& - \frac{\xi}{2}e^{-\sigma\tau(t)}\int_\Omega w_t^2(t-\tau(t))(1-\tau'(t))dx - \frac{\sigma\xi}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{-\sigma(t-s)}w_t^2(x,s)dxds \\
\le {} & \left(\frac{|\mu_2|}{2\sqrt{1-d}} - \mu_1 + \frac{\xi}{2}\right)\|w_t(t)\|^2 + \left\{\frac{|\mu_2|\sqrt{1-d}}{2} - \frac{\xi(1-d)}{2e^{\sigma\tau_1}}\right\}\|w_t(t-\tau(t))\|^2 \\
& - k\|\zeta_t(t)\|^2_{L^2(\Gamma_1)} - \frac{\sigma\xi}{2}\int_{t-\tau(t)}^{t}\int_\Omega e^{-\sigma(t-s)}w_t^2(x,s)dxds \\
& + \int_\Omega \big(f(u^1(t)) - f(u^2(t))\big)w_t(t)dx. \qquad (4.25)
\end{aligned}
$$

Next, we define the perturbed energy

$$G_\varepsilon(t) = G(t) + \varepsilon\psi(t), \qquad (4.26)$$

where $\varepsilon > 0$ will be fixed and

$$\psi(t) = \int_\Omega w(t)w_t(t)dx + \int_{\Gamma_1}\zeta(t)\zeta_t(t)d\Gamma + \int_{\Gamma_1} w(t)\zeta(t)d\Gamma. \qquad (4.27)$$

Note that $|\psi(t)| \le 2(1+c_*)G(t)$. It follows that

$$\frac{1}{2}G(t) \le G_\varepsilon(t) \le \frac{3}{2}G(t), \quad \forall\varepsilon \ge 0, \quad \varepsilon \le \varepsilon_0 = \frac{1}{4(1+c_*)}. \qquad (4.28)$$

In what follows, we show the estimate of $\psi'(t)$. By differentiating the function $\psi(t)$ in (4.27) and using (4.19)-(4.20), we have

$$\begin{aligned}
\psi'(t) = &\int_\Omega |w_t(t)|^2 dx + \int_\Omega w(t)w_{tt}(t)dx + \int_{\Gamma_1} |\zeta_t(t)|^2 d\Gamma + \int_{\Gamma_1} \zeta(t)\zeta_{tt}(t)d\Gamma \\
&+ \int_{\Gamma_1} w_t(t)\zeta(t)d\Gamma + \int_{\Gamma_1} w(t)\zeta_t(t)d\Gamma \\
= &\int_\Omega |w_t(t)|^2 dx + \int_\Omega w(t)\Big\{\Delta w(t) - \mu_1 w_t(t) - \mu_2 w_t(t-\tau(t)) \\
&+ f(u^1(t)) - f(u^2(t))\Big\}dx + \int_{\Gamma_1} |\zeta_t(t)|^2 d\Gamma - \int_{\Gamma_1} \zeta(t)\Big\{k\zeta_t(t) \\
&+ \zeta(t) + w_t(t)\Big\}d\Gamma + \int_{\Gamma_1} w_t(t)\zeta(t)d\Gamma + \int_{\Gamma_1} w(t)\zeta_t(t)d\Gamma \\
= &\int_\Omega |w_t(t)|^2 dx - \int_\Omega |\nabla w(t)|^2 dx \\
&- \mu_1 \int_\Omega w_t(t)w(t)dx - \mu_2 \int_\Omega w_t(t-\tau(t))w(t)dx \\
&+ \int_\Omega \Big(f(u^1(t)) - f(u^2(t))\Big)w(t)dx - k\int_{\Gamma_1} \zeta(t)\zeta_t(t)d\Gamma \\
&- \int_{\Gamma_1} |\zeta(t)|^2 d\Gamma + \int_{\Gamma_1} |\zeta_t(t)|^2 d\Gamma + 2\int_{\Gamma_1} w(t)\zeta_t(t)d\Gamma. \qquad (4.29)
\end{aligned}$$

Using (4.24) and (4.29), we get

$$\begin{aligned}
\psi'(t) = &-G(t) + \frac{3}{2}\int_\Omega |w_t(t)|^2 dx - \frac{1}{2}\|\nabla w(t)\|^2 - \mu_1 \int_\Omega w_t(t)w(t)dx \\
&- \mu_2 \int_\Omega w_t(t-\tau(t))w(t)dx + \int_\Omega \big(f(u^1(t)) - f(u^2(t))\big)w(t)dx \\
&- k\int_{\Gamma_1} \zeta(t)\zeta_t(t)d\Gamma - \frac{1}{2}\int_{\Gamma_1} |\zeta(t)|^2 d\Gamma + \frac{3}{2}\int_{\Gamma_1} |\zeta_t(t)|^2 d\Gamma \\
&+ 2\int_{\Gamma_1} w(t)\zeta_t(t)d\Gamma + \frac{\xi}{2}\int_{t-\tau(t)}^t \int_\Omega e^{\sigma(s-t)} w_t^2(x,s)dxds. \qquad (4.30)
\end{aligned}$$

