QUASILINEAR CHOQUARD EQUATIONS INVOLVING N-LAPLACIAN AND CRITICAL EXPONENTIAL NONLINEARITY

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Abstract

In the present paper, we study a class of quasilinear Choquard equations involving N-Laplacian and the nonlinearity with the critical exponential growth. We discuss the existence of positive solutions of such equations.

1. **Introduction**

This paper is concerned with the existence of the positive solutions for the family of quasilinear equations with N-Laplacian and exponential Choquard type nonlinearity

$$\begin{cases} -\Delta_N u - \Delta_N(u^2)u = \left(\displaystyle\int_\Omega \frac{F(y,u)}{|x-y|^\mu}\,dy\right) f(x,u) \text{ in } \Omega, \\ u > 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{P_*}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $0 < \mu < N$ and $\Delta_N := \text{div}\left(|\nabla u|^{N-2}\nabla u\right)$ is called the N-Laplacian. The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is given by $f(x,s) = g(x,s)\exp(|s|^{\frac{2N}{N-1}})$, where $g \in C(\bar{\Omega} \times \mathbb{R})$ satisfies some appropriate assumptions described later.

The problems involving the quasilinear operator $-\Delta_p u - \Delta_p(u^2)u$, $1 < p < \infty$, has been of interest to many researchers for long due to its significant applications in the modeling of the physical phenomenon such as in plasma physics and fluid mechanics [4], in dissipative quantum mechanics [18], etc. Solutions of such equation are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iu_t = -\Delta u + V(x)u - h_1(|u|^2)u - C\Delta h_2(|u|^2)h_2'(|u|^2)u, \quad x \in \mathbb{R}^N, \tag{1.1}$$

where V is a potential, C is a real constant, h_1 and h_2 are real functions. Depending upon the different type of h_2, the quasilinear equations of the form (1.1) appear in the study of mathematical physics. For example, in the modeling of the superfluid film equation in plasma physics, $h_2(s) = s$ (see [35]) and for studying the self-channeling of a high-power ultra short laser in matter h_2 is considered to be $\sqrt{1+s^2}$ (see [35]).

The main mathematical difficulty we face in studying the problem of type (P_*) occurs due to the quasilinear term $\Delta_N(u^2)u$, which doesn't allow the natural energy functional corresponding to the problem (P_*) to be well defined for all $u \in W_0^{1,N}(\Omega)$ (defined in Section 2). Hence, we can not apply variational method directly for such problem. To overcome this inconvenience, several methods and arguments have been developed, such as the perturbation method (see for e.g.,[22, 25]) a constrained minimization technique (see for e.g., [23, 24, 34, 36]), and a change of variables (see for e.g., [6], [10]-[17]). The nonlinearity in the problem (P_*), which is nonlocal in nature, is driven by the Hardy-Littlewood-Sobolev inequality and the Trudinger-Moser inequality. Let us first recall the following well known Hardy-Littlewood-Sobolev inequality [Theorem 4.3, p.106] [20].

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Proposition 1.1. ***(Hardy-Littlewood-Sobolev inequality)*** *Let t, $r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $g_1 \in L^t(\mathbb{R}^n)$ and $g_2 \in L^r(\mathbb{R}^n)$. Then there exists a sharp constant $C(t, N, \mu, r)$, independent of g_1, g_2 such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{g_1(x)g_2(y)}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \le C(t, N, \mu, r)\|g_1\|_{L^t(\mathbb{R}^N)}\|g_2\|_{L^r(\mathbb{R}^N)}. \tag{1.2}$$

If $t = r = \frac{2N}{2N-\mu}$ then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N}{2} - \frac{\mu}{2}\right)}{\Gamma\left(N - \frac{\mu}{2}\right)} \left\{\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)}\right\}^{-1+\frac{\mu}{N}}.$$

In this case there is equality in (1.2) if and only if $g_1 \equiv (constant)g_2$ and

$$g_1(x) = c_0(a^2 + |x-b|^2)^{\frac{-(2N-\mu)}{2}}$$

for some $c_0 \in \mathbb{C}$, $0 \ne a \in \mathbb{R}$ and $b \in \mathbb{R}^N$.

Now a days, an ample amount of attention has been attributed to the study of Choquard type equations, which was started with the seminal work of S. Pekar [32], where the author considered the following nonlinear Schrödinger-Newton equation:

$$-\Delta u + V(x)u = (\mathcal{K}_\mu * u^2)u + \lambda f_1(x, u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $\lambda > 0$, $\mathcal{K}_\mu$ denotes the Riesz potential, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function and $f_1 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with some appropriate growth assumptions. When $\lambda = 0$, the nonlinearity in the right-hand side of (1.3) is termed as Choquard type nonlinearty. In the application point of view, such type of nonlinearity plays an important role in describing the Bose-Einstein condensation (see [7]) and also, appears in the modeling of the self gravitational collapse of a quantum mechanical wave function (see [33]). P. Choquard (see [19]) studied such elliptic equations of type (1.3) for construing the quantum theory of a polaron at rest and for modeling the phenomenon of an electron being trapped in its own hole in the Hartree-Fock theory. When $V(x) = 1, \lambda = 0$, the equations of type (1.3) were studied rigorously in [20, 21]. For more extensive study of Choquard equations, without attempting to provide a complete list, we refer to [27, 28, 29, 30] and the references therein.

The main feature of the problem (P_*) is that the nonlinear term $f(x, t)$ has the maximal growth on t, that is, critical exponential growth with respect to the following Trudinger–Moser inequality (see [31]):

Theorem 1.2. ***(Trudinger–Moser inequality)*** *For $N \ge 2$, $u \in W_0^{1,N}(\Omega)$*

$$\sup_{\|u\|\le 1} \int_\Omega \exp(\alpha|u|^{\frac{N}{N-1}})\, dx < \infty$$

if and only if $\alpha \le \alpha_N$, where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and $\omega_{N-1} = (N-1)$- dimensional surface area of $\mathbb{S}^{N-1}$.

Here the Sobolev space $W_0^{1,N}(\Omega)$ and the corresponding norm $\|\cdot\|$ are defined in Section 2. The critical growth non-compact problems associated to this inequality are initially studied by the work of Adimurthi [1] and de Figueiredo et. al [8]. More recently, authors in [13, 14, 15] studied the existence of multiple positive solutions for quasilinear equations involving exponential nonlinearities. Unlike as in the case of critical exponential problem involving N-Laplacian, where we generally consider the critical exponential growth as $\exp(|t|^{N/(N-1)})$, in our problem (P_*), due to the presence of the quasilinear operator, the critical exponential growth becomes $\exp(|t|^{2N/(N-1)})$. In the case $1 < p < N$, the nonlinearity is of polynomial growth and the critical growth is t^{2p^*}, where $p^* = Np/(N-p)$ (see [9, 11, 26]). When $p = 2$, the problem of type (P_*) without the convolution term, that is the equation

$$-\Delta u - \Delta(u^2)u + V(x)u = f(x, u) \text{ in } \mathbb{R}^2,$$

where $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous potential and $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a continuous function with some suitable assumption and is having critical exponential growth ($\exp(\alpha s^4)$), was studied by Wang et. al in [10]. Later, Wang et. al [37] extended this problem for $p = N \geq 2$.

Involving critical exponential Choquard type nonlinearity and Laplacian operator, Alves et. al [2] studied the following problem

$$-\epsilon^2 \Delta u + V(x)u = \left(\int_{\mathbb{R}^2} \frac{F(y,u)}{|x-y|^\mu}\, dy \right) f(x,u), \quad x \in \mathbb{R}^2,$$

where $\epsilon > 0$, $0 < \mu < 2$, $V : \mathbb{R}^2 \to \mathbb{R}$ is the continuous potential function with some particular properties and the continuous function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ enjoys the critical exponential growth with some appropriate assumptions. Consequently, for the higher dimension, that is for $N \geq 2$, authors in [3] discussed the Kirchhoff- Choquard problems involving the N-Laplacian with critical exponential growth.

Inspired from all the above mentioned works, in this article, we investigate the existence results for the problem (P_*), involving the Choquard type critical nonlinearity motivated by the above inequality (1.2). The main contribution in this work is to identify the first critical level for this problem and study the Palais-Smale sequences below this level. We would like to mention that to the best of our knowledge, there is no work on the existence of positive solutions to the elliptic equations involving quasilinear operator and critical exponential Choquard type nonlinearity. In this article, we study such equations for the first time. Our result is new even for the case $N = 2$.

