Reproducing kernel approach for numerical solutions of fuzzy fractional initial value problems under the Mittag-Leffler kernel differential operator

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Abstract. In this research study, fuzzy fractional differential equations in presence of the Atangana-Baleanu-Caputo differential operators are analytically and numerically treated using extended reproducing Kernel Hilbert space technique. With the utilization of a fuzzy strongly generalized differentiability form, a new fuzzy characterization theorem beside two fuzzy fractional solutions is constructed and computed. To besetment the attitude of fuzzy fractional numerical solutions; analysis of convergence and conduct of error beyond the reproducing kernel theory are explored and debated. In this tendency, three computational algorithms and modern trends in terms of analytic and numerical solutions are propagated. Meanwhile, the dynamical characteristics and mechanical features of these fuzzy fractional solutions are demonstrated and studied during two applications via three-dimensional graphs and tabulated numerical values. In the end, highlights and future suggested research work are eluded.

Keywords: Fuzzy ABC fractional derivative; Fuzzy ABC FFIVP; Characterization theorem; Fuzzy ABC solution; Numerical analytical RKHSM

Abbreviations: FFDE: fuzzy fractional differential equations; FFIVP: fuzzy fractional initial value problem; FDE: fractional differential equation; ABC: Atangana-Baleanu-Caputo; FSGD: fuzzy strongly generalized differentiable; RKHSM: reproducing kernel Hilbert space method

1 Preface and Show

The topic of FFDEs is frequently used as models to describe uncertain complex dynamic processes and uncertain physical phenomenon in various branches of engineering sciences and applied mathematical sciences [1-5]. To understand comprehensively the mechanism of uncertain physical phenomena defined by fuzzy fractional calculus; we must have to determine their analysis and theories in the fuzzy setting. Anyhow, seeking the solutions of FFDEs has now become a blistering subject in modern uncertain science research due to its large application fields. Recently, several approaches have been made for defining and solving FFDEs by a diverse group of scientists depending on Riemann-Liouville, Caputo-Liouville, or conformable issues [6-10]. The most common weakness among those qualifiers is collected in singularity, nonlocality, or limit entity. Now, no one dismisses that, the ultimate normal definition ought to hail from the real-world miracles that have been formulated from uncertain fractional dynamic patterns. To transact of these reversals, a novel version of FFDEs instituted on fuzzy ABC fractional derivative is used to build and formulate new concretes fuzzy mathematical concepts. This new fuzzy fractional ABC derivative seems to be liberating of singularity, nonlocality, or limit entity; because the kernel function depends on the wilderness exponential decay which makes FFDEs more pragmatic in formulating various uncertain physical models [11-21].

In this manuscript, a new kind of FFDEs derivable from ABC fractional calculus theory is constructed based on a new extended type of FSGD. After that, the RKHSM is developed for the first time in the fuzzy ABC setting to finding numerical solutions for such fuzzy differential issues. To justify more, those analyses appointed the discussions on the following underlying FFIVP:

$$\begin{cases} {}^{ABC}_{\ \ 0}\mathcal{D}^{\alpha}_{t} x(t) = f\left(t, x(t)\right), \\ x(0) = \mathcal{U}. \end{cases} \tag{1}$$

In this scope, we will symbolize the following underlying icons: $\mathcal{T} \coloneqq [0,1]$, $t \in \mathcal{T}$, $\mathcal{U} \in \mathcal{F}(\mathbb{R})$, $\alpha \in (0,1]$, $\mathcal{F}(\mathbb{R})$ the set of fuzzy numbers on $\mathbb{R}$, and ${}^{ABC}_{\ \ 0}\mathcal{D}^{\alpha} x(t)$ the fuzzy ABC fractional derivative of x in t over $\mathcal{T}$. Whilst,

$$\begin{cases} x: \mathcal{T} \to \mathcal{F}(\mathbb{R}), \\ f: \mathcal{T} \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}). \end{cases} \tag{2}$$

A quick overview of the reproducing kernel results can be viewed from [22-24] and an overview of its application fields can be collected from [25-41]. This modern numerical method is based on its structure on pointwise evaluation, successive approximations, and the Green functions approach. The RKHSM major territory topic is in numerical simulation of multidimensional problems engender in applied mathematics and engineering researches. Anyhow, properties, characteristics, results on the RKHSM can be viewed and collected with huge acquaintances from [22-41].

In contract of the preface and show, the enduring sections are synopsized sequentially as follows: the principal results: fuzzy calculus and ABC fractional calculus; fuzzy ABC fractional derivative: rules and formulation; fuzzy ABC FFDEs: structures and tools; requirement tools of the RKHSM: appropriate spaces and independency; representation of fuzzy ABC solutions: fuzzy ABC analytical and numerical solutions; convergence and error frameworks: ensuring and exists of fuzzy ABC solutions; algorithms and packages: construct fuzzy ABC solutions and Mathematica software; fuzzy applications on ABC FFIVPs: fuzzy ABC resistance-inductance circuit and fuzzy ABC FFIVP with fuzzy forcing term; results, analysis, and talks: tables and figures; ultimately, highlight and future research.

2 Principal Results

The contents of this segment are portioned into two slices; the first one is about the fuzzy calculus in its traditional form, whilst the results of the crisp ABC fractional approach are the tenor of the last one. The theoretical results about the fuzzy calculus in its integral form can be viewed from [42-46], whilst, the ABC tactic in its crisp theory is examined upon with its certain properties heavily and strongly in the latest times as viewed from [11-21].

Principally, a fuzzy set $\mathcal{U}$ in $\mathbb{R}$ can be distinguished by its membership task as $\mathcal{U}: \mathrm{X} \subset \mathbb{R} \to \mathcal{I}: [0,1]$. Fundamental analytic properties of fuzzy sets such as convexity, upper semicontinuous, normal, and boundedness support can be collected in detail from [42]. The r-cut impersonation play an important and fundamental role in the fuzzy analysis approach, anyhow, $\forall r \in \mathcal{I} - \{0\}$, set $[\mathcal{U}]^{r} = \{s \in \mathbb{R} | \mathcal{U}(s) \geq r\}$ and $[\mathcal{U}]^{0} = \overline{\{s \in \mathbb{R} | \mathcal{U}(s) > 0\}}$. So, $\mathcal{U} \in \mathcal{F}(\mathbb{R})$ iff $[\mathcal{U}]^{1} \neq \phi$ and $[\mathcal{U}]^{r}$ is convex compact in $\mathbb{R}$ [43]. Certainly, if $\mathcal{U} \in \mathcal{F}(\mathbb{R})$, then $[\mathcal{U}]^{r} = [\mathcal{U}_1(r), \mathcal{U}_2(r)]$ providing $\mathcal{U}_1(r) = \min\{s | s \in [\mathcal{U}]^{r}\}$ and $\mathcal{U}_2(r) = \max\{s | s \in [\mathcal{U}]^{r}\}$. Hither, $[\mathcal{U}]^{r}$ denotes the r-cut impersonation of $\mathcal{U}$ and $\mathcal{U}_{1r}$ and $\mathcal{U}_{2r}$ composes to $\mathcal{U}_1(r)$ and $\mathcal{U}_2(r)$, with one another. The space of all crisp differentiable functions on $\mathcal{T}$ is symbolized by $\mathcal{D}^{\mathbb{R}}(\mathcal{T})$.

Definition 1 [43] A fuzzy number $\mathcal{U}$ is a fuzzy subset in $\mathbb{R}$ with normal, convex, and upper semicontinuous with bounded support.

Theorem 1 [43] Suppose that $\mathcal{U}_{1,2}: \mathcal{I} \to \mathbb{R}$ satisfy the following underlying requirements:

i. $\mathcal{U}_1$ nondecreasing bounded and $\mathcal{U}_2$ nonincreasing bounded,
ii. $\lim_{r \to h^-} \mathcal{U}_{(1,2)r} = \mathcal{U}_{(1,2)h}$ and $\lim_{\alpha \to 0^+} \mathcal{U}_{(1,2)r} = \mathcal{U}_{(1,2)0}$,
iii. $\mathcal{U}_{11} \leq \mathcal{U}_{21}$.

Then, $\mathcal{U}: \mathbb{R} \to \mathcal{I}$ constructed as $\mathcal{U}(s) = \sup\left\{r \middle| \mathcal{P}_{1(r)} \leq s \leq \mathcal{P}_{2(r)}\right\}$ belong to $\mathcal{F}(\mathbb{R})$ with impersonation $[\mathcal{U}_{1r}, \mathcal{U}_{2r}]$. Indeed, if $\mathcal{U}_{1,2}: \mathcal{I} \to \mathbb{R}$ belong to $\mathcal{F}(\mathbb{R})$ with impersonation $[\mathcal{U}_{1r}, \mathcal{U}_{1r}]$, then $\mathcal{U}_{1,2}$ are satisfies the aforesaid conditions.

Definition 2 [42] A mappnig $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$ is continuous at $t^* \in \mathcal{T}$, if $\forall \gamma > 0$ and $\forall t \in \mathcal{T}$, $\exists \delta > 0$ such that $\mathcal{D}_{\hbar}\left(x(t), x(t^*)\right) < \gamma$ whenever $|t - t^*| < \delta$, where $\mathcal{D}_{\hbar}$ is the Hausdorff space on $\mathcal{F}(\mathbb{R})$ and can viewed as

$$\begin{cases} \mathcal{D}_{\hbar}: \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathbb{R}^{+}, \\ \mathcal{D}_{\hbar}(\mathcal{U}, \mathcal{V}) = \sup_{r \in \mathcal{I}} \max\{|\mathcal{U}_{1r} - \mathcal{V}_{1r}|, |\mathcal{U}_{2r} - \mathcal{V}_{2r}|\}. \end{cases} \tag{3}$$

Indeed, x is continuous on $\mathcal{T}$ if it is continuous $\forall t \in \mathcal{T}$.

Let $\mathcal{U}, \mathcal{V} \in \mathcal{F}(\mathbb{R})$, if $\exists \mathcal{W} \in \mathcal{F}(\mathbb{R})$ satisfies $\mathcal{U} = \mathcal{V} + \mathcal{W}$, then $\mathcal{W}$ invite as $\mathcal{H}$-difference of $(\mathcal{U}, \mathcal{V})$ and denoted by $\mathcal{U} \ominus \mathcal{V}$. Hither, $\ominus$ attitudes constantly for $\mathcal{H}$-difference. Indeed, $\mathcal{U} \ominus \mathcal{V} \neq \mathcal{U} + (-1)\mathcal{V} = \mathcal{U} - \mathcal{V}$ and if $\mathcal{U} \ominus \mathcal{V}$ exists, then $[\mathcal{U} \ominus \mathcal{V}]^{r} = [\mathcal{U}_{1r} - \mathcal{V}_{1r}, \mathcal{U}_{2r} - \mathcal{V}_{2r}]$.

Definition 3 [44] A mapping $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$ is FSGD at $t \in \mathcal{T}$, if $\exists \mathcal{D}_t x(t) \in \mathcal{F}(\mathbb{R})$ such that one of the following underlying requirements is achieved:

i. $\forall \gamma > 0$ small enough; the $\mathcal{H}$-differences $x(t+\gamma) \ominus x(t)$ and $x(t) \ominus x(t-\gamma)$ exist together

$$\begin{aligned} \mathcal{D}_t x(t) &= \lim_{\gamma \to 0^+} \frac{x(t+\gamma) \ominus x(t)}{\gamma} \\ &= \lim_{\gamma \to 0^+} \frac{x(t) \ominus x(t-\gamma)}{\gamma}. \end{aligned} \tag{4}$$

ii. $\forall \gamma > 0$ small enough; the $\mathcal{H}$-differences $x(t) \ominus x(t+\gamma)$ and $x(t-\gamma) \ominus x(t)$ exist together

$$\begin{aligned} \mathcal{D}_t x(t) &= \lim_{\gamma \to 0^+} \frac{x(t) \ominus x(t+\gamma)}{-\gamma} \\ &= \lim_{\gamma \to 0^+} \frac{x(t-\gamma) \ominus x(t)}{-\gamma}. \end{aligned} \tag{5}$$

In Definition 3; $\lim_{\gamma \to 0^+}(\cdot)$ is considered in $(\mathcal{F}(\mathbb{R}), \mathcal{D}_{\hbar})$ and at $\delta(\mathcal{T})$ we assume unidirectional derivatives. Indeed, x is differentiable on $\mathcal{T}$ as long as x is differentiable $\forall t \in \mathcal{T}$.

