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The dynamic analysis on a class of stochastic impulsive equations with doubly weighted pseudo almost automorphic coefficients on time scales

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Devoting to exploring the translation invariance and convolution invariance of doubly weighted pseudo almost automorphic stochastic processes with impulses on time scales proposed in this paper. Based on these results, taking advantage of a new approach to obtain the existence and uniqueness of the doubly weighted pseudo almost automorphic solutions to a class of stochastic nonlinear impulsive equations on time scales, which enrich the dynamics of doubly weighted pseudo almost automorphic stochastic processes. Finally, an example is researched to illustrate our conclusions.

Keywords: doubly weighted pseudo almost automorphic stochastic processes; time scales; existence and uniqueness

1. INTRODUCTION

In view of the inevitability of random factors in various fields in the authentic world, and a large number of dynamic systems may have structural changes once subjected to small variations in random factors, therefore, qualitative analysis of stochastic differential equations in depicting economical models, electronics, nuclear reactor dynamics, fluid dynamics, biological kinetics and so on have magnetized more and more attention of many mathematicians, see previous studies for details [1]-[5]. At this point it is natural and realistic to investigate a class of stochastic nonlinear equations driven by Brownian motion.

The difference with random factors is that many phenomena characterized by the fact that their states are subject to mutation at certain point, and then can be modeled by impulsive system, which has become an active area of research due to its fully consideration of the influence of instantaneous changes on the whole process [6]-[7], especially, it has the properties of differential and difference equations. In addition, as a link and promotion for the classical theory of differential equations and difference equations, the theory of time scales effectively unify continuous and discrete analysis, and has occupied an irreplaceable position in various fields of application, such as quantum physics, artificial intelligence, economics et al., see the literature [8]-[14] for more details and references therein. In recent years, the perfect match between impulse and time scales has become a hot topic and attracted increasing attention on the existence and uniqueness of almost periodic and almost automorphic mild solutions or its extension of different kinds of abstract equations [15]-[16].

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As one of the important and significant extensions of classical almost automorphic function [17], Diagana introduced a new concept of doubly weighted pseudo almost automorphic functions [18], which simplified as weighted pseudo almost automorphic functions presented by Blot if ρ and q is equivalent [19]. More interesting, Yang and Zhu extends the case of doubly weight to stochastic process in the sense of square-mean [20]. In resent years, there has been tremendous interest in developing the qualitative property of differential equations with almost automorphic coefficient or its promotion, such as the existence, uniqueness, stability of varied differential equations, the detail contributions respect to this topic can be references therein in [21]-[23]. Nevertheless, there are very few authors which have been worked on the almost periodic/automorphic stochastic process and its applications to stochastic equations on time scales with impulses [24]. So far, the research on p-mean doubly weighted pseudo almost automorphic stochastic process with impulses on time scales for $p > 2$ remains unexplored, let alone in exploring its properties or even existence and uniqueness of the doubly weighted pseudo almost automorphic mild solutions for a class of stochastic abstract equations.

As we know, it is pointed out in [25]-[28] that the weighted pseudo almost automorphic mild solution to various stochastic differential equations obtained by classic Banach fixed point theorem, the acquisition of these theorems requires the indispensable lipchitz assumptions for the coefficients of stochastic systems, which is a strong constraint. In order to weaken this condition, we split the doubly weighted pseudo almost automorphic stochastic process with impulses on time scales proposed in this paper into two parts, including almost automorphic and ergodic perturbed composition at infinity. The feature of the required assumptions is that only the former needs the lipchitz condition, while the latter controlled by a bounded functions and a nondecreasing function. Therefore, a natural question is that wether we can get the existence and uniqueness of the doubly weighted pseudo almost automorphic mild solution for a class of impulsive stochastic equations with time scales without utilize Banach fixed point theorem? This is an urgent but unsolved problem.

In order to fill the gap of the foregoing discussion, this paper firstly proposed the concept of p-mean doubly weighted pseudo almost automorphic stochastic processes with impulses on time scales, and further establishes some properties of the space of these stochastic processes, such as the convolution invariance and translation invariance. These results obtained are not appeared in previous papers. Moreover, by taking advantage of a new approach presented in [29] under the non-lipchitz condition, we investigate a class of nonlinear stochastic equations driven by Brownian motion of the form

$$\begin{cases} \Delta x(t) = A(t)x(t)\Delta t + F_1(t, x(t))\Delta t + \displaystyle\int_{-\infty}^{t} B(t-u)F_2(u, x(u))\Delta u\Delta t \\ \qquad\qquad + \displaystyle\int_{-\infty}^{t} C(t-u)F_3(u, x(u))\Delta W(u)\Delta t, \ \ t \in \mathbb{T}, \ \ t \neq t_i, \\ x(t_i^+) - x(t_i^-) = I_i(x(t_i)), \ \ i \in \mathbb{Z}, \end{cases} \tag{1}$$

where $A(t)$ is a family of linear operator; $\mathbb{T}$ is almost periodic time scale; Δx stands for the Δ-stochastic differential of stochastic process x; B and C are convolution-type kernels; $\{B_t : t \in \mathbb{T}\}$ is Brownian motion indexed by time scale defined on a complete probability space, F_1, F_2 and F_3 are stochastic processes that satisfied some appropriate assumptions. In addition, the notations $x(t_i^+)$ and $x(t_i^-)$ represent the right-hand and the left-hand side limits of $x(\cdot)$ at t_i in the sense of time scale respectively.

Finally, we briefly describe the organization and main results of this paper. In section 2, the relevant definitions and lemmas are simply introduced. In section 3, the convolution invariance and translation invariance of the space of doubly weighted pseudo almost automorphic stochastic processes with impulses on time scales are presented. In section 4, for the stochastic equations (1), the existence and uniqueness of the p-mean doubly weighted pseudo almost automorphic mild solution is proved. Finally, an example is explored to illustrate our conclusions.

2. PRELIMINARIES

Throughout this paper, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathbb{H}, \|\cdot\|)$ be real separable Hilbert spaces. Denote by $L^p(\mathbb{H})$ the set of all p-mean integrable $\mathbb{H}$-valued random variables that is a Banach space endowed with the norm $\|X\|_{PC} = \sup_{t\in\mathbb{T}}(\mathbb{E}\|X(t)\|^p)^{\frac{1}{p}} < \infty$. Assume $C_b(\mathbb{T}, L^p(\mathbb{H}))$ represents the space of all stochastically continuous bounded mappings from $\mathbb{T}$ to $L^p(\mathbb{H})$.

Denote the closed nonempty subset of real linear space $\mathbb{R}$ by the time scale $\mathbb{T}$ and the interval $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$. Define the backward jump operator $\varrho \colon \mathbb{T} \to \mathbb{T}$, the forward jump operator $\sigma \colon \mathbb{T} \to \mathbb{T}$ and the graininess function $\mu \colon \mathbb{T} \to \mathbb{R}^+$ as $\varrho(t) = \sup\{s \in \mathbb{T} : s < t\}$, $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\mu(t) = \sigma(t) - t$ respectively. In addition, the point $t \in \mathbb{T}$ is called left-dense or right-dense if $\rho(t) = t$, $t > \inf \mathbb{T}$ or $\sigma(t) = t$, $t < \sup \mathbb{T}$, it is called left-scattered or right-scattered if $\varrho(t) < t$ or $\sigma(t) > t$ separately. Besides, if $\mathbb{T}$ has a left-scattered maximum or a right-scattered minimum m, then define $\mathbb{T}^k = \mathbb{T} - m$ or $\mathbb{T}_k = \mathbb{T} - m$ correspondingly, otherwise, $\mathbb{T}^k = \mathbb{T}_k = \mathbb{T}$.

Definition 2.1.[9] A function $g \colon \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it is continuous at right dense points of $\mathbb{T}$ and its left-side limits exist at left dense points. The set of all rd-continuous functions will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. For any $f \in C_{rd}(\mathbb{T}, \mathbb{R})$, presented by $f^{\Delta}(t)$ as the delta derivative of f at t, which is the number (if it exists) with the property that for any given $\epsilon > 0$, there exists a neighborhood U of t such that for all $s \in U$, it yields

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| < \epsilon|\sigma(t) - s|.$$

Further, if $F^{\Delta}(t) = f(t)$, then delta integral is defined as

$$\int_{r_1}^{r_2} f(t)\Delta t = F(r_2) - F(r_1) \text{ for } r_1,\ r_2 \in \mathbb{T}.$$

Definition 2.2.[9] A function $p \colon \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all such regressive and rd-continuous functions will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}, \mathbb{R})$. Let the set $\mathfrak{R}^+ = \mathfrak{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0,\ t \in \mathbb{T}\}$.