We now state all the hypotheses imposed on the continuous function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, given by $f(x,s) = g(x,s) \exp\left(s^{\frac{2N}{N-1}}\right)$, as following:

- (f_1) $g \in C(\overline{\Omega} \times \mathbb{R})$, $g(x,s) = 0$, for all $s \leq 0$, $g(x,s) > 0$, for all $s > 0$.
- (f_2) $\lim\limits_{s \to 0} \dfrac{f(x,s)}{s^{N-1}} = 0$, uniformly in $x \in \Omega$.
- (f_3) For any $\epsilon > 0$, $\lim\limits_{s \to \infty} \sup\limits_{x \in \overline{\Omega}} g(x,s) \exp\left(-\epsilon |s|^{\frac{2N}{N-1}}\right) = 0$ and $\lim\limits_{s \to \infty} \inf\limits_{x \in \overline{\Omega}} g(x,s) \exp(\epsilon |s|^{\frac{2N}{N-1}}) = \infty$.
- (f_4) There exist positive constants s' and m_0 such that
$$0 < s^{m_0} F(x,s) \leq M_0 f(x,s) \text{ for all } (x,s) \in \mathbb{R}^2 \times [s', +\infty).$$
- (f_5) There exists $\tau > N$ such that $0 < \tau F(x,s) \leq f(x,s)s$, for all $s > 0$.
- (f_6) We assume that
$$\lim_{s \to +\infty} \frac{s f(x,s) F(x,s)}{\exp\left(2|s|^{\frac{2N}{N-1}}\right)} = \infty, \text{ uniformly in } x \in \overline{\Omega}. \tag{1.4}$$

Example 1.3. Consider $f(x,s) = g(x,s) e^{|s|^{\frac{2N}{N-1}}}$, where $g(x,s) = \begin{cases} t^{a_0 + (N-1)} \exp(d_0 s^r), \text{ if } s > 0 \\ 0, \text{ if } s \leq 0 \end{cases}$ for some $a_0 > 0$, $0 < d_0 \leq \alpha_N$ and $1 \leq r < \frac{2N}{N-1}$. Then f satisfies all the conditions from (f_1)-(f_6).

Our main result reads as following:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a smooth bounded domain and let the hypotheses (f_1)-(f_6) hold. Then the problem (P_*) has a non trivial positive weak solution.*

Notation. Throughout this paper, we make use of the following notations:

- If u is a measurable function, we denote the positive and negative parts by $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$, respectively.
- If A is a measurable set in $\mathbb{R}^N$, we denote by $|A|$ the Lebesgue measure of A.
- The arrows $\rightharpoonup$, $\to$ denote weak convergence, strong convergence, respectively.
- $B_r(x)$ denotes the ball of radius $r > 0$ centered at $x \in \Omega$.

- $\overline{B_r(x)}$ denotes the closure of the set $B_r(x)$ with respect to $W_0^{1,N}(\Omega)$-norm topology.
- $\overline{\mathcal{S}}$ denotes the closure of a set $\mathcal{S} \subset \mathbb{R}^N$.
- $A \subset\subset B$ implies $\overline{A}$ is compact in B.
- $c, C_1, C_2, \cdots, \tilde{C}_1, \tilde{C}_2, \cdots, C$ and $\tilde{C}$ denote (possibly different from line to line) positive constants.

2. Preliminaries and Variational Set-up

For $u : \Omega \to \mathbb{R}$, measurable function, and for $1 \le p \le \infty$, we define the Lebesgue space $L^p(\Omega)$ as

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} | \int_\Omega |u|^p dx < \infty\}$$

equipped with the usual norm denoted by $\|u\|_{L^p(\Omega)}$. Now the Sobolev space $W_0^{1,p}(\Omega)$ is defined as

$$W_0^{1,p}(\Omega) = \{u \in L^p(\Omega) | \int_\Omega |\nabla u|^p dx < \infty\}$$

which is endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^p dx \right)^{1/p}.$$

We have that the embedding $W_0^{1,N}(\Omega) \ni u \mapsto \exp(|u|^\beta) \in L^1(\Omega)$ is compact for all $\beta \in \left[1, \frac{N}{N-1}\right)$ and is continuous for $\beta = \frac{N}{N-1}$. Consequently, the map $\mathcal{M} : W_0^{1,N}(\Omega) \to L^q(\Omega)$, for $q \in [1, \infty)$, defined by $\mathcal{M}(u) := \exp\left(|u|^{\frac{N}{N-1}}\right)$ is continuous with respect to the norm topology.
The natural energy functional associated to the problem (P_*) is the following:

$$I(u) = \frac{1}{N} \int_\Omega (1 + 2^{N-1}|u|^N)|\nabla u|^N dx - \frac{1}{2} \int_\Omega \int_\Omega \frac{F(y, u(y))F(x, u(x))}{|x-y|^\mu} dxdy. \tag{2.1}$$

Observe that, the functional I is not well defined in $W_0^{1,N}(\Omega)$ due to the fact that $\displaystyle\int_\Omega u^N |\nabla u|^N dx$ is not finite for all $u \in W_0^{1,N}(\Omega)$. So, it is difficult to apply variational methods directly in our problem (P_*). In order to overcome this difficulty, we apply the following change of variables which was introduced in [6], namely, $w := h^{-1}(u)$, where h is defined by

$$\begin{cases} h'(s) = \dfrac{1}{(1 + 2^{N-1}|h(s)|^N)^{\frac{1}{N}}} \text{ in } [0, \infty), \\ h(s) = -h(-s) \text{ in } (-\infty, 0]. \end{cases} \tag{2.2}$$

Now we gather some properties of h, which we follow throughout in this article. For the detailed proofs of such results, one can see [6, 10].

Lemma 2.1. *The function h satisfies the following properties:*

(h_1) *h is uniquely defined, C^∞ and invertible;*
(h_2) *$h(0) = 0$;*
(h_3) *$0 < h'(s) \le 1$ for all $s \in \mathbb{R}$;*
(h_4) *$\frac{1}{2}h(s) \le sh'(s) \le h(s)$ for all $s > 0$;*
(h_5) *$|h(s)| \le |s|$ for all $s \in \mathbb{R}$;*
(h_6) *$|h(s)| \le 2^{1/(2N)}|s|^{1/2}$ for all $s \in \mathbb{R}$;*
(h_7) *$(h(s))^2 - h(s)h'(s)s \ge 0$ for all $s \in \mathbb{R}$;*
(h_8) *$|h(s)| \ge h(1)|s|$ for $|s| \le 1$ and $|h(s)| \ge h(1)|s|^{1/2}$ for $|s| \ge 1$;*
(h_9) *$h''(s) < 0$ when $s > 0$ and $h''(s) > 0$ when $s < 0$.*

Example 2.2. One of the example of such functions is given in

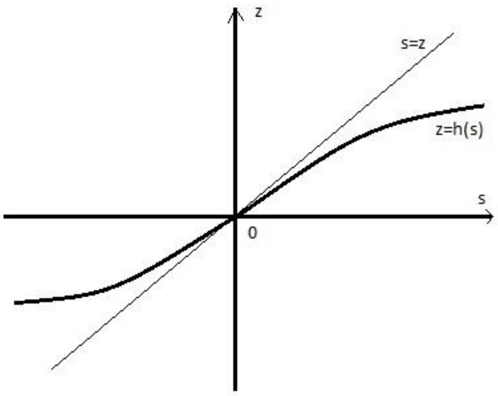

FIGURE 1. Plot of the function h

After employing the change of variable $w = h^{-1}(u)$, we define the new functional $J : W_0^{1,N}(\Omega) \to \mathbb{R}$ as

$$J(w) = \frac{1}{N}\int_\Omega |\nabla w|^N dx - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(y, h(w))F(x, h(w))}{|x-y|^\mu} dxdy. \tag{2.3}$$

From the properties of f, h, it can be derived that the functional J is well defined and $J \in C^1$. We observe that if $w \in W_0^{1,N}(\Omega)$ is a critical point of the functional J, that is for every $v \in W_0^{1,N}(\Omega)$,

$$\int_\Omega |\nabla w|^{N-2}\nabla w \nabla v dx - \frac{1}{2}\int_\Omega\int_\Omega \frac{F(y, h(w)(y))f(x, h(w)(x))}{|x-y|^\mu} h'(w)v(x)dxdy = 0, \tag{2.4}$$

then w is a weak solution (solution, for short) to the following problem:

$$\begin{cases} -\Delta_N w = \left(\int_{\mathbb{R}^2} \frac{F(y, h(w))}{|x-y|^\mu}\, dy\right) f(x, h(w)) \text{ in } \Omega, \\ w > 0 \text{ in } \Omega, \\ w = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

It is easy to see that problem (2.5) is equivalent to our problem (P_*), which takes $u = h(w)$ as its solutions. Thus, our main objective is now reduced to proving the existence of the solution of (2.5).