From Hölder's and Young's inequality, we deduce

$$\left|-\mu_1 \int_\Omega w_t(t)w(t)dx\right| \le \frac{1}{8}\|\nabla w(t)\|^2 + 2\mu_1^2 c_*\|w_t(t)\|^2, \qquad (4.31)$$

$$\left|-\mu_2 \int_\Omega w_t(t-\tau(t))w(t)dx\right| \le \frac{1}{8}\|\nabla w(t)\|^2 + 2|\mu_2|^2 c_* \int_\Omega w_t^2(t-\tau(t))dx, \qquad (4.32)$$

$$\left|\int_{\Gamma_1} w(t)\zeta_t(t)d\Gamma\right| \le \frac{1}{4}\|\nabla w(t)\|^2 dx + \widetilde{c}_* \int_{\Gamma_1} |\zeta_t(t)|^2 d\Gamma, \qquad (4.33)$$

and

$$\left|-k\int_{\Gamma_1}\zeta(t)\zeta_t(t)d\Gamma\right|\le\frac{k^2}{4}\int_{\Gamma_1}|\zeta_t(t)|^2d\Gamma+\int_{\Gamma_1}|\zeta(t)|^2d\Gamma. \tag{4.34}$$

Thus, from (4.30)-(4.34), we arrive at

$$\begin{aligned}\psi'(t)\le&-G(t)+\left(\frac{3}{2}+2\mu_1^2c_*\right)\|w_t(t)\|^2+2|\mu_2|^2c_*\int_\Omega w_t^2(t-\tau(t))dx\\&+\int_\Omega\big(f(u^1(t))-f(u^2(t))\big)w(t)dx+\left(\frac{3}{2}+2\widetilde{c}_*+\frac{k^2}{4}\right)\|\zeta_t(t)\|^2_{L^2(\Gamma_1)}\\&+\frac{\xi}{2}\int_{t-\tau(t)}^t\int_\Omega e^{\sigma(s-t)}w_t^2(x,s)dxds.\end{aligned}\tag{4.35}$$

Consequently, from (4.25), (4.26) and (4.35), we get

$$\begin{aligned}G'_\varepsilon(t)\le&-\varepsilon G(t)-\left\{\mu_1-\frac{|\mu_2|}{2\sqrt{1-d}}-\frac{\xi}{2}-\varepsilon\left(\frac{3}{2}+2\mu_1^2c_*\right)\right\}\|w_t(t)\|^2\\&-\left(\frac{\xi(1-d)}{2e^{\sigma\tau_1}}-\frac{|\mu_2|\sqrt{1-d}}{2}-2\varepsilon|\mu_2|^2c_*\right)\int_\Omega w_t^2(t-\tau(t))dx\\&-\left\{k-\varepsilon\left(\frac{3}{2}+2\widetilde{c}_*+\frac{k^2}{4}\right)\right\}\|\zeta_t(t)\|^2_{L^2(\Gamma_1)}\\&-\frac{\xi}{2}(\sigma-\varepsilon)\int_{t-\tau(t)}^t\int_\Omega e^{-\sigma(t-s)}w_t^2(x,s)dxds\\&+\int_\Omega\big(f(u^1(t))-f(u^2(t))\big)w_t(t)dx\\&+\varepsilon\int_\Omega\big(f(u^1(t))-f(u^2(t))\big)w(t)dx.\end{aligned}\tag{4.36}$$

Furthermore, since $\frac{p}{2(p+1)}+\frac{1}{2(p+1)}+\frac{1}{2}=1$, from the Hölder's and Young's inequality, (2.4) and (2.5), there exists a constant $C_B$ (it may be different from line to line) such that

$$\begin{aligned}&\left|\int_\Omega\big(f(u^1(t))-f(u^2(t))\big)w_t(t)dx\right|\\&\le c_f\int_\Omega(1+|u^1(t)|^p+|u^2(t)|^p)|w(t)||w_t(t)|dx\\&\le c_f\left(|\Omega|^{\frac{p}{2(p+1)}}+\|u^1(t)\|^p_{2(p+1)}+\|u^2(t)\|^p_{2(p+1)}\right)\|w(t)\|_{2(p+1)}\|w_t(t)\|\\&\le C_B\|w(t)\|_{2(p+1)}\|w_t(t)\|\\&\le\frac{\varepsilon}{2}\|w_t(t)\|^2+C_B\|w(t)\|^2_{2(p+1)}.\end{aligned}\tag{4.37}$$