Definition 2.3.[9] If p is a regressive function, then the generalized exponential function e_p is given as the unique solution of the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$, where $s \in \mathbb{T}$. An explicit formula for $e_p(t, s)$ is defined as

$$e_p(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right\} \text{ for all } s, t \in \mathbb{T}$$

with

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & \text{for } h \neq 0, \\ z, & \text{for } h = 0. \end{cases}$$

Lemma 2.1.[10] Let p and q are regressive functions, define

$$p \oplus q = p + q + \mu pq, \quad \ominus p = \frac{-p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Definition 2.4.[10] If $a, b \in \mathbb{T}$ and $a \leq b$, then

$$\vartheta((a, b]_{\mathbb{T}}) = b - a, \qquad \vartheta((a, b)_{\mathbb{T}}) = \varrho(b) - a.$$

If $a, b \in \mathbb{T} \backslash \mathbb{T}^k$ and $a \leq b$, then

$$\vartheta([a, b)_{\mathbb{T}}) = \varrho(b) - \varrho(a), \qquad \vartheta([a, b]_{\mathbb{T}}) = b - \varrho(a),$$

For more details of time scales and Δ-measurability, one is referred to[].

Definition 2.5.[11, 12] A time scale $\mathbb{T}$ is said to almost periodic if

$$\Xi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T},\ \text{for any } t \in \mathbb{T}\} \neq 0.$$

Let Γ be a collection of sets which is constructed by subsets of $\mathbb{R}$. A time scale $\mathbb{T}$ is called an almost periodic time scale with respect to Γ, if

$$\Gamma^* = \{\pm\tau \in \cap\Lambda : \Lambda \in \mathbb{T},\ t \pm \tau \in \mathbb{T},\ \text{for } t \in \mathbb{T}\},$$

and Γ^* is called the smallest almost periodic set of $\mathbb{T}$.

Definition 2.6.[13] A Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ indexed by a time scale $\mathbb{T}$ is an adapted stochastic process $W = \{W(t) : t \in \mathbb{T}\}$ with the following properties:

(i) $W(t_0) = 0$ a.s.;

(ii) if $t_0 \leq s < t$, then the increment $W(t) - W(s)$ is independent of $\mathcal{F}(s)$, and is normally distributed with mean zero and variance $t - s$ for $t, s \in \mathbb{T}$.

Definition 2.7.[13] A stochastic process $f\colon \mathbb{T} \times \Omega \to \mathbb{R}$ belongs to $L^2([0,1]_{\mathbb{T}})$ if f is adapted and $\mathbb{P}\left(\int_0^1 |f(t,\omega)|^2 \Delta t < \infty\right) = 1$.

Lemma 2.2.[13] A Δ-stochastic integral has the following properties:

(i) If $f, g \in L^2([0,1]_{\mathbb{T}})$ and $a_1, a_2 \in \mathbb{R}$, then

$$\int_0^1 [a_1 f(t) + a_2 g(t)] \Delta W(t) = a_1 \int_0^1 f(t) \Delta W(t) + a_2 \int_0^1 g(t) \Delta W(t).$$

(ii) Itô-isometry holds, that is

$$\mathbb{E}\left\{\left[\int_0^1 f(t)\Delta W(t)\right]^2\right\} = \mathbb{E}\left[\int_0^1 f^2(t)\Delta t\right].$$

Next, one introduces some concepts in the case of p-mean, which not investigated in previous papers.

Definition 2.8. $f\colon \mathbb{T} \times L^p(\mathbb{H})$ is said to be rd-piecewise continuous for the increasing sequence $\{t_k\} \subseteq \mathbb{T}$, $k \in \mathbb{Z}$, if f is continuous on $[t_k, t_{k+1})_{\mathbb{T}}$, where $[t_k, t_{k+1})_{\mathbb{T}}$ are called intervals of continuity of the function f.

Denote the space of all such stochastic processes by $PC^{rd}(\mathbb{T}, L^p(\mathbb{H}))$.

In the following, let $\mathcal{D}$ be the unbounded increasing sequences of real numbers set that consists of all sequences $\{t_i\}_{i\in\mathbb{Z}}$ satisfying $\inf\limits_{i\in\mathbb{Z}}(t_{i+1} - t_i) > 0$. For any $\{t_i\}_{i\in\mathbb{Z}} \in \mathcal{D}$, let $BPC^{rd}(\mathbb{T}, L^p(\mathbb{H}))$ be the space of all bounded rd-piecewise continuous functions $f\colon \mathbb{T} \times L^p(\mathbb{H})$ such that f is continuous at t for any $t \notin \{t_i\}_{i\in\mathbb{Z}}$ and $f(t_i) = f(t_i^-)$ for all $i \in \mathbb{Z}$.

Definition 2.9. Let $t_i^j = t_{i+j} - t_i$ for $i, j \in \mathbb{Z}$. The set $\{t_i^j\}$ is called equipotentially almost automorphic on an almost periodic time scale $\mathbb{T}$, if for any sequence of real numbers $\{s_n'\}_{n\in\mathbb{Z}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{Z}}$ such that $\lim\limits_{n\to\infty} t_k^{s_n} = \beta_k$ is well defined and $\lim\limits_{n\to\infty} \beta_k^{-s_n} = t_k$ for $t_k \in \mathcal{D}$.

Definition 2.10. A stochastic process $X \in PC^{rd}(\mathbb{T}, L^p(\mathbb{H}))$ is said to be p-mean piecewise almost automorphic if sequences of impulsive $\{t_k\}$ satisfying $\{t_k^j\}$ is equipotentially almost automorphic and for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{Z}} \subseteq \Xi$, there exists a subsequence $\{s_n\}_{n\in\mathbb{Z}} \subseteq \Xi$ such that for stochastic process $X^*\colon \mathbb{T} \to L^p(\mathbb{H})$ satisfying

$$\lim_{n\to\infty} \mathbb{E}\|X(t+s_n) - X^*(t)\|^p = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}\|X^*(t-s_n) - X(t)\|^p = 0.$$

The familay of all such p-mean stochastic processes is denoted by $AA(\mathbb{T}, L^p(\mathbb{H}))$.

Remark 2.1. $AA(\mathbb{T}, L^p(\mathbb{H}))$ is a Banach space with norm $\|\cdot\|_{PC}$.

Definition 2.11. A jointly continuous stochastic process $f(t,x) \in PC^{rd}(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H}))$, is said to be p-mean piecewise almost automorphic in $t \in \mathbb{T}$ and for all $x \in L^p(\mathbb{H})$ if sequences of impulsive $\{t_k\}$ satisfying $\{t_k^j\}$ is equipotentially almost automorphic and for every sequence of real numbers $\{s_n'\}_{n\in\mathbb{Z}} \subseteq \Xi$, there exists a subsequence $\{s_n\}_{n\in\mathbb{Z}} \subseteq \Xi$ such that for stochastic process $f^*(t,x)\colon \mathbb{T} \times L^p(\mathbb{H}) \to L^p(\mathbb{H})$ satisfying

$$\lim_{n\to\infty} \mathbb{E}\|f(t+s_n,x) - f^*(t,x)\|^p = 0 \text{ and } \lim_{n\to\infty} \mathbb{E}\|f^*(t-s_n,x) - f(t,x)\|^p = 0.$$

The set of all such stochastic processes is denoted by $AA(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H}))$.

From Definition 2.9 and Definition 2.10, one gives the next conclusions.

Lemma 2.3. Let $x \in PC^{rd}(\mathbb{T}, L^p(\mathbb{H}))$ is piecewise almost automorphic on time scales, and $\{t_k\} \subseteq \mathbb{T}$ is equipotentially almost automorphic with $\inf\limits_{i,q\in\mathbb{Z}} t_i^q > 0$, then $\{x(t_k)\}$ is a p-mean almost automorphic sequence.

Lemma 2.4. Let I_k is a sequence of p-mean almost automorphic and $x \in AA(\mathbb{T}, L^p(\mathbb{H}))$, assume there exists a constant $L > 0$ such that

$$\mathbb{E}\|I_k(x) - I_k(y)\|^p \leq L\mathbb{E}\|x-y\|^p, \ \ k \in \mathbb{Z}$$

for any $x, y \in L^p(\mathbb{H})$, then $\{I_k(x(t_k))\}$ is p-mean almost automorphic sequence.

Let $\mathcal{U}$ be the set of Δ-locally integrable ρ: $\mathbb{T} \to (0, +\infty)$, for given $r \in [0, +\infty) \cap \Xi$ and for any $t_0 \in \mathbb{T}$, define

$$m(r, \rho, t_0) := \int_{t_0-r}^{t_0+r} \rho(s)\Delta s.$$

Further, let $\mathcal{U}_\infty := \{\rho \in \mathcal{U} : \lim_{r\to+\infty} m(r, \rho, t_0) = +\infty\}$, $\mathcal{U}_b = \{\rho \in \mathcal{U}_\infty : \rho$ is bounded and $\inf_{x\in\mathbb{T}} \rho(x) > 0\}$. It is clear that $\mathcal{U}_b \subset \mathcal{U}_\infty \subset \mathcal{U}$. In addition, denote by

$$PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H})) := \left\{ x \in BPC^{rd}(\mathbb{T}, L^p(\mathbb{H})) : \lim_{r\to+\infty} \frac{1}{m(r, \rho, t_0)} \int_{t_0-r}^{t_0+r} \mathbb{E}\|x(t)\|^p q(t)\Delta t = 0 \right\},$$
$$PAA^{\rho,q}(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H})) := \{f(t, x) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H})) : \text{uniformly with respect to } x \in L^p(\mathbb{H})\}.$$

Based on Definition 2.10, we propose the following concepts.