3. Proof of the main theorem

3.1. **Mountain pass geometry.** We begin this section with the study of mountain pass structure and Palais-Smale sequences corresponding to the energy functional $J : W_0^{1,N}(\Omega) \to \mathbb{R}$ associated to (2.5). From the assumptions, (f_2)-(f_3), we obtain that for any $\epsilon > 0$, $r \geq 1$, there exist $\tilde{C}(N, \epsilon)$ and $C(N, \epsilon) > 0$ such that

$$|f(x, s)| \leq \epsilon|s|^{N-1} + \tilde{C}(N, \epsilon)|s|^{r-1}\exp\left((1+\epsilon)|s|^{\frac{2N}{N-1}}\right), \text{ for all } (x, s) \in \Omega \times \mathbb{R}, \tag{3.1}$$

$$|F(x, s)| \leq \epsilon|s|^N + C(N, \epsilon)|s|^r \exp\left((1+\epsilon)|s|^{\frac{2N}{N-1}}\right), \text{ for all } (x, s) \in \Omega \times \mathbb{R}. \tag{3.2}$$

Thus, for any $u \in W_0^{1,N}(\Omega)$, in light of Sobolev embedding, we have $u \in L^q(\Omega)$ for all $q \in [1, \infty)$, which implies that

$$F(x, u) \in L^q(\Omega) \text{ for any } q \geq 1. \tag{3.3}$$

Now by (2.2), we have if $w \in W_0^{1,N}(\Omega)$, then $h(w) \in W_0^{1,N}(\Omega)$, which together with Proposition 1.1 implies that

$$\int_\Omega \left(\int_\Omega \frac{F(y, h(w))}{|x-y|^\mu} dy\right) F(x, h(w))\, dx \leq C(N, \mu)\|F(\cdot, h(w))\|^2_{L^{\frac{2N}{2N-\mu}}(\Omega)}. \tag{3.4}$$

Lemma 3.1. *Assume that the conditions* (f_1)-(f_5) *hold and let* h *be defined as in* (2.2). *Then there exist* $\rho_*>0$ *and* $R_*>0$ *such that*

$$J(w)\geq R_*>0 \quad \text{for any } w\in W_0^{1,N}(\Omega) \text{ with } \|w\|=\rho_*.$$

Proof. Let $w\in W_0^{1,N}(\Omega$. From (3.4), (3.2), Hölder inequality, Sobolev inequality and Lemma 2.1-(h_5), (h_6), we have

$$\begin{aligned}
&\int_\Omega\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right)F(x,h(w))\,dx\\
&\leq C(N,\mu)\|\epsilon|h(w)|^N+C(\epsilon)|h(w)|^r\exp\left((1+\epsilon)|h(w)|^{\frac{2N}{N-1}}\right)\|^2_{\frac{2N}{2N-\mu}}\\
&\leq C(\mu)\left[2^{\frac{2N}{2N-\mu}}\left\{\epsilon^{\frac{2N}{2N-\mu}}\int_\Omega|h(w)|^{\frac{2N^2}{2N-\mu}}dx\right.\right. &(3.5)\\
&\left.\left.+\ (C(\epsilon))^{\frac{2N}{2N-\mu}}\int_\Omega|h(w)|^{\frac{2Nr}{2N-\mu}}\exp\left(\frac{2N}{2N-\mu}(1+\epsilon)|h(w)|^{\frac{2N}{N-1}}\right)dx\right\}\right]^{\frac{2N-\mu}{N}}\\
&\leq C(N,\mu)\left[2^{\frac{2N}{2N-\mu}}\left\{\epsilon^{\frac{2N}{2N-\mu}}\int_\Omega|w|^{\frac{2N^2}{2N-\mu}}dx\right.\right. &(3.6)\\
&\left.\left.\quad+\ (C(\epsilon))^{\frac{2N}{2N-\mu}}\int_\Omega 2^{\frac{r}{2N-\mu}}|w|^{\frac{Nr}{2N-\mu}}\exp\left(\frac{2N}{2N-\mu}(1+\epsilon)2^{\frac{1}{N-1}}|w|^{\frac{N}{N-1}}\right)dx\right\}\right]^{\frac{2N-\mu}{N}}\\
&\leq C(N,\mu)4\left[\epsilon^2\|w\|^{2N}_{L^{\frac{2N^2}{2N-\mu}}(\Omega)}+(C(\epsilon))^2 2^{\frac{r}{N}}\|w\|^{2r}_{L^{\frac{2Nr}{2N-\mu}}(\Omega)}\left\{\int_\Omega\exp\left(\frac{4N(1+\epsilon)}{2N-\mu}2^{\frac{1}{N-1}}(|w|)^{\frac{N}{N-1}}\right)dx\right\}^{\frac{2N-\mu}{2N}}\right]\\
&\leq C_1(N,\mu,\epsilon)\left[\|w\|^{2N}+\|w\|^{2r}\left\{\int_\Omega\exp\left(\frac{4N(1+\epsilon)}{2N-\mu}2^{\frac{1}{N-1}}(|w|)^{\frac{N}{N-1}}\right)dx\right\}^{\frac{2N-\mu}{2N}}\right]\\
&\leq C_1(N,\mu,\epsilon)\left[\|w\|^{2N}+\|w\|^{2r}\left\{\int_\Omega\exp\left(\frac{4N(1+\epsilon)}{2N-\mu}2^{\frac{1}{N-1}}\|w\|^{\frac{N}{N-1}}\left(\frac{|w|}{\|w\|}\right)^{\frac{N}{N-1}}\right)dx\right\}^{\frac{2N-\mu}{2N}}\right] &(3.7)
\end{aligned}$$

So, for sufficiently small $\epsilon>0$ if we choose w such that $\|w\|$ is small enough so that

$$\frac{4N2^{\frac{1}{N-1}}(1+\epsilon)\|w\|^{\frac{N}{N-1}}}{2N-\mu}<\alpha_N,$$

then using Theorem 1.2 in (3.5), we obtain

$$\int_\Omega\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right)F(x,h(w))\,dx\leq C_1(N,\mu,\epsilon)\left(\|w\|^{2N}+\|w\|^{2r}\right). \tag{3.8}$$

Using (2.3), (3.8), we have

$$J(w)\geq\frac{1}{N}\|w\|^N-C_1(N,\mu,\epsilon)\left(\|w\|^{2N}+\|w\|^{2r}\right).$$

Now by taking $r>0$ such that $2r>N$, we can choose $0<\rho_*<1$ sufficiently small so that, we finally obtain $J(w)\geq R_*>0$ for all $w\in W_0^{1,N}(\Omega)$ with $\|w\|=\rho_*$ and for some $R_*>0$ depending on ρ_*. □

Lemma 3.2. *Assume that the conditions* (f_1)-(f_5) *hold and let* h *be defined as in* (2.2). *Then there exists* $v_*\in W_0^{1,N}(\Omega)$ *with* $\|v_*\|>\rho_*$ *such that* $J(v_*)<0$ *, where* ρ_* *is given as in Lemma 3.1.*

Proof. The condition (f_5) implies that there exist some positive constant $C_1,C_2>0$ such that

$$F(x,s)\geq C_1s^\tau-C_2 \quad \text{for all } (x,s)\in\Omega\times[0,\infty). \tag{3.9}$$

Let $\phi(\geq 0) \in C_c^\infty(\Omega)$ such that $\|\phi\|= 1$. Now by Lemma 2.1-(h_6),(h_8) and (3.9), for large $t > 1$, we obtain

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega \frac{F(y,h(t\phi))}{|x-y|^\mu} dy\right) F(x,h(t\phi))\, dx \\
&\geq \int_\Omega \int_\Omega \frac{(C_1(h(t\phi(y)))^\tau - C_2)(C_1(h(t\phi(x)))^\tau - C_2)}{|x-y|^\mu}\, dxdy \\
&= C_1^2 \int_\Omega \int_\Omega \frac{(h(t\phi(y)))^\tau (h(t\phi(x)))^\tau}{|x-y|^\mu}\, dxdy \\
&\quad - 2C_1C_2 \int_\Omega \int_\Omega \frac{(h(t\phi(x)))^\tau}{|x-y|^\mu}\, dxdy + C_2^2 \int_\Omega \int_\Omega \frac{1}{|x-y|^\mu}\, dxdy \\
&\geq C_1^2 (h(1))^{2\tau} t^\tau \int_\Omega \int_\Omega \frac{(\phi(y))^{\frac{\tau}{2}} (\phi(x))^{\frac{\tau}{2}}}{|x-y|^\mu}\, dxdy \\
&\quad - 2C_1C_2 2^{\frac{\tau}{2N}} t^{\frac{\tau}{2}} \int_\Omega \int_\Omega \frac{(\phi(x))^{\frac{\tau}{2}}}{|x-y|^\mu}\, dxdy + C_2^2 \int_\Omega \int_\Omega \frac{1}{|x-y|^\mu}\, dxdy.
\end{aligned}$$

From the last relation and (2.3) , we obtain

$$\begin{aligned}
J(t\phi) &\leq \|t\phi\|^N - \frac{1}{2} \int_\Omega \left(\int_\Omega \frac{F(y,h(t\phi))}{|x-y|^\mu} dy\right) F(x,h(t\phi))\, dx \\
&\leq C_3 t^N - C_4 t^\tau + C_5 t^{\frac{\tau}{2}} - C_6,
\end{aligned} \tag{3.10}$$

where $C_i's$ are positive constants for $i = 3, 4, 5, 6$. From (3.10), we infer that $J(t\phi) \to -\infty$ as $t \to \infty$, since $\tau > N$. Thus, there exists $t_0(> 0) \in \mathbb{R}$ so that $v_*(:= t_0\phi) \in W_0^{1,N}(\Omega)$ with $\|v_*\|> \rho_*$ such that $J(v_*) < 0$. □

From the above two lemmas, we get that J satisfies the mountain pass geometry near 0. Let $\Gamma = \{\gamma \in C([0, 1], W_0^{1,N}(\Omega)) : \gamma(0) = 0, J(\gamma(1)) < 0\}$ and define the mountain pass critical level

$$\beta_* = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)). \tag{3.11}$$

Then by Lemma 3.1, Lemma 3.2 and the mountain pass theorem we know that there exists a Palais-Smale sequence $\{w_k\} \subset W_0^{1,N}(\Omega)$ for J at level β_*, that is, as $k \to \infty$

$$J(w_k) \to \beta_*; \text{ and } J'(w_k) \to 0 \quad \text{in } \left(W_0^{1,N}(\Omega)\right)^*.$$

Moreover, Lemma 3.1 guarantees that $\beta^* > 0$.

