POSITIVE ALMOST PERIODIC SOLUTIONS OF NONAUTONOMOUS EVOLUTION EQUATIONS AND APPLICATION TO LOTKA–VOLTERRA SYSTEMS

KAMAL KHALIL

ABSTRACT. Consider the nonautonomous semilinear evolution equation of type: $(\star)$ $u'(t) = A(t)u(t) + f(t, u(t))$, $t \in \mathbb{R}$, where $A(t)$, $t \in \mathbb{R}$ is a family of closed linear operators on a Banach space X, the nonlinear term f, acting on some real interpolation spaces, is assumed to be almost periodic only in a weak sense (i.e. in Stepanov sense) with respect to t and Lipschitzian in bounded sets with respect to the second variable. We prove the existence and uniqueness of positive almost periodic solutions in the strong sense (Bohr sense) for equation $(\star)$ using the exponential dichotomy approach. Then, we establish a new composition result of Stepanov almost periodic functions by assuming only the continuity of f in the second variable. Moreover, we provide an application to a nonautonomous system of reaction–diffusion equations describing a Lotka–Volterra predator–prey model with diffusion and time–dependent parameters in a generalized almost periodic environment.

1. Introduction

The concept of almost periodicity introduced by H. Bohr in [10] is a trivial generalization of the well-known periodicity. Both concepts are given in the literature by means of continuous bounded functions. Moreover, W. Stepanov in [33] investigated a more general definition of almost periodicity through locally integrable functions with certain weak appropriate boundedness conditions. Hence, an almost periodic function in Stepanov sense is not necessarily continuous or bounded. The motivation for considering almost periodic functions in Stepanov sense is the existence of a more wide class of unbounded and discontinuous functions describing periodic and almost periodic motions, see Section 5 for more details about examples from this class.

Now, consider the following nonautonomous semilinear evolution equation,

$$u'(t) = A(t)u(t) + f(t, u(t)) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where $(A(t), D(A(t)))$, $t \in \mathbb{R}$ is a family of linear closed operators defined on a Banach space X (not necessarily densely defined) that generates a positive analytic evolution family $(U(t,s))_{t \geq s}$ which has an exponential dichotomy on $\mathbb{R}$. The nonlinear term $f : \mathbb{R} \times X^t_\alpha \longrightarrow X$ is only almost periodic in Stepanov sense (i.e. in the weak sense) of order $1 \leq p < \infty$ with respect to t and Lipschitzian in bounded sets

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with respect to the second variable, where X^t_α, $0 < \alpha < 1$, $t \in \mathbb{R}$ are some continuous interpolation spaces with respect to the linear operators $(A(t), D(A(t)))$, $t \in \mathbb{R}$, such that the following continuous embedding hold

$$D(A(t)) \hookrightarrow X^t_\beta \hookrightarrow X^t_\alpha \hookrightarrow X \quad \text{ for all } 0 < \alpha < \beta < 1,\ t \in \mathbb{R}.$$

In this work, we establish sufficient weak conditions on the forcing terms $A(\cdot)$ and f insuring the existence and uniqueness of almost periodic solutions (in the strong sense) to equation (1.1). Moreover, we introduce a new composition result of Stepanov almost periodic functions of order $1 \leq p < \infty$. That is, for given any function $f : \mathbb{R} \times Y \longrightarrow X$ which is Stepanov almost periodic with respect to t and continuous with respect to x and $u : \mathbb{R} \longrightarrow Y$ is Stepanov almost periodic for any Banach spaces X and Y. Then, $f(\cdot, u(\cdot))$ is also Stepanov almost periodic. Our composition result require only the continuity of f with respect to the second argument while the literature's works assume the uniform Lipschitz condition which is very stronger as hypothesis than ours, see [2, 5, 15, 26]. To our knowledge, in the instance where the input f belongs to a broader class of Stepanov almost periodic functions (with respect to t) which is not necessarily uniformly globally Lipschitzian (in the second variable), for this reason, there are no consistent results in the literature devoted to the existence (and/or the uniqueness) of almost periodic solutions, since the composition results for Stepanov μ-pseudo almost periodic functions established in the literature up to here require the uniform Lipschitz condition. Technically, our strategy concerns to study at first the following linear inhomogeneous equation:

$$u'(t) = A(t)u(t) + h(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where h is almost automorphic in Stepanov sense of order $1 \leq p < \infty$. We show that the unique mild solution given by:

$$u(t) = \int_{\mathbb{R}} G(t,s)h(s)ds, \quad t \in \mathbb{R},$$

where $G(\cdot,\cdot)$ is the associated Green function, is almost periodic in the strong sense. Hence, using our suitable composition result, we prove, in view of the Banach contraction principle on positive cones, that equation (1.1) has a unique positive (Bohr) almost periodic solution $u : \mathbb{R} \longrightarrow X^t_\alpha$ defined by:

$$u(t) = \int_{\mathbb{R}} G(t,s)f(s,u(s))ds, \quad t \in \mathbb{R}.$$

Furthermore, an illustrating application to a nonautonomous system of reaction–diffusion equations describing a Lotka–Volterra predator–prey model in a bounded domain (the habitat) $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ with Lipschitz boundary $\partial\Omega$ and time-dependent parameters in a generalized almost periodic environment is provided. More precisely, we introduce sufficient weak conditions insuring the existence and uniqueness of positive almost periodic solutions to our model (5.2) where the parameters are weakly (not regular) almost periodic and also the terms of interactions depend to the density (resp. the gradient of the density). For more details about the study of models of type (5.1), we refer to [11, 22, 25, 30, 31] and references therein.

In the literature, we found several works devoted to the existence and uniqueness of almost periodic (resp. periodic) solutions semilinear evolution equation in infinite dimensional setting, see [2, 5, 8, 13, 15, 19, 23, 26, 28, 29, 34]. In [8, 19, 34] the authors proved the existence and uniqueness of almost periodic solutions to equation (1.2) (i.e. equation (1.1) in the linear inhomogeneous case) where $A(\cdot)$ is periodic and h is almost periodic in the strong sense. Moreover, in [28] the authors proved that equation (1.2) has a unique almsot periodic solution provided that $R(\omega, A(\cdot))$, for some $\omega \in \mathbb{R}$, and h are almost periodic in the strong sense. Furthermore, in [2, 5, 15, 26] the authors proved the existence and uniqueness to equation (1.1) where f is Stepanov almost periodic with respect to t and globally Lipschitzian in the second variable.

The organization of this paper is as follows: Section 2 is devoted to preliminaries, notations and main hypotheses of this work. In Section 3 we give our new composition result of Stepanov almost periodic functions of order $1 \leq p < \infty$. Section 4 is provided to the existence and uniqueness results of almost periodic solutions to equations (1.2) and (1.1) respectively. In Section 5, we study the existence and uniqueness of almost periodic solutions to our Lotka-Volterra model (5.1).

2. Notations, Preliminaries and main hypotheses

In this section, we recall notations, definitions and preliminary results needed in the following.

Notations. Throughout this work, $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ are two any Banach lattices with the embedding $Y \hookrightarrow X$ and the positive cones which are defined by $X^+ := \{x \in X : x \geq 0\}$ and $Y^+ = X^+ \cap Y$. Let $(\mathcal{L}(X), \|\cdot\|_{\mathcal{L}(X)})$ be the Banach space of bounded linear operators in X and let $BC(\mathbb{R}, X)$ equipped with the sup norm, denoted by $\|\cdot\|_\infty$, is the Banach space of bounded continuous functions f from $\mathbb{R}$ into X. Moreover, for $1 \leq p < \infty$, q denotes its conjugate exponent defined by $\dfrac{1}{p} + \dfrac{1}{q} = 1$ if $p \neq 1$ and $q = \infty$ if $p = 1$. By $L^p_{loc}(\mathbb{R}, X)$ (resp. $L^p(\mathbb{R}, X)$), we designate the space (resp. the Banach space) of all equivalence classes of measurable functions f from $\mathbb{R}$ into X such that $\|f(\cdot)\|^p$ is locally integrable (resp. integrable). We denote by $\Gamma(\cdot)$ the gamma function defined by $\Gamma(z) := \displaystyle\int_0^\infty s^{z-1}e^{-s}ds$ for $z > 0$. Moreover, in the following it is assumed that the resolvent set $\rho(A)$ of $(A, D(A))$ is defined by $\rho(A) := \{\lambda \in \mathbb{C} : (\lambda - A)^{-1}$ exists in $\mathcal{L}(X)\} \subset \mathbb{C}$ and the spectrum $\sigma(A)$ of $(A, D(A))$ is defined as: $\sigma(A) := \mathbb{C} \setminus \rho(A)$. For $\lambda \in \rho(A)$, the resolvent operator $R(\lambda, A)$ is defined by $R(\lambda, A) := (\lambda - A)^{-1}$.

2.1. **Evolution families and intermediate spaces.** Let $(A(t), D(A(t)))$, $t \in \mathbb{R}$ be a family of linear closed operators on a Banach space X that satisfies the following conditions: there exist constants $\omega \in \mathbb{R}, \theta \in (\frac{\pi}{2}, \pi), M > 0$ and $\eta, \nu \in (0, 1]$ with $\eta + \nu > 1$ such that

$$\begin{cases} \Sigma_{\omega,\theta} := \{z \in \mathbb{C} : z \neq 0, |\, arg(z) \,| \leq \theta\} \subset \rho(A(t) - \omega) \\ \|\lambda R(\lambda, A(t) - \omega)\|_{\mathcal{L}(X)} \leq L, \end{cases} \tag{2.1}$$

$$\|(A(t) - \omega)R(\lambda, A(t) - \omega)[R(\omega, A(t)) - R(\omega, A(s))]\|_{\mathcal{L}(X)} \leq \frac{M|t-s|^\eta}{|\lambda|^\nu} \tag{2.2}$$

for all $t \geq s$, $t, s \in \mathbb{R}$ and $\lambda \in \Sigma_{\omega,\theta}$. The domains $D(A(t))$ of the operators $A(t)$ may change with respect to t and do not required to be dense in X. The condition (2.1) means that each operator $A(t)$ generates a bounded analytic semigroup $(T_t(s))_{s\geq 0}$ where the semigroups may be not strongly continuous at 0. The condition (2.2) provides some stability in the dependence on t of the operators $A(t)$. Note that, the above conditions (2.1) and (2.2) was introduced in [1] to solve the following Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t), & t \geq s \\ u(s) = x \in X, & . \end{cases} \tag{2.3}$$

in the parabolic context, where the unique solution (in the classical sense) is given by a two-parameter family of bounded linear operators in X, i.e., $u(t) = U(t,s)x$ with $(U(t,s))_{t\geq s} \subset \mathcal{L}(X)$ is an evolution family. More precisely, for $t > s$ the map $(t,s) \longmapsto U(t,s) \in \mathcal{L}(X)$ is continuous and continuously differentiable in t, $U(t,s)$ maps X into $D(A(t))$ and it holds that $\dfrac{\partial U(t,s)}{\partial t} = A(t)U(t,s)$. Moreover, $U(t,s)$ and $(t-s)A(t)U(t,s)$ are exponentially bounded and that satisfy

$$U(t,s)U(s,r) = U(t,r) \quad \text{and} \quad U(t,t) = I \quad \text{for} \quad t \geq s \geq r.$$

Furthermore, for $s \in \mathbb{R}$ and $x \in \overline{D(A(s))}$, the unique solution $t \longmapsto u(t) = U(t,s)x$ in $C([s,\infty),X) \cap C^1((s,\infty),X)$ is continuous at $t = s$. For more details, we refer to [1, 17, 27].

Now, we introduce the interpolation spaces for the operators $A(t)$, $t \in \mathbb{R}$. Let A be a sectorial operator, i.e., A satisfy (2.1) instead of $A(t)$ (it is well known that A generates an analytic semigroup $(T_A(t))_{t\geq 0}$ on X). For $\alpha \in (0,1)$, we give the real interpolation spaces:

$$X_\alpha := \overline{D(A)}^{\|\cdot\|_\alpha}, \quad \text{where} \quad \|x\|_\alpha := \sup_{\lambda>0} \|\lambda^\alpha (A-\omega)x\| \quad \text{for all } x \in D(A).$$

Then, $(X_\alpha, \|\cdot\|_\alpha)$ are Banach spaces. Let $X_0 := X$, $X_1 := D(A)$ and $\|x\|_0 = \|x\|$, $\|x\|_1 = \|(A-\omega)x\|$ be the corresponding norms respectively.

$$D(A) \hookrightarrow X_\beta \hookrightarrow X_\alpha \hookrightarrow X \tag{2.4}$$

for all $0 < \alpha < \beta < 1$. Let $A(t)$, $t \in \mathbb{R}$ which satisfies (2.1), we set $X_\alpha^t := X_\alpha$ where $A(t)$ is taken instead of A in the definition of the spaces X_α, $0 < \alpha < 1$, and the corresponding norms. Then, the embedding in (2.4) holds with uniformly bounded norms in $t \in \mathbb{R}$. Moreover, in the case of a constant domain, i.e., $D := D(A(t))$, $t \in \mathbb{R}$, we can replace assumption (2.2) with the following:

There exist constants $\omega \in \mathbb{R}$, $L \geq 0$ and $0 < \mu \leq 1$ such that

$$\|(A(t) - A(s))R(\omega, A(r))\| \leq L|t-s|^\mu \quad \text{for } t, s, r \in \mathbb{R}. \tag{2.5}$$

A sufficient condition ensuring (2.5) is the next:

$$\|(\omega - A(t))R(\omega, A(s)) - I_X\| \leq L_0|t-s|^{\mu_0} \quad \text{for } t, s \in \mathbb{R} \tag{2.6}$$

for some $\omega \in \mathbb{R}$, $L_0 \geq 0$ and $0 < \mu_0 \leq 1$.