Definition 2.12. Let $\rho,\ q \in \mathcal{U}_\infty$. A stochastic process $f \in PC^{rd}(\mathbb{T}, L^p(\mathbb{H}))$ is said to be doubly weighted piecewise pseudo almost automorphic on time scales provided $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{T}, L^p(\mathbb{H}))$ and $f_2 \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$.

Denote by $DWPAA(\mathbb{T}, L^p(\mathbb{H}), \rho, q)$ the set of all such processes.

Similarly, we can introduce

$$\begin{aligned} &DWPAA(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H}), \rho, q) \\ :=& \left\{ f = f_1 + f_2 \in PC^{rd}(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H})) : f_1 \in AA(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H})) \text{ and } f_2 \in PAA^{\rho,q}(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H})) \right\}. \end{aligned}$$

Remark 2.2. If $\rho/q \in \mathcal{U}_b$, then ρ and q is equivalent. Further, it follows $DWPAA(\mathbb{T}, L^p(\mathbb{H}), \rho, q) = DWPAA(\mathbb{T}, L^p(\mathbb{H}), \rho) = DWPAA(\mathbb{T}, L^p(\mathbb{H}), q)$.

Remark 2.3. In this paper, one investigates the doubly weighted piecewise pseudo almost automorphic stochastic process for $p > 2$ and that ρ is not equivalent to q, which is more difficult and possesses complex qualitative properties than the case $p = 2$ and $\rho,\ q$ is equivalent.

Next, we will present a indispensable Krasnoselskii's fixed point theorem used in section 4.

Lemma 2.5.[29] Let $\mathbb{B}$ be a bounded closed and convex subset of X, J_1, J_2 be two maps of $\mathbb{B}$ into X such that

$$J_1x + J_2y \in B \text{ for } x, y \in \mathbb{B}.$$

If J_1 is a contraction and J_2 is completely continuous, then the equation

$$J_1x + J_2x = x$$

has a solution on $\mathbb{B}$.

3. TRANSLATION INVARIANCE AND CONVOLUTION INVARIANCE

This section mainly establishes the translation and convolution invariance of the doubly weighted piecewise pseudo almost automorphic stochastic processes on time scales for the nonequivalence weights functions ρ_1, q_1 and ρ_2, q_2, which play an important role in the research of next section.

Denote by

$$\mathcal{U}_\infty^* = \left\{ \rho,\ q \in \mathcal{U}_\infty : \sup_{t\notin M_\varepsilon(\cdot)} \frac{q(t)}{\rho(t)} < +\infty,\ \sup_{t\in M_\varepsilon(\cdot)} q(t) < +\infty,\ \text{for any } \cdot \in BPC^{rd}(\mathbb{T}, L^p(\mathbb{H})) \right\},$$

where $M_\varepsilon(\cdot) := \{t \in \mathbb{T} : \mathbb{E}\| \cdot (t)\|^p \geq \varepsilon\}$.

Inspired by Lemma 3.2 in [16] and Lemma 3.1 in [20], one presents the following lemma.

Lemma 3.1. Let $\rho,\ q\in\mathcal{U}_\infty^*$, then $f\in PAA_0(\mathbb{T},L^p(\mathbb{H}),\rho,q)$ if and only if for each $\varepsilon>0$, it yields

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho,t_0)}\vartheta_\Delta([t_0-r,t_0+r]_\mathbb{T}\cap M_\varepsilon(f))=0.$$

Denote by $M_{r,t_0,\varepsilon}(\cdot)=[t_0-r,t_0+r]_\mathbb{T}\cap M_\varepsilon(\cdot)$. Based on Lemma 3.1, under some suitable conditions, one gives some results as follows.

Theorem 3.1. Let $\rho,\ q\in\mathcal{U}_\infty^*$ and

$$\overline{\lim}_{|t|\to\infty}\frac{\rho(t+\tau)}{\rho(t)}<+\infty,\quad \sup_{t\in M_\varepsilon(\cdot)}\frac{q(t+\tau)}{q(t)}<+\infty,\quad \sup_{t\notin M_\varepsilon(\cdot)}\frac{q(t+\tau)}{\rho(t)}<+\infty\ \text{ for }\tau\in\mathbb{T},$$

then $DWPAA(\mathbb{T},L^p(\mathbb{H}),\rho,q)$ is translation invariant with respect to Ξ.

Proof. It is clear that in order to complete the proof, it only needs to show $PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$ is translation invariant with respect to Ξ, that is, for any $f\in PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$, it follows

$$f_\tau(t)=f(t-\tau)\in PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))\text{ for }\tau\in\mathbb{T}.$$

Similar to the proof of Theorem 2.1 in [22], it follows $\overline{\lim}_{|t|\to+\infty}\frac{\rho(t+\tau)}{\rho(t)}<+\infty$ leads to

$$\overline{\lim}_{r\to+\infty}\frac{m(r+\tau,\rho,t_0)}{m(r,\rho,t_0)}<+\infty.$$

Without loss of generality, let $\tau>0$, one can calculate as follows:

$$\int_{t_0-r}^{t_0+r}\mathbb{E}\|f(t-\tau)\|^pq(t)\Delta t=\int_{t_0-r-\tau}^{t_0+r-\tau}\mathbb{E}\|f(t)\|^pq(t+\tau)\Delta t=J_1+J_2,$$

where

$$\begin{aligned}J_1&=\int_{[t_0-r-\tau,t_0+r-\tau]\cap M_\varepsilon(f)}\mathbb{E}\|f(t)\|^pq(t+\tau)\Delta t,\\ J_2&=\int_{[t_0-r-\tau,t_0+r-\tau]\setminus([t_0-r-\tau,t_0+r-\tau]\cap M_\varepsilon(f))}\mathbb{E}\|f(t)\|^pq(t+\tau)\Delta t.\end{aligned}$$

From the definition of $M_\varepsilon(f)$, one deduces

$$J_1\le\int_{M_{r+\tau,t_0,\varepsilon}(f)}\mathbb{E}\|f(t)\|^pq(t+\tau)\Delta t\le\sup_{t\in M_\varepsilon(f)}\frac{q(t+\tau)}{q(t)}\int_{M_{r+\tau,t_0,\varepsilon}(f)}\mathbb{E}\|f(t)\|^pq(t)\Delta t,\tag{2}$$

and

$$\begin{aligned}J_2&\le\int_{[t_0-r-\tau,t_0+r+\tau]\setminus M_{r+\tau,t_0,\varepsilon}(f)}\mathbb{E}\|f(t)\|^pq(t+\tau)\Delta t\\ &\le\sup_{t\notin M_\varepsilon(f)}\frac{q(t+\tau)}{\rho(t)}\int_{[t_0-r-\tau,t_0+r+\tau]\setminus M_{r+\tau,t_0,\varepsilon}(f)}\mathbb{E}\|f(t)\|^p\rho(t)\Delta t.\end{aligned}\tag{3}$$

According to (2)-(3), one obtains

$$\begin{aligned}&\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\mathbb{E}\|f_\tau(t)\|^pq(t)\Delta t\\ \le&\sup_{t\in\mathbb{R}}\mathbb{E}\|f(t)\|^p\sup_{t\in M_\varepsilon(f)}q(t)H_1(\rho,q,\tau,t_0)\frac{\vartheta_\Delta(M_{r+\tau,t_0,\varepsilon}(f))}{m(r+\tau,\rho,t_0)}+H_2(\rho,q,\tau,t_0)\varepsilon,\end{aligned}$$

where

$$H_1(\rho,q,\tau,t_0)=\frac{m(r+\tau,\rho,t_0)}{m(r,\rho,t_0)}\sup_{t\in M_\epsilon(f)}\frac{q(t+\tau)}{q(t)},\qquad H_2(\rho,q,\tau,t_0)=\frac{m(r+\tau,\rho,t_0)}{m(r,\rho,t_0)}\sup_{t\notin M_\epsilon(f)}\frac{q(t+\tau)}{\rho(t)},$$

therefore,

$$\overline{\lim}_{|t|\to+\infty}H_i(\rho,q,\tau,t_0)<\infty,\ \ i=1,2.$$

Since $f\in PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$ and $\rho,\ q\in\mathcal{U}_\infty^*$, thus, from Lemma 3.1, it follows that for each $\varepsilon>0$,

$$\frac{\vartheta_\Delta(M_{r+\tau,t_0,\varepsilon}(f))}{m(r+\tau,\rho,t_0)}\to 0 \text{ as } r\to+\infty.$$

In view of $\varepsilon\to 0$, then

$$\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\mathbb{E}\|f_\tau(t)\|^p q(t)\Delta t\to 0 \text{ as } r\to+\infty,$$

which implies $f_\tau\in PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$. □

In order to investigate the convolution invariance of the space of the doubly weighted piecewise pseudo almost automorphic stochastic processes, one define

$$(\mathcal{K}\xi)(t)=(\xi*k)(t):=\int_{\mathbb{R}}k(t-s)\xi(s)\Delta W(s),\quad k\in L^2(\mathbb{T}).$$

Theorem 3.2. Let $\rho,\ q\in\mathcal{U}_\infty^*$ and the space $DWPAA(\mathbb{T},L^p(\mathbb{H}),\rho,q)$ is translation invariant, then $DWPAA(\mathbb{T},L^p(\mathbb{H}),\rho,q)$ is convolution invariant.