3.2. Analysis of Palais-Smale sequence.

Lemma 3.3. *Let (f_1) and (f_5) hold. Let h be given as in* (2.2)*. Then every Palais-Smale sequence of J is bounded in $W_0^{1,N}(\Omega)$.*

Proof. Let $\{w_k\} \subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence of J at level $c \in \mathbb{R}$, that is, as $k \to \infty$

$$J(w_k) \to c, \ J'(w_k) \to 0 \,\text{in} W{-}1, N(\Omega).$$

Then we have

$$\begin{aligned}
&\frac{1}{N}\|w_k\|^N - \frac{1}{2} \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k)}{|x-y|^\mu} dy\right) F(x,h(w_k))\, dx \to c \text{ as } k \to \infty, \\
&\bigg| \int_\Omega \nabla |w_k(x)|^{N-2} \nabla w_k \nabla \phi dx \\
&\qquad - \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k)) h'(w_k)\phi\, dx \bigg| \leq \epsilon_k \|\phi\|,
\end{aligned} \tag{3.12}$$

where $\epsilon_k \to 0$ as $k \to \infty$. In the last relation, taking $\phi = w_k$ we get

$$\left| \|w_k\|^N - \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k)h'(w_k)w_k\, dx \right| \leq \epsilon_k \|w_k\|. \tag{3.13}$$

Now set

$$v_k := \frac{h(w_k)}{h'(w_k)}.$$

Since by Lemma 2.1-(h_4), we have $|v_k|\leq 2|w_k|$ and by using (2.2) and Lemma 2.1-(h_5), we get

$$|\nabla v_k|= \left(1+\frac{2^{N-1}|h(w_k)|^N}{1+2^{N-1}|h(w_k)|^N}\right)|\nabla w_k|^N\leq 2|\nabla w_k|,$$

combining these, we obtain

$$\|v_k\|\leq 2\|w_k\|.$$

Therefore, $v_k \in W_0^{1,N}(\Omega)$. Then by choosing $\phi = v_k$ and inserting this into (3.12), we deduce

$$\begin{aligned}\langle J'(w_k), v_k\rangle &= \int_\Omega \left(1+\frac{2^{N-1}|h(w_k)|^N}{1+2^{N-1}|h(w_k)|^N}\right)|\nabla w_k|^N dx \\ &\quad - \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h(w_k)dx \\ &= \epsilon_k\|w_k\|. \end{aligned}\tag{3.14}$$

Using (3.14) and (f_5), for some $q > 2N$, we get

$$\begin{aligned} C+\epsilon_k\|w_k\| &\geq J(w_k) - \frac{1}{q}\langle J'(w_k), v_k\rangle \\ &\geq \int_\Omega \left[\frac{1}{N}-\frac{1}{q}\left(1+\frac{2^{N-1}|h(w_k)|^N}{1+2^{N-1}|h(w_k)|^N}\right)\right]|\nabla w_k|^N dx \\ &\quad -\frac{1}{2}\int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)\left[F(x,h(w_k)) - \frac{2}{q}f(x,h(w_k))h(w_k)\right]dx \\ &\geq \int_\Omega \left(\frac{1}{N}-\frac{2}{q}\right)|\nabla w_k|^N dx = \left(\frac{1}{N}-\frac{2}{q}\right)\|w_k\|^N, \end{aligned}\tag{3.15}$$

the above relation yields that $\{w_k\}$ must be bounded in $W_0^{1,N}(\Omega)$. □

Next, we have the following lemma:

Lemma 3.4. *If f satisfies (f_1)-(f_6) and h is defined as in (2.2), then*

$$0<\beta^* < \frac{1}{N(h(1))^{2N}}\left(\frac{2N-\mu}{2N}\alpha_N\right)^{N-1},$$

where β^* is given as in (3.11).

Proof. Since for $w \not\equiv 0$, by Lemma 3.2, $J(tw) \to -\infty$ as $t \to +\infty$, hence, from (3.11), we get

$$\beta^* \leq \max_{t\in[0,1], w\in W_0^{1,N}(\Omega)\setminus\{0\}} J(tw).$$

So, it is sufficient to prove that there exists $w \in W_0^{1,N}(\Omega)$ such that $\|w\|= 1$ and

$$\max_{t\in[0,\infty)} J(tw) < \frac{1}{N(h(1))^{2N}}\left(\frac{2N-\mu}{2N}\alpha_N\right)^{N-1}.$$

To show this, let us consider the sequence of Moser functions $\{\mathcal{M}_k\}$ defined as

$$\mathcal{M}_k(x) = \frac{1}{\omega_{N-1}^{\frac{1}{N}}}\begin{cases}(\log k)^{\frac{N-1}{N}}, \ 0\leq |x|\leq \dfrac{\delta}{k}, \\ \dfrac{\log\left(\frac{\delta}{|x|}\right)}{(\log k)^{\frac{1}{N}}}, \ \dfrac{\delta}{k}\leq |x|\leq \delta \\ 0, \ |x|\geq \delta.\end{cases}$$

Then it is easy to see that $\mathrm{supp}(\mathcal{M}_k) \subset B_\delta(0)$ and $\|\mathcal{M}_k\|= 1$ for all $k \in \mathbb{N}$. Now we assert that there exists some $k \in \mathbb{N}$ such that

$$\max_{t\in[0,\infty)} J(t\mathcal{M}_k) < \frac{1}{N(h(1))^{2N}} \left(\frac{2N-\mu}{2N}\alpha_N\right)^{N-1}. \tag{3.16}$$

Indeed, if this doesn't hold then for all $k \in \mathbb{N}$, there exists $t_k > 0$ such that

$$\max_{t\in[0,\infty)} J(t\mathcal{M}_k) = J(t_k\mathcal{M}_k) \geq \frac{1}{N(h(1))^{2N}} \left(\frac{2N-\mu}{2N}\alpha_N\right)^{N-1}. \tag{3.17}$$

Since (f_1) implies that $F(x, h(t_k\mathcal{M}_k)) \geq 0$ for all $k \in \mathbb{N}$ so by the definition of $J(t_k w_k)$ together with (3.17), we obtain

$$t_k^N \geq \frac{1}{(h(1))^{2N}} \left(\frac{2N-\mu}{2N}\alpha_N\right)^{N-1}. \tag{3.18}$$

Also, in view of (3.17), we have $\frac{d}{dt}(J(t\mathcal{M}_k))|_{t=t_k}= 0$. This combining with Lemma 2.1-(h_4) yields that

$$\begin{aligned} t_k^N &= \int_\Omega \left(\int_\Omega \frac{F(y, h(t_k\mathcal{M}_k))}{|x-y|^\mu} dy\right) f(x, h(t_k\mathcal{M}_k))h'(t_k\mathcal{M}_k)t_k\mathcal{M}_k \, dx \\ &\geq \frac{1}{2}\int_{B_{\delta/k}(0)} f(x, h(t_k\mathcal{M}_k))h(t_k\mathcal{M}_k) \left(\int_{B_{\delta/k}(0)} \frac{F(y, h(t_k\mathcal{M}_k))}{|x-y|^\mu} \, dy\right) \, dx. \end{aligned} \tag{3.19}$$

Now from (1.4), we know that for each $b > 0$ there exists a constant R_b such that

$$sf(x, s)F(x, s) \geq b \exp\left(2|s|^{\frac{2N}{N-1}}\right), \text{ whenever } s \geq R_b.$$

From (3.18), we infer that $t_k\mathcal{M}_k \to \infty$ as $k \to \infty$ in $B_{\delta/k}(0)$. Now by Lemma 2.1-(h_8), we get $h(t_k\mathcal{M}_k) \to \infty$ as $k \to \infty$, uniformly in $B_{\delta/k}(0)$. So, we can choose $s_b \in \mathbb{N}$ such that in $B_{\delta/k}(0)$,

$$h(t_k\mathcal{M}_k) \geq R_b, \text{ for all } k \geq s_b.$$

In addition, using the same idea as in [2] (see equation (2.11)), we can have

$$\int_{B_{\delta/k}(0)}\int_{B_{\delta/k}(0)} \frac{dxdy}{|x-y|^\mu} \geq C_{\mu,N}\left(\frac{\delta}{k}\right)^{2N-\mu},$$

where $C_{\mu,N}$ is a positive constant depending on μ and N. Using these estimates in (3.19) together with (3.18) and Lemma 2.1-(h_8), for sufficiently large b, we get

$$\begin{aligned} t_k^N &\geq \frac{b}{2}\int_{B_{\delta/k}(0)}\int_{B_{\delta/k}(0)} \frac{dxdy}{|x-y|^\mu} \exp\left(2(h(t_k\mathcal{M}_k))^{\frac{2N}{N-1}}\right) \\ &\geq \frac{b}{2}\int_{B_{\delta/k}(0)}\int_{B_{\delta/k}(0)} \frac{dxdy}{|x-y|^\mu} \exp\left(2((h(1))^2(t_k\mathcal{M}_k))^{\frac{N}{N-1}}\right) \\ &= \frac{b}{2}\exp\left(\log k\left(\frac{2(h(1))^{\frac{2N}{N-1}}(t_k)^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{1}{N-1}}}\right)\right)\int_{B_{\delta/k}(0)}\int_{B_{\delta/k}(0)} \frac{dxdy}{|x-y|^\mu} \\ &\geq \frac{b}{2}\exp\left(\log k\left(\frac{2(h(1))^{\frac{2N}{N-1}}t_k^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{1}{N-1}}}\right)\right) C_{\mu,N}\left(\frac{\delta}{k}\right)^{2N-\mu} \\ &= \frac{b}{2}\exp\left(\log k\left[\left(\frac{2(h(1))^{\frac{2N}{N-1}}t_k^{\frac{N}{N-1}}}{\omega_{N-1}^{\frac{1}{N-1}}}\right) - (2N-\mu)\right]\right) C_{\mu,N}\,(\delta)^{2N-\mu} \\ &\geq \frac{b}{2}C_{\mu,N}\delta^{2N-\mu}. \end{aligned} \tag{3.20}$$

