UNBOUNDED GENERALIZATION OF LOGARITHMIC REPRESENTATION OF INFINITESIMAL GENERATORS

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ABSTRACT. The logarithmic representation of infinitesimal generators is generalized to the cases when the evolution operator is unbounded. The generalized result is applicable to the representation of infinitesimal generators of unbounded evolution operators, where unboundedness of evolution operator is an essential ingredient of nonlinear analysis. In conclusion a general framework for the identification between the infinitesimal generators with evolution operators is established. A mathematical framework for such an identification is indispensable to the rigorous treatment of nonlinear transforms: e.g., transforms appearing in the theory of integrable systems.

Keywords: logarithm of operators, resolvent operator, Dunford-Taylor integral

1. INTRODUCTION

The theory of analytic semigroup [16, 23, 24] provides an integral representation for a certain class of semigroups of operators. It actually clarifies the analytic property and the resulting regularity of solutions for evolution equations of parabolic type. In the theory of analytic semigroup of operators, the exponential functions of unbounded operators are defined by means of the Dunford-Taylor integral [26], where note that the Riesz-Dunford integral [4] and the Dunford-Taylor integral are the generalized concept of the Cauchy integral representation in complex analysis. In the preceding work [8], the logarithmic function of bounded operators are shown to be well-defined, while the theory for defining the logarithmic function of unbounded operators has not been established yet, except for the logarithms of sectorial operators [1, 2, 5, 6, 19, 20, 21].

The aim of this article is to generalize the logarithmic representation of infinitesimal generators by defining the logarithmic functions of unbounded operators without assuming the sectorial property of the evolution operators. Although the boundedness of evolution operator is assumed in the standard theory of abstract evolution equations, this restriction prevents us to have access to the abstract and general treatments of nonlinear transforms in which the evolution operators (i.e. solutions) are often identified with the infinitesimal generators of another equations (for example, see [12, 14]). That is, the standard theory of abstract evolution equations is generalized in this article. In conclusion the unbounded generalization of the logarithmic representation is presented by means of newly-proposed doubly-implemented resolvent representation.

2. Mathematical settings

Let X be a Banach space and t and s are real numbers included in a finite interval $[-T, T]$. Let $U(t, s)$ be the evolution operator in X. Two parameter semigroup $U(t, s)$, which is continuous with respect to both parameters t and s, is assumed to be a closed operator in X, and not necessarily a bounded operator on X. That is, the boundedness condition

$$(1) \qquad\qquad \|U(t, s)\| \le M e^{\omega(t-s)},$$

is not assumed. Furthermore the resolvent set of $U(t, s)$ is assumed to be non-empty. Following the standard theory of abstract evolution equations [17, 18, 25] the semigroup property:

$$U(t, r)U(r, s) = U(t, s)$$

is assumed to be satisfied for arbitrary t and s included in a finite interval $[-T, T]$. For any $t, r, s \in [-T, T]$ satisfying $s \le r \le t$, the domain space of $U(t, s)$ is assumed to be

$$D(U(t, s)) = \{x \in D(U(r, s)); \ D(U(r, s))x \in D(U(t, r)), \}.$$

In the following $D(U(t, s))$ is denoted by $D(U)$ if there is no ambiguity. Compared to the standard treatment, this condition is additionally assumed. All the above conditions are weaker than those for the preceding work dealing with the logarithmic representation of infinitesimal generator of bounded $U(t, s)$, as the domain space is fixed to be X in case of bounded $U(t, s)$ [8]. The continuity assumption for $U(t, s)$ makes sense in terms of describing a time evolution within a fixed time, because $U(t, s)$ is not assumed to be a linear operator. Note that $U(t, s)$ can be either linear or nonlinear semigroup.

3. Main result

3.1. Logarithm of unbounded two-parameter semigroup.
Let X and $U(t, s)$ be a Banach space and a two-parameter semigroup respectively, as defined in the previous section. In particular, $U(t, s)$ with its non-empty resolvent set is assumed to be only a closed operator but a bounded operator. For the definition of the logarithm of $U(t, s)$, the resolvent operator of $U(t, s)$

$$(2) \qquad\qquad I_\eta(t, s) := (I - \eta^{-1}U(t, s))^{-1}$$

is utilized. Here $\eta \in \mathbf{C}$ is assumed to be included in the resolvent set of $U(t, s)$, so that $I_\eta(t, s)$ is necessarily bounded on X. It also follows according to the validity of

$$(3) \qquad I_\eta(t, s) - I = (I - I_\eta(t, s)^{-1})I_\eta(t, s) = \eta^{-1}U(t, s)I_\eta(t, s)$$