Proof. Clearly, it only needs to prove $PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$ is convolution invariant provided it is translation invariant. It follows $\xi*k\in BPC^{rd}(\mathbb{T},L^p(\mathbb{H}))$ for any $\xi\in BPC^{rd}(\mathbb{T},L^p(\mathbb{H}))$ and $k\in L^2(\mathbb{T})$. By using Burkholder-Davis-Gundy inequality, it follows

$$\begin{aligned}\mathbb{E}|(\xi*k)(t)|^p\le&\bar{\sigma}^pC_p\mathbb{E}\left(\int_{\mathbb{R}}|k(t-s)|^2\|\xi(s)\|^2\Delta s\right)^{\frac{p}{2}}\\ \le&\bar{\sigma}^pC_p\|k\|_{L^2}^{p-2}\int_{\mathbb{T}}|k(t-s)|^2\mathbb{E}\|\xi(s)\|^p\Delta s.\end{aligned}$$

Therefore

$$\begin{aligned}&\lim_{r\to+\infty}\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\mathbb{E}\|(\xi*k)(t)\|^p q(t)\Delta t\\ \le&\lim_{r\to+\infty}\frac{\bar{\sigma}^pC_p\|k\|_{L^2}^{p-2}}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\int_{\mathbb{R}}\mathbb{E}\|\xi(s)\|^p|k(t-s)|^2\Delta s q(t)\Delta t\\ =&\lim_{r\to+\infty}\frac{\bar{\sigma}^pC_p\|k\|_{L^2}^{p-2}}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\int_{\mathbb{R}}\mathbb{E}\|\xi(t-s)\|^p|k(s)|^2\Delta s q(t)\Delta t.\end{aligned}$$

From Fibini theorem, it follows

$$\begin{aligned}&\lim_{r\to+\infty}\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\mathbb{E}\|(\xi*k)(t)\|^p q(t)\Delta t\\ \le&\bar{\sigma}^pC_p\|k\|_{L^2}^{p-2}\int_{\mathbb{R}}|k(s)|^2\lim_{r\to+\infty}\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\mathbb{E}\|\xi(t-s)\|^p q(t)\Delta t\Delta s.\end{aligned}$$

Combining the translation invariant of $PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$ and the Lebesgue dominated convergence theorem, it deduces

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho,t_0)}\int_{t_0-r}^{t_0+r}\mathbb{E}\|(\xi*k)(t)\|^p q(t)\Delta t=0,$$

which yields the convolution invariant of $PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$. Further, $DWPAA(\mathbb{T},L^p(\mathbb{H}),\rho,q)$ is convolution invariant. □

Corollary 3.1. If define

$$(\mathcal{K}\xi)(t) = (\xi * k)(t) := \int_{\mathbb{R}} k(t-s)\xi(s)\Delta s, \quad k \in L^1(\mathbb{T}),$$

then the Theorem 3.2 still holds.

Remark 3.1. The translation invariance of the space of $PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ is the sufficient condition of its convolution invariance.
Remark 3.2. Denote by $\mathcal{U}^{\rho,q} := \{\rho,\ q \in \mathcal{U}_\infty : PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ is translation invariant$\}$ in the rest of this paper.

4. EXISTENCE AND UNIQUENESS

In order to state the main results, one demands the following conditions.

(H_1) $A(t)$ generates an exponential stable evolution system $\{T(t,s)\}_{t\geq s}$, that is, there exist $K>0$ and $\delta>0$ satisfy

$$\|T(t,s)\| \leq Ke_{\ominus\delta}(t,s) \text{ for } t \geq s,$$

and for any sequence $\{s_n'\}_{n=1}^\infty \subseteq \Xi$, there exist a subsequence $\{s_n\}_{n=1}^\infty$ such that for any $\varepsilon>0$, there exists $N>0$ satisfying

$$\|T(t+s_n, s+s_n) - T_1(t,s)\| \leq \varepsilon e_{\ominus\delta}(t,s) \text{ and } \|T_1(t-s_n, s-s_n) - T(t,s)\| \leq \varepsilon e_{\ominus\delta}(t,s) \text{ for } t \geq s.$$

(H_2) Assume $F_i = \varphi_i + \eta_i \in DWPAA(\mathbb{T}\times L^p(\mathbb{H}), L^p(\mathbb{H}), \rho, q)$ and $\rho, q \in \mathcal{U}^{\rho,q}$, where $\varphi_i \in AA(\mathbb{T}\times L^p(\mathbb{H}), L^p(\mathbb{H}))$ and $\eta_i \in PAA^{\rho,q}(\mathbb{T}\times L^p(\mathbb{H}), L^p(\mathbb{H}))$ such that there exists positive constant L satisfies

$$\mathbb{E}\|\varphi_i(t,x) - \varphi_i(t,z)\|^p \leq L\mathbb{E}\|x-z\|^p,\ \ i=1,2,3 \tag{4}$$

for any $x, z \in L^p(\mathbb{H})$. Moreover, there exist $\gamma(t) \in PAA^{\rho,q}(\mathbb{T}, \mathbb{T}^+)$ with $\zeta_2 := \sup_{t\in\mathbb{T}} \gamma(t)$, and a nondecreasing function Φ: $\mathbb{T}^+ \to \mathbb{T}^+$ such that for all $x \in L^p(\mathbb{H})$ with $\|x\|_{PC} \leq h$, it follows

$$\mathbb{E}\|\eta_i(t,x)\|^p \leq \gamma(t)\Phi(h^p) \text{ and } \liminf_{h\to+\infty} \frac{\Phi(h)}{h} = \zeta_1. \tag{5}$$

(H_3) Let $I_i \in PC^{rd}(L^p(\mathbb{H}), L^p(\mathbb{H}))$ is a p-mean almost automorphic sequence and there exists a constant $L>0$ such that

$$\mathbb{E}\|I_k(x) - I_k(z)\|^p \leq L\mathbb{E}\|x-z\|^p,\ \ k \in \mathbb{Z},$$

for any $x, z \in L^p(\mathbb{H})$.

Lemma 4.1. Assume (H_1) holds and $\varphi \in AA(\mathbb{T}, L^p(\mathbb{H}))$, then

$$\chi(t) := \int_{-\infty}^t T(t,\sigma(s)) \int_{-\infty}^s C(s-u)\varphi(u)\Delta W(u)\Delta s,\ \ t \geq \sigma(s)$$

lies in $AA(\mathbb{T}, L^p(\mathbb{H}))$.

Proof. For any given sequence $\{s_n'\}_{n\in\mathbb{N}} \subseteq \Xi$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and stochastic process $\widetilde{\varphi}$: $\mathbb{T} \to L^p(\mathbb{H})$ such that

$$\lim_{n\to\infty} \mathbb{E}\|\varphi(t+s_n) - \widetilde{\varphi}(t)\|^p = 0. \tag{6}$$

Let $\widetilde{W}(m) = W(s+m) - W(s)$ for each $m \in \mathbb{R}$, then $\widetilde{W}$ also is a Brownian motion and has the same distribution as W. Define

$$\widetilde{\chi}_1(t) = \int_{-\infty}^t T_1(t,\sigma(s)) \int_{-\infty}^s C(s-u)\widetilde{\varphi}(u)\Delta W(u)\Delta s,$$

then

$$\mathbb{E}\|\chi_1(t+s_n) - \widetilde{\chi}_1(t)\|^p \leq 2^{p-1}[\Psi^1_{t,s_n}(t) + \Psi^2_{t,s_n}(t)],$$

where

$$\Psi^1_{t,s_n}(t) := \mathbb{E}\left\|\int_{-\infty}^{t} T(t+s_n,\sigma(r)+s_n)\int_{-\infty}^{0} C(-m)[\varphi(r+s_n+m)-\widetilde{\varphi}(r+m)]\Delta\widetilde{W}(m)\Delta r\right\|^p,$$
$$\Psi^2_{t,s_n}(t) := \mathbb{E}\left\|\int_{-\infty}^{t} [T(t+s_n,\sigma(r)+s_n)-T_1(t,r)]\int_{-\infty}^{0} C(-m)\widetilde{\varphi}(r+m)\Delta\widetilde{W}(m)\Delta r\right\|^p.$$

From the well-known Hölder and Burkholder-Davis-Gundy inequality, it yields

$$\begin{aligned}\Psi^1_{t,s_n}(t) \leq &\frac{K^p}{(-\ominus\delta)^{p-1}}\int_{-\infty}^{t} e_{\ominus\delta}(t,\sigma(r))\mathbb{E}\left\|\int_{-\infty}^{0} C(-m)[\varphi(r+s_n+m)-\widetilde{\varphi}(r+m)]\Delta\widetilde{W}(m)\right\|^p dr\\
\leq &\frac{K^pC_p}{(-\ominus\delta)^{p-1}}\int_{-\infty}^{t} e_{\ominus\delta}(t,\sigma(r))\mathbb{E}\left\|\int_{-\infty}^{0} [C(-m)\varphi(r+s_n+m)-\widetilde{\varphi}(r+m)]^2\Delta m\right\|^{\frac{p}{2}} dr\\
\leq &\frac{K^p\bar{C}_p(1+\bar{\mu}\delta)^{p-1}}{\delta^{p-1}}\int_{-\infty}^{t} e_{\ominus\delta}(t,\sigma(r))\int_{0}^{\infty}|C(u)|^2\mathbb{E}\|\varphi(r+s_n-u)-\widetilde{\varphi}(r-u)\|^p\Delta u\Delta r\\
\leq &\frac{K^p\bar{C}_p(1+\bar{\mu}\delta)^{p}\|C\|^2_{L^2(0,+\infty)}}{\delta^p}\sup_{t\in\mathbb{T}}\mathbb{E}\|\varphi(t+s_n)-\widetilde{\varphi}(t)\|^p,\end{aligned}$$

where $\bar{C}_p = C_p\|C\|^{p-2}_{L^2(0,+\infty)}$ and C_p is a positive constant.