Definition 4 [44] For a map $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$, the following underlying is achieved:

i. x is called (1)-fuzzy differentiable on $\mathcal{T}$ if x is differentiable in status (i) of Definition 3 and its related derivative symbolized as $\mathcal{D}_t^1 x(t)$.

ii. x is called (2)-fuzzy differentiable on $\mathcal{T}$ if x is differentiable in status (ii) of Definition 3 and its related derivative symbolized as $\mathcal{D}_t^2 x(t)$.

Theorem 2 [44] For a map $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$ the following underlying is achieved:

i. If x is (1)-fuzzy differentiable on $\mathcal{T}$, then $x_{1r}, x_{2r} \in \mathcal{D}^{\mathbb{R}}(\mathcal{T})$ together
$$[\mathcal{D}_t^1 x(t)]^{r} = [x_{1r}'(t), x_{2r}'(t)]. \tag{6}$$

ii. If x is (2)-fuzzy differentiable on $\mathcal{T}$, then $x_{1r}, x_{2r} \in \mathcal{D}^{\mathbb{R}}(\mathcal{T})$ together
$$[\mathcal{D}_t^2 x(t)]^{r} = [x_{2r}'(t), x_{1r}'(t)]. \tag{7}$$

The space of Sobolev of first order on $\mathcal{T}$ of a map $x: \mathcal{T} \to \mathbb{R}$ is defined insomuch as $\mathcal{S}(\mathcal{T}) = \{x \in L^2(\mathcal{T}): x, x' \in L^2(\mathcal{T})\}$. The function of Gösta-Leffler in one parameter can be expanded as $\mathcal{L}_\alpha(t) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} t^n, \alpha > 0, t \in \mathbb{R}$.

Definition 5 [11] Let $x: \mathcal{T} \to \mathbb{R}$, $x \in \mathcal{S}(\mathcal{T})$, and $\alpha \in (0,1]$. Then the crisp ABC fractional derivative of order α of x at the base node $\mathrm{t} = 0$ is defined as

$${}^{ABC}_{0}\mathcal{D}_t^{\alpha} x(t) = \frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t x'(s) \mathcal{L}_\alpha\left(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\right) ds. \tag{8}$$

As long as $\mathcal{N}(\alpha)$ is the normalizing ABC function and is erected as $\mathcal{N}(0) = \mathcal{N}(1) = 1$. Hither, $\mathcal{N}(\alpha)$ is fixed accordingly to $\mathcal{N}(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$. More theoretical results on normalizing the ABC function can be viewed in [21].

3 Fuzzy ABC Fractional Derivative

To describe the fundamental steps in the fuzzy ABC approach; we firstly should present the body mathematical structure of the fuzzy ABC fractional derivative. After that, based on FSGD equivalent statements for fuzzy ABC fractional derivative are utilized.

The FSGD allows us not to lose the possible fuzzy ABC solutions when solving FFDEs in ABC emotion. In the remains of this analysis, we will refer to the following characters: $\mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T})$ the space of all fuzzy continuous functions on $\mathcal{T}$, $\mathcal{L}^{\mathcal{F}(\mathbb{R})}(\mathcal{T})$ the space of all fuzzy integrable functions on $\mathcal{T}$, and $\mathcal{D}_\alpha^{\mathbb{R}}(\mathcal{T})$ the space of all crisp ABC α-differentiable functions on $\mathcal{T}$.

Definition 6 Let $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$, $x, x' \in \mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T}) \cap \mathcal{L}^{\mathcal{F}(\mathbb{R})}(\mathcal{T})$, and $\alpha \in (0,1]$. Then the fuzzy ABC fractional derivative of x at the base node $\mathrm{t} = 0$, symbolize by ${}^{ABC}_{0}\mathcal{D}_t^{\alpha} x(t)$, is defined as

$$ {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x(t) = \frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t \mathcal{D}_s x(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds. \tag{9} $$

Definition 7 For a map ${}^{ABC}_{0}\mathcal{D}_t^{\alpha} x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$; we say that x is $\alpha(1)$-fuzzy ABC fractional differentiable when x is (1)-fuzzy differentiable in view of Eq. (4) and x is $\alpha(2)$-fuzzy ABC fractional differentiable when x is (2)-fuzzy differentiable in view of Eq. (5).

Theorem 3 Let $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$, $x, {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x \in \mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T}) \cap \mathcal{L}^{\mathcal{F}(\mathbb{R})}(\mathcal{T})$, and $\alpha \in (0,1]$. Then the following underlying is achieved:

i. If x is (1)-fuzzy differentiable on $\mathcal{T}$, then we have $\alpha(1)$-fuzzy ABC fractional derivative jointly with $x_{1r}, x_{2r} \in \mathcal{D}_\alpha^{\mathbb{R}}(\mathcal{T})$ and
$$ \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha(1)} x(t) \right]^r = \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{1r}(t), {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{2r}(t) \right]. \tag{10} $$

ii. If x is (2)-fuzzy differentiable on $\mathcal{T}$, then we have $\alpha(2)$-fuzzy ABC fractional derivative jointly with $x_{1r}, x_{2r} \in \mathcal{D}_\alpha^{\mathbb{R}}(\mathcal{T})$ and
$$ \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha(2)} x(t) \right]^r = \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{2r}(t), {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{1r}(t) \right]. \tag{11} $$

Proof. Remind that, $[\mathcal{D}_t^1 x(t)]^r = [x'_{1r}(t), x'_{2r}(t)]$ and $[\mathcal{D}_t^2 x(t)]^r = [x'_{2r}(t), x'_{1r}(t)]$. Thus, besides of $\mathcal{L}_\alpha(-t), \frac{\mathcal{N}(\alpha)}{1-\alpha} \geq 0$ in the zone of $\forall t \in \mathcal{W}$, $\forall r \in \mathcal{I}$, and $\forall \alpha \in (0,1]$, we view status (i) as

$$ \begin{aligned} \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha(1)} x(t) \right]^r &= \left[\frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t \mathcal{D}_s^1 x(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds \right]^r \\ &= \left[\frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t x'_{1r}(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds, \frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t x'_{2r}(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds \right] \\ &= \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{1r}(t), {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{2r}(t) \right]. \end{aligned} \tag{12} $$

Likewise, we view status (ii) as

$$ \begin{aligned} \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha(2)} x(t) \right]^r &= \left[\frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t \mathcal{D}_s^2 x(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds \right]^r \\ &= \left[\frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t x'_{2r}(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds, \frac{\mathcal{N}(\alpha)}{1-\alpha} \int_0^t x'_{1r}(s) \mathcal{L}_\alpha \left(-\frac{\alpha}{1-\alpha} (t-s)^\alpha \right) ds \right] \\ &= \left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{2r}(t), {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x_{1r}(t) \right]. \blacksquare \end{aligned} \tag{13} $$

4 Fuzzy ABC FDEs

The formulation of FFDEs by employed fuzzy characterization theorem and FSGD is a very important task in utilizing the numerical analysis methods in the fuzzy emotion. Next, characterization based and theoretical results that are related to $\alpha(1)$- and $\alpha(2)$-fuzzy ABC fractional solutions are derivable and inferred.

Let us foremost focusing on the functional fuzzy structure of ABC FFIVP of the form

$$ \begin{cases} {}^{ABC}_{0}\mathcal{D}_t^{\alpha} x(t) = f(t, x(t)), \\ x(0) = \mathcal{U}. \end{cases} \tag{14} $$

Fundamentally the r-cut impersonation of $\left({}^{ABC}_{0}\mathcal{D}_t^{\alpha} x(t), x(t), \mathcal{U}, f(t, x(t)) \right)$ should be acquired, whilst the most important one is $\left[f(t, x(t)) \right]^r = \left[f_{1r}(t, x_{1r}(t), x_{2r}(t)), f_{2r}(t, x_{1r}(t), x_{2r}(t)) \right]$. In effect, to deal with ABC FFIVP in a realistic approach, one can find the underlying coupled crisp systems of ABC FDE performed by $\alpha(1)$- or $\alpha(2)$-fuzzy ABC fractional derivative, with one another, as follows:

- System of $\alpha(1)$-crisp ABC FDE:
$$ \begin{cases} {}^{ABC}_{0}\mathcal{D}_t^{\alpha(1)} x(t) = f(t, x(t)), \\ x(0) = \mathcal{U}. \end{cases} \tag{15} $$
- System of $\alpha(2)$-crisp ABC FDE:

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(2)} x(t) = f(t, x(t)), \\ x(0) = \mathcal{U}. \end{cases} \tag{16}$$

Definition 8 Let $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$, $x, {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha} x \in \mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T}) \cap \mathcal{L}^{\mathcal{F}(\mathbb{R})}(\mathcal{T})$, and $\alpha \in (0,1]$ and x be such that $\mathcal{D}^{\alpha(1)} x(t)$ or $\mathcal{D}^{\alpha(2)} x(t)$ exists. Then the following underlying is achieved:

i. If $x(t)$ and ${}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x(t)$ satisfy Eq. (15), then $x(t)$ is said a $\alpha(1)$-fuzzy ABC solution of Eq. (14).
ii. If $x(t)$ and ${}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(2)} x(t)$ satisfy Eq. (16), then $x(t)$ is said a $\alpha(2)$-fuzzy ABC solution of Eq. (14).

The subsequent result is about the characterization theorem of ABC FFIVP of Eq. (14). Hither, we can apply this theorem to find or construct numerical or analytical fuzzy ABC solutions in general.

Theorem 4 Let $x, y: \mathcal{T} \to \mathcal{F}(\mathbb{R})$, $x, y, {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha} x, {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha} y \in \mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T}) \cap \mathcal{L}^{\mathcal{F}(\mathbb{R})}(\mathcal{T})$, $f \in \mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T} \times \mathcal{F}(\mathbb{R}))$, and $\alpha \in (0,1]$. Assume that

i. $[f(t, x(t))]^{r} = [f_{1r}(t, x_{1r}(t), x_{2r}(t)), f_{2r}(t, x_{1r}(t), x_{2r}(t))]$,
ii. $f_{(1,2)r}$ are equicontinuous and uniformly bounded on any bounded set,
iii. $\exists \mathcal{K} > 0$ with

$$\begin{aligned} &|f_{(1,2)r}(t, x_{1r}(t), x_{2r}(t)) - f_{(1,2)r}(t, y_{1r}(t), y_{2r}(t))| \\ &\qquad \le \mathcal{K} \max\{|x_{1r}(t) - y_{1r}(t)|, |x_{2r}(t) - y_{2r}(t)|\}. \end{aligned} \tag{17}$$

Then, the following underlying is achieved:

i. For $\alpha(1)$-fuzzy ABC fractional derivative; the ABC FFIVP of Eq. (14) and the system of $\alpha(1)$-crisp ABC FDE of Eq. (15) are equivalent.
ii. For $\alpha(2)$-fuzzy ABC fractional derivative; the ABC FDE of Eq. (14) and the system of $\alpha(2)$-crisp ABC FDE of Eq. (16) are equivalent.