Now, we introduce exponential dichotomy of an evolution family which is an important tool in our study, see [2, 17, 32]

Definition 2.1. [2] *An evolution family $(U(t,s))_{s\leq t}$ on a Banach space X is called has an exponential dichotomy (or hyperbolic) in $\mathbb{R}$ if there exists a family of projections $P(t)\in\mathcal{L}(X)$, $t\in\mathbb{R}$, being strongly continuous with respect to t, and constants $\delta, N>0$ such that*

(i) $U(t,s)P(s)=P(t)U(t,s)$.
(ii) $U(t,s)\ :\ Q(s)X\longrightarrow Q(t)X$ *is invertible with the inverse* $\tilde{U}(t,s)$ *(i.e.,* $\tilde{U}(t,s)=U(s,t)^{-1}$*).*
(iii) $\|U(t,s)P(s)\|\leq Ne^{-\delta(t-s)}$ *and* $\|\tilde{U}(s,t)Q(t)\|\leq Ne^{-\delta(t-s)}$

for all $t,s\in\mathbb{R}$ with $s\leq t$, where, $Q(t):=I-P(t)$.

Hence, given a hyperbolic evolution family $(U(t,s))_{s\leq t}$, then its associated Green function is defined by:

$$G(t,s)=\begin{cases} U(t,s)P(s), & t,s\in\mathbb{R},\ s\leq t,\\ -\tilde{U}(t,s)Q(s), & t,s\in\mathbb{R},\ s>t.\end{cases} \tag{2.7}$$

Next, we show that exponential dichotomy can be characterized in many cases, for more details, see [21].

Remark 2.2. ***(a)*** *If $U(t,s)=T(t-s)$, $t\geq s$ and $(T(t))_{t\geq 0}\subset\mathcal{L}(X)$ is a semigroup of bounded linear operators. In that case, the generator of $(U(t,s))_{t\geq s}$ is a time constant sectorial operator $(A,D(A))$ such that $\sigma(A)\cap\{\lambda\in\mathbb{C}:\ |Re(\lambda)|\leq\beta\}=\emptyset$ for some $\beta>0$. Then $(U(t,s))_{t\geq s}$ has an exponential dichotomy on $\mathbb{R}$ with constant projections $P(t)=P$ and $Q(t):=Q$, $t\in\mathbb{R}$ and exponent $\delta:=\beta$.*

(b) *If $(A(t))_{t\in\mathbb{R}}$ is p-periodic (with $p>0$). In that case the evolution family $(U(t,s))_{t\geq s}$ is also p-periodic in t and s and the spectrum of 'the period map' $\mathbb{U}_p:=U(t+p,t)$, $t\in\mathbb{R}$ is independent of t. Moreover if $\sigma(\mathbb{U}_p)\cap\{\lambda\in\mathbb{C}: e^{-p\beta}\leq|\lambda|\leq e^{p\beta}\}=\emptyset$, for some $\beta>0$. Then $(U(t,s))_{t\geq s}$ has an exponential dichotomy on $\mathbb{R}$ with p-periodic projections $P(t)$ and $Q(t)$, $t\in\mathbb{R}$ and exponent $\delta:=\beta$.*

(c) *If $(U(t,s))_{t\geq s}$ is exponentially stable i.e., $\omega(U)<0$, where*

$$\omega(U):=\inf\{\omega\in\mathbb{R}:\exists\ M_\omega\geq 1\ \text{with}\ \|U(t,s)\|\leq M_\omega e^{\omega(t-s)},\ t\geq s,\ s\in\mathbb{R}\}.$$

Then $(U(t,s))_{t\geq s}$ has an exponential dichotomy on $\mathbb{R}$ with projections $P(t)=I_X$, $t\in\mathbb{R}$, (the identity operator of X) and exponent $\delta:=-\omega(U)$.

More generally, from [32], the exponential dichotomy holds if the following is true:

Assume (2.5) holds and the semigroups $(T^t(\tau))_{\tau\geq 0}$ are hyperbolic with projections P_t and constants $N,\delta>0$ such that $\|A(t)T^t(\tau)P_t\|\leq\psi(\tau)$ and $\|A(t)T^t_Q(\tau)Q_t\|\leq\psi(-\tau)$ for $\tau>0$ and a function ψ such that the mapping $\mathbb{R}\ni s\longmapsto\varphi(s):=|s|^\mu\psi(s)$ is integrable with $r:=L\|\varphi\|_{L^1(\mathbb{R})}<1$. Then $(U(t,s))_{t\geq s}$ is hyperbolic with an exponent $0<\delta'<\delta(1-r)/2N$.

Now, we give some dichotomy estimates of the evolution family $(U(t,s))_{s\leq t}$ through the interpolation spaces X^t_α, $0\leq\alpha\leq 1$, see [4] and [6] for a more general result.

Theorem 2.3. *Let $x\in X$, $0<\alpha\leq 1$. Then, the following hold:*

(i) *There exists a constant* $c(\alpha)$*, such that*

$$\|\tilde{U}(t,s)Q(t)x\|_\alpha^s \quad \leq \quad c(\alpha)e^{-\delta(s-t)}\|x\| \quad \text{for } t < s. \tag{2.8}$$

(ii) *There exists a constant* $m(\alpha)$*, such that*

$$\|U(t,s)P(s)x\|_\alpha^t \quad \leq \quad m(\alpha)(t-s)^{\alpha-1}e^{-\gamma(t-s)}\|x\| \quad \text{for } t > s. \tag{2.9}$$

2.2. **Almost periodic functions.** In this section, we recall important properties on almost periodic functions in the sense of Bohr and that of Stepanov, [3, 10, 33].

Definition 2.4 (H. Bohr [10])**.** *A continuous function* $f : \mathbb{R} \longrightarrow X$ *is said to be almost periodic, if for every* $\varepsilon > 0$*, there exists* $l_\varepsilon > 0$*, such that for every* $a \in \mathbb{R}$*, there exists* $\tau \in [a, a + l_\varepsilon]$ *satisfying:*

$$\|f(t+\tau) - f(t)\| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $AP(\mathbb{R}, X)$*.*

Theorem 2.5 (S. Bochner [9])**.** *A continuous function* $f : \mathbb{R} \longrightarrow X$ *is almost periodic if and only if for every sequence* $(\sigma_n)_{n\geq 0}$ *of real numbers, there exists a subsequence* $(s_n)_{n\geq 0} \subset (\sigma_n)_{n\geq 0}$ *and a continuous function* $g : \mathbb{R} \longrightarrow X$*, such that*

$$g(t) =: \lim_n f(t + s_n) \text{ uniformly on } t \in \mathbb{R}. \tag{2.10}$$

Definition 2.6 (S. Bochner [9])**.** *A continuous function* $f : \mathbb{R} \longrightarrow X$ *is said to be almost automorphic if for every sequence* $(\sigma_n)_{n\geq 0}$ *of real numbers, there exist a subsequence* $(s_n)_{n\geq 0} \subset (\sigma_n)_{n\geq 0}$ *and a function* $g \in L^\infty(\mathbb{R}, X)$*, such that the following limits*

$$g(t) =: \lim_n f(t + s_n) \quad \text{and} \quad f(t) = \lim_n g(t - s_n)$$

are well-defined for each $t \in \mathbb{R}$*. The space of all such functions will be denoted* $AA(\mathbb{R}, X)$*.*

Definition 2.7. *Let* $(Z, \|\cdot\|_Z)$ *be any Banach space. A continuous function* $F : \mathbb{R} \times \mathbb{R} \to Z$ *is said to be bi-almost periodic if for every* $\varepsilon > 0$*, there exists* $l_\varepsilon > 0$*, such that for every* $a \in \mathbb{R}$*, there exists* $\tau \in [a, a + l_\varepsilon]$ *satisfying:*

$$\|F(t+\tau, s+\tau) - F(t,s)\|_Z < \varepsilon \quad \text{for all } t, s \in \mathbb{R}.$$

The space of all such functions is denoted by $bAP(\mathbb{R}, X)$*.*

Definition 2.8. *Let* $1 \leq p < \infty$*. A function* $f \in L^p_{loc}(\mathbb{R}, X)$ *is said to be bounded in Stepanov sense if*

$$\sup_{t\in\mathbb{R}} \left(\int_{[t,t+1]} \|f(s)\|^p ds \right)^{\frac{1}{p}} = \sup_{t\in\mathbb{R}} \left(\int_{[0,1]} \|f(t+s)\|^p ds \right)^{\frac{1}{p}} < \infty.$$

The space of all such functions is denoted by $BS^p(\mathbb{R}, X)$ *and it is provided with the following norm:*

$$\begin{aligned} \|f\|_{BS^p} \quad &:= \quad \sup_{t\in\mathbb{R}} \left(\int_{[t,t+1]} \|f(s)\|^p ds \right)^{\frac{1}{p}} \\ &= \quad \sup_{t\in\mathbb{R}} \|f(t+\cdot)\|_{L^p([0,1],X)}. \end{aligned}$$

Definition 2.9 (Bochner transform). *Let $f \in L^p_{loc}(\mathbb{R}, X)$ for $1 \leq p < \infty$. The Bochner transform of f is the function $f^b : \mathbb{R} \longrightarrow L^p([0,1], X)$ defined by*

$$(f^b(t))(s) = f(t+s) \quad \textit{for } s \in [0,1],\ t \in \mathbb{R}.$$

Now, we give the definition of almost periodicity in Stepanov sense.

Definition 2.10. *Let $1 \leq p < \infty$. A function $f \in L^p_{loc}(\mathbb{R}, X)$ is said to be almost periodic in the sense of Stepanov (or S^p-almost periodic), if for every $\varepsilon > 0$, there exists $l_\varepsilon > 0$, such that for every $a \in \mathbb{R}$, there exists $\tau \in [a, a + l_\varepsilon]$ satisfying*

$$\left(\int_{[t,t+1]} \|f(s+\tau) - f(s)\|^p ds \right)^{\frac{1}{p}} < \varepsilon \quad \textit{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $APS^p(\mathbb{R}, X)$.

Remark 2.11. ***(a)*** *Every (Bohr) almost periodic function is S^p-almost periodic for $1 \leq p < \infty$. The converse is not true in general (see Proposition 2.13).*

(b) *For all $1 \leq p_1 \leq p_2 < \infty$, if f is S^{p_2}-almost periodic, then f is S^{p_1}-almost periodic.*

(c) *The Bochner transform of an X-valued function is a $L^p([0,1], X)$-valued function. Moreover, a function f is S^p-almost periodic if and only if f^b is (Bohr) almost periodic.*

Using the Bochner transform i.e. Definition 2.9 and Theorem 2.5, we can deduce easily the following characterization of Stepanov almost periodicity using sequences.

Theorem 2.12. *Let $f \in L^p_{loc}(\mathbb{R}, X)$. The function f is S^p-almost periodic if and only if for every sequence $(s_n)_n$ of real numbers there exists a subsequence $(\sigma_n)_n \subset (s_n)_n$ and a function $g \in BS^p(\mathbb{R}, X)$ such that*

$$\lim_n \left(\int_{[t,t+1]} \|f(s+\sigma_n) - g(s)\|^p ds \right)^{\frac{1}{p}} = 0, \tag{2.11}$$

uniformly in $t \in \mathbb{R}$.

A sufficient condition for a Stepanov almost periodic function to be Bohr almost periodic is given in the next.

Proposition 2.13. *Let $f \in L^p_{loc}(\mathbb{R}, X)$ for $1 \leq p < \infty$. If f is S^p-almost periodic and uniformly continuous, then f is almost periodic.*

Definition 2.14. *Let $1 \leq p < \infty$. A function $f : \mathbb{R} \times X \longrightarrow Y$ such that $f(\cdot, x) \in L^p_{loc}(\mathbb{R}, Y)$ for each $x \in X$ is said to be S^p-almost periodic in t uniformly with respect to x in X if for each compact set K in X, for all $\varepsilon > 0$ there exists $l_{\varepsilon,K} > 0$, such that for every $a \in \mathbb{R}$ there exists $\tau \in [a, a + l_{\varepsilon,K}]$ satisfying:*

$$\sup_{x \in K} \left(\int_{[t,t+1]} \|f(s+\tau, x) - f(s, x)\|^p ds \right)^{\frac{1}{p}} < \varepsilon \quad \textit{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $APS^pU(\mathbb{R} \times X, Y)$.