Based on the translation invariance of $AA(\mathbb{T},L^p(\mathbb{H}))$, (6) and the Lebesgue dominated convergence theorem, one deduces $\Psi^1_{t,s_n}(t)\to 0$ as $n\to\infty$.

Similarly, for any $\varepsilon>0$, there exists $N>0$ such that

$$\Psi^2_{t,s_n}(t) \leq \frac{\varepsilon^p\bar{C}_p(1+\bar{\mu}w)^p\|C\|^2_{L^2(0,+\infty)}}{\delta^p}\sup_{t\in\mathbb{T}}\mathbb{E}\|\widetilde{\varphi}(t)\|^p,$$

therefore, $\Psi^2_{t,s_n}(t)\to 0$ as $n\to\infty$. Further, $\mathbb{E}\|\chi_1(t+s_n)-\widetilde{\chi}_1(t)\|^p\to 0$ as $n\to\infty$, likewise, $\lim_{n\to\infty}\mathbb{E}\|\widetilde{\chi}_1(t-s_n)-\chi_1(t)\|^p=0$ can be checked.

Taking a Taking a analogous method as the proof of Lemma 4.1, the following conclusion is established successfully.

Corollary 4.1. Assume (H_1) holds and $\phi\in AA(\mathbb{T},L^p(\mathbb{H}))$, then

$$\Phi_1(t)=\int_{-\infty}^{t} T(t,\sigma(s))\int_{-\infty}^{s} B(s-u)\phi(u)\Delta u\Delta s$$

and

$$\Phi_2(t)=\int_{-\infty}^{t} T(t,\sigma(s))\varphi(s)\Delta s$$

are piecewise almost automorphic.

Lemma 4.2. Assume (H_1) holds and $t_i\in AA(L^p(\mathbb{H}),L^p(\mathbb{H}))$ and $x(t_i)\in AA(\mathbb{T},L^p(\mathbb{H}))$, if $x:\mathbb{T}\to L^p(\mathbb{H}))$ is defined by

$$x(t):=\sum_{t_i<t} T(t,t_i)x(t_i),$$

then $x\in AA(\mathbb{T},L^p(\mathbb{H}))$.

Proof. Since $x(t_i)\in AA(\mathbb{T},L^p(\mathbb{H}))$, then for any given sequence $\{\tau'_n\}_{n\in\mathbb{N}}\subseteq\Xi$, there exists a subsequence $\{\tau_n\}_{n\in\mathbb{N}}$ and stochastic processes $\widetilde{x}(t_i)$: $\mathbb{T}\to L^p(\mathbb{H})$ such that

$$\lim_{n\to\infty}\mathbb{E}\|x(t_i+\tau_n)-\widetilde{x}(t_i)\|^p=0 \text{ for } t_i\in\mathbb{T},\ i\in\mathbb{Z}. \tag{7}$$

Let

$$x^*(t)=\sum_{t_i<t} T(t,t_i)\widetilde{x}(t_i),$$

it obtains

$$
\begin{aligned}
&\mathbb{E}\|x(t+\tau_n)-x^*(t)\|^p\\
\le&2^{p-1}\left\{\mathbb{E}\left\|\sum_{t_i<t}[T(t+\tau_n,t_i+\tau_n)-T(t,t_i)]\,x(t_i+\tau_n)\right\|^p+\mathbb{E}\left\|\sum_{t_i<t}T(t,t_i)\,[x(t_i+\tau_n)-\widetilde{x}(t_i)]\right\|^p\right\}\\
\le&2^{p-1}\left(\sum_{t_i<t}e_{\ominus\delta}(t,t_i)\right)^{p-1}\left\{\sum_{t_i<t}e_{\ominus\delta}(t,t_i)\left[\varepsilon^p\mathbb{E}\left\|x(t_i+\tau_n)\right\|^p+K^p\mathbb{E}\left\|x(t_i+\tau_n)-\widetilde{x}(t_i)\right\|^p\right]\right\}\\
\le&\frac{2^{p-1}}{(1-e_{\ominus\delta}(\phi_0,0))^p}\left[\varepsilon^p\sup_{t_i\in\mathbb{T}}\mathbb{E}\left\|x(t_i+\tau_n)\right\|^p+K^p\sup_{t_i\in\mathbb{T}}\mathbb{E}\left\|x(t_i+\tau_n)-\widetilde{x}(t_i)\right\|^p\right].
\end{aligned}
$$

Since $e_{\ominus\delta}(\phi_0,0)<1$ with $\phi_0=\inf_{i\in\mathbb{Z}}(t_{i+1}-t_i)$, therefore $\lim_{n\to+\infty}\mathbb{E}\|x(t+\tau_n)-x^*(t)\|^p=0$ by utilizing (7) and $\varepsilon\to 0$. Similarly, it yields $\lim_{n\to+\infty}\mathbb{E}\|x^*(t-\tau_n)-x(t)\|^p=0$, further, $x\in AA(\mathbb{T},L^p(\mathbb{H}))$.

Theorem 4.1. Let (H_1)-(H_3) hold, then Eq.(1) admits a unique p-mean doubly weighted piecewise pseudo almost automorphic mild solution provided that

$$
\frac{(1+\bar{\mu}\delta)^p}{\delta^p}\left(1+\|B\|^p_{L^1(0,+\infty)}+C_p\|C\|^p_{L^2(0,+\infty)}\right)(L+\zeta_1\zeta_2)+\frac{L}{(1-e_{\ominus\delta}(\phi_0,0))^p}<\frac{1}{4^{p-1}K^p}. \tag{8}
$$

Proof. Next, we will divide the proof into five steps.

Step 1. Assume

$$
\Theta_h:=\{\omega(\cdot)\in PAA^{p,q}(\mathbb{T},L^p(\mathbb{H})):\|\omega\|_{PC}\le h\}.
$$

Moreover, for any $\phi\in AA(\mathbb{T},L^p(\mathbb{H}))$, set the operator $(\Gamma_1\omega)(t):=\sum_{i=1}^{7}(\ell_i\omega)(t)$ with

$$
\begin{aligned}
(\ell_1\omega)(t):=&\int_{-\infty}^{t}T(t,\sigma(s))[\varphi_1(s,\phi(s)+\omega(s))-\varphi_1(s,\phi(s))]\Delta s,\\
(\ell_2\omega)(t):=&\int_{-\infty}^{t}T(t,\sigma(s))\eta_1(s,\phi(s)+\omega(s))\Delta s,\\
(\ell_3\omega)(t):=&\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}B(s-u)[\varphi_2(u,\phi(u)+\omega(u))-\varphi_2(u,\phi(u))]\Delta u\Delta s,\\
(\ell_4\omega)(t):=&\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}B(s-u)\eta_2(u,\phi(u)+\omega(u))\Delta u\Delta s,\\
(\ell_5\omega)(t):=&\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}C(s-u)[\varphi_3(u,\phi(u)+\omega(u))-\varphi_3(u,\phi(u))]\Delta W(u)\Delta s,\\
(\ell_6\omega)(t):=&\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}C(s-u)\eta_3(u,\phi(u)+\omega(u))\Delta W(u)\Delta s,\\
(\ell_7\omega)(t):=&\sum_{t_i<t}T(t,t_i)[I_i(\phi(t_i)+\omega(t_i))-I_i(\phi(t_i))].
\end{aligned}
$$

Based on (5) and (8), it is not difficult to obtain that there exists a positive constant h_0 satisfies

$$
\frac{4^{p-1}K^p(1+\bar{\mu}\delta)^p}{\delta^p}\left(1+\|B\|^p_{L^1(0,+\infty)}+C_p\|C\|^p_{L^2(0,+\infty)}\right)\{Lh_0^p+\zeta_2\Phi[(h_0+\|x\|_{PC})^p]\}+\frac{4^{p-1}K^pL}{(1-e_{\ominus\delta}(\phi_0,0))^p}\le h_0^p. \tag{9}
$$

For above given positive constant h_0, then we claim that $\Gamma_{11}:=\ell_1+\ell_3+\ell_5+\ell_7$ and $\Gamma_{12}:=\ell_2+\ell_4+\ell_6$ maps Θ_{h_0} into itself. In fact, by applying (H_2), it yields for $i=1,2,3$ that

$$
\max_{t\in\mathbb{T}}\{\mathbb{E}\|\varphi_i(t,x(t)+\omega(t))-\varphi_i(t,x(t))\|^p,\mathbb{E}\|I_i(\phi(t_i)+\omega(t_i))-I_i(\phi(t_i))\|^p\}\le L\mathbb{E}\|\omega(t)\|^p, \tag{10}
$$

$$
\mathbb{E}\|\eta_i(t,x(t)+\omega(t))\|^p\le\gamma(t)\Phi[(h+\|x\|_{PC})^p]. \tag{11}
$$

For any $\omega_1 \in \Theta_{h_0}$, in view of $\omega_1(\cdot) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ and $\gamma(\cdot) \in PAA^{\rho,q}(\mathbb{T}, \mathbb{T}^+)$, then $\varphi_i(\cdot, x(\cdot) + \omega_1(\cdot)) - \varphi_i(\cdot, x(\cdot)) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$, $\eta_i(\cdot, x(\cdot) + \omega_1(\cdot)) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ and $I_i(\phi(t_i) + \omega(t_i)) - I_i(\phi(t_i)) \in PAA^{\rho,q}(L^p(\mathbb{H}), L^p(\mathbb{H}))$. Further, by utilizing Theorem 3.2 and Corollary 3.1, it obtains $(\ell_i\omega_1)(t) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ for $i = 1, \cdots, 7$, that is, $(\Gamma_{1j}\omega_1)(t) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ for $j = 1, 2$.