Proof. Herein, we will view status (i) (similar analysis can be exercised for status (ii)). Assume that x is $\alpha(1)$-fuzzy ABC fractional derivative. Condition (ii) on $f_{(1,2)r}$ reveals the continuity of f. The Lipchitzian of Condition (iii) confirms f is Lipchitzian in $(\mathcal{F}(\mathbb{R}), \mathcal{D}_{\hbar})$ as

$$\begin{aligned} \mathcal{D}_{\hbar}\left(f(t, x(t)), f(t, y(t))\right) &= \sup_{r \in \mathcal{I}} \max\{|f_{1r}(t, x(t)) - f_{1r}(t, y(t))|, |f_{2r}(t, x(t)) - f_{2r}(t, y(t))|\} \\ &= \sup_{r \in \mathcal{I}} \max\{|f_{1r}(t, x_{1r}(t), x_{2r}(t)) - f_{1r}(t, y_{1r}(t), y_{2r}(t))|, |f_{2r}(t, x_{1r}(t), x_{2r}(t)) \\ &\quad - f_{2r}(t, y_{1r}(t), y_{2r}(t))|\} \\ &\le \mathcal{K} \sup_{r \in \mathcal{I}} \max\{|x_{1r}(t) - y_{1r}(t)|, |x_{2r}(t) - y_{2r}(t)|\} \\ &= \mathcal{K}\mathcal{D}_{\hbar}(x(t), y(t)). \end{aligned} \tag{18}$$

As $f \in \mathcal{C}^{\mathcal{F}(\mathbb{R})}(\mathcal{T} \times \mathcal{F}(\mathbb{R}))$, Lipchitzian and boundedness, harvest that ABC FFIVP of Eq. (14) has a unique fuzzy ABC solution on $\mathcal{T}$. By $\alpha(1)$-fuzzy ABC fractional derivative; $x_{1r}(t), x_{2r}(t) \in \mathcal{D}^{\mathbb{R}}(\mathcal{T})$. So, $[x_{1r}(t), x_{2r}(t)]$ is a fuzzy r-cut solution of ABC FFIVP of Eq. (14). Conversely, if one has a fuzzy solution $[x_{1r}(t), x_{2r}(t)]$ for ABC FFIVP of Eq. (14). The Lipschitzian in Eq. (17) gives the uniqueness and existence of fuzzy ABC solution $\tilde{x}(t)$. But $\tilde{x}$ is $\alpha(1)$-fuzzy ABC fractional derivative, then $\tilde{x}_{(1,2)r}(t)$ with $[\tilde{x}]^{r} = [\tilde{x}_{1r}(t), \tilde{x}_{2r}(t)]$ is a solution for the system of crisp ABC FDE of Eq. (15). But the solution of the system of crisp ABC FDE of Eq. (15) is unique, we gained $[\tilde{x}(t)]^{r} = [x_{1r}(t), x_{2r}(t)] = [x(t)]^{r}$. As long as, ABC FFIVP of Eq. (14) and the system of crisp ABC FDE of Eq. (15) are equivalent. ■

5 Requirement Tools of the RKHSM

With the growth of technique and science, many phenomena cannot be well utilized by the fuzzy differential problems. For example, various uncertain physical processes own memory and hereditary ownerships and cannot be well-drawn unless if one used FFDEs. Anyhow, when we have faced these challenges; we must build up a new excellent numerical tool. Hither, the RKHSM is presented as a novel solver for ABC FFIVPs.

Firstly, we set the following subordinate important determinants: $[x(t)]_{r} = (x_{1r}(t), x_{2r}(t))$, $[y(t)]_{r} = (y_{1r}(t), y_{2r}(t))$, and $|\mathcal{C}|^{\mathbb{R}}(\mathcal{T})$ to symbolizes the set of absolutely continuous functions on $\mathcal{T}$. A continuous mapping $\mathbb{P}: \mathcal{T} \times \mathcal{T} \to \mathbb{R}$ is a reproducing kernel of $\mathbb{K} \neq \emptyset$ if the following underlying requirements are satisfied, wheresoever $\mathbb{H}$ is a Hilbert space of functions defined on $\mathcal{T}$:

i. $\forall t \in \mathcal{T}: \mathbb{P}(\cdot, t) \in \mathbb{H}$,

ii. $\forall \psi \in \mathbb{H}$ and $\forall t \in \mathcal{T}: \langle \psi(\cdot), \mathbb{P}(\cdot, t) \rangle_{\mathbb{H}} = \psi(t)$.

Definition 9 [6] The space $\mathbb{W}(\mathcal{T})$ is harmonious as

$$\begin{cases} \mathbb{W}(\mathcal{T}) = \left\{ [x(t)]_r^{\mathcal{T}}: x_{(1,2)r} \in |\mathcal{C}|^{\mathbb{R}}(\mathcal{T}), x''_{(1,2)r} \in L^2(\mathcal{T}), \text{and } x_{(1,2)r}(0) = 0 \right\}, \\ \langle [x(t)]_r, [y(t)]_r \rangle_{\mathbb{W}} = \sum_{i=1}^{2} \left(x_{ir}(0) y_{ir}(0) + x'_{ir}(0) y'_{ir}(0) + \int_{\mathcal{T}} x''_{ir}(t) y''_{ir}(t) dt \right), \\ \| [x(t)]_r \|_{\mathcal{W}} = \sqrt{\langle [x(t)]_r, [x(t)]_r \rangle_{\mathbb{W}}}. \end{cases} \tag{19}$$

Definition 10 [6] The space $\mathbb{V}(\mathcal{T})$ is harmonious as

$$\begin{cases} \mathbb{V}(\mathcal{T}) = \left\{ [x(t)]_r: x_{(1,2)r} \in |\mathcal{C}|^{\mathbb{R}}(\mathcal{T}), x'_{(1,2)r} \in L^2(\mathcal{T}) \right\}, \\ \langle [x(t)]_r, [y(t)]_r \rangle_{\mathbb{V}} = \sum_{i=1}^{2} \left(\int_{\mathcal{T}} x_{ir}(t) y_{ir}(t) dt + \int_{\mathcal{T}} x'_{ir}(t) y'_{ir}(t) dt \right), \\ \| [x(t)]_r \|_{\mathbb{V}} = \sqrt{\langle [x(t)]_r, [x(t)]_r \rangle_{\mathbb{V}}}. \end{cases} \tag{20}$$

Theorem 5 [6] The space $\mathbb{W}(\mathcal{T})$ is a complete reproducing kernel with $\overline{\mathbb{G}}_t(s) = \left(\mathbb{G}_t(s), \mathbb{G}_t(s) \right)$ and

$$\mathbb{G}_t(s) = \frac{1}{6} \begin{cases} s(-s^2 + 6t + 3ts), s \leq t, \\ t(-t^2 + 6s + 3ts), s > t. \end{cases} \tag{21}$$

Theorem 6 [6] The space $\mathbb{V}(\mathcal{T})$ is a complete reproducing kernel with $\overline{\mathbb{F}}_t(s) = \left(\mathbb{F}_t(s), \mathbb{F}_t(s) \right)$ and

$$\mathbb{F}_t(s) = \frac{1}{2} \sinh^{-1}(1) \left(\cosh(t + s - 1) + \cosh(|t - s| - 1) \right). \tag{22}$$

When the RKHSM is used, we must partition the interval $\mathcal{T}$ into uniform pieces of t_i. This will be gained the set $\{t_i\}_{i=1}^{\infty}$ which be dense in $\mathcal{T}$. Thither, we seek to cover the $\mathcal{T}$ set as well as the approximation procedure ought to finish up in limited phases.

Theorem 7 In $\mathbb{W}(\mathcal{T})$, the set $\left\{ \overline{\mathbb{G}}_{t_i}(s) \right\}_{i=1}^{\infty}$ is linearly independent.

Proof. If $\{\rho_i\}_{i=1}^{p}$ is picked as $\sum_{i=1}^{p} \rho_i \overline{\mathbb{G}}_{t_i}(s) = 0$ and taking $\hbar_k(s) \in \mathbb{W}(\mathcal{T})$ such that $\hbar_k(s_l) = \delta_{l,k}$ at $l = 1,2, \cdots, p$, then

$$\begin{aligned} 0 &= \left\langle \hbar_k(s), \sum_{i=1}^{p} \sigma_i \overline{\mathbb{G}}_{t_i}(s) \right\rangle_{\mathbb{W}} \\ &= \sum_{i=1}^{p} \rho_i \left\langle \hbar_k(s), \overline{\mathbb{G}}_{t_i}(s) \right\rangle_{\mathbb{W}} \\ &= \sum_{i=1}^{p} \rho_i \hbar_k(s_i) \\ &= \rho_i, \end{aligned} \tag{23}$$

where $k = 1,2, \cdots, p$. This validation that $\left\{ \overline{\mathbb{G}}_{t_i}(s) \right\}_{i=1}^{p}$ is linearly independent $\forall p \geq 1$. ■

6 Representation of Fuzzy ABC Solutions

The fuzzy ABC fractional problem formalism, homogenized fuzzy initial condition, orthogonal function system, completeness, exemplification of fuzzy ABC analytic and numerical solutions in the adequate Hilpert spaces $\mathbb{W}(\mathcal{T})$ and $\mathbb{V}(\mathcal{T})$ are the main significations of the following subordinate part.

In the following execution, we just theorize the $\alpha(1)$-fuzzy ABC solution exclusively (similar execution can be utilized for the $\alpha(1)$-fuzzy ABC solution). Before continuance more, we must homogenize the fuzzy initial condition in Eq. (14) to be in $\mathbb{W}(\mathcal{T})$ as the surrogate $x(t): \to x(t) \ominus \mathcal{U}$. Eventually, we will still use $x(t)$ as

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x(t) = f(t, x(t)), \\ x(0) = 0. \end{cases} \tag{24}$$

Now, locate the operator $\mathbb{Q}: \mathbb{W}(\mathcal{T}) \to \mathbb{V}(\mathcal{T})$ such that $\mathbb{Q}x(t) = {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x(t)$. So, Eq. (15) can be turned into the following subordinate tantamount form:

$$\begin{cases} \mathbb{Q}x(t) = f(t, x(t)), \\ x(0) = 0. \end{cases} \tag{25}$$

Hither, we will designate $[\mathbb{Q}x(t)]_r = \left[{}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x(t)\right]_r$ which intends that $\mathbb{Q}_1 x_{1r}(t) = {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{1r}(t)$ and $\mathbb{Q}_2 x_{2r}(t) = {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{2r}(t)$. Next, we will assign a system of orthogonal functions using the following subordinate junctures: put $\mathfrak{T}_{ij}(t) = \mathbb{F}_{t_i}(t)\mathbb{e}_j$ and $u_{ij}(t) = \mathbb{Q}^* \mathfrak{T}_{ij}(t)$ at $i = 1,2,3, \ldots$ and $j = 1,2$, where $\mathbb{e}_1 = (1,0)^T$ and $\mathbb{e}_2 = (0,1)^T$. Hither, $\mathbb{Q}^* = \operatorname{diag}(\mathbb{Q}_1^*, \mathbb{Q}_2^*)$ and $\{t_i\}_{i=1}^{\infty}$ is dense on $\mathcal{T}$. Notice here that, the system of orthonormal functions $\{\bar{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ of $\mathbb{W}(\mathcal{T})$ can be formulated as follows:

$$\bar{u}_{ij}(t) = \sum_{l=1}^{i} \sum_{k=1}^{j} z_{lk}^{ij} u_{lk}(t), i = 1,2,3, \ldots, j = 1,2, \tag{26}$$

where z_{lk}^{ij} are orthogonalization coefficients of $\{\bar{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Theorem 8 The orthonormal system $\{u_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is complete with $u_{ij}(t) = \mathbb{Q}_s \mathbb{G}_t(s)|_{s=t_i}$.