Hypotheses. Here, we list our main hypotheses:
(H1) The operators $A(t)$, $t \in \mathbb{R}$ satisfy the assumptions (2.1) and (2.2).
(H2) The evolution family $(U(t,s))_{t\geq s}$ generated by $A(t)$, $t \in \mathbb{R}$ has an exponential dichotomy on $\mathbb{R}$ with constants $N, \delta > 0$, projections $P(t), t \in \mathbb{R}$, and Green's function G.
(H3) For each $x \in X$, $G(\cdot,\cdot)x$ is bi-almost periodic.
(H4) There exist $0 \leq \alpha < \beta < 1$ such that $X^t_\alpha = X_\alpha$ and $X^t_\beta = X_\beta$ for every $t \in \mathbb{R}$ with uniformly equivalent norms.
(H5) f is locally Lipschitzian with respect to the second argument i.e., for all $\rho > 0$ there exists a nonegative scalar function $L_\rho(\cdot)$ such that

$$\|f(t,x) - f(t,y)\| \leq L_\rho(t)\|x-y\|_\alpha, \quad x, y \in B(0,\rho),\, t \in \mathbb{R}.$$

Moreover, for the positivity, we need the following additional conditions:
(C) The functions G and f satisfy the invariance conditions:

$$G(t,s)x \in X^+ \text{ if } x \in X^+, \quad t, s \in \mathbb{R}, t \geq s,$$

and

$$f(t,x) \in X^+ \text{ if } x \in X^+_\alpha, \quad t \in \mathbb{R},$$

where $X^+ := \{x \in X : x \geq 0\}$.

Remark 2.15. ***(a)*** *In the case of an exponentially stable evolution family $(U(t,s))_{t\geq s}$ i.e., the case of Remark 2.2-**(c)**, we have $G(t,s) = U(t,s)$ for $t \geq s$. Hence, hypothesis **(H3)** is reduced to the bi-almost periodicity of $(U(t,s))_{t\geq s}$ and this is hold for example if $A(\cdot)$ is p-periodic.*
(b) *In* [28]*, the authors proved that if $R(\omega, A(\cdot))$ is almost periodic for some $\omega \in \mathbb{R}$. Then, the associated Green function is bi-almost periodic. Here, in this paper, we improve this assumption by proving that **(H3)** holds, if only $A(\cdot)$ is S^1-almost periodic, see Section 5.*

Remark 2.16. *It is true that the assumption of the positivity of the Green function G in **(C)** is looking abstract, but in the case of an exponentially stable evolution family where $G(t,s) = U(t,s)$ for $t \geq s$ the condition **(C)** is satisfied if the evolution family $(U(t,s))_{t\geq s}$ is positive.*

3. New composition results of Stepanov almost periodic functions

In this section, we prove a new composition result of S^p-almost periodic functions (for $1 \leq p < \infty$). We begin by the following useful characterization.

Lemma 3.1. *Let $1 \leq p < +\infty$ and $f : \mathbb{R} \times Y \longrightarrow X$ be a function such that $f(\cdot, x) \in L^p_{loc}(\mathbb{R}, X)$ for each $x \in X$. Then, $f \in APS^pU(\mathbb{R} \times Y, X)$ if and only if the following hold:*

- **(i)** *For each $x \in Y$, $f(\cdot, x) \in APS^p(\mathbb{R}, X)$.*
- **(ii)** *f is S^p-uniformly continuous with respect to the second argument on each compact subset K in Y in the following sense: for all $\varepsilon > 0$ there exists $\delta_{K,\varepsilon}$ such that for all $x_1, x_2 \in K$ one has*

$$\|x_1 - x_2\| \leq \delta_{K,\varepsilon} \Longrightarrow \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p ds\right)^{\frac{1}{p}} \leq \varepsilon \quad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Proof. Let $f \in APS^pU(\mathbb{R}\times X, Y)$ and $f^b : \mathbb{R}\times Y \longrightarrow L^p([0,1], X)$ be the Bochner transform associated to f. It follows in view of [16, Lemma 2.6], that **(i)** is clearly satisfied and for each compact subset K in X, for all $\varepsilon > 0$ there exists $\delta_{K,\varepsilon}$ such that for all $x_1, x_2 \in K$ one has

$$\|x_1 - x_2\| \le \delta_{K,\varepsilon} \quad \Longrightarrow \quad \|f^b(t,x_1) - f^b(t,x_2)\| \le \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

Since,

$$\begin{aligned} \|f^b(t,x_1) - f^b(t,x_2)\| &= \left(\int_{[0,1]} \|(f^b(t,x_1))(s) - (f^b(t,x_2))(s)\|^p ds\right)^{\frac{1}{p}} \\ &= \left(\int_t^{t+1} \|f(s,x_1) - f(s,x_2)\|^p ds\right)^{\frac{1}{p}} \quad \text{for all } t \in \mathbb{R}. \end{aligned}$$

It follows that (3.1) holds and then **(ii)** is achieved.
Conversely, let $f : \mathbb{R}\times X \longrightarrow Y$ be a function such that $f(\cdot, x) \in L^p_{loc}(\mathbb{R}, X)$ for each $x \in Y$. Assume that f satisfies **(i)-(ii)**. Let us fix a compact subset K in Y and $\varepsilon > 0$. Since K is compact, it follows that there exists a finite subset $\{x_1, ..., x_n\} \subset K$ $(n \in \mathbb{N}^*)$ such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{K,\varepsilon})$. Therefore, for $x \in K$, there exist $i = 1, ..., n$ satisfying $\|x - x_i\| \le \delta_{K,\varepsilon}$. Let $\tau \in \mathbb{R}$. Then, we obtain that

$$\begin{aligned} &\left(\int_t^{t+1} \|f(s+\tau,x) - f(s,x)\|^p ds\right)^{\frac{1}{p}} \le \left(\int_t^{t+1} \|f(s+\tau,x) - f(s+\tau,x_i)\|^p ds\right)^{\frac{1}{p}} \\ &+ \left(\int_t^{t+1} \|f(s+\tau,x_i) - f(s,x_i)\|_Y^p ds\right)^{\frac{1}{p}} \\ &+ \left(\int_t^{t+1} \|f(s,x_i) - f(s,x)\|^p ds\right)^{\frac{1}{p}}, \; t \in \mathbb{R}. \end{aligned} \tag{3.2}$$

Using **(i)**, we have for each $i = 1, \dots, n$, $f(\cdot, x_i) \in AAS^p(\mathbb{R}, X)$. Hence, there exists $l_{K,\varepsilon} > 0$ such that for all $a \in \mathbb{R}$ there exists $\tau \in [a, a + l_{K,\varepsilon}]$ satisfying

$$\left(\int_t^{t+1} \|f(s+\tau,x_i) - f(s,x_i)\|^p ds\right)^{\frac{1}{p}} \le \frac{\varepsilon}{3} \quad \text{for all } t \in \mathbb{R}. \tag{3.3}$$

Since $\|x - x_i\| \le \delta_{K,\delta}$ and by **(ii)**, we claim that

$$\left(\int_t^{t+1} \|f(s+\tau,x) - f(s+\tau,x_i)\|^p ds\right)^{\frac{1}{p}} \le \frac{\varepsilon}{3} \quad \text{for all } t \in \mathbb{R}, \tag{3.4}$$

and

$$\left(\int_t^{t+1} \|f(s,x) - f(s,x_i)\|^p ds\right)^{\frac{1}{p}} \le \frac{\varepsilon}{3} \quad \text{for all } t \in \mathbb{R}. \tag{3.5}$$

Consequently, we replace (3.3), (3.4) and (3.5) in (3.2), we obtain that

$$\sup_{x\in K} \left(\int_t^{t+1} \|f(s+\tau,x) - f(s,x)\|^p ds\right)^{\frac{1}{p}} \le \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

□

By Lemma 3.1, we deduce the following composition result.

Theorem 3.2. *Let* $1 \leq p < +\infty$ *and* $f \in APS^pU(\mathbb{R} \times Y, X)$. *Assume that* $x \in AP(\mathbb{R}, Y)$. *Then,* $f(\cdot, x(\cdot)) \in APS^p(\mathbb{R}, X)$.

Proof. Let $t, \tau \in \mathbb{R}$. Then, we have

$$\begin{aligned} & \left(\int_t^{t+1} \|f(s+\tau, x(s+\tau)) - f(s, x(s))\|^p ds \right)^{\frac{1}{p}} \\ \leq & \left(\int_t^{t+1} \|f(s+\tau, x(s+\tau)) - f(s+\tau, x(s))\|^p ds \right)^{\frac{1}{p}} \\ & + \left(\int_t^{t+1} \|f(s+\tau, x(s)) - f(s, x(s))\|^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Moreover, given $K := \overline{\{x(t) : \ t \in \mathbb{R}\}}$ a compact subset of Y and $\varepsilon > 0$ be fixed. Using Lemma 3.1-**(ii)** it follows that there exists $\delta_{\varepsilon,K} > 0$ such that (3.1) holds. Let $\varepsilon > 0$, since $u \in AP(\mathbb{R}, Y)$, it follows that, there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains an element τ such that

$$\|x(s+\tau) - x(s)\| \leq \delta_{\varepsilon,K} \ \text{ for all } s \in \mathbb{R}.$$

Moreover, for each $s \in \mathbb{R}$, we have $x(s) \in K$. Hence,

$$\left(\int_t^{t+1} \|f(s+\tau, x(s+\tau)) - f(s+\tau, x(s))\|^p ds \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4} \tag{3.6}$$

Furthermore, since K is compact, it follows that, there exists a finite subset $\{x_1, ..., x_n\} \subset K$ $(n \in \mathbb{N}^*)$ such that $K \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{K,\varepsilon})$. Then, for all $t \in \mathbb{R}$ there exists $i(t) = 1, ..., n$ such that $\|x(t) - x_{i(t)}\| \leq \delta_{K,\varepsilon}$. Thus

$$\left(\int_t^{t+1} \|f(s+\tau, x(s)) - f(s+\tau, x_{i(t)}))\|^p ds \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}, \tag{3.7}$$

and

$$\left(\int_t^{t+1} \|f(s, x(s)) - f(s, x_{i(t)}))\|^p ds \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}. \tag{3.8}$$

Using Lemma 3.1-**(i)**, we get that

$$\left(\int_t^{t+1} \|f(s+\tau, x_{i(t)}) - f(s, x_{i(t)}))\|^p ds \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4}. \tag{3.9}$$

Consequently, by (3.6), (3.7), (3.8) and (3.9), we obtain that

$$\left(\int_t^{t+1} \|f(s+\tau, x(s+\tau)) - f(s, x(s))\|^p ds \right)^{\frac{1}{p}} \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

This proves the result. $\square$

4. Almost periodic solutions to semilinear evolution equations

In this section, we prove the existence and uniqueness of almost periodic solutions for our semilinear evolution equation (1.1). For that purpose, we set the following associated inhomogeneous linear equation:

$$u'(t) = A(t)u(t) + h(t) \quad \text{for } t \in \mathbb{R} \tag{4.1}$$

where $h : \mathbb{R} \longrightarrow X$ is locally integrable. A mild solution of equation (4.1) is the continuous function $u : \mathbb{R} \longrightarrow X_\alpha^t$ which satisfies the following variation of constants formula:

$$u(t) = U(t,\sigma)u(\sigma) + \int_\sigma^t U(t,s)h(s)ds \qquad \text{for all } t \geq \sigma. \tag{4.2}$$

Similarly, we define mild solution of equation (1.1) by taking $f(\cdot, u(\cdot)) = h(\cdot)$.

Now, we prove the existence and uniqueness of almost periodic mild solutions to (4.1) under the assumption that h is only S^p-almost periodic ($1 \leq p < \infty$). For technical reasons, we distinguish two cases, $p = 1$ and $1 < p < \infty$.