On the other hand, using the fact that $J(tw) \to -\infty$ as $t \to \infty$ and (3.17), we get that the sequence $\{t_k\}$ is bounded, which contradicts (3.20), since b is arbitrary. This establishes our claim (3.16) and hence, we conclude the proof. □

3.3. **Convergence results.** Let $\{w_k\} \subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence for J. Now Lemma 3.3 yields that $\{w_k\}$ is bounded in $W_0^{1,N}(\Omega)$. Thus, there exists $w \in W_0^{1,N}(\Omega)$ such that up to a subsequence, still denoted by $\{w_k\}$,

$$\begin{aligned} &w_k \rightharpoonup w \quad \text{weakly in } W_0^{1,N}(\Omega), \\ &w_k \to w \quad \text{strongly in } L^q(\Omega),\ q \in [1,\infty), \\ &w_k(x) \to w(x) \quad \text{point-wise a.e. in } \Omega, \end{aligned}$$

as $k \to \infty$. Also, from (3.12) and (3.13), we obtain that for some positive constants C', C'',

$$\int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) F(x,h(w_k))\ dx \le C', \tag{3.21}$$

$$\int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k))h'(w_k)w_k\ dx \le C''. \tag{3.22}$$

Now we have the next two results, where we consider $\{w_k\}$ to be satisfying all the above facts.

Lemma 3.5. *Assume that the assumptions (f_1)-(f_5) hold and let h be defined as in (2.2). Let $\{w_k\} \subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence for J. Then we have*

$$\lim_{k\to\infty} \int_\Omega \left(\int_\Omega \frac{|F(y,h(w_k)) - F(y,h(w))|}{|x-y|^\mu} dy\right) |F(x,h(w_k)) - F(x,h(w))|\ dx = 0. \tag{3.23}$$

Proof. We know that if a function $\mathcal{F} \in L^1(\Omega)$ then for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\left|\int_{\Omega'} \mathcal{F}(x)\ dx\right| < \epsilon,$$

for any measurable set $\Omega' \subset \Omega$ with $|\Omega'| \le \delta(\epsilon)$. Also, if $\mathcal{F} \in L^1(\Omega)$ then for any fixed $\delta_0 > 0$ there exists $\alpha > 0$ such that

$$|\{x \in \Omega : |\mathcal{F}(x)| \ge \alpha\}| \le \delta_0.$$

Now (3.21) gives that

$$\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) F(\cdot,h(w_k)) \in L^1(\Omega)$$

and similarly, (3.4) gives that

$$\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu} dy\right) F(\cdot,h(w)) \in L^1(\Omega).$$

Now we fix $\delta_* > 0$ and choose $\alpha > \max\left\{1, \left(\frac{2C''M_0}{\delta_*}\right)^{\frac{1}{m_0+1}}, s'\right\}$. Then by using (f_4), Lemma 2.1-(h_4) and (3.21), we deduce

$$\begin{aligned} &\int_{\Omega\cap\{h(w_k)\ge\alpha\}} \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) F(x,h(w_k))\ dx \\ &\le M_0 \int_{\Omega\cap\{h(w_k)\ge\alpha\}} \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) \frac{f(x,h(w_k))}{(h(w_k))^{m_0}}\ dx \\ &\le M_0 \int_{\Omega\cap\{h(w_k)\ge\alpha\}} \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) \frac{f(x,h(w_k))h(w_k)}{(h(w_k))^{m_0+1}}\ dx \\ &\le \frac{2M_0}{\alpha^{m_0+1}} \int_{\Omega\cap\{h(w_k)\ge\alpha\}} \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k))h'(w_k)w_k\ dx < \delta_*. \end{aligned} \tag{3.24}$$

Using the similar argument as above in addition with Fatou's lemma, we have

$$\int_{\Omega\cap\{h(w)\ge\alpha\}} \left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu} dy\right) F(x,h(w))\ dx \le \delta_*. \tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\begin{aligned}
&\left|\int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx - \int_\Omega \left(\int_\Omega \frac{f(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx\right|\\
&\leq 2\delta_* + \left|\int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\right.\\
&\qquad\left. - \int_{\Omega\cap\{h(w)\leq\alpha\}} \left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx\right|.
\end{aligned}$$

Next, we show that as $k\to\infty$

$$\int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx \to \int_{\Omega\cap\{h(w)\leq\alpha\}} \left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx.$$

Since $\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(\cdot,h(w)) \in L^1(\Omega)$, Fubini's theorem yields that

$$\begin{aligned}
&\lim_{\Lambda\to\infty} \int_{\Omega\cap\{h(w)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w)\geq\Lambda\}} \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx\\
&= \lim_{\Lambda\to\infty} \int_{\Omega\cap\{h(w)\geq\Lambda\}} \left(\int_{\Omega\cap\{h(w)\leq\alpha\}} \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx = 0.
\end{aligned}$$

Thus, we can fix $\Lambda > \max\left\{\left(\frac{2C'M_0}{\delta_*}\right)^{\frac{1}{m_0+1}}, s'\right\}$ such that, using (3.21), (f_4) and Lemma 2.1-(h_4), we deduce

$$\begin{aligned}
&\int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w_k)\geq\Lambda\}} \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\\
&\leq M_0 \int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w_k)\geq\Lambda\}} \frac{f(y,h(w_k))}{(h(w_k))^{m_0}|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\\
&\leq \frac{M_0}{\Lambda^{m_0+1}} \int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w_k)\geq\Lambda\}} \frac{f(y,h(w_k))h(w_k)(y)}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\\
&\leq \frac{2M_0}{\Lambda^{m_0+1}} \int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w_k)\geq\Lambda\}} \frac{f(y,h(w_k))h'(w_k)w_k(y)}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\\
&\leq \frac{2M_0}{\Lambda^{m_0+1}} \int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h'(w_k)w_k\,dx \leq \delta_*.
\end{aligned}$$

Again, following the similar argument as above and employing Fatou's lemma, we can derive

$$\int_{\Omega\cap\{h(w)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w)\geq\Lambda\}} \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx \leq \delta_*.$$

Thus, we obtain

$$\begin{aligned}
&\left|\int_{\Omega\cap\{h(w)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w)\geq\Lambda\}} \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx\right.\\
&\qquad\left. - \int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w_k)\geq\Lambda\}} \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\right| \leq 2\delta_*.
\end{aligned}$$

Now we claim that as $k\to\infty$, for fixed positive real numbers α and Λ, the following holds:

$$\begin{aligned}
&\left|\int_{\Omega\cap\{h(w_k)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w_k)\leq\Lambda\}} \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\,dx\right.\\
&\qquad\left. - \int_{\Omega\cap\{h(w)\leq\alpha\}} \left(\int_{\Omega\cap\{h(w)\leq\Lambda\}} \frac{F(y,h(w))}{|x-y|^\mu}dy\right) F(x,h(w))\,dx\right| \to 0.
\end{aligned} \tag{3.26}$$