Since $L^{2(p+1)}(\Omega) \hookrightarrow L^2(\Omega)$, similar to (4.37), we deduce

$$\int_\Omega \big(f(u^1(t)) - f(u^2(t))\big)w(t)dx \le C_B\|w(t)\|^2_{2(p+1)}. \tag{4.38}$$

Combining with (4.36)-(4.38), we obtain

$$\begin{aligned}G'_\varepsilon(t) \le & -\varepsilon G(t) - \left\{\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2} - \varepsilon(2+2\mu_1^2 c_*)\right\}\|w_t(t)\|^2 \\ & - \left(\frac{\xi(1-d)}{2e^{\sigma\tau_1}} - \frac{|\mu_2|\sqrt{1-d}}{2} - 2\varepsilon|\mu_2|^2 c_*\right)\int_\Omega w_t^2(t-\tau(t))dx \\ & - \left\{k - \varepsilon\left(\frac{3}{2} + 2\tilde{c}_* + \frac{k}{4}\right)\right\}\|\zeta_t(t)\|^2_{L^2(\Gamma_1)} \\ & - \frac{\xi}{2}(\sigma-\varepsilon)\int_{t-\tau(t)}^t\int_\Omega e^{-\sigma(t-s)}w_t^2(x,s)dxds + C_B\|w(t)\|^2_{2(p+1)}.\end{aligned}$$

At this point, using (4.6), (4.7) and choosing $\varepsilon$ sufficiently small such that

$$\mu_1 - \frac{|\mu_2|}{2\sqrt{1-d}} - \frac{\xi}{2} - \varepsilon(2+2\mu_1^2c_*) > 0,$$

$$\frac{\xi(1-d)}{2e^{\sigma\tau_1}} - \frac{|\mu_2|\sqrt{1-d}}{2} - 2\varepsilon|\mu_2|^2c_* > 0,$$

$$k - \varepsilon\left(\frac{3}{2} + 2\tilde{c}_* + \frac{k}{4}\right) > 0,$$

and

$$\sigma - \varepsilon > 0,$$

then we have

$$G'_\varepsilon(t) \le -\varepsilon G(t) + C_B\|w(t)\|^2_{2(p+1)}, \quad \forall t \ge 0, \quad \forall \varepsilon > 0. \tag{4.39}$$

Using (4.28) and (4.39), we have

$$G'_\varepsilon(t) \le -\frac{2\varepsilon}{3}G_\varepsilon(t) + C_B\|w(t)\|^2_{2(p+1)}, \quad \forall t \ge 0, \quad \forall \varepsilon > 0.$$

By using of Gronwall's inequality, we deduce

$$G_\varepsilon(t) \le G_\varepsilon(0)e^{-\frac{2\varepsilon}{3}t} + C_B\int_0^t e^{-\frac{2\varepsilon}{3}(t-s)}\|w(s)\|^2_{2(p+1)}ds.$$

Then, taking $\gamma = \frac{2\varepsilon}{3}$ and using again (4.28) to replace $G_\varepsilon(t)$ by $G(t)$, we get

$$G(t) \le 3G(0)e^{-\gamma t} + 2C_B\int_0^t e^{-\gamma(t-s)}\|w(s)\|^2_{2(p+1)}ds.$$

By the definition (4.24), we obtain (4.18).

**Lemma 4.3.** Under the hypotheses of Theorem 3.3, the dynamical system $(\mathcal{H}, S(t))$ corresponding to problem (1.1) is asymptotically smooth.

**Proof.** We apply Theorem 3.2. Let $B$ be a bounded positively invariant subset of $\mathcal{H}$ with respect to $S(t)$. For initial data $z_0^1$, $z_0^2$ in set $B$, we write

$$S(t)z_0^i = (u^i(t), u_t^i(t), \delta^i(t), \delta_t^i(t)), \quad i = 1, 2.$$

Given $\varepsilon > 0$, we choose $T$ sufficiently large such that $C_B e^{-\gamma T} < \varepsilon$, where $C_B$ is given in Lemma 4.2. We claim that there exists a constant $C_{BT} > 0$ such that

$$\|S(T)z_0^1 - S(T)z_0^2\|_{\mathcal{H}} \le \varepsilon + \phi_T(z_0^1, z_0^2), \quad \forall z_0^1, z_0^2 \in B, \tag{4.40}$$

where

$$\phi_T(z_0^1, z_0^2) \le C_{BT}\left(\int_0^T \|u^1(s) - u^2(s)\|^{\vartheta} ds\right)^{\frac{1}{4}} \tag{4.41}$$

for some constant $\vartheta > 0$.