Theorem 4.1. *Let $h \in L^\infty(\mathbb{R}, X)$ and assume that **(H1)-(H2)** and **(H4)** are satisfied. Then, equation (4.1) has a unique bounded mild solution $u : \mathbb{R} \longrightarrow X_\alpha$ given by*

$$u(t) = \int_{\mathbb{R}} G(t,s)h(s)ds \quad \textit{for all } t \in \mathbb{R}. \tag{4.3}$$

Proof. Let $h \in L^\infty(\mathbb{R}, X)$. First, we show that the integral formula in (4.3) is well defined and yields a continuous bounded function in X_α. Indeed, using the estimates (2.8) and (2.9), one has

$$\begin{aligned}
\|\int_{\mathbb{R}} G(t,s)h(s)ds\|_\alpha &= \|\int_{-\infty}^t U(t,s)P(s)h(s)ds - \int_t^{+\infty} U(t,s)Q(s)h(s)ds\|_\alpha \\
&\leq \int_{-\infty}^t \|U(t,s)P(s)h(s)\|_\alpha ds + \int_t^{+\infty} \|U(t,s)Q(s)h(s)\|_\alpha ds \\
&\leq m(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\gamma(t-s)}\|h(s)\|ds \\
&+ c(\alpha)\int_t^{+\infty} e^{-\delta(s-t)}\|h(s)\|ds \\
&\leq \left(m(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha) + c(\alpha)\delta^{-1}\right)\|h\|_\infty, \ t \in \mathbb{R}.
\end{aligned}$$

Thus $v := \displaystyle\int_{\mathbb{R}} G(\cdot,s)h(s)ds$ defines a bounded function from $\mathbb{R}$ to X_α and it is clear that it is also continuous. Now, we show that v is the unique mild solution of equation (4.1). Let $\sigma \in \mathbb{R}$ be fixed and $t \geq \sigma$. Then, by the property of evolution families, we claim that

$$\begin{aligned} v(t) &= \int_{-\infty}^{t} U(t,s)P(s)h(s)ds - \int_{t}^{+\infty} \tilde{U}(t,s)Q(s)h(s)ds \\ &= \int_{-\infty}^{\sigma} U(t,s)P(s)h(s)ds + \int_{\sigma}^{t} U(t,s)P(s)h(s)ds \\ &\quad - \int_{t}^{\sigma} \tilde{U}(t,s)Q(s)h(s)ds - \int_{\sigma}^{+\infty} \tilde{U}(t,s)Q(s)h(s)ds \\ &= U(t,\sigma)v(\sigma) + \int_{\sigma}^{t} U(t,s)h(s)ds \end{aligned}$$

where $P(t)$ and $Q(t)$, $t \in \mathbb{R}$ are the corresponding projections of the exponential dichotomy. Conversely, let u be the bounded mild solution of equation (4.1) defined by

$$u(t) = U(t,\sigma)u(\sigma) + \int_{\sigma}^{t} U(t,s)h(s)ds.$$

Then,

$$P(t)u(t) = U(t,\sigma)P(\sigma)u(\sigma) + \int_{\sigma}^{t} U(t,s)P(s)h(s)ds$$

and

$$Q(t)u(t) = U(t,\sigma)Q(\sigma)u(\sigma) + \int_{\sigma}^{t} \tilde{U}(t,s)Q(s)h(s)ds$$

Hence, by the boundedness of u and in view of the estimates (2.8) and (2.9), it follows using the dominated convergence Theorem, by letting $\sigma \to -\infty$ and $\sigma \to -\infty$ respectively, that

$$P(t)u(t) = \int_{-\infty}^{t} U(t,s)P(s)h(s)ds$$

and

$$Q(t)u(t) = -\int_{t}^{+\infty} U(t,s)Q(s)h(s)ds$$

Since the decomposition $u(t) = P(t)u(t) + Q(t)u(t)$ is unique, we obtain that u is uniquely determined by the integral formula given by v. □

For $1 < p < \infty$, we dot not need to assume the boundedness of h, only to be S^p-bounded is sufficient. Indeed, we have the following main result.

Theorem 4.2. *Let $h \in BS^p(\mathbb{R}, X)$ and assume that **(H1)-(H2)** and **(H4)** hold. Then, equation* (4.1) *has a unique bounded mild solution $u : \mathbb{R} \longrightarrow X_\alpha$ defined by the integral formula* (4.3).

Proof. The fact that the mild solution of equation (4.1) is defined uniquely by (4.3) is similar as in the proof of Theorem 4.1. Hence, it suffices to prove that

$u \in BC(\mathbb{R}, X_\alpha)$. Let $h \in BS^p(\mathbb{R}, X)$, using Hölder inequality, we obtain that

$$\begin{aligned}
& \| \int_{\mathbb{R}} G(t,s)h(s)ds \|_\alpha \\
= & \| \int_{-\infty}^{t} U(t,s)P(s)h(s)ds - \int_{t}^{+\infty} U(t,s)Q(s)h(s)ds \|_\alpha \\
\leq & \int_{-\infty}^{t} \|U(t,s)P(s)h(s)\|_\alpha ds + \int_{t}^{+\infty} \|U(t,s)Q(s)h(s)\|_\alpha ds \\
\leq & m(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|h(s)\| ds + c(\alpha) \int_{t}^{+\infty} e^{-\delta(s-t)} \|h(s)\| ds \\
\leq & m(\alpha) \left(\int_{-\infty}^{t} (t-s)^{-q\alpha} e^{-q\frac{\gamma}{2}(t-s)} ds \right)^{\frac{1}{q}} \left(\int_{-\infty}^{t} e^{-p\frac{\gamma}{2}(t-s)} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\
+ & c(\alpha) \left(\int_{t}^{+\infty} e^{-q\frac{\delta}{2}(s-t)} ds \right)^{\frac{1}{q}} \left(\int_{t}^{+\infty} e^{-p\frac{\delta}{2}(s-t)} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\
= & m(\alpha) \left(\frac{2}{q\gamma} \right)^{\alpha} \Gamma(q(1-\alpha))^{\frac{1}{q}} \left(\sum_{k\geq 1} \int_{t-k}^{t-k+1} e^{-p\frac{\gamma}{2}(t-s)} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\
+ & c(\alpha) \left(\frac{2}{q\delta} \right)^{\frac{1}{q}} \left(\sum_{k\geq 1} \int_{t+k-1}^{t+k} e^{-p\frac{\delta}{2}(s-t)} \|h(s)\|^p ds \right)^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
\leq & m(\alpha) \left(\frac{2}{q\gamma} \right)^{\alpha} \Gamma(q(1-\alpha))^{\frac{1}{q}} \left(\sum_{k\geq 1} e^{-p\frac{\gamma}{2}(k-1)} \int_{t-k}^{t-k+1} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\
+ & c(\alpha) \left(\frac{2}{q\delta} \right)^{\frac{1}{q}} \left(\sum_{k\geq 1} e^{-p\frac{\delta}{2}(k-1)} \int_{t+k-1}^{t+k} \|h(s)\|^p ds \right)^{\frac{1}{p}} \\
= & \left[m(\alpha) \left(\frac{2}{q\gamma} \right)^{\alpha} \Gamma(q(1-\alpha))^{\frac{1}{q}} \frac{e^{\frac{\gamma}{2}}}{e^{\frac{\gamma}{2}} - 1} + c(\alpha) \left(\frac{2}{q\delta} \right)^{\frac{1}{q}} \frac{e^{\frac{\delta}{2}}}{e^{\frac{\delta}{2}} - 1} \right] \|h\|_{BS^p}, \ t \in \mathbb{R}.
\end{aligned}$$

This proves the result. □

Next result, we show that the unique bounded mild solution u is almost periodic.

Theorem 4.3. *Let $h \in APS^1(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X)$ and assume that **(H1)-(H4)** are satisfied. Then, equation* (4.1) *has a unique almost periodic mild solution $u : \mathbb{R} \longrightarrow X_\alpha$ given by the integral formula* (4.3).

Proof. Let $h \in APS^1(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X)$ and u be the unique bounded mild solution of equation (4.1) provided by Theorem 4.1. Then, we have

$$\begin{aligned}
u(t) &= \int_{-\infty}^{t} U(t,s)P(s)h(s)ds - \int_{t}^{+\infty} \tilde{U}(t,s)Q(s)h(s)ds \\
&:= u^P(t) + u^Q(t), \quad t \in \mathbb{R}
\end{aligned}$$

where $u^P(t) = \int_{-\infty}^{t} U(t,s)P(s)h(s)ds$ and $u^Q(t) = -\int_{t}^{+\infty} \tilde{U}(t,s)Q(s)h(s)ds$.
Let $t,\ \tau \in \mathbb{R}$. Thus, in view of Cauchy–Schwarz inequality, we have

$$\begin{aligned}
& \|u^P(t+\tau) - u^P(t)\|_\alpha \\
= \; & \| \int_{-\infty}^{t+\tau} U(t+\tau,s)P(s)h(s)ds - \int_{-\infty}^{t} U(t,s)P(s)h(s)ds\|_\alpha \\
\leq \; & \int_{-\infty}^{t} \|U(t,s)P(s)\left[h(s+\tau)-h(s)\right]\|_\alpha ds \\
+ \; & \int_{-\infty}^{t} \|\left[U(t+\tau,s+\tau)P(s+\tau) - U(t,s)P(s)\right]h(s+\tau)\|_\alpha ds \\
\leq \; & m(\alpha)\sqrt{2\gamma^{-1}\|h\|_\infty \Gamma(2\alpha+1)} \left(\sum_{k\geq 1} e^{-\gamma(k-1)} \int_{t-k}^{t-k+1} \|h(s+\tau)-h(s)\|ds \right)^{\frac{1}{2}} \\
+ \; & m(\alpha) \sum_{k\geq 1} \left(\int_{t-k}^{t-k+1} \|\left[U(t+\tau,s+\tau)P(s+\tau) - U(t,s)P(s)\right]h(s+\tau)\|_\alpha ds \right)^{\frac{1}{2}} \\
\times \; & e^{-\frac{\gamma}{4}(k-1)} \left(2\int_{t-k}^{t-k+1} (t-s)^{-\alpha} e^{-\frac{\gamma}{2}((t-s)} \|h(s+\tau)\|ds \right)^{\frac{1}{2}} \\
\\
\leq \; & m(\alpha)\sqrt{2\gamma^{-1}\|h\|_\infty \Gamma(2\alpha+1)} \left(\sum_{k\geq 1} e^{-\gamma(k-1)} \int_{t-k}^{t-k+1} \|h(s+\tau)-h(s)\|ds \right)^{\frac{1}{2}} \\
+ \; & m(\alpha)\sqrt{2\left(\frac{2}{\gamma}\right)^{1-\alpha} \Gamma(1+\alpha)\|h\|_\infty} \\
\times \; & \sum_{k\geq 1} e^{-\frac{\gamma}{4}(k-1)} \left(\int_{t-k}^{t-k+1} \|\left[U(t+\tau,s+\tau)P(s+\tau) - U(t,s)P(s)\right]h(s+\tau)\|_\alpha ds \right)^{\frac{1}{2}}
\end{aligned}$$

Let $\varepsilon > 0$, since $h \in APS^1(\mathbb{R}, X)$, it follows that, there exists $l_\varepsilon > 0$ such that every interval of length $l_\varepsilon > 0$ contains an element τ such that

$$\int_{t}^{t+1} \|h(s+\tau)-h(s)\|ds \leq \frac{\varepsilon^2\sqrt{e^\gamma - 1}}{4m(\alpha)\sqrt{2\gamma^{-1}\|h\|_\infty \Gamma(2\alpha+1)e^\gamma}} \text{ uniformly for } t \in \mathbb{R}. \tag{4.4}$$

Furthermore, by hypothesis **(H3)**, we can find the same almost period τ such that for each $x \in X$, we have

$$\|U(t+\tau,s+\tau)P(s+\tau)x - U(t,s)P(s)x\|_\alpha \leq \frac{\varepsilon^2(e^{\frac{\gamma}{4}}-1)}{4m(\alpha)\|h\|_\infty\sqrt{2\left(\frac{2}{\gamma}\right)^{1-\alpha}\Gamma(1+\alpha)}}\|x\| \tag{4.5}$$

uniformly for $t, s \in \mathbb{R}$. Therefore,

$$\|u^P(t+\tau) - u^P(t)\|_\alpha \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \text{ uniformly for } t \in \mathbb{R}.$$

By the same way, we prove that

$$\|u^Q(t+\tau)-u^Q(t)\|_\alpha \leq \frac{\varepsilon}{2} \text{ uniformly for } t\in\mathbb{R}.$$

Hence,

$$\|u(t+\tau)-u(t)\|_\alpha \leq \varepsilon \text{ uniformly for } t\in\mathbb{R}.$$

This proves the almost periodicity of the mild solution u. $\square$

Similarly, for $1<p<\infty$, we can prove easily the following existence result of a unique almost periodic solution for equation (4.1).

Theorem 4.4. *Let $h\in APS^p(\mathbb{R},X)$ and assume that **(H1)-(H4)** are satisfied. Then, equation* (4.1) *has a unique almost periodic mild solution $u:\mathbb{R}\longrightarrow X_\alpha$ given by the integral formula* (4.3).

Now, we return to equation (1.1) and we give our main results on the existence and uniqueness of almost periodic solutions. Furthermore, consider $u\in AP(\mathbb{R},X_\alpha)$, then using Theorem 3.2, the function $h=f(\cdot,u(\cdot))\in APS^p(\mathbb{R},X)$ for all $1\leq p<\infty$. Moreover, let $f(\cdot,x)\in APS^1(\mathbb{R},X)\cap L^\infty(\mathbb{R},X)$ (resp. $f(\cdot,x)\in APS^p(\mathbb{R},X)$, $1<p<\infty$) for each $x\in X_\alpha$. Then, in view of Theorems 4.3 and 4.4, the map $F:AP(\mathbb{R},X_\alpha)\longrightarrow AP(\mathbb{R},X_\alpha)$ given by

$$Fu(t)=\int_{\mathbb{R}} G(t,s)f(s,u(s))ds, \quad t\in\mathbb{R}$$

is well-defined. Further on, the following Lemma which shows the positive invariance of the solution, is also needed.