Proof. If $\langle [x(t)]_r^T, u_{ij}(t) \rangle_{\mathbb{W}} = 0$ at $i = 1,2, \ldots, j = 1,2$, then

$$\begin{aligned} \langle [x(t)]_r^T, u_{ij}(t) \rangle_{\mathbb{W}} &= \langle [x(t)]_r^T, \mathbb{Q}^* \mathfrak{T}_{ij}(t) \rangle_{\mathbb{W}} \\ &= \langle \mathbb{Q}[x(t)]_r^T, \mathfrak{T}_{ij}(t) \rangle_{\mathbb{V}} \\ &= \mathbb{Q}(t_i) = 0. \end{aligned} \tag{27}$$

But, $[x(t)]_r^T = \sum_{j=1}^2 x_{jr}(t)\mathbb{e}_j = \sum_{j=1}^2 \langle [x(\cdot)]_r^T, \mathbb{G}_t(\cdot)\mathbb{e}_j \rangle_{\mathbb{W}} \mathbb{e}_j$ and $\mathbb{Q}[x(t)]_r^T = \sum_{j=1}^2 \langle \mathbb{Q}[x(t)]_r^T, \mathfrak{T}_{ij}(t) \rangle_{\mathbb{W}} \mathbb{e}_j = 0$. By the density of $\{t_i\}_{i=1}^{\infty}$ on $\mathcal{T}$, we gained $\mathbb{Q}[x(t)]_r^T = 0$. Through the existence of $\mathbb{Q}^{-1}$, produces that $[x(t)]_r^T = 0$. Posteriorly, $\{u_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ is a complete on $\mathbb{W}(\mathcal{T})$. Again, visibly one has

$$\begin{aligned} u_{ij}(t) = \mathbb{Q}^* \mathfrak{T}_{ij}(t) &= \langle \mathbb{Q}^* \mathfrak{T}_{ij}(s), \mathbb{G}_t(s) \rangle_{\mathbb{W}} \\ &= \langle \mathfrak{T}_{ij}(s), \mathbb{Q}_s \mathbb{G}_t(s) \rangle_{\mathbb{V}} \\ &= \mathbb{Q}_s \mathbb{G}_t(s)|_{s=t_i}. \blacksquare \end{aligned} \tag{28}$$

The readers should know that $\left[f(t, x(t))\right]_r = \left(f_{1r}(t, [x(t)]_r^T), f_{2r}(t, [x(t)]_r^T)\right)$ beside of $\left[f(t, x(t))\right]^r = \left[f_{1r}(t, x_{1r}(t), x_{2r}(t)), f_{2r}(t, x_{1r}(t), x_{2r}(t))\right]$ and we will establish and employ this in the next analysis.

Theorem 9 Let z_{lk}^{ij} are orthogonalization coefficients for $\{\bar{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$. Then the analytic solution of Eq. (24) fulfill well

$$[x(t)]_r^T = \sum_{i=1}^{\infty} \sum_{j=1}^{2} \sum_{l=1}^{i} \sum_{k=1}^{j} z_{lk}^{ij} f_{kr}(t_l, [x(t_l)]_r^T) \bar{u}_{ij}(t). \tag{29}$$

Proof. Because $\langle [x(t)]_r^T, \mathfrak{S}_{ij}(t) \rangle_{\mathbb{W}} = x_{jr}(t_i)$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{2} \langle [x(t)]_r^T, \bar{u}_{ij}(t) \rangle_{\mathbb{W}} \bar{u}_{ij}(t)$ is the Fourier series of $\{\bar{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$. Then the mentioned series is convergent in the feeling of $\|\cdot\|_{\mathbb{W}}$. From here,

$$\begin{aligned} [x(t)]_r^T &= \sum_{i=1}^{\infty} \sum_{j=1}^{2} \langle [x(t)]_r^T, \bar{u}_{ij}(t) \rangle_{\mathbb{W}} \bar{u}_{ij}(t) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{2} \left\langle [x(t)]_r^T, \sum_{l=1}^{i} \sum_{k=1}^{j} z_{lk}^{ij} u_{lk}(t) \right\rangle_{\mathbb{W}} \bar{u}_{ij}(t) \end{aligned} \tag{30}$$

$$= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij}\langle [x(t)]_r^T, \mathbb{Q}^*\mathfrak{T}_{lk}(t)\rangle_{\mathbb{W}}\overline{u}_{ij}(t)$$
$$= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij}\langle \mathbb{Q}[x(t)]_r^T, \mathfrak{T}_{lk}(t)\rangle_{\mathbb{V}}\overline{u}_{ij}(t)$$
$$= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij}\langle f_{kr}(t,[x(t)]_r^T), \mathfrak{T}_{lk}(t)\rangle_{\mathbb{V}}\overline{u}_{ij}(t)$$
$$= \sum_{i=1}^{\infty}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij} f_{kr}(t_l,[x(t_l)]_r^T)\overline{u}_{ij}(t).$$

So, $[x(t)]_r^T$ in Eq. (29) symbolize the analytic solution of Eq. (25). ■

Remark 1 For the numerical calculations on the software package used, we should amputate the expression in Eq. (29) to engendering the n-term crisp ABC numerical solution of $[x(t)]_r^T$ as

$$[x^n(t)]_r^T = \sum_{i=1}^{n}\sum_{j=1}^{2}\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij} f_{kr}(t_l,[x(t_l)]_r^T)\overline{u}_{ij}(t). \tag{31}$$

7 Convergence and Error Frameworks

To control the fuzzy ABC numerical solution obtained from the RKHSM and to know its behavior and limits; we ought to study its convergence and error frameworks. These concepts will discuss and derive in detail in the following part. Next, $\mathcal{C}^{\mathbb{R}}(\mathcal{T}\times\mathbb{R}\times\mathbb{R})$ denotes the set of all continuous crisp functions on $\mathcal{T}\times\mathbb{R}\times\mathbb{R}$.

Hither, we assume the following subordinate: $\|[x^{n-1}]_r^T\|_{\mathbb{W}}$ is bounded as $n\to\infty$, $\{t_i\}_{i=1}^{\infty}$ is dense on $\mathcal{T}$, and solution of Eq. (24) unique and exists in $\mathbb{W}(\mathcal{T})$. Now, we will show the convergence of $[x(t)]_r^T$ in $\mathcal{T}$ over $\mathbb{W}(\mathcal{T})$.

Theorem 10 Let $[f(t,[x(t)]_r^T)]_r \in \mathcal{C}^{\mathbb{R}}(\mathcal{T}\times\mathbb{R}\times\mathbb{R})$. If $\|[x^{n-1}]_r^T - [x]_r^T\|_{\mathbb{W}} \to 0$ and $t_n \to s$ as $n\to\infty$, then as $n\to\infty$, one can get

$$[f(t_n,[x^{n-1}(t_n)]_r^T)]_r \to [f(s,[x^{n-1}(s)]_r^T)]_r. \tag{32}$$

Proof. Firstly, we have to start by proving that $[x^{n-1}(t_n)]_r^T \to [x(s)]_r^T$. Since,

$$\begin{aligned} |[x^{n-1}(t_n)]_r^T - [x(s)]_r^T| &= |[x^{n-1}(t_n)]_r^T - [x^{n-1}(s)]_r^T + [x^{n-1}(s)]_r^T - [x(s)]_r^T| \\ &\le |[x^{n-1}(t_n)]_r^T - [x^{n-1}(s)]_r^T| + |[x^{n-1}(s)]_r^T - [u(s)]_r^T| \\ &\le |([x^{n-1}(\tau)]_r^T)'||t_n - s| + |[x^{n-1}(s)]_r^T - [x(s)]_r^T|, \end{aligned} \tag{33}$$

where $\tau \in (\min\{t_n,s\},\max\{t_n,s\})$. So, $|[x^{n-1}(t_n)]_r^T - [x(s)]_r^T| \to 0$ as $n\to\infty$. By the fact that $[f(t,[x(t)]_r^T)]_r \in \mathcal{C}^{\mathbb{R}}(\mathcal{T}\times\mathbb{R}\times\mathbb{R})$ one has the demanded requirements. ■

Next, we stand for $\mathbb{A}_{(n,j)r} = \sum_{l=1}^{n}\sum_{k=1}^{j} z_{lk}^{ij} f_{kr}(t_l,[x(t_l)]_r^T)$. De facto, this authorizes one to put $[x^n(t)]_r^T$ as

$$[x^n(t)]_r^T = \sum_{i=1}^{n}\sum_{j=1}^{2}\mathbb{A}_{(i,j)r}\,\overline{u}_{ij}(t). \tag{34}$$

Theorem 11 In the monotonous recipe of Eqs. (34), one has $[x^n(t)]_r^T \to [x(t)]_r^T$ as $n\to\infty$.

Proof. From Eq. (34), we infer that $[x^{n+1}(t)]_r^T = [x^n(t)]_r^T + \sum_{j=1}^{2}\mathbb{A}_{(n+1,j)r}\,\overline{u}_{(n+1)j}(t)$. By the orthogonality of $\{\overline{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$ one win this

$$\begin{aligned} \|[x^{n+1}]_r^T\|_{\mathbb{W}}^2 &= \|[x^n]_r^T\|_{\mathbb{W}}^2 + \sum_{j=1}^{2}\mathbb{A}_{(n+1,j)r}^2 \\ &= \|[x^{n-1}]_r^T\|_{\mathbb{W}}^2 + \sum_{j=1}^{2}\mathbb{A}_{(n,j)r}^2 + \sum_{j=1}^{2}\mathbb{A}_{(n+1,j)r}^2 \\ &= \cdots \end{aligned} \tag{35}$$

$$= \|[x^0]_r^T\|_{\mathbb{W}}^2 + \sum_{i=1}^{n+1}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}^2.$$

Thereafter, $\|[x^{n+1}]_r^T\|_{\mathbb{W}} \geq \|[x^n]_r^T\|_{\mathbb{W}}$ and $\exists \mathbb{c} \in \mathbb{R}$ with $\sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}^2 = \mathbb{c}$, which reap that $\left\{\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}^2\right\}_{i=1}^{\infty} \in l^2$. Over and above,

$$[x^m(t)]_r^T - [x^{m-1}(t)]_r^T \perp [x^{m-1}(t)]_r^T - [x^{m-2}(t)]_r^T \perp \cdots \perp [x^{n+1}(t)]_r^T - [x^n(t)]_r^T, \tag{36}$$

it pull off for $m > n$ that

$$\begin{aligned} \|[x^m]_r^T - [x^n]_r^T\|_{\mathbb{W}}^2 &= \|[x^m]_r^T - [x^{m-1}]_r^T + [x^{m-1}]_r^T - \cdots + [x^{n+1}]_r^T - [x^n]_r^T\|_{\mathbb{W}}^2 \\ &= \|[x^m]_r^T - [x^{m-1}]_r^T\|_{\mathbb{W}}^2 + \|[x^{m-1}]_r^T - [x^{m-2}]_r^T\|_{\mathbb{W}}^2 + \ldots + \|[x^{n+1}]_r^T - [x^n]_r^T\|_{\mathbb{W}}^2. \end{aligned} \tag{37}$$

While, $\|[x^m]_r^T - [x^{m-1}]_r^T\|_{\mathbb{W}}^2 = \sum_{j=1}^{2} \mathbb{A}_{(m,j)r}^2$. So, as $n, m \to \infty$, we get $\|[x^m]_r^T - [x^n]_r^T\|_{\mathbb{W}}^2 = \sum_{l=n+1}^{m}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}^2 \to 0$. As the completeness, $\exists [x^n(t)]_r^T \in \mathbb{W}(\mathcal{T})$ with $[x^n(t)]_r^T \to [x(t)]_r^T$ as $n \to \infty$ in the feeling of $\|\cdot\|_{\mathbb{W}}$. ■

Theorem 12 In the monotonous recipe of Eqs. (34), one has $[x(t)]_r^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\, \overline{u}_{ij}(t)$ as $n \to \infty$.