In addition, together (10) with (11), based on (H_1)-(H_3), by using Hölder inequality and Burkholder-Davis-Gundy inequality, it yields for $\omega_1 \in \Theta_{h_0}$ that

$$\begin{aligned}
&\mathbb{E}\|(\Gamma_{11}\omega_1)(t)\|^p \\
\le& 4^{p-1}\left(\mathbb{E}\|(\ell_1\omega_1)(t)\|^p + \mathbb{E}\|(\ell_3\omega_1)(t)\|^p + \mathbb{E}\|(\ell_5\omega_1)(t)\|^p + \mathbb{E}\|(\ell_7\omega_1)(t)\|^p\right) \\
\le& \frac{4^{p-1}K^p}{(-\ominus\delta)^{p-1}}\Bigg[\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\mathbb{E}\|\varphi_1(s,x(s)+\omega_1(s)) - \varphi_1(s,x(s))\|^p\Delta s \\
&+ \|B\|_{L^1(0,+\infty)}^{p-1}\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\int_{-\infty}^s |B(s-u)|\mathbb{E}\|\varphi_2(u,x(u)+\omega_1(u)) - \varphi_2(u,x(u))\|^p\Delta u\Delta s \\
&+ C_p\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\mathbb{E}\left(\int_{-\infty}^s |C(s-u)|^2\|\varphi_3(u,x(u)+\omega_1(u)) - \varphi_3(u,x(u))\|^2\Delta u\right)^{\frac{p}{2}}\Delta s\Bigg] \\
&+ 4^{p-1}K^p\left(\sum_{t_i<t} e_{\ominus\delta}(t,t_i)\right)^{p-1}\sum_{t_i<t} e_{\ominus\delta}(t,t_i)\mathbb{E}\|I_i(\phi(t_i)+\omega_1(t_i)) - I_i(\phi(t_i))\|^p \\
\le& 4^{p-1}K^pL\left[\frac{(1+\bar{\mu}\delta)^p}{\delta^p}\left(1 + \|B\|_{L^1(0,+\infty)}^p + C_p\|C\|_{L^2(0,+\infty)}^p\right) + \frac{1}{(1-e_{\ominus\delta}(\phi_0,0))^p}\right]h_0^p,
\end{aligned} \tag{12}$$

and

$$\begin{aligned}
\mathbb{E}\|(\Gamma_{12}\omega_1)(t)\|^p \le& 3^{p-1}\left(\mathbb{E}\|(\ell_2\omega_1)(t)\|^p + \mathbb{E}\|(\ell_4\omega_1)(t)\|^p + \mathbb{E}\|(\ell_6\omega_1)(t)\|^p\right) \\
\le& \frac{3^{p-1}K^p(1+\bar{\mu}\delta)^p}{\delta^p}\left(1 + \|B\|_{L^1(0,+\infty)}^p + C_p\|C\|_{L^2(0,+\infty)}^p\right)\zeta_2\Phi[(h_0 + \|x\|_{PC})^p].
\end{aligned}$$

By applying (9), it follows

$$\|\Gamma_{1j}\omega_1\|_{PC} \le h_0 \text{ for } j = 1, 2,$$

this indicates Γ_{1j}: $\Theta_{h_0} \to \Theta_{h_0}$ hold for $j = 1, 2$.

Step 2. Show the operator Γ_2 is a contraction mapping on $AA(\mathbb{T}, L^p(\mathbb{H}))$, where

$$\begin{aligned}
(\Gamma_2\tilde{x})(t) :=& \int_{-\infty}^t T(t,\sigma(s))\varphi_1(s,\tilde{x}(s))\Delta s + \int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s B(s-u)\varphi_2(u,\tilde{x}(u))\Delta u\Delta s \\
&+ \int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s C(s-u)\varphi_3(u,\tilde{x}(u))\Delta W(u)\Delta s + \sum_{t_i<t} T(t,t_i)I_i(\tilde{x}(t_i)),
\end{aligned}$$

for any $\tilde{x} \in AA(\mathbb{T}, L^p(\mathbb{H}))$. In fact, for any x_1, $x_2 \in AA(\mathbb{T}, L^p(\mathbb{H}))$, it deduces

$$\begin{aligned}
&(\Gamma_2x_1)(t) - (\Gamma_2x_2)(t) \\
=& \int_{-\infty}^t T(t,\sigma(s))[\varphi_1(s,x_1(s)) - \varphi_1(s,x_2(s))]\Delta s + \sum_{t_i<t} T(t,t_i)[I_i(x_1(t_i)) - I_i(x_2(t_i))] \\
&+ \int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s B(s-u)[\varphi_2(u,x_1(u)) - \varphi_2(u,x_2(u))]\Delta u\Delta s \\
&+ \int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s C(s-u)[\varphi_3(u,x_1(u)) - \varphi_3(u,x_2(u))]\Delta W(u)\Delta s \\
:=& \nabla(\varphi_1,x_1,x_2,t) + \nabla(I_i,x_1,x_2,t) + \nabla(\varphi_2,x_1,x_2,t) + \nabla(\varphi_3,x_1,x_2,t).
\end{aligned}$$

From (H_1), (4) and (H_3), similar to the calculation of (12), it obtains

$$\begin{aligned}&\mathbb{E}\|(\Gamma_2 x_1)(t)-(\Gamma_2 x_2)(t)\|^p\\ \leq& 4^{p-1}\left(\mathbb{E}\|\nabla(\varphi_1,x_1,x_2,t)\|^p+\mathbb{E}\|\nabla(I_i,x_1,x_2,t)\|^p+\mathbb{E}\|\nabla(\varphi_2,x_1,x_2,t)\|^p+\mathbb{E}\|\nabla(\varphi_3,x_1,x_2,t)\|^p\right)\\ \leq& 4^{p-1}K^pL\left\{\frac{(1+\bar{\mu}\delta)^p}{\delta^p}\left(1+\|B\|^p_{L^1(0,+\infty)}+C_p\|C\|^p_{L^2(0,+\infty)}\right)+\frac{1}{(1-e_{\ominus\delta}(\phi_0,0))^p}\right\}\sup_{t\in\mathbb{T}}\mathbb{E}\|x_1(t)-x_2(t)\|^p.\end{aligned}$$

Therefore, based on the condition (8), it yields

$$\|(\Gamma_2 x_1)-(\Gamma_2 x_2)\|_{PC}<\|x_1-x_2\|_{PC}.$$

Step 3. Show Γ_{11} is a contraction mapping on Θ_{h_0}.

For any ω_1, $\omega_2\in\Theta_{h_0}$, by using (4) it follows

$$\begin{aligned}&\mathbb{E}\left\|[\varphi_i(\cdot,x(\cdot)+\omega_1(\cdot))-\varphi_i(\cdot,x(\cdot))]-[\varphi_i(\cdot,x(\cdot)+\omega_2(\cdot))-\varphi_i(\cdot,x(\cdot))]\right\|^p\leq L\mathbb{E}\left\|\omega_1(\cdot)-\omega_2(\cdot)\right\|^p,\\ &\mathbb{E}\left\|[I_i(\phi(\cdot)+\omega_1(\cdot))-I_i(\phi(\cdot))]-[I_i(\phi(\cdot)+\omega_2(\cdot))-I_i(\phi(\cdot))]\right\|^p\leq L\mathbb{E}\left\|\omega_1(\cdot)-\omega_2(\cdot)\right\|^p.\end{aligned}$$

Together this with Hölder and Burkholder-Davis-Gundy inequality, it is not difficult to deduce

$$\begin{aligned}&\mathbb{E}\|(\Gamma_{11}\omega_1)-(\Gamma_{11}\omega_2)\|^p\\ \leq& 4^{p-1}K^pL\left\{\frac{(1+\bar{\mu}\delta)^p}{\delta^p}\left(1+\|B\|^p_{L^1(0,+\infty)}+C_p\|C\|^p_{L^2(0,+\infty)}\right)+\frac{1}{(1-e_{\ominus\delta}(\phi_0,0))^p}\right\}\sup_{t\in\mathbb{T}}\mathbb{E}\|\omega_1(t)-\omega_2(t)\|^p,\end{aligned}$$

this implies based on (8) that

$$\|(\Gamma_{11}\omega_1)-(\Gamma_{11}\omega_2)\|_{PC}<\|\omega_1-\omega_2\|_{PC}.$$

Step 4. Prove $\{(\Gamma_{12}\omega)(t):\omega(t)\in\Theta_{h_0}\}$ is a relatively compact set in $L^p(\mathbb{H})$.