that the product of operator $U(t, s)I_\eta(t, s)$ is bounded on X. The operator $U(t, s)I_\eta(t, s)$ is regarded as the resolvent approximation of $U(t, s)$, so that any difference between certain resolvent operator and a scalar-multiplied identity operator can be regarded as the resolvent approximation. In the following the resolvent operator $I_\eta(t, s)$ is denoted simply by I_η, if there is no ambiguity. Let Log be a principal branch of logarithm being defined by

$$\mathrm{Log} z = \log|z| + i\,\mathrm{arg} Z,$$

where Z is a complex number chosen to satisfy $|Z| = |z|$, $-\pi < \arg Z \le \pi$, Formal calculation of logarithm leads to the definition of $\mathrm{Log}U(t,s)$

(4)
$$\mathrm{Log}U(t,s) := \mathrm{Log}[U(t,s)I_\eta] - \mathrm{Log}[I_\eta]$$
$$= \int_{\Gamma_1} \mathrm{Log}\lambda(\lambda I - U(t,s)I_\eta)^{-1}d\lambda - \int_{\Gamma_2} \mathrm{Log}\lambda(\lambda I - I_\eta)^{-1}d\lambda,$$

where the relation $U(t,s) = U(t,s)I_\eta I_\eta^{-1}$ is adopted for the logarithmic rule. The right hand side is always regarded as the Riesz-Dunford integral if it is possible to define integral paths Γ_1 and Γ_2 excluding the origin and including the spectral sets of $U(t,s)I_\eta$ and I_η respectively. That is, by utilizing the logarithmic rule, there is no need to introduce the Dunford-Taylor integral for defining the logarithm of generally unbounded operators, although the Dunford-Taylor integral is a standard method of operator theory to define the functions of unbounded operators. The representation shown in Eq. (4) leads to the logarithmic representation of infinitesimal generators simply using the Riesz-Dunford integral, where note that the Dunford-Taylor integral is a generalization of Riesz-Dunford integral in terms of defining the functions of unbounded operators. Since the representation using Riesz-Dunford integral is associated with the operators included in the $B(X)$-module [10, 15], here is an advantage of the proposed method only using the Riesz-Dunford integral representation.

In order to obtain the logarithmic representation of infinitesimal generators, it is necessary to deal with the t-differential of logarithm. Using Eq. (3),

(5)
$$\partial_t \mathrm{Log}U(t,s) = \partial_t \mathrm{Log}[\eta(I_\eta - I)] - \partial_t \mathrm{Log}[I_\eta]$$
$$= (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \mathrm{Log}[\eta I_\eta + (\nu - \eta)I)] - (I + \nu I_\eta^{-1})\partial_t \mathrm{Log}[I_\eta + \nu I]$$

follows, where ν is a certain complex number and t-differential is defined using a kind of weak topology (i.e. the locally-strong topology defined in [13]). Here two different resolvent operators with parameters λ and ν are doubly implemented. It is notable here that there is no need to take the limit such as $\eta \to \infty$ and/or $\nu \to \infty$. It realizes the pure Riesz-Dunford treatment of logarithm of unbounded operators; the right hand side of Eq. (5) consists of the bounded type of logarithmic representation

$$\partial_t \mathrm{Log}[\eta(I_\eta - I)] := (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \mathrm{Log}[\eta I_\eta + (\nu - \eta)I)]$$
$$= (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \int_{\Gamma_3} \mathrm{Log}\lambda(\lambda I - \eta I_\eta - (\nu - \eta)I))^{-1}d\lambda,$$

$$\partial_t \mathrm{Log}[I_\eta] := (I + \nu I_\eta^{-1})\partial_t \mathrm{Log}[I_\eta + \nu I]$$
$$= (I + \nu I_\eta^{-1})\partial_t \int_{\Gamma_4} \mathrm{Log}\lambda(\lambda I - I_\eta - \nu I))^{-1}d\lambda,$$

which can be defined similar to the preceding work [8], Indeed, according to the boundedness of $U(t,s)I_\eta = \eta(I_\eta - I)$ and I_η, there always exist certain ν for integral paths Γ_3 and Γ_4 excluding the origin and including the spectral sets of $\eta I_\eta + (\nu - \eta)I$ and $I_\eta + \nu I$ respectively. The introduction of $\nu \in \mathbf{C}$ is based on the idea presented in Ref. [8]; i.e., the shift on the complex plain. Regardless of the choice of η, the above two logarithms of operators in the right hand side are always well-defined by choosing appropriate ν with sufficiently-large $|\nu|$. In this way the logarithmic representation of infinitesimal generators is generalized to unbounded two-parameter semigroup.