It is easy to compute that

$$\left(\int_{\Omega\cap\{h(w_k)\le\Lambda\}}\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)F(x,h(w_k))\chi_{\Omega\cap\{h(w_k)\le\alpha\}}$$
$$\to\left(\int_{\Omega\cap\{h(w)\le\Lambda\}}\frac{F(y,h(w))}{|x-y|^\mu}dy\right)F(x,h(w))\chi_{\Omega\cap\{h(w)\le\alpha\}} \tag{3.27}$$

point-wise a.e. as $k\to\infty$. Now using $N=r$ in (3.2), Lemma 2.1-(h_5) and (1.2), we get a constant $C_{\alpha,\Lambda}>0$ depending on α and Λ such that

$$\begin{aligned}
&\int_{\Omega\cap\{h(w_k)\le\alpha\}}\left(\int_{\Omega\cap\{h(w_k)\le\Lambda\}}\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)F(x,h(w_k))dx\\
&\le C_{\alpha,\Lambda}\int_{\Omega\cap\{h(w_k)\le\alpha\}}\left(\int_{\{h(w_k)\le\Lambda\}}\frac{|h(w_k)(y)|^r}{|x-y|^\mu}dy\right)|h(w_k)(x)|^r dx\\
&\le C_{\alpha,\Lambda}\int_{\Omega\cap\{h(w_k)\le\alpha\}}\left(\int_{\{h(w_k)\le\Lambda\}}\frac{|w_k(y)|^r}{|x-y|^\mu}dy\right)|w_k(x)|^r dx\\
&\le C_{\alpha,\Lambda}\int_\Omega\int_\Omega\left(\frac{|w_k(y)|^r}{|x-y|^\mu}\ dy\right)|w_k(x)|^r\ dx\\
&\le C_{\alpha,\Lambda}C(N,\mu)\|w_k\|^{2r}_{L^{\frac{2Nr}{2N-\mu}}(\Omega)}\to C_{\alpha,\Lambda}C(N,\mu)\|w\|^{2r}_{L^{\frac{2Nr}{2N-\mu}}(\Omega)}\text{ as }k\to\infty,
\end{aligned}\tag{3.28}$$

since $w_k\to w$ strongly in $L^q(\Omega)$ for each $q\in[1,\infty)$. This combining with Theorem 4.9 in [5] implies that there exists $\mathcal{G}\in L^1(\Omega)$ such that up to a subsequence, for each $k\in\mathbb{N}$, we have

$$\left|\left(\int_{\Omega\cap\{h(w_k)\le\Lambda\}}\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)F(x,h(w_k))\chi_{\Omega\cap\{h(w_k)\le\alpha\}}\right|\le|\mathcal{G}(x)|.$$

Therefore, using (3.27) and employing the Lebesgue dominated convergence theorem, we obtain (3.26). Thus, $\Lambda\to\infty$

$$\int_\Omega\int_{\Omega\cap\{|h(w)|\ge\Lambda\}}\frac{F(y,h(w))}{|x-y|^\mu}F(x,h(w))dy\ dx=o(\Lambda), \tag{3.29}$$

$$\int_\Omega\int_{\{|h(w_k)|\ge\Lambda\}}\frac{F(y,h(w_k))}{|x-y|^\mu}F(x,h(w_k))dy\ dx=o(\Lambda), \tag{3.30}$$

$$\int_\Omega\int_{\{|h(w)|\ge\Lambda\}}\frac{F(y,h(w_k))}{|x-y|^\mu}F(x,h(w))dy\ dx=o(\Lambda), \tag{3.31}$$

and

$$\int_\Omega\int_{\{|h(w_k)|\ge\Lambda\}}\frac{F(y,h(w_k))}{|x-y|^\mu}F(x,h(w))dy\ dx=o(\Lambda). \tag{3.32}$$

So,

$$
\begin{aligned}
&\int_\Omega\left(\int_\Omega \frac{|F(y,h(w_k))-F(y,h(w))|}{|x-y|^\mu}dy\right)|F(x,h(w_k))-F(x,h(w))|\ dx\\
&\leq 2\int_\Omega\left(\int_\Omega \frac{\chi_{\{h(w_k)\geq\Lambda\}}(y)F(y,h(w_k))}{|x-y|^\mu}dy\right)F(x,h(w_k))\ dx\\
&+4\int_\Omega\left(\int_\Omega \frac{F(y,h(w_k))\chi_{\{h(w)\geq\Lambda\}}(x)F(x,h(w))}{|x-y|^\mu}dy\right)\ dx\\
&+4\int_\Omega\left(\int_\Omega \frac{\chi_{\{h(w_k)\geq\Lambda\}}(y)F(y,h(w_k))F(x,h(w))}{|x-y|^\mu}dy\right)\ dx\\
&+2\int_\Omega\left(\int_\Omega \frac{\chi_{\{h(w)\geq\Lambda\}}(y)F(y,h(w))}{|x-y|^\mu}dy\right)F(x,h(w))\ dx\\
&+\int_\Omega\Bigg[\left(\int_\Omega \frac{|F(y,h(w_k))\chi_{\{h(w_k)\leq\Lambda\}}(y)-F(y,h(w))\chi_{\{h(w)\leq\Lambda\}}(y)|}{|x-y|^\mu}dy\right)\\
&\qquad\qquad |F(x,h(w_k))\chi_{\{h(w_k)\leq\Lambda\}}(x)-F(x,h(w))\chi_{\{h(w)\leq\Lambda\}}(x)|\Bigg]\ dx.
\end{aligned}
$$

Then from Lebesgue dominated convergence theorem we infer that the last integration tends to 0 as $k\to\infty$. Hence, making use of (3.29)-(3.32), we finally conclude (3.23). $\square$

Lemma 3.6. *Assume that (f_1)-(f_5) hold and h is defined as in (2.2). Let $\{w_k\}\subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence for J. Then for all $\varphi\in W_0^{1,N}(\Omega)$, we have*

$$\lim_{k\to\infty}\int_\Omega\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)\varphi\ dx=\int_\Omega\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right)f(x,h(w))h'(w)\varphi\ dx.$$

Proof. Let $\Omega'\subset\subset\Omega$ and $\psi\in C_c^\infty(\Omega)$ such that $0\leq\psi\leq 1$ and $\psi\equiv 1$ in Ω'. One can easily compute that

$$\left\|\frac{\psi}{1+w_k}\right\|^N=\int_\Omega\left|\frac{\nabla\psi}{1+w_k}-\psi\frac{\nabla w_k}{(1+w_k)^2}\right|^N\ dx\leq 2^{N-1}(\|\psi\|^N+\|w_k\|^N), \tag{3.33}$$

which yields that $\frac{\psi}{1+w_k}\in W_0^{1,N}(\Omega)$. Now taking $\phi=\frac{\psi}{1+w_k}$ in (3.12) as a test function and using Lemma 2.1-(h_3) and (3.33), we obtain

$$
\begin{aligned}
&\int_{\Omega'}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)\frac{f(x,h(w_k))}{1+u_k}h'(w_k)\ dx\\
&\leq\int_\Omega\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)\frac{f(x,h(w_k)h'(w_k)\psi}{1+w_k}\ dx\\
&\leq\int_\Omega\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)\frac{f(x,h(w_k)\psi}{1+w_k}\ dx\\
&=\epsilon_k\left\|\frac{\psi}{1+w_k}\right\|+\int_\Omega|\nabla w_k|^{N-2}\nabla w_k\nabla\left(\frac{\psi}{1+w_k}\right)\ dx\\
&\leq\epsilon_k 2^{\frac{N-1}{N}}(\|\psi\|+\|w_k\|)+\int_\Omega|\nabla w_k|^{N-2}\nabla w_k\left(\frac{\nabla\psi}{1+w_k}-\varphi\frac{\nabla w_k}{(1+w_k)^2}\right)\ dx\\
&\leq\epsilon_k 2^{\frac{N-1}{N}}(\|\varphi\|+\|w_k\|)+\int_\Omega|\nabla w_k|^{N-1}(|\nabla\psi|+|\nabla w_k|)\ dx\\
&\leq\epsilon_k 2^{\frac{N-1}{N}}(\|\psi\|+\|w_k\|)+[\|\psi\|\|w_k\|^{N-1}+\|w_k\|^N]\leq C_1,
\end{aligned}
\tag{3.34}
$$

where C_1 is a positive constant and in the last line we used the fact that $\{w_k\}$ is bounded in $W_0^{1,N}(\Omega)$. Again, using the boundedness of the sequence $\{w_k\}$, from (3.13), we get

$$\begin{aligned}
&\int_{\Omega'}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h'(w_k)w_k\, dx\\
&\le \int_{\Omega}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h'(w_k)w_k\, dx\\
&\le \epsilon_k\|w_k\|+\|w_k\|^N\le C_2 \qquad (3.35)
\end{aligned}$$

for some constant $C_2>0$. Combining (3.34) and (3.35), we deduce

$$\begin{aligned}
&\int_{\Omega'}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h'(w_k)\, dx\\
&\le 2\int_{\Omega'\cap\{w_k<1\}}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) \frac{f(x,h(w_k))h'(w_k)}{1+w_k}\, dx\\
&+\int_{\Omega'\cap\{w_k\ge 1\}}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) w_kf(x,h(w_k))h'(w_k)\, dx\\
&\le 2\int_{\Omega'}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) \frac{f(x,h(w_k))h'(w_k)}{1+w_k}\, dx\\
&+\int_{\Omega'}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) w_kf(x,h(w_k))h'(w_k)\, dx\\
&\le 2C_1+C_2:=C_3.
\end{aligned}$$

Thus, the sequence $\{v_k\}:=\left\{\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)\right\}$ is bounded in $L^1_{\rm loc}(\Omega)$. Therefore, there exists a radon measure ζ such that, up to a subsequence, $v_k\rightharpoonup\zeta$ in the *weak**-topology as $k\to\infty$. Hence, we have

$$\lim_{k\to\infty}\int_\Omega\int_\Omega\left(\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h'(w_k)\eta\, dx=\int_\Omega \eta\, d\zeta,\ \forall \eta\in C_c^\infty(\Omega).$$