Proof. By taking $\lim_{n\to\infty}(\cdot)$ on two sides of Eq. (34), one has $[x(t)]_r^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\overline{u}_{ij}(t)$. While $\mathbb{Q}[x(t)]_r^T = \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\, \mathbb{Q}\overline{u}_{ij}(t)$, then

$$\begin{aligned} \mathbb{Q}_k[x(t_l)]_r^T &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\langle \mathbb{Q}\overline{u}_{ij}(t), \mathfrak{T}_{lk}(t)\rangle_{\mathbb{W}} \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\langle \overline{u}_{ij}(t), \mathbb{Q}^*\mathfrak{T}_{lk}(t)\rangle_{\mathbb{W}} \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\langle \overline{u}_{ij}(t), u_{lk}(t)\rangle_{\mathbb{W}}. \end{aligned} \tag{38}$$

$$\begin{aligned} \sum_{l'=1}^{l}\sum_{k'=1}^{k} z_{l'k'}^{lk}\mathbb{Q}_{k'}[u(t)]_r^T(t_{l'}) &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\left\langle \overline{u}_{ij}(t), \sum_{l'=1}^{l}\sum_{k'=1}^{k} z_{l'k'}^{lk}u_{l'k'}(t)\right\rangle_{\mathbb{W}} \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{2} \mathbb{A}_{(i,j)r}\langle \overline{u}_{ij}(t), \overline{u}_{l'k'}(t)\rangle_{\mathbb{W}} \\ &= \mathbb{A}_{(l,k)r}. \end{aligned} \tag{39}$$

Vindicatory, if $l = 1$, then $\mathbb{Q}_j[x(t_1)]_r^T = \mathscr{f}_{jr}(t_1, [x^0(t_1)]_r^T)$. So as to, $\mathbb{Q}[x(t_1)]_r^T = \left[\mathscr{f}\left(t_1, x^0(t_1)\right)\right]_r$. If $l = 2$, then $\mathbb{Q}_j[x(t_2)]_r^T = \mathscr{f}_{jr}(t_2, [x^1(t_2)]_r^T)$. So as to, $\mathbb{Q}[x(t_2)]_r^T = \left[\mathscr{f}\left(t_2, x^1(t_2)\right)\right]_r$. In like trajectory, the shape form is $\mathbb{Q}[x(t_n)]_r^T = \left[\mathscr{f}\left(t_n, x^{n-1}(t_n)\right)\right]_r$. Through the density, $\forall s \in \mathcal{T}$; $\exists \left\{t_{n_q}\right\}_{q=1}^{\infty}$ such that $t_{n_q} \to s$ as $q \to \infty$ or $\mathbb{Q}\left[x\left(t_{n_q}\right)\right]_r^T = \left[\mathscr{f}\left(t_{n_q}, u^{n_q-1}\left(t_{n_q}\right)\right)\right]_r$. Let $j \to \infty$, by Theorem 9, one has $\mathbb{Q}[x(s)]_r^T = \left[\mathscr{f}\left(s, x(s)\right)\right]_r$. Likewise, since $\overline{u}_{ij}(t) \in \mathbb{W}(\mathcal{T})$, then $[x(t)]_r^T$ fulfill Eq. (25). Ultimately, the unique solution of Eq. (25) harness the desired score.

Next, we will debate the attitude of errors for large n. In the extreme, we will denote $\mathfrak{R}_n = \|[x]_r^T - [x^n]_r^T\|_{\mathbb{W}}$ on $\mathcal{T}$ provided that $[x(t)]_r^T$ and $[x^n(t)]_r^T$ are taken away from Eqs. (29) and (31), with one another.

Theorem 13 The sequence $\{\mathfrak{R}_n\}_{n=1}^{\infty}$ is monotone decreasing in $\mathbb{W}(\mathcal{T})$ with $\mathfrak{R}_n \to 0$ as $n \to \infty$.

Proof. It's visible that

$$\mathfrak{R}_n^2 = \left\|\sum_{i=n+1}^{\infty}\sum_{j=1}^{2}\langle [x(t)]_r^T, \overline{u}_{ij}(t)\rangle_{\mathbb{W}}\overline{u}_{ij}(t)\right\|_{\mathbb{W}}^2 \tag{40}$$

$$= \sum_{i=n+1}^{\infty} \sum_{j=1}^{2} \left\langle [x(t)]_r^T, \bar{u}_{ij}(t) \right\rangle_{\mathbb{W}}^2$$

$$\leq \sum_{i=n}^{\infty} \sum_{j=1}^{2} \left\langle [x(t)]_r^T, \bar{u}_{ij}(t) \right\rangle_{\mathbb{W}}^2$$

$$= \left\| \sum_{i=n}^{\infty} \sum_{j=1}^{2} \left\langle [x(t)]_r^T, \bar{u}_{ij}(t) \right\rangle_{\mathbb{W}} \bar{u}_{ij}(t) \right\|_{\mathbb{W}}^2$$

$$= \Re_{n-1}^2.$$

Thus, $\{\Re_n\}_{n=1}^{\infty}$ is monotone decreasing in the feeling of $\|\cdot\|_{\mathbb{W}}$. From Theorem 8 and convergent fact on $\sum_{i=1}^{\infty} \sum_{j=1}^{2} \left\langle [x(t)]_r^T, \bar{u}_{ij}(t) \right\rangle_{\mathbb{W}} \bar{u}_{ij}(t)$ one harvest that $\Re_n^2 \to 0$ as $n \to \infty$. ■

8 Algorithms and Packages

Computational algorithms are given to laying the groundwork for the solution methods. Anyhow, to perform the numerical operations using the RKHSM the following three underlying algorithms must be applied and controlled. The first one is to check and find the validity of fuzzy ABC analytical solutions, the second one is to gain the orthonormal functions systems $\left\{\bar{u}_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$, whiles the last one is to how one can apply the RKHSM steps.

The inputs for the three algorithms are as pursues, with one another: $[x^n]_r^T$, $[x]_r^T$, $u_{lk}(t)$, $\bar{u}_{ij}(t)$, types of fuzzy ABC fractional derivative, truth interval $\mathcal{I}$, truth values r, order α of ABC fractional derivative, n collocation points, and the domain $\mathcal{T}$.

Algorithm 1 Finding and ensuring the validity of the $\alpha(1)$- or a $\alpha(2)$-fuzzy ABC solutions of Eq. (25):

Phase I. If $x(t)$ is $\alpha(1)$-fuzzy ABC fractional differentiable on $\mathcal{T}$, then applying the underlying steps:

- **i.** Solve $\alpha(1)$-crisp ABC FDE system to the references $[x_{1r}(t), x_{2r}(t)]$,
- **ii.** Assure $[x_{1r}(t), x_{2r}(t)]$ and $\left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha(1)} x_{1r}(t), {}^{ABC}_{0}\mathcal{D}_t^{\alpha(1)} x_{2r}(t)\right]$ are righteous sets,
- **iii.** Forecast $\alpha(1)$-fuzzy ABC solution of $x(t)$ as $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$.

Phase II. If $x(t)$ is $\alpha(2)$-fuzzy ABC fractional differentiable on $\mathcal{T}$, then applying the underlying steps:

- **i.** Solve the $\alpha(2)$-crisp ABC FDE system to the references $[x_{1r}(t), x_{2r}(t)]$,
- **ii.** Assure $[x_{1r}(t), x_{2r}(t)]$ and $\left[{}^{ABC}_{0}\mathcal{D}_t^{\alpha(2)} x_{2r}(t), {}^{ABC}_{0}\mathcal{D}_t^{\alpha(2)} x_{1r}(t)\right]$ are righteous sets,
- **iii.** Forecast a $\alpha(2)$-fuzzy ABC solution of $x(t)$ as $[x(t)]^r = [x_{1r}(t), x_{2r}(t)]$.

Algorithm 2 Applying the Gram-Schmidt process to finding z_{lk}^{ij} and $\left\{\bar{u}_{ij}(t)\right\}_{(i,j)=(1,1)}^{(\infty,2)}$:

Phase 1: At $l = 1,2, \ldots, k = 1,2, \ldots, l$, $i = 1,2,3, \ldots$, and $j = 1,2$ applying the underlying:

$$z_{lk}^{ij} = \begin{cases} \dfrac{1}{\|u_{11}\|_{\mathbb{W}}}, l = k = 1, \\ \dfrac{1}{\sqrt{\|u_{lk}\|_{\mathbb{W}}^2 - \sum_{p=1}^{l-1} \langle u_{lk}(t), \bar{u}_{ij}(t) \rangle_{\mathbb{W}}^2}}, l = k \neq 1, \\ -\dfrac{\sum_{p=k}^{l-1} \langle u_{lk}(t), \bar{u}_{ij}(t) \rangle_{\mathbb{W}} z_{pk}^{ij}}{\sqrt{\|u_{lk}\|_{\mathbb{W}}^2 - \sum_{p=1}^{l-1} \langle u_{lk}(t), \bar{u}_{ij}(t) \rangle_{\mathbb{W}}^2}}, l > k, \end{cases} \tag{41}$$

Output: The orthogonalization coefficients z_{lk}^{ij}.

Phase 2: At $i = 1,2,3, \ldots$ and $j = 1,2$ put

$$\bar{u}_{ij}(t)=\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij}\, u_{lk}(t), \tag{42}$$

Output: systems of orthonormal functions $\{\bar{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Algorithm 3 Finding $\alpha(1)$-fuzzy ABC RKHSM numerical solution for Eq. (25):

Phase I: Fixed t, s in $\mathcal{T}$ and do the underlying steps:

- **i.** Put $t_i=\frac{1}{n}i$ at $i=0,1,\dots,n$,
- **ii.** Put $r_\eta=\frac{\eta}{m}$ at $\eta=0,1,\dots,m$,
- **iii.** Put $u_{ij}(t)=\mathbb{Q}_s\mathbb{G}_t(s)|_{s=t_i}$ at $i=1,2,\dots,n$ and $j=1,2$,

Output: The orthogonal function system $\{\bar{u}_{ij}(t)\}_{(i,j)=(1,1)}^{(\infty,2)}$.

Phase II: At $l=1,2,\dots$ and $k=1,2,\dots,l$ do Algorithm 2;

Output: The orthogonalization coefficients z_{lk}^{ij}.

Phase III: Set $\bar{u}_{ij}(t)=\sum_{l=1}^{i}\sum_{k=1}^{j} z_{lk}^{ij}\, u_{lk}(t)$ at $i=1,2,\dots,n$ and $j=1,2$;

Output: The orthonormal function system $\bar{u}_{ij}(t)$.

Phase IV: Put $[x^0(t_1)]_r^T=0$ and at $i=1,2,\dots,n$ do the underlying steps:

- **i.** Put $\left[x^i(t_i)\right]_r^T=\left[x^{i-1}(t_i)\right]_r^T$,
- **ii.** Put $\mathbb{A}_{(i,j)r}=\sum_{l=1}^{i}\sum_{k=1}^{j}\omega_{lk}^{ij} f_{kr}(t_l,[x(t_l)]_r^T)$,
- **iii.** Put $\left[x^i(t)\right]_r^T=\sum_{k=1}^{i}\sum_{j=1}^{2}\mathbb{A}_{(i,j)r}\,\bar{u}_{ij}(t)$;

Output: The n-term numerical reckoning $[x^n(t)]_r^T$ of $[x(t)]_r^T$.