Indeed, applying Gagliardo-Nirenberg inequality we get

$$\|u^1(t) - u^2(t)\|_{2(p+1)} \le C_\theta \|\nabla u^1(t) - \nabla u^2(t)\|^{\theta} \|u^1(t) - u^2(t)\|^{1-\theta}$$

with $\theta = \frac{n}{2}(1 - \frac{1}{p+1})$. Then we cam rewrite (4.18) as

$$\|z^1(t) - z^2(t)\|_{\mathcal{H}}^2 \le C_B e^{-\gamma t} + C_B\left(\int_0^T (\|\nabla u^1(s)\| + \|\nabla u^2(s)\|)^{4\theta} ds\right)^{\frac{1}{2}}$$
$$\times \left(\int_0^T \|u^1(s) - u^2(s)\|^{4(1-\theta)} ds\right)^{\frac{1}{2}}$$

for $t < T$. Since $u^1, u^2 \in L^\infty_{\text{loc}}(0, \infty)$, we get that there exists $C_{BT} > 0$ such that

$$\|z^1(T) - z^2(T)\|_{\mathcal{H}} \le C_B e^{-\frac{\gamma}{2}T} + C_{BT}\left(\int_0^T \|u^1(t) - u^2(t)\|^{4(1-\theta)} ds\right)^{\frac{1}{4}},$$

which implies that (4.40) and (4.41) hold.

It remains to show that $\phi_T$ satisfies (3.1).

Indeed, given a sequence of initial data $(z_0^n)$ in $B$, we denote $S(t)(z_0^n) = (u^n(t), u_t^n(t), \delta^n(t), \delta_t^n(t)) \in H^1(\Omega) \times L^2(\Omega) \times L^2(\Gamma_1) \times L^2(\Gamma_1)$, $T > 0$, then from the compact embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the Aubin's lemma implies that there exists a subsequence $(u^{n_k})$ that converges strongly in $C([0, T], L^2(\Omega))$. Therefore we see that

$$\lim_{k\to\infty} \lim_{l\to\infty} \int_0^T \|u^{n_k}(s) - u^{n_l}(s)\|^{\vartheta} ds = 0,$$

which shows (3.1) holds. The asymptotically smoothness property of $(\mathcal{H}, S(t))$ follows Theorem 3.2.

**Proof of Theorem 3.3.**

Lemma 4.3 imply that $(\mathcal{H}, S(t))$ is an asymptotically smooth dissipative dynamical system. Then the existence of a compact global attractor to problem (1.1) follows from Theorem 3.1.

## References


[1] R. Adams and J. J. F. Fournier, *Sobolev Spaces*, 2nd edition, Elsevier, Academic Press, Amsterdam, 2003.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, Studies in Mathematics and its Application, Vol. 25, North-Holland, Amaterdam, 1992.

[3] J. T. Beale and S. I. Rosenerans, *Acoustic boundary conditions*, Bull. Amer. Math. Soc. **80**(1974), 1276–1278.

[4] I. Chueshov and I. Lasiecka, *Long-Time Behavier of Second Order Evolution Equations with Nonlinear Damping*, American Mathematical Society, Province, Vol. 195, 2008.

[5] I. Chueshov and I. Lasiecka, *Von Karman Evolution Equations: Well-Posedness and Long-Time Dynamics*, Springer Monographs in Mathematics, Springer, New York, 2010.

[6] R. Datko, J. Lagnese and M. P. Polis, *An example on the effect of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24**(1)(1986), 152–156.

[7] B. Feng, *Long-time dynamics of a plate equation with memory and time delay*, Bull. Braz. Math. Soc. New series, **49**(2)(2018), 395–418.

[8] S. Frigeri, *Attractors for semilinear damped wave equations with an acoustic boundary condition*, J. Evol. Equ. **10**(2010), 29–58.

[9] A. Guesmia, *Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay*, IMA J. Math. Control Inform. **30**(4)(2013), 507–526.

[10] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs, American Mathematical Society, Province, Vol. 25, 1988.

[11] G. Liu, H. Yue and H. Zhang, *Long-time behavior for a wave equation with time delay*, Taiwanese J. Mathematics, **21**(1)(2017), 107–129.

[12] T. F. Ma and T. M. Souza, *Pullback dynamics of nonautonomous wave equation with acoustic boundary condition*, Differential Integral Equations, **30**(5-6)(2017), 443–462.

[13] P. M. Morse and K. U. Ingard, *Theoretical Acoustics*, McGrow-Hill, New York, 1968.

[14] S. Nicaise and C. Pignotti, *Interior feedback stabilization of wave equations with time delay*, Electron. J. Differential Equations, **2011**(41)(2011), 1-20.

[15] S. Nicaise and C. Pignotti, *Exponential stability of absract evolution equations with time delay*, J. Evol. Equ. **15**(1)(2015), 107–129.

[16] S. H. Park, *Long-time behavior for suspension bridge equations with time delay*, Z. Angew. Math. Phys. **69**(2)(2018), 1–12.