Lemma 4.5. *Assume that **(H1)-(H2)**, **(H4)** and **(C)** hold. Then the operator given by*

$$Fu(t)=\int_{\mathbb{R}} G(t,s)f(s,u(s))ds, \quad t\in\mathbb{R},$$

maps $BC(\mathbb{R},X_\alpha^+)$ into itself.

Proof. Let $u\in BC(\mathbb{R},X_\alpha^+)$ it is well-known that F is well-defined and that $Fu(\cdot)\in BC(\mathbb{R},X_\alpha)$. To conclude, it suffices to show that $Fu(t)\in X_\alpha^+$ for every $t\in\mathbb{R}$. That is, by assumption **(C)** we have $f(s,u(s))\in X_\alpha^+$ and that $G(t,s)f(s,u(s))\in X_\alpha^+$ for $t,s\in\mathbb{R}$. So the result holds by integrating over $\mathbb{R}$. $\square$

Consequently from Lemma 4.5 it is clear that F maps $AP(\mathbb{R},X_\alpha^+)$ into itself.

Theorem 4.6. *Let $f(\cdot,x)\in APS^1(\mathbb{R},X)\cap L^\infty(\mathbb{R},X)$ for each $x\in X_\alpha$ satisfying **(H5)** with $L_\rho\in L^\infty(\mathbb{R},\mathbb{R}^+)$. Assume that **(H1)-(H4)** and **(C)** hold. If there exists $\rho>0$ such that*

$$\rho>\left[m(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)+c(\alpha)\delta^{-1}\right]\|f(\cdot,0)\|_\infty>0, \tag{4.6}$$

and

$$\|L_\rho\|_\infty\leq\left[m(\alpha)\gamma^{\alpha-1}\Gamma(1-\alpha)+c(\alpha)\delta^{-1}\right]^{-1}-\rho^{-1}\|f(\cdot,0)\|_\infty. \tag{4.7}$$

Then, equation (1.1) *has a unique almost periodic solution $u:\mathbb{R}\longrightarrow X_\alpha$ with $0\leq u(t)\leq\rho$ for all $t\in\mathbb{R}$.*

Proof. Consider the set $\Lambda_\rho^{AP} := \{v \in AP(\mathbb{R}, X_\alpha^+) : \|v\|_\infty \leq \rho\}$ and define the map $F : \Lambda_\rho^{AP} \longrightarrow AP(\mathbb{R}, X_\alpha^+)$ by

$$Fu(t) = \int_{\mathbb{R}} G(t,s) f(s,u(s)) ds, \quad t \in \mathbb{R}.$$

First, we show that $F\Lambda_\rho^{AP} \subset \Lambda_\rho^{AP}$. Let $u \in \Lambda_\rho^{AP}$. Then, by assumptions on f, we obtain that

$$\begin{aligned}
& \|Fu(t)\|_\alpha \\
\leq\ & \int_{\mathbb{R}} \|G(t,s) f(s,u(s))\|_\alpha ds \\
\leq\ & m(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|f(s,u(s)) - f(s,0)\| ds \\
+\ & m(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|f(s,0)\| ds \\
+\ & c(\alpha) \int_{t}^{+\infty} e^{-\delta(s-t)} \|f(s,u(s)) - f(s,0)\| ds + c(\alpha) \int_{t}^{+\infty} e^{-\delta(s-t)} \|f(s,0)\| ds \\
\leq\ & \rho m(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \left[L_\rho(s) + \rho^{-1} \|f(s,0)\| \right] ds \\
+\ & \rho c(\alpha) \int_{t}^{+\infty} e^{-\delta(s-t)} \left[L_\rho(s) + \rho^{-1} \|f(s,0)\| \right] ds \\
\leq\ & \rho m(\alpha) \left[\|L_\rho\|_\infty + \rho^{-1} \|f(\cdot,0)\|_\infty \right] \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} ds \\
+\ & \rho c(\alpha) \left[\|L_\rho\|_\infty + \rho^{-1} \|f(\cdot,0)\|_\infty \right] \int_{t}^{+\infty} e^{-\delta(s-t)} ds \\
=\ & \rho \left[m(\alpha) \gamma^{\alpha-1} \Gamma(1-\alpha) + c(\alpha) \delta^{-1} \right] \left[\|L_\rho\|_\infty + \rho^{-1} \|f(\cdot,0)\|_\infty \right]
\end{aligned}$$

Hence, $F\Lambda_\rho^{AP} \subset \Lambda_\rho^{AP}$. Therefore, let $u,\ v \in \Lambda_\rho^{AP}$. Then, a straightforward calculation yields

$$\begin{aligned}
& \|Fu(t) - Fv(t)\|_\alpha \\
\leq\ & \int_{\mathbb{R}} \|G(t,s) \left[f(s,u(s)) - f(s,u(s)) \right] \|_\alpha ds \\
\leq\ & m(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} \|f(s,u(s)) - f(s,v(s))\| ds \\
+\ & c(\alpha) \int_{t}^{+\infty} e^{-\delta(s-t)} \|f(s,u(s)) - f(s,v(s))\| ds \\
\leq\ & m(\alpha) \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\gamma(t-s)} L_\rho(s) ds \|u-v\|_\infty \\
+\ & c(\alpha) \int_{t}^{+\infty} e^{-\delta(s-t)} L_\rho(s) ds \|u-v\|_\infty \\
=\ & \left[m(\alpha) \gamma^{\alpha-1} \Gamma(1-\alpha) + c(\alpha) \delta^{-1} \right] \|L_\rho\|_\infty \|u-v\|_\infty,
\end{aligned}$$

for all $t \in \mathbb{R}$. Thus the conditions (4.6) and (4.6) yields that

$$\left[m(\alpha) \gamma^{\alpha-1} \Gamma(1-\alpha) + c(\alpha) \delta^{-1} \right] \|L_\rho\|_\infty < 1.$$

Therefore the map F defines a strict contraction in Λ_ρ^{AP}. Consequently, by Banach fixed point Theorem, we obtain the existence and uniqueness of a solution $u \in \Lambda_\rho^{AP}$. This proves the result. $\square$

Theorem 4.7. *Let $f(\cdot, x) \in APS^p(\mathbb{R}, X)$ for each $x \in X_\alpha$ satisfying **(H5)** with $L_\rho \in BS^p(\mathbb{R}, \mathbb{R}^+)$ for $1 < p < \infty$. Assume that **(H1)**-**(H4)** and **(C)** hold. If there exists $\rho > 0$ such that*

$$\rho > \left[m(\alpha) \left(\frac{2}{q\gamma} \right)^{\alpha} \Gamma(q(1-\alpha))^{\frac{1}{q}} \frac{e^{\frac{\gamma}{2}}}{e^{\frac{\gamma}{2}} - 1} + c(\alpha) \left(\frac{2}{q\delta} \right)^{\frac{1}{q}} \frac{e^{\frac{\delta}{2}}}{e^{\frac{\delta}{2}} - 1} \right] \|f(\cdot, 0)\|_{BS^p} > 0, \tag{4.8}$$

and

$$\|L_\rho\|_{BS^p} \leq \left[m(\alpha) \left(\frac{2}{q\gamma} \right)^{\alpha} \Gamma(q(1-\alpha))^{\frac{1}{q}} \frac{e^{\frac{\gamma}{2}}}{e^{\frac{\gamma}{2}} - 1} + c(\alpha) \left(\frac{2}{q\delta} \right)^{\frac{1}{q}} \frac{e^{\frac{\delta}{2}}}{e^{\frac{\delta}{2}} - 1} \right]^{-1} - \rho^{-1} \|f(\cdot, 0)\|_{BS^p}. \tag{4.9}$$

Then, equation (1.1) has a unique almost periodic solution $u : \mathbb{R} \longrightarrow X_\alpha$ with $0 \leq u(t) \leq \rho$ for all $t \in \mathbb{R}$.

Proof. Consider the set $\Lambda_\rho^{AP} := \{v \in AP(\mathbb{R}, X_\alpha^+) : \|v\|_\infty \leq \rho\}$ and define the map $F : \Lambda_\rho^{AP} \longrightarrow AP(\mathbb{R}, X_\alpha^+)$ by

$$Fu(t) = \int_{\mathbb{R}} G(t,s) f(s, u(s)) ds, \quad t \in \mathbb{R}.$$

First, we show that $F\Lambda_\rho^{AP} \subset \Lambda_\rho^{AP}$. Indeed, let $u \in \Lambda_\rho^{AP}$. Then, by assumptions on f, we obtain that

$$\begin{aligned}
& \|Fu(t)\|_\alpha \\
\leq\ & \int_{\mathbb{R}} \|G(t,s)f(s,u(s))\|_\alpha ds \\
\leq\ & m(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\gamma(t-s)}\|f(s,u(s))-f(s,0)\|ds \\
+\ & m(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\gamma(t-s)}\|f(s,0)\|ds \\
+\ & c(\alpha)\int_{t}^{+\infty} e^{-\delta(s-t)}\|f(s,u(s))-f(s,0)\|ds + c(\alpha)\int_{t}^{+\infty} e^{-\delta(s-t)}\|f(s,0)\|ds \\
\leq\ & \rho m(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\gamma(t-s)}\left[L_\rho(s)+\rho^{-1}\|f(s,0)\|\right]ds \\
+\ & \rho c(\alpha)\int_{t}^{+\infty} e^{-\delta(s-t)}\left[L_\rho(s)+\rho^{-1}\|f(s,0)\|\right]ds \\
\leq\ & \rho m(\alpha)\left(\int_{-\infty}^{t}(t-s)^{-q\alpha}e^{-q\frac{\gamma}{2}(t-s)}ds\right)^{\frac{1}{q}}\left(\int_{-\infty}^{t} e^{-p\frac{\gamma}{2}(t-s)}\left[|L_\rho(s)|^p+\rho^{-p}\|f(s,0)\|^p\right]ds\right)^{\frac{1}{p}} \\
+\ & \rho c(\alpha)\left(\int_{t}^{+\infty}e^{-q\frac{\delta}{2}(s-t)}ds\right)^{\frac{1}{q}}\left(\int_{t}^{+\infty} e^{-p\frac{\delta}{2}(s-t)}\left[|L_\rho(s)|^p+\rho^{-p}\|f(s,0)\|^p\right]ds\right)^{\frac{1}{p}} \\
\leq\ & \rho\left[m(\alpha)\left(\frac{2}{q\gamma}\right)^{\alpha}\Gamma(q(1-\alpha))^{\frac{1}{q}}\frac{e^{\frac{\gamma}{2}}}{e^{\frac{\gamma}{2}}-1}+c(\alpha)\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\frac{e^{\frac{\delta}{2}}}{e^{\frac{\delta}{2}}-1}\right] \\
& \times\left[\|L_\rho\|_{BS^p}+\rho^{-1}\|f(\cdot,0)\|_{BS^p}\right] \\
\leq\ & \rho, \quad t\in\mathbb{R}.
\end{aligned}$$

Hence, $F\Lambda_\rho^{AP} \subset \Lambda_\rho^{AP}$. Therefore, let $u,\ v \in \Lambda_\rho^{AP}$. Then, a straightforward calculation yields

$$\begin{aligned}
& \|Fu(t)-Fv(t)\|_\alpha \\
\leq\ & \int_{\mathbb{R}} \|G(t,s)\left[f(s,u(s))-f(s,u(s))\right]\|_\alpha ds \\
\leq\ & m(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\gamma(t-s)}\|f(s,u(s))-f(s,v(s))\|ds \\
+\ & c(\alpha)\int_{t}^{+\infty} e^{-\delta(s-t)}\|f(s,u(s))-f(s,v(s))\|ds \\
\leq\ & m(\alpha)\int_{-\infty}^{t} (t-s)^{-\alpha}e^{-\gamma(t-s)}L_\rho(s)ds\|u-v\|_\infty \\
+\ & c(\alpha)\int_{t}^{+\infty} e^{-\delta(s-t)}L_\rho(s)ds\|u-v\|_\infty \\
\leq\ & \|L_\rho\|_{BS^p}\left[m(\alpha)\left(\frac{2}{q\gamma}\right)^{\alpha}\Gamma(q(1-\alpha))^{\frac{1}{q}}\frac{e^{\frac{\gamma}{2}}}{e^{\frac{\gamma}{2}}-1}+c(\alpha)\left(\frac{2}{q\delta}\right)^{\frac{1}{q}}\frac{e^{\frac{\delta}{2}}}{e^{\frac{\delta}{2}}-1}\right]\|u-v\|_\infty,
\end{aligned}$$

for all $t \in \mathbb{R}$. Thus, by assumption, the map F defines a contraction in Λ_ρ^{AP}. Consequently, by Banach fixed point argument, we obtain the existence and uniqueness of a solution $u \in \Lambda_\rho^{AP}$. This proves the result. $\square$

5. Application

In this section we study the dynamics of a two-species competition model with diffusion and time dependent parameters, represented by a system of two non-autonomous reaction-diffusion equations over two interacting density species $u(t,\xi)$ and $v(t,\xi)$ at the location (a unit of space) ξ which belongs to the habitat $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, in a generalized almost periodic environment, namely

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,\xi) = d_1(t)\Delta u(t,\xi) + a(t)u(t,\xi) - c_1(t)\dfrac{v(t,\xi)u(t,\xi)}{1+v(t,\xi)}, \ t \in \mathbb{R}, \xi \in \Omega, \\ \dfrac{\partial}{\partial t}v(t,\xi) = d_2(t)\Delta v(t,\xi) - b(t)v(t,\xi) + c_2(t)\dfrac{u(t,\xi)v(t,\xi)}{1+|\nabla u(t,\xi)|}, \ t \in \mathbb{R}, \xi \in \Omega, \\ u(t,\xi)|_{\partial\Omega} = 0; \ v(t,\xi)|_{\partial\Omega} = 0, \ t \in \mathbb{R}, \xi \in \partial\Omega \end{cases} \tag{5.1}$$

where,

• $u(t,\xi)$ and $v(t,\xi)$ are respectively the local densities of the preys and the predators at time t and at location ξ.