Software packages are the foundation stone in the fields of numerical analysis; it is a task where the reader chooses the type of programming that he masters and wants to use and available to him. Anyhow, all numerical results and graphical exemplifications in this analysis are made and done with Mathematics 9 software package.

9 Fuzzy Applications on ABC FFIVPs

To navigate more in the utilized fuzzy analyses, we must add some applications to show the strength of the presented study and the strength of the presented numerical method. Anyhow, two ABC FFIVPs are discussed and utilized for the first time in this section by displaying some tables, figures, and analyzes.

To discuss our utilized outcomes in the shape of realistic fuzzy models; duo applications are debated here. The first one focuses on the fuzzy resistance-inductance circuit, whilst, the last focuses on fuzzy forcing term effects.

Application 1 Look firstly for the underlying fuzzy ABC resistance-inductance circuit:

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha} x(t)=f\big(t,x(t)\big),\\ x(0)=\mathcal{U}, \end{cases} \tag{43}$$

indeed $x:\mathcal{T}\to\mathcal{F}(\mathbb{R})$ and $f:\mathcal{T}\times\mathcal{F}(\mathbb{R})\to\mathcal{F}(\mathbb{R})$ with

$$f\big(t,x(t)\big)=-RL^{-1}x(t)+E(t), \tag{44}$$

$$\mathcal{U}(s)=\begin{cases} 25s-24, s\in[0.96,1],\\ -100s+101, s\in[1,1.01],\\ 0, s\in\mathbb{R}-[0.96,1.01], \end{cases} \tag{45}$$

where $\alpha\in(0,1]$, $t\in\mathcal{T}$, $R=1$ Ohm, $L=1$ Henry, and $E(t)=\sin(t)$.

The underlying subsequent coupled crisp systems of ABC FDE in term of r-cut impersonation that are linked to $\alpha(1)$- and $\alpha(1)$-fuzzy ABC FFIVP of Eqs. (43), (44), and (45) can be appearing with one another as

Status 1. The system of $\alpha(1)$-crisp ABC FDE corresponding to $\alpha(1)$-fuzzy ABC fractional derivative is

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{1r}(t) = -x_{2r}(t) + \sin(t), \\ {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{2r}(t) = -x_{1r}(t) + \sin(t), \\ x_{1r}(0) = 0.96 + 0.04r, \\ x_{2r}(0) = 1.01 - 0.01r. \end{cases} \tag{46}$$

The analytic fuzzy solutions of Eqs. (46) when $\alpha = 1$ is

$$\begin{cases} x_{1r}(t) = 0.5\big(\sin(t) - \cos(t)\big) + 0.5e^{-t} + (0.96 + 0.04r)\cosh(t) - (1.01 - 0.01r)\sinh(t), \\ x_{2r}(t) = 0.5\big(\sin(t) - \cos(t)\big) + 0.5e^{-t} + (1.01 - 0.01r)\cosh(t) - (0.96 + 0.04r)\sinh(t). \end{cases} \tag{47}$$

In the fuzzy tactic and in terms of $\mathcal{U}$ one can collect and represent the expression in Eq. (47) as

$$x(t) = 0.5\big(\sin(t) - \cos(t)\big) + 0.5e^{-t} + \mathcal{U}\odot\cosh(t) - \mathcal{U}\odot \sinh(t). \tag{48}$$

Status 2. The system of $\alpha(2)$-crisp ABC FDE corresponding to $\alpha(2)$-fuzzy ABC fractional derivative is

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{1r}(t) = -x_{1r}(t) + \sin(t), \\ {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{2r}(t) = -x_{2r}(t) + \sin(t), \\ x_{1r}(0) = 0.96 + 0.04r, \\ x_{2r}(0) = 1.01 - 0.01r. \end{cases} \tag{49}$$

The analytic fuzzy solutions of Eqs. (49) when $\alpha = 1$ is

$$\begin{cases} x_{1r}(t) = 0.5\big(\sin(t) - \cos(t)\big) + 0.5e^{-t} + (0.96 + 0.04r)e^{-t}, \\ x_{2r}(t) = 0.5\big(\sin(t) - \cos(t)\big) + 0.5e^{-t} + (1.01 - 0.01r)e^{-t}. \end{cases} \tag{50}$$

In the fuzzy tactic and terms of $\mathcal{U}$ one can collect and represent the expression in Eq. (50) as

$$x(t) = 0.5\big(\sin(t) - \cos(t)\big) + 0.5e^{-t} + \mathcal{U}\odot e^{-t}. \tag{51}$$

Application 2 Now, look for the underlying fuzzy ABC FFIVP with fuzzy forcing term:

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha} x(t) = f\big(t, x(t)\big), \\ x(0) = \mathcal{U}, \end{cases} \tag{52}$$

indeed $x: \mathcal{T} \to \mathcal{F}(\mathbb{R})$ and $f: \mathcal{T} \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R})$ with

$$f\big(t, x(t)\big) = 2tx(t) + t\mathcal{U}, \tag{53}$$

$$\mathcal{P}(s) = \max(0,1 - |s|), \tag{54}$$

where $\alpha \in (0,1]$, $t \in \mathcal{T}$, and $s \in \mathbb{R}$.

The underlying subsequent coupled crisp systems of ABC FDE in term of r-cut impersonation that are linked to $\alpha(1)$- and $\alpha(1)$-fuzzy ABC FFIVP of Eqs. (52), (53), and (54) can be appearing with one another as

Status 1. The system of $\alpha(1)$-crisp ABC FDE corresponding to $\alpha(1)$-fuzzy ABC fractional derivative is

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{1r}(t) = 2tx_{1r}(t) + t(r - 1), \\ {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{2r}(t) = 2tx_{2r}(t) + t(1 - r), \\ x_{1r}(0) = r - 1, \\ x_{2r}(0) = 1 - r. \end{cases} \tag{55}$$

The analytic fuzzy solutions of Eqs. (55) when $\alpha = 1$ is

$$\begin{cases} x_{1r}(t) = 0.5(r - 1)\big(3e^{t^2} - 1\big), \\ x_{2r}(t) = 0.5(1 - r)\big(3e^{t^2} - 1\big). \end{cases} \tag{56}$$

In the fuzzy tactic and terms of $\mathcal{U}$ one can collect and represent the expression in Eq. (56) as

$$x(t) = 0.5\odot\mathcal{U}\odot\big(3e^{t^2} - 1\big). \tag{57}$$

Status 2. The system of $\alpha(2)$-crisp ABC FDE corresponding to $\alpha(2)$-fuzzy ABC fractional derivative is

$$\begin{cases} {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{1r}(t) = 2tx_{2r}(t) + t(1 - r), \\ {}^{ABC}_{\ 0}\mathcal{D}_t^{\alpha(1)} x_{2r}(t) = 2tx_{1r}(t) + t(r - 1), \\ x_{1r}(0) = r - 1, \\ x_{2r}(0) = 1 - r. \end{cases} \tag{58}$$

The analytic fuzzy solutions of Eqs. (58) when $\alpha = 1$ is

$$\begin{cases} x_{1r}(t) = 0.5(r-1)\left(3e^{-t^2}-1\right), \\ x_{2r}(t) = 0.5(1-r)\left(3e^{-t^2}-1\right). \end{cases} \tag{59}$$

In the fuzzy tactic and terms of $\mathcal{U}$ one can collect and represent the expression in Eq. (59) as

$$x(t) = 0.5 \odot \mathcal{U} \odot \left(3e^{-t^2}-1\right). \tag{60}$$

10 Results, Analysis, and Talks

In the previous two fuzzy applications, the readers should memorandum that the non-attendance of fuzzy ABC analytic solutions for various $\alpha \in (0,1]$ does not affect the obtained numerical results; because we have plotted the fuzzy ABC numerical solutions at different values of α which have been guaranteed from the prior convergence theorems.

Anyhow, by catching the following underlying inputs: $t_i = \frac{i}{n}$ at $i = 0,1,\dots,n = 21$ on $\mathcal{T}$ and $r_\eta = \frac{\eta}{m}$ at $\eta = 0,1,3, m = 4$ on $\mathcal{I}$ in $[x^n(t_i)]_{r_\eta}^T$ with using of Algorithms 1, 2, and 3 in all computations effects over $\alpha \in (0,1]$ and $t \in \mathcal{T}$; a set of numerical data are listed in the following attached tables side by side with attached figures.

In the running tables; several numerical effectiveness has been gained and exhibited for both presented applications. Anyhow, Tables 1 and 2 related to Application 1 and utilized the absolute errors for approximating the fuzzy ABC solutions in phases of $\alpha(1)$- and $\alpha(2)$-fuzzy ABC fractional derivative, with one another. Whilst, Tables 3 and 4 related to Application 2 and utilized the absolute errors for approximating the fuzzy ABC solutions in phases of $\alpha(1)$- and $\alpha(2)$-fuzzy ABC fractional derivative, with one another.

Table 1. Numerical outcomes in form of absolute errors in Applications 1 in phase of $\alpha(1)$-fuzzy ABC fractional derivative using RKHSM.

	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{1r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	6.5316019×10^{-6}	6.4963275×10^{-6}	6.4610530×10^{-6}	6.4257787×10^{-6}	6.3905043×10^{-6}
	0.4	4.7281578×10^{-6}	4.6720920×10^{-6}	4.6160263×10^{-6}	4.5599606×10^{-6}	4.5038949×10^{-6}
	0.6	3.3436333×10^{-6}	3.2633070×10^{-6}	3.1829808×10^{-6}	3.1026546×10^{-6}	3.0223283×10^{-6}
	0.8	2.3053850×10^{-6}	2.1958916×10^{-6}	2.0863983×10^{-6}	1.9769050×10^{-6}	1.8674116×10^{-6}
	1.0	1.5600204×10^{-6}	1.4147544×10^{-6}	1.2694885×10^{-6}	1.1242225×10^{-6}	9.7895651×10^{-7}
	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{2r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	6.0984787×10^{-6}	6.1714850×10^{-6}	6.2444915×10^{-6}	6.3174979×10^{-6}	6.3905043×10^{-6}
	0.4	4.1654746×10^{-6}	4.2500797×10^{-6}	4.3346850×10^{-6}	4.4192898×10^{-6}	4.5038949×10^{-6}
	0.6	2.6152480×10^{-6}	2.7170181×10^{-6}	2.8187882×10^{-6}	2.9205583×10^{-6}	3.0223283×10^{-6}
	0.8	1.3655063×10^{-6}	1.4909826×10^{-6}	1.6164590×10^{-6}	1.7419353×10^{-6}	1.8674116×10^{-6}
	1.0	3.5070339×10^{-7}	5.0776668×10^{-7}	6.6482996×10^{-7}	8.2189324×10^{-7}	9.7895651×10^{-7}

Table 2. Numerical outcomes in form of absolute errors in Applications 1 in phase of $\alpha(2)$-fuzzy ABC fractional derivative using RKHSM.