Since w_k satisfies (3.12), we achieve

$$\int_A \eta d\zeta=\lim_{k\to\infty}\int_A|\nabla w_k|^{N-2}\nabla w_k\nabla\eta\, dx,\quad \forall A\subset\Omega,$$

which together with Lemma 4.1 yields that the Radon measure ζ is absolutely continuous with respect to the Lebesgue measure. So, there exists a function $\varrho\in L^1_{\rm loc}(\Omega)$ such that for any $\eta\in C_c^\infty(\Omega)$, it holds that $\int_\Omega\eta\, d\zeta=\int_\Omega\eta\varrho\, dx$, thanks to Radon-Nikodym theorem,. Therefore, we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\int_{\Omega}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) f(x,h(w_k))h'(w_k)\eta(x)\, dx\\
&=\int_\Omega\eta\varrho\, dx=\int_{\Omega}\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right) f(x,h(w))h'(w)\eta(x)\, dx,\quad \forall\eta\in C_c^\infty(\Omega),
\end{aligned}$$

since $C_c^\infty(\Omega)$ is dense in $W_0^{1,N}(\Omega)$, this completes the proof. □

Proof of Theorem 1.4: Let $\{w_k\}$ be a Palais-Smale sequence at the level β^*. Then $\{w_k\}$ can be obtained as a minimizing sequence associated to the variational problem (3.11). Then by Lemma 3.3, there exists $w\in W_0^{1,N}(\Omega)$ such that, up to a subsequence, $w_k\rightharpoonup w$ weakly in $W_0^{1,N}(\Omega)$ as $k\to\infty$.
Now by using Lemma 4.1 and Lemma 3.6, we infer that w forms a weak solution of (2.5). We claim that $w\not\equiv 0$. Indeed, if not, that is, if $w\equiv 0$ then using Lemma 3.5, we have

$$\int_{\Omega}\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right) F(x,h(w_k))\, dx\to 0 \text{ as } k\to\infty,$$

which yields that $\lim\limits_{k\to\infty}J(w_k)=\dfrac{1}{N}\lim\limits_{k\to\infty}\|w_k\|^N=\beta^*$. That is,

$$\lim_{k\to\infty}\|w_k\|^N=\beta^*N.$$

Therefore, for any real number $l>0$, there exists $k_0\in\mathbb{N}$ such that
$$\|w_k\|^{\frac{N}{N-1}}<\frac{2N-\mu}{2N(h(1))^{2N}}\alpha_N-l,\quad\text{for all }\ k\geq k_0. \tag{3.36}$$
Next, we show that
$$\int_\Omega\left(\int_\Omega\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)w_k\ dx\to 0 \text{ as } k\to\infty. \tag{3.37}$$
Now using Proposition 1.1, (3.1), Lemma 2.1-(h_4)-(h_6), Hölder inequality and Sobolev inequality, we deduce
$$\begin{aligned}
&\int_\Omega\left(\int_\Omega\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)w_kdx\\
&\leq \tau\int_\Omega\left(\int_\Omega\frac{f(y,h(w_k))h(w_k)}{|x-y|^\mu}dy\right)f(x,h(w_k))h(w_k)dx\\
&\leq \tau C(N,\mu)\left(\int_\Omega|f(x,h(w_k))h(w_k)|^{\frac{2N}{2N-\mu}}\ dx\right)^{\frac{2N-\mu}{N}}\\
&\leq \tilde{C}_1(N,\mu,\epsilon)\left(\int_\Omega\left(|h(w_k)|^N+|h(w_k)|^r\exp((1+\epsilon)|h(w_k)|^{\frac{2N}{N-1}})\right)^{\frac{2N}{2N-\mu}}\ dx\right)^{\frac{2N-\mu}{N}}\\
&\leq \tilde{C}_1(N,\mu,\epsilon)\left[\|w_k\|^{2N}_{L^{\frac{2N^2}{2N-\mu}}(\Omega)}+\|w_k\|^{2r}_{L^{\frac{2Nr}{2N-\mu}}(\Omega)}\left(\int_\Omega\exp\left(2(1+\epsilon)\frac{2N}{2N-\mu}h|(w_k)|^{\frac{2N}{N-1}}\right)\ dx\right)^{\frac{2N-\mu}{2N}}\right]\\
&\leq \tilde{C}_2(N,\mu,\epsilon)\left[\|w_k\|^{2N}+\|w_k\|^{2r}\left(\int_\Omega\exp\left(2(1+\epsilon)\frac{2N}{2N-\mu}2^{\frac{1}{N-1}}\|w_k\|^{\frac{N}{N-1}}\left(\frac{|w_k|}{\|w_k\|}\right)^{\frac{N}{N-1}}\right)\ dx\right)^{\frac{2N-\mu}{2N}}\right].
\end{aligned} \tag{3.38}$$
Using (3.36), we can choose l (for example, take $l=\frac{2N-\mu}{(h(1))^{2N}2N}\alpha_N-\frac{2N-\mu}{2(1+\epsilon)2^{\frac{1}{N-1}}2N}\alpha_N>0$, as $h(1)\leq 1$) such that, for sufficiently large $k\in\mathbb{N}$, $2(1+\epsilon)\frac{2N}{2N-\mu}2^{\frac{1}{N-1}}\|w_k\|^{\frac{N}{N-1}}<\alpha_N$. Therefore, in the light of Theorem 1.2 along with Lemma 3.4, for sufficiently large $k\in\mathbb{N}$, from (3.38), it follows that
$$\int_\Omega\left(\int_\Omega\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)w_kdx<C.$$
Now by employing Vitali's convergence theorem, we obtain
$$\int_\Omega\left(\int_\Omega\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)w_k\ dx\to 0 \text{ as } k\to\infty.$$
Since, $\{v_n\}$ is a Palais-Smale sequence for J, hence we have $\lim\limits_{k\to\infty}\langle J'(w_k),w_k\rangle=0$ which together with (3.37) gives that $\lim_{k\to\infty}\|w_k\|^N=0$. Thus, from Lemma 3.5, it follows that $\lim\limits_{k\to\infty}J(w_k)=0=\beta^*$ which is a contradiction to the fact that $\beta^*>0$. Hence, $w\not\equiv 0$.

Next, we prove that $w>0$ in Ω. Now Lemma 3.3 yields that $\{w_k\}$ is bounded. Therefore, there exists a constant $a^*>0$ such that, up to a subsequence, $\|w_k\|\to a^*$ as $k\to\infty$. From the fact that $J'(w_k)\to 0$ strongly in $\left(W_0^{1,N}(\Omega)\right)^*$ and using Lemma 3.6 and Lemma 4.1, when $k\to\infty$, we obtain
$$\int_\Omega\left(\int_\Omega\frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)\varphi\ dx\to\int_\Omega\left(\int_\Omega\frac{F(y,h(w))}{|x-y|^\mu}dy\right)f(x,h(w))h'(w)\varphi\ dx$$
and
$$\int_\Omega|\nabla w|^{N-2}\nabla w\nabla\varphi\ dx=\int_\Omega\left(\int_\Omega\frac{F(y,h(w))}{|x-y|^\mu}dy\right)f(x,h(w))h'(w)\varphi\ dx,$$
for all $\varphi\in W_0^{1,N}(\Omega)$. In particular, taking $\varphi=w^-$ in the above equation, we obtain $\|w^-\|=0$ which implies that $w^-=0$ a.e. in Ω. Therefore, $w\geq 0$ a.e. in Ω.

From Theorem 1.2, we have $f(\cdot,h(w))\in L^q(\Omega)$, for $1\leq q<\infty$. From (3.3), we know that $F(x,h(w))\in$

$L^q(\Omega)$, for any $q \in [1,\infty)$. Since $\mu \in (0,N)$ and Ω is bounded then using the fact that $y \to |x-y|^{-\mu} \in L^q(\Omega)$ for all $q \in (1, \frac{N}{\mu})$ uniformly in $x \in \Omega$ and applying Hölder's inequality, we can deduce

$$\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu} dy \in L^\infty(\Omega). \tag{3.39}$$

Therefore, using the fact that $h'(s) \le 1$, we get

$$\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}\ dy\right) f(x,h(w))h'(w) \in L^q(\Omega),$$

for $1 \le q < \infty$. Now by employing the regularity results for the elliptic equations, we infer that $w \in L^\infty(\Omega)$ and $w \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$. Finally, from the strong maximum principle, we draw the conclusion that $w > 0$ in Ω and $w \not\equiv 0$. This completes the proof of Theorem 1.4.

4. Appendix A

In this section, we give prove of the following almost everywhere convergence of gradients of Palais-Smale sequences using Concentration compactness arguments.