• $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is an open bounded subset with Lipschitz boundary conditions.

• $\Delta := \sum_{k=1}^{n} \dfrac{\partial^2}{\partial \xi_i^2}$ is the Laplace operator on Ω, $d_i \in C^\mu(\mathbb{R},\mathbb{R}^+)$, $0 < \mu \leq 1$ (eventually here $\mu = 1$), $i = 1,2$ are the diffusion terms corresponding to the preys and the predators respectively, such that $0 < d_i^0 := \inf_{t\in\mathbb{R}}(d_i) \leq d_i^1 := |d_i|_\infty < \infty$.

• $a,\ b \in L^1_{loc}(\mathbb{R},\mathbb{R}^+)$ correspond to the growth terms in the absence of interaction, of the populations $u(t,\xi)$ and $v(t,\xi)$ respectively.

•. The nonlinear terms $g_i : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N \longrightarrow \mathbb{R}$ are defined by

$$g_1(t,u(t,\xi),v(t,\xi),\nabla v(t,\xi)) = a(t)u(t,\xi) - c_1(t)\frac{v(t,\xi)u(t,\xi)}{1+v(t,\xi)}$$

and

$$g_2(t,u(t,\xi),v(t,\xi),\nabla u(t,\xi)) = c_2(t)\frac{u(t,\xi)v(t,\xi)}{1+|\nabla u(t,\xi)|}$$

where $c_i \in L^1_{loc}(\mathbb{R},\mathbb{R}^+)$ for $i = 1,2$ are the interaction terms of the preys and the predators respectively.

The abstract formulation. Consider the Banach space $X := C_0(\overline{\Omega}) \times C_0(\overline{\Omega})$, equipped with the given norm: $\| \begin{pmatrix}\varphi \\ \psi\end{pmatrix} \| = \|\varphi\|_\infty + \|\psi\|_\infty$, where $C_0(\overline{\Omega})$ is the space of continuous functions $\varphi : \overline{\Omega} \longleftrightarrow \mathbb{R}$ such that $\varphi|_{\partial\Omega} = 0$. We define the closed linear operators $(A(t), D(A(t)))$, $t \in \mathbb{R}$, by

$$\begin{cases} A(t) := \begin{pmatrix} d_1(t)\Delta & 0 \\ 0 & d_2(t)\Delta - b(t)\end{pmatrix}, \\ D(A(t)) = C_0^2(\overline{\Omega}) \times C_0^2(\overline{\Omega}) := D \end{cases} \tag{5.2}$$

where $C_0^2(\overline{\Omega}) := \left\{\varphi \in C_0(\overline{\Omega}) \cap H_0^1(\Omega) : \ \Delta\varphi \in C_0(\overline{\Omega})\right\}$. Let $\alpha \in (1/2,1)$ and X_α be the real interpolation space between X and $D(A(t))$ given by $X_\alpha^t = X_\alpha :=$

$C_\alpha(\overline{\Omega}) \times C_\alpha(\overline{\Omega})$ where $C_\alpha(\overline{\Omega}) = \{\varphi \in C^{2\alpha}(\overline{\Omega}) : \varphi|_{\partial\Omega} = 0\}$ with the norm $\|\cdot\|_{0,\alpha} := \|\cdot\|_\infty + \|\nabla \cdot\|_\infty + [\cdot]_\theta$ where $[\varphi]_\theta := \sup_{\xi,\eta\in\overline{\Omega},\ \xi\neq\eta} \dfrac{|\varphi(\xi)-\varphi(\eta)|}{|\xi-\eta|^\theta}$, $\theta = 2\alpha - 1$. Hence, $\| \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \|_\alpha = \|\varphi\|_{0,\alpha} + \|\psi\|_{0,\alpha}$ defines a norm on X_α, see [27] for more details about the spaces of Hölder continuous functions $C^{2\alpha}(\overline{\Omega})$. It is clear that the following embedding hold

$$X_\alpha \hookrightarrow X_1 \hookrightarrow X,$$

where $X_1 = C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$. Therefore, the nonlinear term $f : \mathbb{R} \times X_\alpha \longrightarrow X$ is defined by

$$f(t, \begin{pmatrix}\varphi\\ \psi\end{pmatrix})(\xi) = \begin{pmatrix} g_1(t, \varphi(\xi), \psi(\xi), \nabla\psi(\xi)) \\ g_2(t, \varphi(\xi), \psi(\xi), \nabla\varphi(\xi)) \end{pmatrix}.$$

Then, (5.1) takes the following abstract form that corresponds to equation (1.1):

$$u'(t) = A(t)u(t) + f(t, u(t)), \qquad t \in \mathbb{R}.$$

In order to prove the existence and uniqueness of almost periodic solutions to (1.1), we check forward our main hypotheses **(H1)-(H5)** and **(C)**. One can remark that the domains are independent of t i.e., $D(A(t)) = D$ which yields that the real interpolation spaces X_α are also indepedent of t and then hypothesis **(H4)** holds. Now, we prove that $(A(t), D(A(t))),\ t \in \mathbb{R}$ satisfy hypotheses **(H1)-(H3)**. Define the operators $A_1(t) = d_1(t)\Delta$ and $A_2(t) = d_2(t)\Delta$. By [14], the operator $A_0 := -\Delta$ on $C_0(\overline{\Omega})$ is sectorial with constant $M \geq 1$ and angle of sectoriality $\theta \in (\frac{\pi}{2}, \pi)$ such that

$$\|\lambda R(\lambda, A_0)\|_{\mathcal{L}(X)} \leq M \quad \text{ for all } \lambda \in \Sigma_{0,\theta}. \tag{5.3}$$

Then, using (5.3) and by assumptions on d_i we claim that

$$\|\lambda R(\lambda, A_i(t))\| = \|\frac{\lambda}{d_i(t)} R(\frac{\lambda}{d_i(t)}, A_0)\| \leq M \quad \text{ for all } t \in \mathbb{R}.$$

Hence, for each $t \in \mathbb{R}$, $A_i(t)$ generates a bounded analytic semigroup $(T_t^i(\tau))_{\tau\geq 0}$ (with uniform bound M with respect to t and the same angle θ) on $C_0(\overline{\Omega})$ such that

$$\| T_t^i(\tau) \| \leq M e^{-d_i^0 \lambda_1 \tau} \qquad \text{for } \tau \geq 0, \tag{5.4}$$

where $\lambda_1 := \min\{\lambda :\ \lambda \in \sigma(A_0)\} > 0$ and $\sigma(A_0)$ is the spectrum of $-\Delta$ in $H_0^1(\Omega)$ and $M = e^{\lambda_1 |\Omega|^{2/N}(4\pi)^{-1}}$, see [20] for more details. Moreover,

$$\sup_{t,s\in\mathbb{R}} \|A_i(t)A_i(s)^{-1}\| = \sup_{t,s\in\mathbb{R}} \frac{d_i(t)}{d_i(s)} < \infty.$$

Furthermore, by

$$\|A_i(t)A_i(s)^{-1} - I_X\| = d_i(s)^{-1}\,|d_i(t) - d_i(s)| \leq L_i\,|d_i(t) - d_i(s)| \leq L_i |t-s|^\mu$$

where $L_i := L_{0,i}(d_i^0)^{-1}$ with $L_{0,i}$ is the Hölder constants of d_i. Thus, by (2.6) we obtain that, for each $i = 1, 2$, $(A_i(t))_{t\in\mathbb{R}}$ generates an evolution family $(U_i(t,s))_{t\geq s}$ on $C_0(\overline{\Omega})$. In otherwise, by (5.4), the semigroups $(T_t^i(\tau))_{\tau\geq 0}$ are hyperbolic with projections $P(t) = I_X$ and $Q(t) = 0$, $t \in \mathbb{R}$ with

$$\|\tau A_i(t) T_t^i(\tau) x\| \leq M e^{-d_i^0 \lambda_1 \tau} \quad \text{ for } \tau > 0.$$

So by taking $\phi_i(\sigma) := M\sigma e^{-d_i^0\lambda_1\sigma}\chi_{\sigma>0}$ we obtain that $L_i\|\phi_i\|_1 := L_iM(d_i^0\lambda_1)^{-2}$. Then, the condition $L_iM(d_i^0\lambda_1)^{-2} < 1$ yields that each evolution family $(U_i(t,s))_{t\geq s}$ is hyperbolic with the same projection I_0 and exponent δ_i satisfying

$$0 < \delta_i < d_i^0\lambda_1 - L_iM(d_i^0\lambda_1)^{-1},$$

and the associated Green functions $G_i(t,s) = U_i(t,s)$, $t \geq s$. Hence, the family of matrix-valued operators $\begin{pmatrix} A_1(t) & 0 \\ 0 & A_2(t)\end{pmatrix}_{t\in\mathbb{R}}$ generates the hyperbolic evolution family $\left(V(t,s) = \begin{pmatrix} U_1(t,s) & 0 \\ 0 & U_2(t,s)\end{pmatrix}\right)_{t\geq s}$ with projections $\left(P(t) = \begin{pmatrix} I_0 & 0 \\ 0 & I_0\end{pmatrix}\right)_{t\in\mathbb{R}}$ and exponent $\delta_0 = \min\{\delta_1;\delta_2\}$. Moreover, by rescaling, we obtain that $(A(t))_{t\in\mathbb{R}}$ generates the hyperbolic evolution family

$$\left(U(t,s) = \begin{pmatrix} U_1(t,s) & 0 \\ 0 & e^{-\int_s^t b(\sigma)d\sigma}U_2(t,s)\end{pmatrix}\right)_{t\geq s}$$

with projections $\left(P(t) = \begin{pmatrix} I_0 & 0 \\ 0 & I_0\end{pmatrix}\right)_{t\in\mathbb{R}}$ and exponent $\delta_0 = \min\{\delta_1;\delta_2 + |b|^{BS^1}\}$. Thus, hypotheses **(H1)** and **(H2)** hold with Green function $G(t,s) := U(t,s)$, $t \geq s$ provided that $L_iM(d_i^0\lambda_1^{-2}) < 1$. To check hypothesis **(H3)**, we need the following preliminary result.

Lemma 5.1. *For each $i = 1,2$, if $A_i(\cdot) \in AP(\mathbb{R}, \mathcal{L}(C_0^2(\overline{\Omega}), C_0(\overline{\Omega})))$. Then, the associated evolution family $U_i(\cdot,\cdot)$ is bi-almost periodic.*

Proof. Let $i = 1,2$ be fixed and $t,\tau \in \mathbb{R}$. Then, we have

$$A_i(t+\tau)^{-1} - A_i(t)^{-1} = A_i(t+\tau)^{-1}(A_i(t+\tau) - A_i(t))A_i(t)^{-1} \tag{5.5}$$

Following [28, Theorem 4.5.], it suffices to show that $A_i^{-1}(\cdot) \in AP(\mathbb{R}, \mathcal{L}(C_0(\overline{\Omega})))$. Let $\varepsilon > 0$ and $\varphi \in C_0(\overline{\Omega})$, from (5.5) and $A_i(\cdot) \in AP(\mathbb{R}, \mathcal{L}(C_0^2(\overline{\Omega}), C_0(\overline{\Omega})))$, it follows that there exists $l_\varepsilon > 0$ such that every interval of length l_ε contains an element τ such that

$$\begin{aligned}
& \|A_i(t+\tau)^{-1}\varphi - A_i(t)^{-1}\varphi\| \\
= \; & \|A_i(t+\tau)^{-1}(A_i(t+\tau) - A_i(t))A_i(t)^{-1}\varphi\| \\
\leq \; & \|A_i(t+\tau)^{-1}\|_{\mathcal{L}(C_0(\overline{\Omega}))}\|A_i(t+\tau) - A_i(t)\|_{\mathcal{L}(C_0^2(\overline{\Omega}),C_0(\overline{\Omega}))}\|A_i(t)^{-1}\varphi\|_{C_0^2(\overline{\Omega})} \\
\leq \; & C\varepsilon.
\end{aligned}$$

□

Consequently, we have the following main result.