Theorem 1. *Let t and s satisfy $-T \leq s \leq t \leq T$, and $D(U)$ and Y be dense subspaces of X. Let $U(t,s)$ satisfy the condition shown in the previous section, and $U(t,s)$ and its infinitesimal generator $A(t)$ are assumed to commute. Furthermore $U(t,s)$ is assumed to be invertible (i.e., $U(s,t) = U(t,s)^{-1}$ exists). For a given two-parameter closed operator $U(t,s) : D(U) \to X$, its infinitesimal generator $A(t) : Y \to X$ is represented by*

$$(6) \qquad A(t) = (I_\eta(t,s)^2 - I_\eta(t,s))^{-1}(I_\eta(t,s) + \nu I)\partial_t \mathrm{Log}[I_\eta(t,s) + \nu I],$$

where ν and η are certain complex numbers satisfying

$$(7) \qquad\qquad\qquad \nu = \frac{\eta}{1-\eta}.$$

Note that the resolvent operator $I_\eta = I_\eta(t,s)$ is a function of t and s.

Proof. The logarithmic relation with $U(t,s) = U(t,s)I_\eta I_\eta^{-1} = \eta(I_\eta - I)I_\eta^{-1}$ lead to

$$A(t) = \partial_t \mathrm{Log}[\eta(I_\eta - I)I_\eta^{-1}] = \partial_t \mathrm{Log}[\eta(I_\eta - I)] - \partial_t \mathrm{Log}[I_\eta]$$

where all the logarithms in this equality are always well-defined by choosing η property, although $\mathrm{Log}U(t,s)$ is not necessarily well-defined. The well-definedness follows from the boundedness of $\eta(I_\eta - I)$ and I_η. Using the logarithmic representation for bounded evolution operator [8], it is further calculated as

$$A(t) = (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \mathrm{Log}[\eta I_\eta + (\nu - \eta)I)] - (I + \nu I_\eta^{-1})\partial_t \mathrm{Log}[I_\eta + \nu I],$$

$$= (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \mathrm{Log}[\eta(I_\eta + \tfrac{\nu-\eta}{\eta}I)] - (I + \nu I_\eta^{-1})\partial_t \mathrm{Log}[I_\eta + \nu I]$$

$$= (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \left\{ \mathrm{Log}[\eta I] + \mathrm{Log}[I_\eta + \tfrac{\nu-\eta}{\eta}I] \right\} - (I + \nu I_\eta^{-1})\partial_t \mathrm{Log}[I_\eta + \nu I]$$

$$= (I + \nu\eta^{-1}(I_\eta - I)^{-1})\partial_t \mathrm{Log}[I_\eta + \tfrac{\nu-\eta}{\eta}I] - (I + \nu I_\eta^{-1})\partial_t \mathrm{Log}[I_\eta + \nu I],$$

where $I_\eta = I_\eta(t,s)$ is a function of t and s. It is possible to take $\nu = \frac{\eta}{1-\eta}$ by assuming $\nu > \max((M+1)\eta, M)$ for a certain real number M satisfying $\|I_\eta\| \leq M$. The logarithms in the right hand side is always well-defined fFor such ν. It follows that

$$A(t) = [(I + \nu\eta^{-1}(I_\eta - I)^{-1}) - (I + \nu I_\eta^{-1})]\partial_t \mathrm{Log}[I_\eta + \nu I]$$

$$= [I_\eta + \eta(1-\eta)^{-1}I]\eta^{-1}(1-\eta)\nu I_\eta^{-1}(I_\eta - I)^{-1}\partial_t \mathrm{Log}[I_\eta + \nu I]$$

so that the logarithmic representation is obtained as

$$A(t) = (I_\eta^2 - I_\eta)^{-1}(I_\eta + \nu I)\partial_t \mathrm{Log}[I_\eta + \nu I],$$

where Eq. (7) is utilized.

For the existence of

$$(8) \quad (I_\eta^2 - I_\eta)^{-1}(I_\eta + \nu I) = \eta U(t,s)^{-1}(I - \eta^{-1}U(t,s))\left((1+\nu)I - \nu\eta^{-1}U(t,s)\right)$$

as an closed operator in X, it is necessary for 0 to be included in the resolvent set of $U(t,s)$. The condition $0 \in \rho(U(t,s))$ is satisfied by assumption. $\qquad\square$

The operator $(I_\eta(t,s) + \nu I)$ is the inverse of the resolvent operator of $I_\eta(t,s)$, here is a reason why the resolvent representation is doubly-imposed in the resulting logarithmic representation. Note that the right hand side of Eq. (6) plays a role of pre-infinitesimal generator being defined in Ref. [8]. A pre-infinitesimal generator is an operator possible to be an infinitesimal generator if an ideal domain is given.