	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{1r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	6.1892670×10^{-6}	6.2395763×10^{-6}	6.2898856×10^{-6}	6.3401950×10^{-6}	6.3905042×10^{-6}
	0.4	4.3516851×10^{-6}	4.3897375×10^{-6}	4.4277900×10^{-6}	4.4658424×10^{-6}	4.5038948×10^{-6}
	0.6	2.9079611×10^{-6}	2.9365529×10^{-6}	2.9651447×10^{-6}	2.9937365×10^{-6}	3.0223283×10^{-6}
	0.8	1.7821690×10^{-6}	1.8034797×10^{-6}	1.8247903×10^{-6}	1.8461010×10^{-6}	1.8674116×10^{-6}
	1.0	9.1603756×10^{-7}	9.3176729×10^{-7}	9.4749704×10^{-7}	9.6322677×10^{-7}	9.7895651×10^{-7}
	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{2r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	6.4408136×10^{-6}	6.4282360×10^{-6}	6.4156589×10^{-6}	6.4030816×10^{-6}	6.3905043×10^{-6}
	0.4	4.5419473×10^{-6}	4.5324342×10^{-6}	4.5229210×10^{-6}	4.5134080×10^{-6}	4.5038948×10^{-6}
	0.6	3.0509201×10^{-6}	3.0437722×10^{-6}	3.0366242×10^{-6}	3.0294763×10^{-6}	3.0223283×10^{-6}
	0.8	1.8887223×10^{-6}	1.8833946×10^{-6}	1.8780670×10^{-6}	1.8727393×10^{-6}	1.8674116×10^{-6}
	1.0	9.9468626×10^{-7}	9.9075382×10^{-7}	9.8682139×10^{-7}	9.8288895×10^{-7}	9.7895651×10^{-7}

Table 3. Numerical outcomes in form of absolute errors in Applications 2 in phase of $\alpha(1)$-fuzzy ABC fractional derivative using RKHSM.

	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{1r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	3.0463929×10^{-6}	2.2847946×10^{-6}	1.5231964×10^{-6}	7.6159820×10^{-7}	0
	0.4	8.9454505×10^{-6}	6.7090879×10^{-6}	4.4727253×10^{-6}	2.2363626×10^{-6}	0
	0.6	2.3237533×10^{-5}	1.7428150×10^{-5}	1.1618767×10^{-5}	5.8093833×10^{-6}	0
	0.8	5.9996031×10^{-5}	4.4997023×10^{-5}	2.9998016×10^{-5}	1.4999008×10^{-5}	0
	1.0	1.5758910×10^{-5}	1.1819183×10^{-5}	7.8794551×10^{-5}	3.9397276×10^{-5}	0
	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{2r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	3.0463929×10^{-6}	2.2847946×10^{-6}	1.5231964×10^{-6}	7.6159820×10^{-7}	0
	0.4	8.9454505×10^{-6}	6.7090879×10^{-6}	4.4727253×10^{-6}	2.2363626×10^{-6}	0
	0.6	2.3237533×10^{-5}	1.7428150×10^{-5}	1.1618767×10^{-5}	5.8093833×10^{-6}	0
	0.8	5.9996031×10^{-5}	4.4997023×10^{-5}	2.9998016×10^{-5}	1.4999008×10^{-5}	0
	1.0	1.5758910×10^{-5}	1.1819183×10^{-5}	7.8794551×10^{-5}	3.9397276×10^{-5}	0

Table 4. Numerical outcomes in form of absolute errors in Applications 2 in phase of $\alpha(2)$-fuzzy ABC fractional derivative using RKHSM.

	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{1r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	2.3624048×10^{-6}	$1.7718036 \times 10^{-}$	1.1812024×10^{-6}	5.9060119×10^{-7}	0
	0.4	2.9107354×10^{-6}	2.1830515×10^{-6}	1.4553677×10^{-6}	7.2768384×10^{-7}	0
	0.6	1.0383274×10^{-6}	7.7874556×10^{-7}	5.1916371×10^{-7}	2.5958186×10^{-7}	0
	0.8	2.0213038×10^{-6}	1.5159778×10^{-6}	1.0106520×10^{-6}	5.0532593×10^{-7}	0
	1.0	4.3374140×10^{-6}	3.2530605×10^{-6}	2.1687070×10^{-6}	1.0843535×10^{-6}	0
	t_i	$r_0 = 0$	$r_1 = 0.25$	$r_2 = 0.5$	$r_3 = 0.75$	$r_4 = 1$
$x_{2r_\eta}(t_i)$	0	0	0	0	0	0
	0.2	2.3624048×10^{-6}	1.7718036×10^{-6}	1.1812024×10^{-6}	5.9060119×10^{-7}	0
	0.4	2.9107354×10^{-6}	2.1830515×10^{-6}	1.4553677×10^{-6}	7.2768384×10^{-7}	0
	0.6	1.0383274×10^{-6}	7.7874556×10^{-7}	5.1916371×10^{-7}	2.5958186×10^{-7}	0
	0.8	2.0213038×10^{-6}	1.5159778×10^{-6}	1.0106520×10^{-6}	5.0532593×10^{-7}	0
	1.0	4.3374140×10^{-6}	3.2530605×10^{-6}	2.1687070×10^{-6}	1.0843535×10^{-6}	0

Finally, the geometric dynamical behaviors over the heritage and memory characteristics are seeking. In the running individual figures; geometrical attributives have been gained and exhibited for both presented applications over $\alpha \in (0,1]$, $t \in \mathcal{T}$, and $r \in \mathcal{I}$. Anyhow, Figures 1 and 2 related to Application 1 and plotted the fuzzy ABC numerical solution in phases of $\alpha(1)$- and $\alpha(2)$-fuzzy ABC fractional derivative with one another. Whilst, Figures 3 and 4 related to Application 2 and plotted the fuzzy ABC numerical solution in phases of $\alpha(1)$- and $\alpha(2)$-fuzzy ABC fractional derivative with one another.

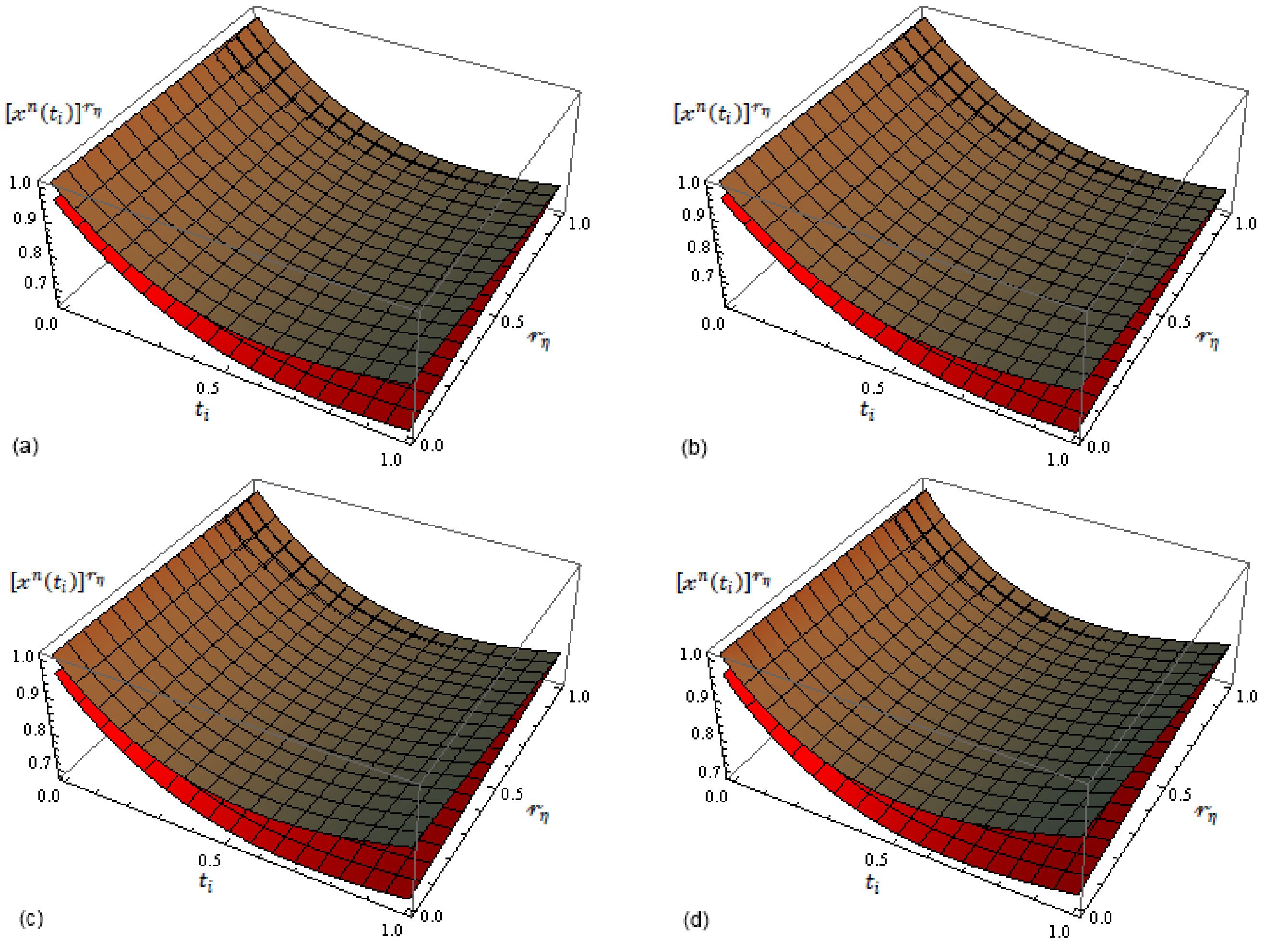
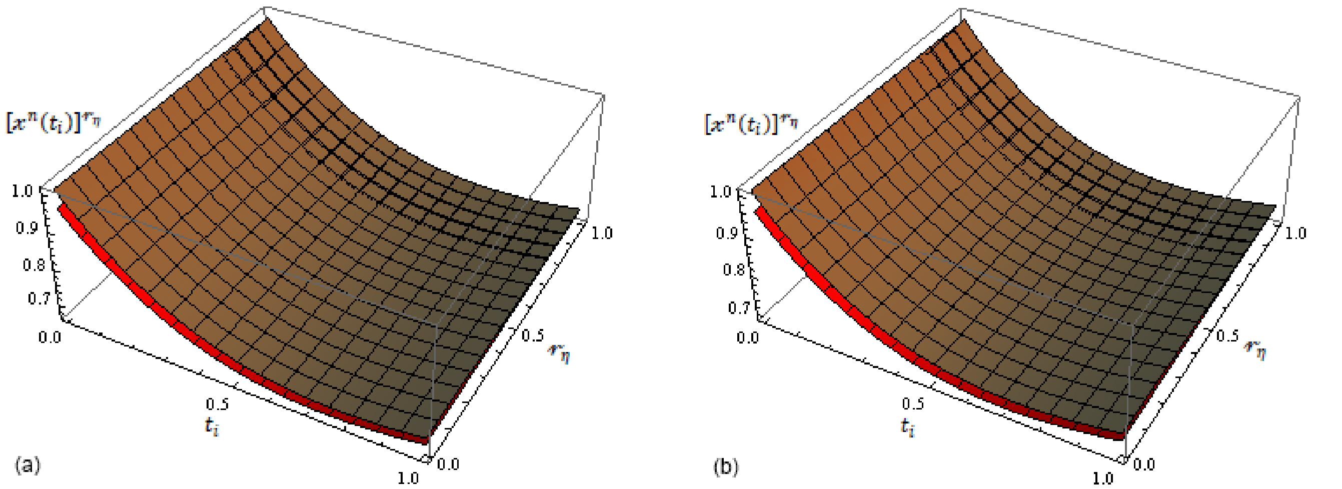


Figure 1: Computation of $\alpha(1)$-fuzzy ABC solution of Application 1 obtained from the RKHSM: (a) $\alpha = 1$, (b) $\alpha = 0.95$, (c) $\alpha = 0.90$, and (d) $\alpha = 0.85$ wheresoever $x_{1r}(t)$: red offshoot and $x_{2r}(t)$: brown offshoot.