Lemma 4.1. *Suppose the assumptions (f_1)-(f_6) hold and h is defined as in (2.2). Let $\{w_k\} \subset W_0^{1,N}(\Omega)$ be a Palais-Smale sequence for J. Then $\nabla w_k \to \nabla w$ a.e. in Ω. Moreover, we have, as $k \to \infty$*

$$|\nabla w_k|^{N-2}\nabla w_k \rightharpoonup |\nabla w|^{N-2}\nabla w \text{ weakly in } (L^{\frac{N}{N-1}}(\Omega))^N. \tag{4.1}$$

Proof. Since Lemma 3.3 yields that $\{w_k\}$ is bounded in $W_0^{1,N}(\Omega)$, there exists $w \in W_0^{1,N}(\Omega)$ such that, in the sense of subsequence, we have $w_k \rightharpoonup w$ weakly in $W_0^{1,N}(\Omega)$, $w_k \to w$ strongly in $L^q(\Omega)$, $q \in [1,\infty)$, $w_k(x) \to w(x)$ point-wise a.e. in Ω, as $k \to \infty$. From the properties of the sequence $\{w_k\}$, it is evident that the sequence $\{|\nabla w_k|^{N-2}\nabla w_k\}$ must be bounded in $(L^{\frac{N}{N-1}}(\Omega))^N$, which implies that there exists $u \in (L^{\frac{N}{N-1}}(\Omega))^N$ such that,

$$|\nabla w_k|^{N-2}\nabla w_k \rightharpoonup u \text{ weakly in } (L^{\frac{N}{N-1}}(\Omega))^N \text{ as } k \to \infty. \tag{4.2}$$

Also we have, $\{|\nabla w_k|^N\}$ is bounded in $L^1(\Omega)$, which yields that there exists a non-negative radon measure σ such that, up to a subsequence, we have

$$|\nabla w_k|^N \to \sigma \text{ in } (C(\overline{\Omega}))^* \text{ as } k \to \infty. \tag{4.3}$$

Our aim is to show $u = |\nabla w|^{N-2}\nabla w$. For that, first we take $\nu > 0$ and set $X_\nu := \{x \in \overline{\Omega} : \sigma(B_l(x) \cap \overline{\Omega}) \ge \nu, \text{ for all } l > 0\}$.

Claim 1: X_ν is a finite set.

Indeed, if not, then there exists a sequence of distinct points $\{z_k\}$ in X_ν such that, $\sigma(B_l(z_k) \cap \overline{\Omega}) \ge \nu$ for all $l > 0$ and for all $k \in \mathbb{N}$. This gives that $\sigma(\{z_k\}) \ge \nu$ for all k. Therefore, $\sigma(X_\nu) = +\infty$. But this is a contradiction to the fact that

$$\sigma(X_\nu) = \lim_{k\to\infty} \int_{X_\nu} |\nabla w_k|^N\ dx \le C.$$

Hence, the claim holds. Thus, we can take $X_\nu = \{z_1, z_2, \dots, z_n\}$.

Claim 2: We can choose $\nu > 0$, such that $\nu^{\frac{1}{N-1}} < \frac{2N-\mu}{2N}\alpha_N$ and we have

$$\begin{aligned}\lim_{k\to\infty} \int_S \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k))h'(w_k)w_k\ dx \\ = \int_S \left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu} dy\right) f(x,h(w))h'(w)w\ dx,\end{aligned} \tag{4.4}$$

where S is any compact subset of $\overline{\Omega} \setminus X_\nu$.

Let $z_0 \in S$ and $l_0 > 0$ be such that $\sigma(B_{l_0}(z_0) \cap \overline{\Omega}) < \nu$ that is $z_0 \notin X_\sigma$. Also, we consider $\phi \in C_c^\infty(\Omega)$

satisfying $0 \le \phi(x) \le 1$ for $x \in \Omega$, $\phi \equiv 1$ in $B_{\frac{l_0}{2}}(z_0) \cap \overline{\Omega}$ and $\phi \equiv 0$ in $\overline{\Omega} \setminus (B_{l_0}(z_0) \cap \overline{\Omega})$. Then

$$\lim_{k\to\infty} \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |\nabla w_k|^N dx \le \lim_{k\to\infty} \int_{B_{l_0}(z_0)\cap\overline{\Omega}} |\nabla w_k|^N \phi dx \le \sigma(B_{l_0}(z_0) \cap \overline{\Omega}) < \nu.$$

Hence, for sufficiently large $k \in \mathbb{N}$ and sufficiently $\epsilon > 0$ small, it follows that

$$\int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |\nabla w_k|^N \le \nu(1-\epsilon). \tag{4.5}$$

Now we estimate the following using (f_3), (4.5) and Lemma 2.1-(h_6):

$$\begin{aligned}
&\int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |f(x,h(w_k))|^q \, dx = \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |g(x,h(w_k))|^q \exp\left(q|h(w_k)|^{\frac{2N}{N-1}}\right) \, dx \\
&\le C \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} \exp\left((1+\epsilon)2^{\frac{1}{N-1}} q|h(w_k)|^{\frac{2N}{N-1}}\right) \, dx \\
&\le C \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} \exp\left((1+\epsilon)2^{\frac{1}{N-1}} q|w_k|^{\frac{N}{N-1}}\right) \, dx \\
&\le C \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} \exp\left((1+\epsilon)q 2^{\frac{1}{N-1}} \nu^{\frac{1}{N-1}} (1-\epsilon)^{\frac{1}{N-1}} \left(\frac{|w_k|^N}{\int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |\nabla w_k|^N dx}\right)^{\frac{1}{N-1}}\right) \, dx
\end{aligned} \tag{4.6}$$

Thus, we can choose $q > 1$ and $\epsilon > 0$ such that $(1+\epsilon)2^{\frac{1}{N-1}} q\nu^{\frac{1}{N-1}} < \alpha_N$ and then using Theorem 1.2 in the last relation, we obtain

$$\int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |f(x,h(w_k))|^q \, dx \le C. \tag{4.7}$$

Next, we consider

$$\begin{aligned}
&\int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} \left| \left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu} dy\right) f(x,h(w_k))h'(w_k)w_k - \left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu} dy\right) f(x,h(w))h'(w)w \right| \, dx \\
&\le \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} \left| \left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu} dy\right) (f(x,h(w_k))h'(w_k)w_k - f(x,h(w))h'(w))w \right| \, dx \\
&+ \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} \left| \left(\int_\Omega \frac{F(y,h(w_k)) - F(y,h(w))}{|x-y|^\mu} dy\right) f(x,h(w_k))h'(w_k)w_k \right| \, dx \\
&:= J_1 + J_2.
\end{aligned}$$

From the asymptotic growth assumptions on f, we obtain

$$\lim_{s\to\infty} \frac{f(x,s)t}{(f(x,s))^r} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } r > 1. \tag{4.8}$$

Using (3.39), we get

$$J_1 \le C \int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}} |f(x,h(w_k))h'(w_k)w_k - f(x,h(w))h'(w)w| \, dx,$$

where $C > 0$ is a constant. Moreover, (4.8) and (4.6) imply that $\{f(x,h(w_k))h'(w_k)w_k\}$ is an equi-integrable family of functions over $B_{\frac{l_0}{2}}(z_0) \cap \overline{\Omega}$. Also, the continuity of f and h implies the pointwise convergence of $f(x,h(w_k))h'(w_k)w_k$ to $f(x,h(w))h'(w)w$ as $k \to \infty$. Hence, by applying Vitali's convergence theorem, we obtain $J_1 \to 0$ as $k \to \infty$. Next, we show that $J_2 \to 0$ as $k \to \infty$.
First by using the semigroup property of the Riesz Potential, we get the following for some constant $C > 0$,

independent of k:

$$\begin{aligned}
&\int_\Omega \left(\int_\Omega \frac{F(y,h(w_k))-F(y,h(w))}{|x-y|^\mu}dy\right)\chi_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}(x)f(x,h(w_k))h'(w_k)w_k\ dx\\
&\leq C\left(\int_\Omega\left(\int_\Omega \frac{|F(y,h(w_k))-F(y,h(w))|dy}{|x-y|^\mu}\right)|F(x,h(u_k))-F(x,h(u))|\ dx\right)^{\frac{1}{2}}\\
&\quad\times\left(\int_\Omega\left(\int_\Omega \chi_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}(y)\frac{f(y,h(w_k))h'(w_k)w_k}{|x-y|^\mu}dy\right)\chi_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}(x)f(x,h(w_k))h'(w_k)w_k\ dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining (4.6), (4.8) and the fact that ν can be chosen sufficiently small with $\nu^{\frac{1}{N-1}} < \frac{2N-\mu}{2N}\alpha_N$, it follows that

$$\begin{aligned}
&\left(\int_\Omega\left(\int_\Omega \chi_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}(y)f(y,h(w_k))h'(w_k)w_k dy\right)\chi_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}(x)f(x,h(w_k))h'(w_k)w_k\ dx\right)^{\frac{1}{2}}\\
&\leq \|\chi_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}f(\cdot,h(w_k))h'(w_k)w_k\|_{L^{\frac{2N}{2N-\mu}}(\Omega)}\leq C.
\end{aligned}$$

Now using Lemma 3.5, we get $J_2\to 0$ as $k\to\infty$. This yields that

$$\begin{aligned}
\lim_{k\to\infty}\int_{B_{\frac{l_0}{2}}(z_0)\cap\overline{\Omega}}\Bigg|&\left(\int_\Omega \frac{F(y,h(w_k))}{|x-y|^\mu}dy\right)f(x,h(w_k))h'(w_k)w_k\\
&-\left(\int_\Omega \frac{F(y,h(w))}{|x-y|^\mu}dy\right)f(x,h(w))h'(w)w\Bigg|\ dx=0.
\end{aligned}$$

Since S is compact, we can repeat this procedure over a finite covering of balls and hence, achieve (4.4). Finally, the proof of (4.1) can be concluded by the similar standard arguments as in the proof of Lemma 4 in [12]. □

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