Proposition 5.2. *Let $d_i \in AP(\mathbb{R})$, $i = 1,2$ and $b \in APS^1(\mathbb{R}, \mathbb{R}^+)$. Then, for each $\begin{pmatrix}\varphi \\ \psi\end{pmatrix} \in X$, the Green function $G(\cdot,\cdot)\begin{pmatrix}\varphi \\ \psi\end{pmatrix}$ is bi-almost periodic. Hence, hypothesis **(H3)** is satisfied.*

Proof. Since $d_i \in AP(\mathbb{R}, \mathbb{R}^+)$, it follows that $A_i(\cdot) \in AP(\mathbb{R}, \mathcal{L}(C_0^2(\overline{\Omega}), C_0(\overline{\Omega})))$ for $i = 1,2$. Then, by Lemma 5.1, we obtain that $U_i(\cdot,\cdot)$ are bi-almost periodic. Now, using Theorem 2.5 and Theorem 2.12, we show that $e^{-\int_\cdot^\cdot b(\sigma)d\sigma}U_2(\cdot,\cdot)$ is almost periodic . Let $(\sigma_n)_n$ be any sequence of real numbers, since $b \in APS^1(\mathbb{R})$, we can find a subsequence $(\tau_n)_n \subset (\sigma_n)_n$ and functions $\tilde{b}$ and $\tilde{U}_2(\cdot,\cdot)$ such that (2.10) and (2.11) hold for b and $U_2(\cdot,\cdot)$ respectively. Define the function $e^{-\int_{\cdot}^{\cdot} \tilde{b}(\sigma)d\sigma}\tilde{U}_2(\cdot,\cdot)$. Then, we obtain that

$$\begin{aligned}
& \|e^{-\int_{s+\tau_n}^{t+\tau_n} b(\sigma)d\sigma} U_2(t+\tau_n, s+\tau_n) - e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\tilde{U}_2(t,s)\| \\
\leq\ & e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\|U_2(t+\tau_n, s+\tau_n)\| \left| e^{-\int_s^t \left[b(\sigma+\tau_n)-\tilde{b}(\sigma)\right]d\sigma} - 1 \right| \\
+\ & e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\|U_2(t+\tau_n, s+\tau_n) - \tilde{U}_2(t,s)\| \\
\leq\ & Me^{-|\tilde{b}|_{BS^1}(t-s)} \left| e^{-\int_s^t \left[b(\sigma+\tau_n)-\tilde{b}(\sigma)\right]d\sigma} - 1 \right| \\
+\ & e^{-|\tilde{b}|_{BS^1}(t-s)}\|U_2(t+\tau_n, s+\tau_n) - \tilde{U}_2(t,s)\|.
\end{aligned}$$

Therefore, we have $\|U_2(t+\tau_n, s+\tau_n) - \tilde{U}_2(t,s)\| \to 0$ as $n\to\infty$ and

$$\begin{aligned}
e^{-\int_s^t \left[b(\sigma+\tau_n)-\tilde{b}(\sigma)\right]d\sigma} &\leq e^{-\sum_{k=[s]}^{[t]}\int_k^{k+1} |b(\sigma+\tau_n)-\tilde{b}(\sigma)|d\sigma} \\
&= e^{-\sup_k \int_k^{k+1} |b(\sigma+\tau_n)-\tilde{b}(\sigma)|d\sigma(t-s)} \\
&\leq e^{-\sup_k \int_k^{k+1} |b(\sigma+\tau_n)-\tilde{b}(\sigma)|d\sigma} \to 1 \text{ as } n\to\infty,
\end{aligned}$$

uniformly in $t,s\in\mathbb{R}$, $t\geq s$. Thus,

$$\|e^{-\int_{s+\tau_n}^{t+\tau_n} b(\sigma)d\sigma} U_2(t+\tau_n, s+\tau_n) - e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\tilde{U}_2(t,s)\| \to 0 \text{ as } n\to\infty,$$

uniformly in $t,s\in\mathbb{R}$, $t\geq s$. Hence $e^{-\int_{\cdot}^{\cdot} b(\sigma)d\sigma}U_2(\cdot,\cdot)$ is bi-almost periodic. Consequently, if we consider $\tilde{G}(\cdot,\cdot) = \begin{pmatrix} \tilde{U}_1(\cdot,\cdot) & 0 \\ 0 & e^{-\int_{\cdot}^{\cdot}\tilde{b}(\sigma)d\sigma}\tilde{U}_2(\cdot,\cdot)\end{pmatrix}$. Then, for $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in X$, we obtain that

$$\begin{aligned}
& \|G(t+\tau_n, s+\tau_n)\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \tilde{G}(t,s)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\| \leq \|U_1(t+\tau_n, s+\tau_n) - \tilde{U}_1(t,s)\|\|\varphi\|_\infty \\
+\ & \|e^{-\int_{s+\tau_n}^{t+\tau_n} b(\sigma)d\sigma} U_2(t+\tau_n, s+\tau_n) - e^{-\int_s^t \tilde{b}(\sigma)d\sigma}\tilde{U}_2(t,s)\|\|\psi\|_\infty \to 0 \text{ as } n\to\infty,
\end{aligned}$$

uniformly in $t,s\in\mathbb{R}$, $t\geq s$. This proves the result. $\square$

Proposition 5.3. *The function f satisfies **(H5)** with*

$$L_\rho(t) = a(t) + c_1(t) + (c_1(t) + c_2(t))\rho + c_2(t)\rho^2, \quad t \in \mathbb{R}.$$

.

Proof. Let $\begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix} \in X_\alpha$ and $\rho > 0$ be such that $\| \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix} \|_\alpha, \; \| \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix} \|_\alpha \le \rho$. Then, for g_2 we have

$$\begin{aligned}
& |g_2(t, \varphi(\xi), \psi(\xi), \nabla\psi(\xi)) - g_2(t, \varphi(\xi), \psi(\xi), \nabla\psi(\xi))| \\
= \;& c_2(t) \left| \left[\frac{\varphi_1(\xi)\psi_1(\xi)}{1 + |\nabla\varphi_1(\xi)|} - \frac{\varphi_2(\xi)\psi_2(\xi)}{1 + |\nabla\varphi_2(\xi)|} \right] \right| \\
= \;& c_2(t) \left| \left[\frac{\varphi_1(\xi)\psi_1(\xi) - \varphi_2(\xi)\psi_2(\xi)}{1 + |\nabla\varphi_2(\xi)|} \right] \right. \\
- \;& \left. \left[\frac{\varphi_1(\xi)\psi_1(\xi)(|\nabla\varphi_2(\xi)| - |\nabla\varphi_1(\xi)|)}{(1 + |\nabla\varphi_1(\xi)|)(1 + |\nabla\varphi_2(\xi)|)} \right] \right| \\
= \;& -c_2(t) \left| \frac{\psi_2(\xi)}{1 + |\nabla\varphi_2(\xi)|} [\varphi_1(\xi) - \varphi_2(\xi)] - \frac{\varphi_1(\xi)}{1 + |\nabla\varphi_2(\xi)|} [\psi_1(\xi) - \psi_2(\xi)] \right. \\
- \;& \left. \frac{\varphi_1(\xi)\psi_1(\xi)}{(1 + |\nabla\varphi_1(\xi)|)(1 + |\nabla\varphi_2(\xi)|)} (|\nabla\varphi_2(\xi)| - |\nabla\varphi_1(\xi)|) \right| \\
\le \;& c_2(t)(\rho^2 + \rho) \left(\|\varphi_1 - \varphi_2\|_{0,\alpha} + \|\psi_1 - \psi_2\|_{0,\alpha} \right) \\
\le \;& c_2(t)(\rho^2 + \rho) \| \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix} - \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix} \|_\alpha, \quad \xi \in \overline{\Omega}, \; t \in \mathbb{R}.
\end{aligned}$$

Arguing as above, we obtain that

$$\begin{aligned}
& |g_1(t, \varphi_1(\xi), \psi_1(\xi), \nabla\varphi_1(\xi)) - g_1(t, \varphi_2(\xi), \psi_2(\xi), \nabla\varphi_2(\xi))| \\
\le \;& (a(t) + c_1(t)(\rho + 1)) \| \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix} - \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix} \|_\alpha,
\end{aligned}$$

for $\xi \in \overline{\Omega}, \; t \in \mathbb{R}$. Hence, we obtain that

$$\|f(t, \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix}) - f(t, \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix})\| \le \left(a(t) + c_1(t) + (c_1(t) + c_2(t))\rho + c_2(t)\rho^2\right) \| \begin{pmatrix}\varphi_1\\ \psi_1\end{pmatrix} - \begin{pmatrix}\varphi_2\\ \psi_2\end{pmatrix} \|_\alpha, \;\; t \in \mathbb{R}.$$

Therefore, f satisfies **(H5)** with $L_\rho(t) = a(t) + c_1(t) + (c_1(t) + c_2(t))\rho + c_2(t)\rho^2$. □

Proposition 5.4. *Assume that $a, c_1, c_2 \in APS^1(\mathbb{R}, \mathbb{R}^+) \cap L^\infty(\mathbb{R})$. Then, for each $x \in X_\alpha$, $f(\cdot, x) \in APS^1(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X)$.*

For the positivity condition **(C)**, we notice first that X is a Banach lattice (since $C_0(\overline{\Omega})$ it is) such that for $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in X$ we have $\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(\xi) \ge 0$ if and only if $\varphi(\xi) \ge 0$ and $\psi(\xi) \ge 0$ for all $\xi \in \overline{\Omega}$. So, we define $X^+ := \{\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in X : \begin{pmatrix}\varphi\\ \psi\end{pmatrix}(\xi) \ge 0 \;\text{ for all } \xi \in \overline{\Omega}\}$. Moreover, the evolution family $(U(t,s))_{t \ge 0}$ is positive. That is for $\begin{pmatrix}\varphi\\ \psi\end{pmatrix} \in X^+$, we have $U(t,s)\begin{pmatrix}\varphi\\ \psi\end{pmatrix}(\xi) \ge 0$ if and only if $U_1(t,s)\varphi(\xi) \ge 0$ and $U_2(t,s)\psi(\xi) \ge 0$ for all $t \ge s$ and $\xi \in \overline{\Omega}$. Notice that the positivity of the evolution families $(U_i(t,s))_{t \ge 0}$ holds immediately. More precisely, from [24] we have $(U_i(t,s))_{t \ge s}$ for $i = 1, 2$ is positive in $L^2(\Omega)$. Therefore, since the Banach space $C_0(\overline{\Omega})$ is a subset of $L^2(\Omega)$ with continuous embedding and $U_i(t,s)C_0(\overline{\Omega}) \subset C_0(\overline{\Omega})$, it follows that for each $i = 1, 2$ and for all $\varphi \in C_0(\overline{\Omega})$ with $\varphi(\xi) \geq 0$, we have $U_i(t,s)\varphi(\xi) \geq 0$ for all $t \geq s$ and $\xi \in \overline{\Omega}$. Furthermore, for $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in X^+$, it is clear that

$$g_2(t, \varphi(\xi), \psi(\xi), \nabla\varphi(\xi) = c_2(t)\frac{\varphi(\xi)\psi(\xi)}{1 + |\nabla\varphi(\xi)|} \geq 0, \quad \xi \in \overline{\Omega}.$$

Moreover,

$$g_1(t, \varphi(\xi), \psi(\xi), \nabla\psi(\xi)) = a(t)\varphi(\xi) - c_1(t)\frac{\psi(\xi)\varphi(\xi)}{1 + \psi(\xi)} \geq \varphi(\xi)(a_0 - |c_1|_\infty), \quad t \in \mathbb{R}, \xi \in \overline{\Omega}.$$

So $g_1(t, \varphi(\xi), \psi(\xi), \nabla\psi(\xi)) \geq 0$ if $a_0 \geq |c_1|_\infty$ where $a_0 := \inf_{t\in\mathbb{R}} |a(t)|$.

Hence, we conclude by the following main Theorem that gives a sufficient condition on the parameters that allow equation (5.1) to obey a unique almost periodic solution in the strong sense.

Theorem 5.5. *Assume that:*

- $L_i M(d_i^0 \lambda_1)^{-2} < 1$ *for* $i = 1, 2$.
- $a_0 \geq |c_1|_\infty$.
- $|a|_\infty + |c_1|_\infty < \dfrac{\delta}{c(\alpha)}$ *where* $c(\alpha)$ *is the constant from* (2.8).