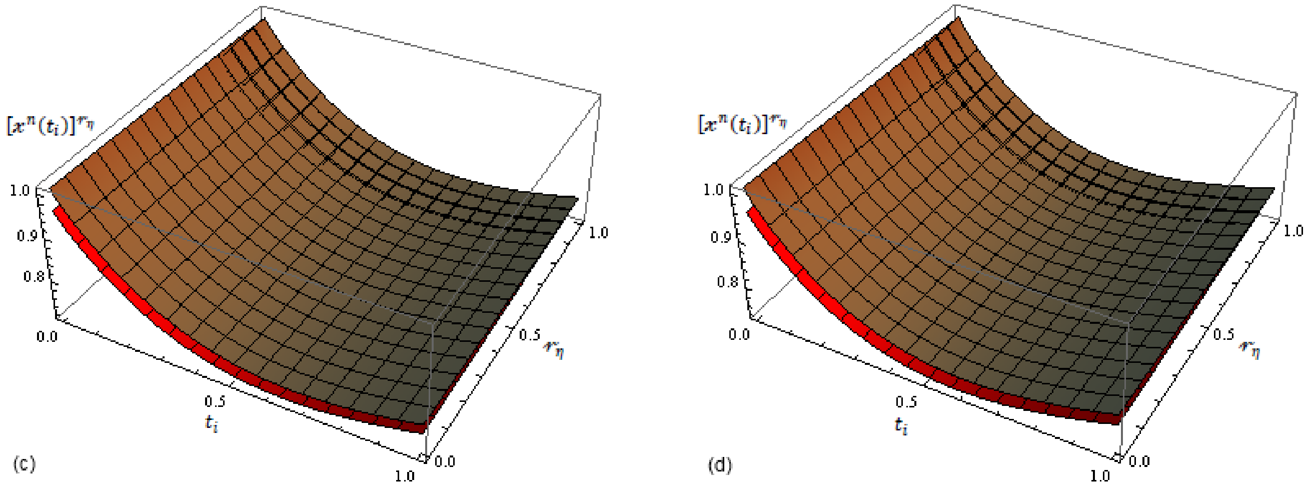


Figure 2: Computation of $\alpha(2)$-fuzzy ABC solution of Application 1 obtained from the RKHSM: (a) $\alpha = 1$, (b) $\alpha = 0.95$, (c) $\alpha = 0.90$, and (d) $\alpha = 0.85$ wheresoever $x_{1r}(t)$: red offshoot and $x_{2r}(t)$: brown offshoot.

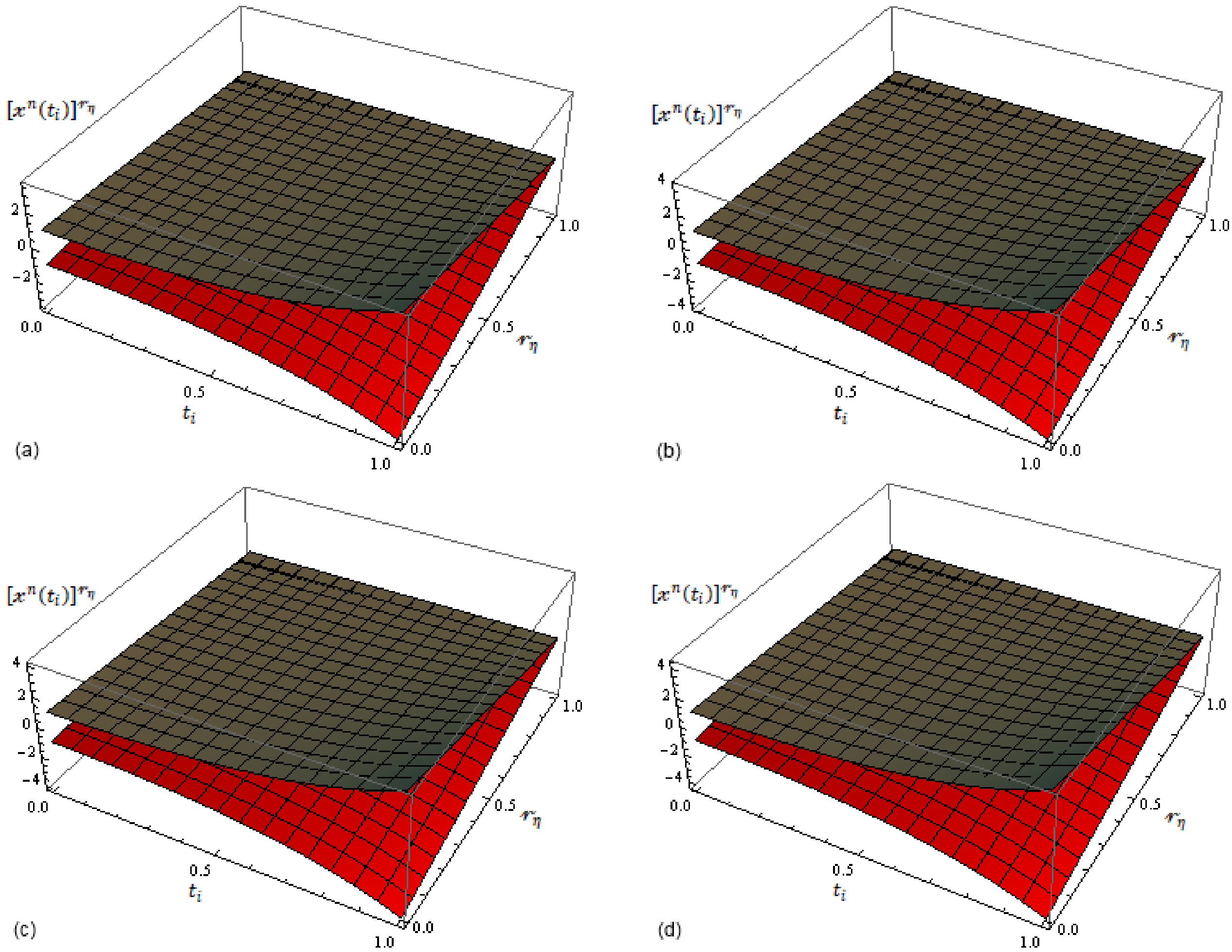


Figure 3: Computation of $\alpha(1)$-fuzzy ABC solution of Application 2 obtained from the RKHSM: (a) $\alpha = 1$, (b) $\alpha = 0.95$, (c) $\alpha = 0.90$, and (d) $\alpha = 0.85$ wheresoever $x_{1r}(t)$: red offshoot and $x_{2r}(t)$: brown offshoot.

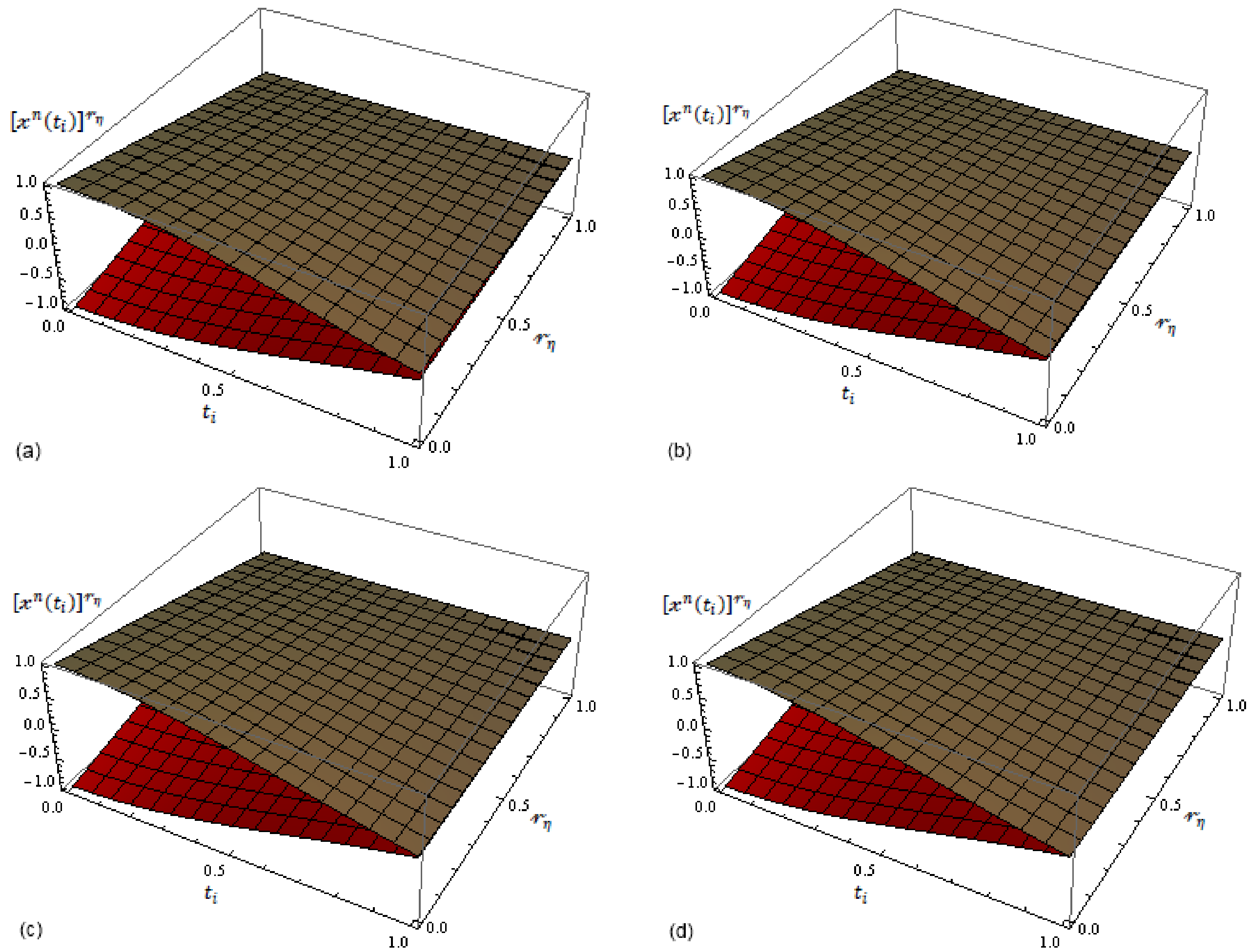


Figure 4: Computation of $\alpha(2)$-fuzzy ABC solution of Application 2 obtained from the RKHSM: (a) $\alpha = 1$, (b) $\alpha = 0.95$, (c) $\alpha = 0.90$, and (d) $\alpha = 0.85$ wheresoever $x_{1r}(t)$: red offshoot and $x_{2r}(t)$: brown offshoot.

One can be observed from the graphs the underlying notes: all plots are almost matched and are analogous in their behaviors; the plots are in pretty contract with each other, essentially when theorizing the fuzzy ABC traditional derivative of $\alpha = 1$; and the fuzzy ABC FIVPs have powerful belongings on the model profiles.

11 Highlight and Future Research

In this novel analysis, fuzzy ABC fractional derivative, fuzzy ABC FIVPs, and fuzzy ABC solutions are discussed and utilized for the first time. Indeed, FSGD in the ABC sense and characterization theorem are likewise discussed for the first time as well. Also, the RKHSM is offered in detail as a novel version solver for such fuzzy ABC FIVPs. Over and above, a computational algorithm concerned with characterizing fuzzy ABC solutions is given. In this direction, two applications on fuzzy ABC FIVPs are fitted to conform to the approaching theoretical analysis in the fuzzy ABC calculus. Hereafter, those proposed extended can be used efficaciously as a substitution planner in the formulation of various kinds of uncertain differential problems under investigation in engineering and applied sciences. Our future research will be analyzed and arrange fuzzy ABC fractional integrodifferential equations.

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