Let $t\in\mathbb{T}$ be fixed, for given $\varepsilon_0>0$, (11) suggests

$$(\Gamma_{12}^{\varepsilon_0}\omega)(t):=\Gamma_1^2(t-\varepsilon_0)+\Gamma_1^4(t-\varepsilon_0)+\Gamma_1^6(t-\varepsilon_0)$$

is uniformly bounded for any $\omega(t)\in\Theta_{h_0}$. Combining with the compactness of evolution family $T(t,t-\varepsilon_0)$, it claims

$$\{T(t,t-\varepsilon_0)(\Gamma_{12}^{\varepsilon_0}\omega)(t):\omega(t)\in\Theta_{h_0}\}$$

is a relatively compact set in $L^p(\mathbb{H})$. Since

$$\begin{aligned}&(\Gamma_{12}\omega)(t)-T(t,t-\varepsilon_0)(\Gamma_{12}^{\varepsilon_0}\omega)(t)\\ =&\int_{t-\varepsilon_0}^t T(t,\sigma(s))\eta_1(s,x(s)+\omega(s))\Delta s+\int_{t-\varepsilon_0}^t T(t,\sigma(s))\int_{-\infty}^s B(s-u)\eta_2(u,x(u)+\omega(u))\Delta u\Delta s\\ &+\int_{t-\varepsilon_0}^t T(t,\sigma(s))\int_{-\infty}^s C(s-u)\eta_3(u,x(u)+\omega(u))\Delta W(u)\Delta s,\end{aligned}$$

therefore, by using some analysis techniques, it deduces

$$\mathbb{E}\left\|(\Gamma_{12}\omega)(t)-T(t,t-\varepsilon_0)(\Gamma_{12}^{\varepsilon_0}\omega)(t)\right\|^p\to 0\text{ as }\varepsilon_0\to 0.$$

Step 5. Claim that $\{(\Gamma_{12}\omega)(t):\omega(t)\in\Theta_{h_0}\}$ is equicontinuous.

For any $\varepsilon_1>0$, based on (11), there exists $\delta_1>0$, for $\omega(t)\in\Theta_{h_0}$, $0\leq t_2-t_1<\delta_1$, it has

$$\mathbb{E}\left\|\int_{t_1}^{t_2}T(t_2,\sigma(s))\int_{-\infty}^s C(s-u)\eta_3(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p<\frac{\varepsilon_1}{12^p},\tag{13}$$

$$\mathbb{E}\left\|\int_{t_1-\delta_1}^{t_1}[T(t_2,\sigma(s))-T(t_1,\sigma(s))]\int_{-\infty}^{s}C(s-u)\eta_3(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p<\frac{\varepsilon_1}{12^p}. \tag{14}$$

Choosing a sufficiently large and suitable constant $r_0>0$ satisfies

$$\mathbb{E}\left\|\int_{-\infty}^{t_1-r_0}[T(t_2,\sigma(s))-T(t_1,\sigma(s))]\int_{-\infty}^{s}C(s-u)\eta_3(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p<\frac{\varepsilon_1}{12^p}. \tag{15}$$

On the other hand, the compactness of the $\{T(t,s)\}_{t\geq s}$ indicates its norm continuity, that is, there exists $0<\delta'<\delta_1$ such that for all $\omega(t)\in\Theta_{h_0}$ and $0\leq t_2-t_1<\delta'$, it follows

$$\mathbb{E}\left\|\int_{t_1-r_0}^{t_1-\delta'}[T(t_2,\sigma(s))-T(t_1,\sigma(s))]\int_{-\infty}^{s}C(s-u)\eta_3(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p<\frac{\varepsilon_1}{12^p}. \tag{16}$$

By applying (13)-(16), it calculates

$$\begin{aligned}
&\mathbb{E}\|(\Gamma_1^6\omega)(t_2)-(\Gamma_1^6\omega)(t_1)\|^p\\
\leq&4^{p-1}\left(\mathbb{E}\left\|\int_{t_1}^{t_2}T(t_2,s)\int_{-\infty}^{s}C(s-u)g_2(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p\right.\\
&+\mathbb{E}\left\|\int_{t_1-\delta'}^{t_1}[T(t_2,s)-T(t_1,s)]\int_{-\infty}^{s}C(s-u)g_2(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p\\
&+\mathbb{E}\left\|\int_{-\infty}^{t_1-r_0}[T(t_2,s)-T(t_1,s)]\int_{-\infty}^{s}C(s-u)g_2(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p\\
&\left.+\mathbb{E}\left\|\int_{t_1-r_0}^{t_1-\delta'}[T(t_2,s)-T(t_1,s)]\int_{-\infty}^{s}C(s-u)g_2(u,x(u)+\omega(u))\Delta W(u)\Delta s\right\|^p\right)<\frac{\varepsilon_1}{3^p}.
\end{aligned}$$

Similarly, we get $\mathbb{E}\|(\Gamma_1^2\omega)(t_2)-(\Gamma_1^2\omega)(t_1)\|^p<\frac{\varepsilon_1}{3^p}$ and $\mathbb{E}\|(\Gamma_1^4\omega)(t_2)-(\Gamma_1^4\omega)(t_1)\|^p<\frac{\varepsilon_1}{3^p}$. Therefore, for any $\varepsilon_1>0$, there exists $\delta_1>0$ such that for any $\omega(t)\in\Theta_{h_0}$ and $0\leq t_2-t_1<\delta'<\delta_1$, it follows

$$\begin{aligned}
&\mathbb{E}\|(\Gamma_{12}\omega)(t_2)-(\Gamma_{12}\omega)(t_1)\|^p\\
\leq&3^{p-1}\left[\mathbb{E}\|(\Gamma_1^2\omega)(t_2)-(\Gamma_1^2\omega)(t_1)\|^p+\mathbb{E}\|(\Gamma_1^4\omega)(t_2)-(\Gamma_1^4\omega)(t_1)\|^p+\mathbb{E}\|(\Gamma_1^6\omega)(t_2)-(\Gamma_1^6\omega)(t_1)\|^p\right]<\varepsilon_1.
\end{aligned}$$

From above discussion, we claim that Eq.(1) admits a unique doubly weighted piecewise pseudo almost automorphic mild solution. In fact, for any $\tilde{x}\in AA(\mathbb{T},L^p(\mathbb{H}))$, under the lipschitz conditions (4), it follows $\varphi_i(t,\tilde{x}(t))\in AA(\mathbb{T},L^p(\mathbb{H}))$. From Lemma 2.3, Lemma 2.4 and (H_3), it gives $I_i(\tilde{x}(t_i))\in AA(L^p(\mathbb{H}),L^p(\mathbb{H}))$. Further, from the lemma 4.1-4.2, corollary 4.1 and step 2, by using the contraction mapping principle in Banach space, it yields that Γ_2 has as least one fixed point $x^*(t)\in AA(\mathbb{T},L^p(\mathbb{H}))$. Moreover, combining step 4 with step 5, it follows the operator Γ_{12} is completely continuous, this together with step 1 and step 3, then the Krasnoselskii's fixed point theorem indicates that there exists one fixed point $\omega^*(t)\in\Theta_{h_0}$, clearly, $\omega^*(t)\in PAA^{\rho,q}(\mathbb{T},L^p(\mathbb{H}))$. Further, consider the coupled system

$$\begin{cases}
x^*(t)=\int_{-\infty}^{t}T(t,\sigma(s))\varphi_1(s,x^*(s))ds+\int_{-\infty}^{t}T(t,s)\int_{-\infty}^{s}B(s-u)\varphi_2(u,x^*(u))\Delta u\Delta s\\
\qquad+\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}C(s-u)\varphi_3(u,x^*(u))\Delta W(u)\Delta s+\sum\limits_{t_i<t}T(t,t_i)I_i(x^*(t_i)),\\
\omega^*(t)=\int_{-\infty}^{t}T(t,\sigma(s))[\varphi_1(s,x^*(s)+\omega^*(s))-\varphi_1(s,x^*(s))]\Delta s\\
\qquad+\int_{-\infty}^{t}T(t,\sigma(s))\eta_1(s,x^*(s)+\omega^*(s))\Delta s\\
\qquad+\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}B(s-u)[\varphi_2(u,x^*(u)+\omega^*(u))-\varphi_2(u,x^*(u))]\Delta u\Delta s\\
\qquad+\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}B(s-u)\eta_2(u,x^*(u)+\omega^*(u))\Delta u\Delta s\\
\qquad+\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}C(s-u)[\varphi_3(u,x^*(u)+\omega^*(u))-\varphi_3(u,x^*(u))]\Delta W(u)\Delta s\\
\qquad+\int_{-\infty}^{t}T(t,\sigma(s))\int_{-\infty}^{s}C(s-u)\eta_3(u,x^*(u)+\omega^*(u))\Delta W(u)\Delta s\\
\qquad+\sum\limits_{t_i<t}T(t,t_i)I_i(x^*(t_i)+\omega^*(t_i))-\sum\limits_{t_i<t}T(t,t_i)I_i(x^*(t_i)),\ \ t\in\mathbb{T},
\end{cases}$$

obviously, $(x^*(t), \omega^*(t)) \in AA(\mathbb{T}, L^p(\mathbb{H})) \times PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$ is a solution to this coupled system. Further $x(t) = x^*(t) + \omega^*(t)) \in DWPAA(\mathbb{T}, L^p(\mathbb{H}), \rho, q)$, it is not difficult to check that $x(t)$ is a solution to (1). □

If the condition (H_2) replaced by $(\bar{H}_2)$, where $(\bar{H}_2)$ presented as follows.