Then, there exists $\rho > 0$ *satisfying*

$$\rho \leq \frac{-(|c_1|_\infty + |c_2|_\infty) + \Big((|c_1|_\infty + |c_2|_\infty)^2 + 4(\frac{\delta}{c(\alpha)} - |a|_\infty - |c_1|_\infty)|c_2|_\infty\Big)^{\frac{1}{2}}}{2|c_2|_\infty}, \tag{5.6}$$

such that our system (5.1) *has a unique almost periodic mild solution fulfilling* $0 \leq u(t,\xi) \leq \rho$ *and* $0 \leq v(t,\xi) \leq \rho$ *for all* $\xi \in \overline{\Omega}$, $t \in \mathbb{R}$.

Proof. Under the above considerations, hypotheses **(H1)-(H5)** and **(C)** hold. Furthermore, using Proposition 5.6, for each $\begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in X_\alpha$, we have that $f(\cdot, \begin{pmatrix} \varphi_2 \\ \psi_2 \end{pmatrix})$ is $APS^1(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X)$. Moreover, we have $L_\rho \in L^\infty(\mathbb{R})$ with

$$\|L_\rho\|_\infty = |c_2|_\infty \rho^2 + (|c_1|_\infty + |c_2|_\infty)\rho + |a|_\infty + |c_1|_\infty.$$

Consider the following second-order algebraic equation with unknown ρ:

$$|c_2|_\infty \rho^2 + (|c_1|_\infty + |c_2|_\infty)\rho + |a|_\infty + |c_1|_\infty - \frac{\delta}{c(\alpha)} = 0,$$

which has as positive solution

$$\rho_0 := \frac{-(|c_1|_\infty + |c_2|_\infty) + \Big((|c_1|_\infty + |c_2|_\infty)^2 + 4(\frac{\delta}{c(\alpha)} - |a|_\infty - |c_1|_\infty)|c_2|_\infty\Big)^{\frac{1}{2}}}{2|c_2|_\infty} > 0.$$

So we can find a real number $\rho \in (0, \rho_0]$ that yields condition (5.6). Therefore, in view of (5.6) we deduce that conditions (4.6) and (4.7) hold. Then, we conclude using Theorem 4.6. $\square$

Illustrations. Our subject here is to illustrate using examples the conditions and the parameters of our model (5.1). Let us define the following scalar functions:

$$d_i(t) = \tilde{d}_i + \hat{d}_i(\cos(t) + \cos(\pi t)),\ t \in \mathbb{R},$$

where $\tilde{d}_i, \hat{d}_i > 0$ for $i = 1, 2$ with $\tilde{d}_i > 2\hat{d}_i$ and

$$\frac{\hat{d}_i}{(\tilde{d}_i - 2\hat{d}_i)^2} < \frac{M}{(1+\pi)}.$$

Moreover, for $\tilde{a}, \hat{a}, \tilde{b}, \hat{b}, \tilde{c}, \hat{c} > 0$, we define

$$a(t) = \begin{cases} \tilde{a} + \hat{a}(\cos(t) + \cos\left(\sqrt{2}t\right)), & t \geq 0, \\ \tilde{a} + \hat{a}(\sin(t) + \sin\left(\sqrt{2}t\right)), & t < 0, \end{cases}$$

$$b(t) = \begin{cases} \tilde{b} + \hat{b}(\cos(t) + \cos\left(\sqrt{2}t\right)), & t \geq 0, \\ \tilde{b} + \hat{b}(\sin(t) + \sin\left(\sqrt{2}t\right)), & t < 0, \end{cases}$$

and

$$c_1(t) = \tilde{c}_1 + \hat{c}_1 \sin(1/p(t)) \text{ and } c_2(t) = \tilde{c}_2 + \hat{c}_2 \cos(1/p(t)),$$

where $p(t) = 2 + \sin(t) + \sin(\sqrt{2}t),\ t \in \mathbb{R}$. We assume that $\tilde{a} \geq 2\hat{a}$, $\tilde{a} \geq 2\hat{b}$ and $\tilde{c}_i \geq \hat{c}_i$, $i = 1, 2$ that guarantee the positivity of our parameters.

Proposition 5.6. *The following assertions are true:*
(i) *For each $i = 1, 2$, the functions $d_i \in AP(\mathbb{R}, \mathbb{R}^+)$ are such that $\inf_{t\in\mathbb{R}} |d_i(t)| = \tilde{d}_i - 2\hat{d}_i > 0$ and $|d_i|_\infty = \tilde{d}_i + 2\hat{d}_i < \infty$. Moreover, d_i are Hölderian with exponent $\mu = 1$ and constant $\hat{d}_i(1 + \pi)$.*
(ii) *The functions $a, b \in APS^1(\mathbb{R}, \mathbb{R}^+) \cap L^\infty(\mathbb{R})$, but, $a, b \notin AP(\mathbb{R}, \mathbb{R}^+)$ since they are discontinuous in $\mathbb{R}$ (at $t = 0$). However, for all $t \in \mathbb{R}$, $0 \leq a(t) \leq |a|_\infty = \tilde{a} + 2\hat{a}$ and $0 \leq b(t) \leq |b|_\infty = \tilde{b} + 2\hat{b}$.*
(iii) *For each $i = 1, 2$, the function $c_i \in APS^1(\mathbb{R}, \mathbb{R}^+) \cap L^\infty(\mathbb{R})$, but, $c_i \notin AP(\mathbb{R})$ since it is not uniformly continuous. Moreover, for all $t \in \mathbb{R}$, $0 \leq c_i(t) \leq |c_i|_\infty = \tilde{c}_i + \hat{c}_i$.*

Proof. **(i)** The function $t \longmapsto \cos(t) + \cos(\pi t)$ is almost periodic in Bohr sense since it is in particular the sum of two periodic functions with the ratio of the two periods is $\dfrac{2\pi}{2} = \pi \notin \mathbb{Q}$. Then, in view of [10], $t \longmapsto \cos(t) + \cos(\pi t)$ is almost periodic not periodic. Hence, the other facts now are obvious.
(ii) The functions a, b are piecewise defined by almost periodic functions in Stepanov sense. Hence, using Definition 2.10, we can prove easily that the entire function belongs to $APS^1(\mathbb{R})$. Moreover, it is clear that a and b are bounded and discontinuous at $t = 0$.
(iii) See [7, Example 3.1]. □

CONFLICT OF INTEREST

The current work does not have any conflicts of interest.

References

[1] P. Acquistapace and B. Terreni; *A unified approach to abstract linear nonautonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova **78** (1987), 47–107.

[2] A. N. Akdad, B. Es-sebbar and K. Ezzinbi; *Composition theorems of Stepanov μ-pseudo almost automorphic functions and applications to nonautonomous neutral evolution equations*, Differential Equations and Dynamical Systems, (2015), 1-20.

[3] L. Amerio and G. Prouse.;*Almost-periodic Functions and Functional Equations*, Van Nostrand Reinhold, 1971.

[4] M. Baroun, S. Boulite, T. Diagana and L. Maniar; *Almost periodic solutions to some semilinear non-autonomous thermoelastic plate equations*, J. Math. Anal. Appl. **349** (2009) 74-84.

[5] M. Baroun, K. Ezzinbi, K. Khalil and L. Maniar; *Pseudo almost periodic solutions for some parabolic evolution equations with Stepanov-like pseudo almost periodic forcing terms*, Journal of Mathematical Analysis and Applications, **462**, (1), (2018), 233-262.

[6] M. Baroun, K. Ezzinbi, K. Khalil and L. Maniar; *Almost automorphic solutions for nonautonomous parabolic evolution equations*, Semigroup Forum **99**, (2019), 525-567.

[7] B. Basit, and H. Günzler; *Spectral criteria for solutions of evolution equations and comments on reduced spectra*, arXiv preprint arXiv:1006.2169 (2010).

[8] C. J. K. Batty, W. Hutter, F. Räbiger; *Almost periodicity of mild solutions of inhomogeneous Cauchy problems*, J. Diff. Equ. **156** (1999), 309–327.

[9] S. Bochner; *A new approach to almost periodicity*, Proceedings of the National Academy of Sciences of the United States of America, **48**, (12), (1962), 2039-2043.

[10] H. Bohr; *Almost periodic functions*, Courier Dover Publications, 2018.

[11] R. S. Cantrell, C. Cosner; *Spatial Ecology via Reaction- Diffusion Equations*, Wiley Ser. Math. Comput. Biol. John Wiley-Sons, Ltd. 2003.

[12] C. Chicone, Y. Latushkin; *Evolution Semigroups in Dynamical Systems and Differential Equations*, Amer. Math. Soc., 1999.

[13] G. D'Agui, J. Mawhin, & A. Sciammetta, *Positive solutions for a discrete two point nonlinear boundary value problem with p-Laplacian*, Journal of Mathematical Analysis and Applications, **447**(1), (2017), 383–397.

[14] G. Da Prato and E. Sinestradi; *Differential operators with non dense domain*, Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, **14** (1987), 285-344.

[15] H-S. Ding, W. Long and G. M. N'Guérékata; *Almost periodic solutions to abstract semilinear evolution equations with Stepanov almost periodic coefficients*, Journal of Computational Analysis and Applications, **13** (2011), 231–243.

[16] P. Cieutat, S. Fatajou, and G.M. N'Guérékata; *Composition of pseudo-almost periodic and pseudo-almost automorphic functions and applications to evolution equations*, Appl. Anal. **89**(1) (2010), 11–27.

[17] K. J. Engel, R. Nagel; *One–Parameter Semigroups for Linear Evolution Equations*, Springer–Verlag, 2000.

[18] B. Es-sebbar, K. Ezzinbi & K. Khalil, *Compact almost automorphic solutions for semilinear parabolic evolution equations*, Applicable Analysis, 1-27 (2020).

[19] G. Gühring, F. Räbiger; *Asymptotic properties of mild solutions of nonautonomous evolution equations with applications to retarded differential equations*, Abstr. Appl. Anal. **4** (1999), 169–194.

[20] T. Cazenave, and A. Haraux; *An introduction to Semilinear Evolution Equations*, Oxford University Press on Demand, **13**, 1998.

[21] D. Henry; *Geometric Theory of Semilinear Parabolic Equations*, Springer–Verlag, 1981.

[22] G. Hetzer and W. Shen; *Uniform persistence, coexistence, and extinction in almost periodic/nonautonomous competition diffusion systems*, SIAM journal on Mathematical Analysis, **34** (2002), 204-221.

[23] K. Khalil, M. Kostić, & M. Pinto, *Stepanov pseudo-almost periodic functions and applications*, Mathematical Methods in the Applied Sciences, **44**(18), 13894-13919 (2021).

[24] H. Laasri,& D. Mugnolo, *Ultracontractivity and Gaussian bounds for evolution families associated with nonautonomous forms*, Mathematical Methods in the Applied Sciences, **43**(3), 1409-1436 (2020).

[25] J.A. Langa, J.C. Robinson, A. Rodriguez-Bernal and A. Suarez; *Permanence and asymptotically stable complete trajectories for nonautonomous Lotka-Volterra models with diffusion*, SIAM journal on Mathematical Analysis, **40**(6) (2009), 2179-2216.
[26] H-X. Li and L-L. Zhang; *Stepanov-like pseudo-almost periodicity and semilinear differential equations with uniform continuity*, Results in Mathematics, **59**.1-2 (2011), 43-61.
[27] A. Lunardi; *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, 1995.
[28] L. Maniar, R. Schnaubelt; *Almost periodicity of inhomogeneous parabolic evolution equations*, Lecture Notes in Pure and Appl. Math., **234**, Dekker, New York, (2003), 299–318.
[29] J. Mawhin, *Periodic solutions of differential and difference systems with pendulum-type nonlinearities: Variational approaches*, In Differential and Difference Equations with Applications,Springer, New York, NY, (2013), 83-98.
[30] A. Moussaoui, MA A. Alaoui, and R. Yafia; *Permanence and periodic solution for a modified Leslie-Gower type predator-prey model with diffusion and non constant coefficients*, BIOMATH, **6**.1 (2017), 1707107.
[31] B. Perthame; *Parabolic equations in biology*, Springer, Cham, 2015.
[32] R. Schnaubelt; *A sufficient condition for exponential dichotomy of parabolic evolution equations*, in: G. Lumer, L. Weis (Eds.), Evolution Equations and their Applications in Physical and Life Sciences (Proceedings Bad Herrenalb, 1998)", Marcel Dekker, 2000, 149–158.
[33] W. Stepanov; *Ueber einige verallgemeinerungen der fastperiodischen functionen*, Annals of Mathematics, **95**, (1926),473-498.
[34] Q. P. Vũ ; *Stability and almost periodicity of trajectories of periodic processes*, J. Differential Equations **115** (1995), 402–415.
[35] L. Wei and H-S. Ding; *Composition theorems of Stepanov almost periodic functions and Stepanov-like pseudo-almost periodic functions*, Advances in difference equations 2011.**1** (2011), 654-695.

Kamal Khalil
Normandie Univ, UNIHAVRE, LMAH, FR-CNRS-3335, ISCN, Le Havre 76600, France.
Email address: kamal.khalil.00@gmail.com