3.2. **Logarithmic representation using the alternative infinitesimal generator.** Let us begin with a part of the logarithmic representation shown in Eq. (6). For the existence of

$$(9) \qquad (I_\eta^2 - I_\eta)^{-1}(I_\eta + \nu I) = (I - I_\eta^{-1})^{-1} I_\eta^{-2}(I_\eta + \nu I)$$

as an closed operator in X, it is necessary for η to be included in the resolvent set of $U(t,s)$. The condition $\eta \in \rho(U(t,s))$ is satisfied by assumption. On the other hand the invertible assumption in Theorem 1 is removed by introducing the alternative infinitesimal generator [9]. Indeed, it is readily seen that 0 and 1 are included in the resolvent set of I_η^{-1} by replacing I_η with $e^{a(t,s)} - \nu I$ with a certain sufficiently large $|\nu|$. The logarithmic representation for infinitesimal generator of closed operator shown in Theorem 1 is generalized in the following lemma.

Lemma 2. *Let $\partial_t a(t,s)$ be an alternative infinitesimal generator being defined by*

$$(10) \qquad a(t,s) := \mathrm{Log}[I_\eta(t,s) + \nu I].$$

Let t and s satisfy $-T \le t, s \le T$, and $D(U)$ and Y be dense subspaces of X. Let $U(t,s)$ satisfy the condition shown in the previous section, and $U(t,s)$ and its infinitesimal generator $A(t)$ are assumed to commute. For a given two-parameter closed operator $U(t,s) : D(U) \to X$, its infinitesimal generator $A(t)$ is represented by

$$(11) \qquad A(t) = \left(e^{a(t,s)} - (2\nu+1)I + (\nu^2+\nu)e^{-a(t,s)}\right)^{-1} \partial_t a(t,s),$$

where ν and η are certain complex numbers satisfying $\nu = \frac{\eta}{1-\eta}$.

Proof. Following the previous research [9], it is practical to define the alternative infinitesimal generator $\partial_t a(t,s)$ with the relation $a(t,s) = \mathrm{Log}[I_\eta(t,s) + \nu I]$. Since $a(t,s)$ is a bounded operator on X, there is no restriction to define both $e^{a(t,s)}$ and $e^{-a(t,s)}$ simultaneously. The logarithmic representation of the infinitesimal generators shown in Theorem 1 is replaced using the alternative infinitesimal generator.

$$A(t) = \left(e^{a(t,s)} - (2\nu+1)I + (\nu^2+\nu)e^{-a(t,s)}\right)^{-1} \partial_t a(t,s)$$

follows, where $\partial_t a(t,s)$ is the infinitesimal generator of

$$(12) \qquad e^{a(t,s)} = I_\eta(t,s) + \nu I = (I - \eta^{-1}U(t,s))^{-1}\{(\nu+1)I - \nu\eta^{-1}U(t,s))\},$$

so that the infinitesimal generator of the inverse of doubly-implemented resolvent operator $I_\eta(t,s) + \nu I$. $\qquad\square$

The relation $I_\eta(t,s) = e^{a(t,s)} - \nu I$ leads to the boundedness

$$\|e^{a(t,s)}\| \le M + \nu < 2\nu,$$

because ν is assumed to satisfy $\nu > \max((M+1)\eta, M)$. That is, although $U(t,s)$ is not assumed to be bounded, $a(t,s)$ is necessarily bounded. Simultaneously the boundedness of its inverse

$$\|e^{-a(t,s)}\| \le 1/(M+\nu) < \frac{1}{2\nu}$$

follows.

3.3. **Algebraic property.** Under the well-definedness of $a(t,s) := \mathrm{Log}[I_\eta(t,s) + \nu I]$, there are three conditions to be satisfied for $A(t)$ to be an element of $B(X)$-module [10, 15]:

(1) boundedness of $\left(e^{a(t,s)} - (2\nu + 1)I + (\nu^2 + \nu)e^{-a(t,s)}\right)^{-1}$;

(2) continuity of $\left(e^{a(t,s)} - (2\nu + 1)I + (\nu^2 + \nu)e^{-a(t,s)}\right)^{-1}$ with respect to two parameters t and s;

(3) commutation between $\left(e^{a(t,s)} - (2\nu + 1)I + (\nu^2 + \nu)e^{-a(t,s)}.\right)^{-1}$ and $\partial_t a(t,s)$

It is remarkable that a part $\left(e^{a(t,s)} - (2\nu + 1)I + (\nu^2 + \nu)e^{-a(t,s)}\right)^{-1}$ is always bounded on X, as $e^{a(t,s)}$ and $e^{-a(t,s)}$ has been clarified to be bounded on X. Furthermore this part is continuous with respect to t and s, as $U(t,s)$ is assumed to be continuous with respect to t and s. Indeed it is sufficient to be

continuity of $U(t,s)$ $\Rightarrow$ continuity of $I_\eta(t,s)$ $\Rightarrow$ continuity of $a(t,s)$

is true for both t and s, where $\eta \in \mathbf{C}$ is taken from the resolvent set of $U(t,s)$. For the commutation, since the commutation between $U(t,s)$ and $I_\lambda(t,s)$ is always true,

commutation: $U(t,s)$ and $A(t) = \partial_t U(t,s)$

  $\Rightarrow$ commutation: $I_\lambda(t,s)$ and $\partial_t I_\lambda(t,s)$

  $\Rightarrow$ commutation: $I_\lambda(t,s)$ and $\partial_t a(t,s)$,

where the commutation between $U(t,s)$ and $A(t)$ is assumed (cf. assumption of Theorem 1 and Lemma 2). Consequently, the infinitesimal generators of unbounded evolution operators being represented by

$$A(t) = \left(e^{a(t,s)} - (2\nu + 1)I + (\nu^2 + \nu)e^{-a(t,s)}\right)^{-1} \partial_t \mathrm{Log}[I_\eta(t,s) + \nu I].$$

are clarified to be elements of $B(X)$-module; i.e. a module over the Banach algebra $B(X)$. This is a generalization of original logarithmic representation in the sense of extending the applicable evolution operators.

3.4. **Relativistic formulation.** In terms of applying to nonlinear transform such as the Cole-Hopf transform[3, 7], it is useful to introduce the relativistic form of the obtained logarithmic representation [11]. The relativistic formulation is represented using the tensor notation.

Lemma 3. *Let n be a positive integer, and i be the evolution direction ($1 \leq i \leq n$). Let x^i and ξ^i satisfy $-L_{\leq} x^i, \xi^i \leq L$, and $D(U)$ and Y be dense subspaces of X. Let $U(x^i, \xi^i)$ satisfy the condition shown in the previous section, and $U(x^i, \xi^i)$ and its infinitesimal generator $A(x^i)$ are assumed to commute. For a given two-parameter closed operator $U(x^i, \xi^i) : D(U) \to X$, its infinitesimal generator $K(x^i)$ is represented by*

(13) $K(x^i) = \left(e^{a(x^i, \xi^i)} - (2\nu + 1)I + (\nu^2 + \nu)e^{-a(x^i, \xi^i)}\right)^{-1} \partial_{x^i} a(x^i, \xi^i),$

where ν and η are certain complex numbers satisfying $\nu = \frac{\eta}{1-\eta}$, and $a(x^i, \xi^i)$ is an alternative infinitesimal generator being defined by $a(x^i, \xi^i) = \mathrm{Log}[I_\eta(x^i, \xi^i) + \nu I]$.

Proof. The statement follows from Lemma 2.   □

In this formulation the evolution direction x^i (i-th direction in the tensor form) can be either t or others.

4. SUMMARY

In the application to soliton theory, it is practical to consider that the evolution operator $U(t, s)$ to be an infinitesimal generator of another equation. For this purpose, it is often necessary to have a certain kind of identification between infinitesimal generators and evolution operators. This situation is often found in the theory of integrable systems.such as soliton theory (for example, see [22]). The general framework for identifying the evolution operators (i.e. solutions) with the infinitesimal generators of another equations (an operator included in an equation) is established. By focusing on the logarithmic relation, the representation is obtained only using the Riesz-Dunford integral of resolvent operators. From an algebraic point of view, it is an advantage of the presented result.

In conclusion, the standard theory of abstract evolution equations itself is generalized in the sense of weakening the assumptions for evolution operators, and the theory for logarithm of operators is improved in the sense of removing the sectorial assumption. The present framework of treating the logarithmic representation of generally-unbounded two parameter semigroup (evolution operator) will open up a way to analyze

- abstract and general treatments to linear and nonlinear transforms;
- explosions of solutions at a finite time;
- some stochastic differential equations without assuming the bounded time evolution;

where It is remarkable that two parameter semigroup $U(t, s)$ can be either linear or nonlinear semigroups, so that the presented logarithmic representation can be utilized for the mathematical modeling of nonlinear phenomena.

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