$(\bar{H}2)$ Assume $F_i = \varphi_i + \eta_i \in DWPAA(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H}), \rho, q)$ and $\rho, q \in \mathcal{U}^{\rho,q}$, where $\varphi_i \in AA(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H}))$ and $\eta_i \in PAA^{\rho,q}(\mathbb{T} \times L^p(\mathbb{H}), L^p(\mathbb{H}))$, for any x, $z \in L^p(\mathbb{H})$, there exists positive constant L satisfies

$$\mathbb{E}|F_i(t,x) - F_i(t,z)|^p \leq L\mathbb{E}|x-z|^p, \; i = 1,2,3.$$

Further, the following result holds.

Theorem 4.2. Let (H_1), $(\bar{H}_2)$ and (H_3) hold, then Eq.(1) admits a unique p-mean doubly weighted piecewise pseudo almost automorphic mild solution x^* provided that

$$\frac{(1+\bar{\mu}\delta)^p}{\delta^p}\left(1 + \|B\|^p_{L^1(0,+\infty)} + C_p\|C\|^p_{L^2(0,+\infty)}\right) + \frac{1}{(1-e_{\ominus\delta}(\phi_0,0))^p} < \frac{1}{4^{p-1}K^pL}. \tag{17}$$

Moreover, the almost automorphic component x_1^* of x^* is the unique mild solution of

$$\begin{cases} \Delta x_1(t) = A(t)x_1(t)\Delta t + \varphi_1(t, x_1(t))\Delta t + \displaystyle\int_{-\infty}^t B(t-u)\varphi_2(u, x_1(u))\Delta u \Delta t \\ \qquad\qquad + \displaystyle\int_{-\infty}^t C(t-u)\varphi_3(u, x_1(u))\Delta W(u)\Delta t, \;\; t \in \mathbb{T}, \;\; t \neq t_i, \\ x_1(t_i^+) - x_1(t_i^-) = I_i(x_1(t_i)), \;\; i \in \mathbb{Z}, \end{cases} \tag{18}$$

Proof. Define the operator Υ with

$$\begin{aligned}(\Upsilon x)(t) :=& \sum_{t_i<t} T(t,t_i)I_i(x(t_i)) + \int_{-\infty}^t T(t,\sigma(s))F_1(s,x(s))\Delta s + \int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s B(s-u)F_2(u,x(u))\Delta u\Delta s \\ &+ \int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s C(s-u)F_3(u,x(u))\Delta W(u)\Delta s := \Im_{F_1,x}(t) + \Im_{F_2,x}(t) + \Im_{F_3,x}(t) + \Im_{I_i,x}(t)\end{aligned}$$

for $t \in \mathbb{T}$. Assume z_1, z_2 are two mild solution of Eq.(1), by using (H_1), $(\bar{H}_2)$ and (H_3), then

$$\begin{aligned} &4^{1-p}\mathbb{E}\|(\Upsilon z_1)(t) - (\Upsilon z_2)(t)\|^p \\ \leq& \mathbb{E}\left\|\sum_{t_i<t} T(t,t_i)[I_i(z_1(t_i)) - I_i(z_2(t_i))]\right\|^p + \mathbb{E}\left\|\int_{-\infty}^t T(t,\sigma(s))[F_1(s,z_1(s)) - F_1(s,z_2(s))]\Delta s\right\|^p \\ &+ \mathbb{E}\left\|\int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s B(s-u)[F_2(u,z_1(u)) - F_2(u,z_2(u))]\Delta u\Delta s\right\|^p \\ &+ \mathbb{E}\left\|\int_{-\infty}^t T(t,\sigma(s))\int_{-\infty}^s C(s-u)[F_3(u,z_1(u)) - F_3(u,z_2(u))]\Delta W(u)\Delta s\right\|^p \\ \leq& K^p\left(\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\Delta s\right)^{p-1}\left[\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\mathbb{E}\|F_1(s,z_1(s)) - F_1(s,z_2(s))\|^p\Delta s\right. \\ &+ \|B\|^{p-1}_{L^1(0,+\infty)}\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\int_{-\infty}^s |B(s-u)|\mathbb{E}\|F_2(u,z_1(u)) - F_2(u,z_2(u))\|^p\Delta u\Delta s \\ &\left.+ C_p\int_{-\infty}^t e_{\ominus\delta}(t,\sigma(s))\mathbb{E}\left(\int_{-\infty}^s |C(s-u)|^2\|F_3(u,z_1(u)) - F_3(u,z_2(u))\|^2\Delta u\right)^{\frac{p}{2}}\Delta s\right] \\ &+ K^p\left(\sum_{t_i<t} e_{\ominus\delta}(t,t_i)\right)^{p-1}\sum_{t_i<t} e_{\ominus\delta}(t,t_i)\mathbb{E}\,\|I_i(z_1(t_i)) - I_i(z_2(t_i))\|^p \\ \leq& K^pL\left[\frac{(1+\bar{\mu}\delta)^p}{\delta^p}\left(1 + \|B\|^p_{L^1(0,+\infty)} + C_p\|C\|^p_{L^2(0,+\infty)}\right) + \frac{1}{(1-e_{\ominus\delta}(\phi_0,0))^p}\right]\sup_{t\in\mathbb{T}}\mathbb{E}\,\|z_1(t) - z_2(t)\|^p. \end{aligned}$$

Therefore, (17) implies the operator Υ is a contraction mapping. Further, for any $x = x_1 + x_2 \in DWPAA(\mathbb{T}, L^p(\mathbb{H}), \rho, q)$, where $x_1 \in AA(\mathbb{T}, L^p(\mathbb{H}))$ and $x_2 \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$, it follows $F_i(\cdot, x(\cdot)) = \varphi_i(\cdot, x_1(\cdot)) + F_i(\cdot, x(\cdot)) - F_i(\cdot, x_1(\cdot)) + \eta_i(\cdot, x_1(\cdot))$, moreover

$$(\Upsilon x)(t) = \Im_{\varphi_1, x_1}(t) + \Im_{\varphi_2, x_1}(t) + \Im_{\varphi_3, x_1}(t) + \Im_{I_i, x_1}(t) + \Im_0(t),$$

where

$$\begin{aligned}\Im_0(t) = &\int_{-\infty}^{t} T(t, \sigma(s))\Psi_{F_1,\eta_1}(s, x, x_1)\Delta s + \int_{-\infty}^{t} T(t, \sigma(s)) \int_{-\infty}^{s} B(s-u)\Psi_{F_2,\eta_2}(s, x, x_1)\Delta u \Delta s \\ &+ \int_{-\infty}^{t} T(t, \sigma(s)) \int_{-\infty}^{s} C(s-u)\Psi_{F_3,\eta_3}(s, x, x_1)\Delta W(u)\Delta s + \sum_{t_i<t} T(t, t_i)[I_i(x(t_i)) - I_i(x_1(t_i))]\end{aligned}$$

and $\Psi_{F_i,\eta_i}(\cdot, x, x_1) = F_i(\cdot, x(\cdot)) - F_i(\cdot, x_1(\cdot)) + \eta_i(\cdot, x_1(\cdot))$ for $i = 1, 2, 3$.

Similar to the proof of Theorem 2.2 in [25], then $F_i(\cdot, x(\cdot)) \in DWPAA(\mathbb{T}, L^p(\mathbb{H}), \rho, q)$, where $\varphi_i(\cdot, x_1(\cdot)) \in AA(\mathbb{T}, L^p(\mathbb{H}))$ and $F_i(\cdot, x(\cdot)) - F_i(\cdot, x_1(\cdot)) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$, $\eta_i(\cdot, x_1(\cdot)) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$. Combine Lemma 2.3, Lemma 2.4, Lemma 4.1, Lemma 4.2 and Corollary 4.1, it claims $\Im_{\varphi_i, x_1}(\cdot) \in AA(\mathbb{T}, L^p(\mathbb{H}))$ and $\Im_{I_i, x_1}(\cdot) \in AA(L^p(\mathbb{H}), L^p(\mathbb{H}))$. By using Theorem 3.2 and the Corollary 3.1, it yields $\Im_0(\cdot) \in PAA^{\rho,q}(\mathbb{T}, L^p(\mathbb{H}))$. Based on the contraction mapping principle in Banach space, it obtains that Eq.(1) admits a unique p-mean doubly weighted piecewise pseudo almost automorphic mild solution x^*.

Assume the operator

$$(\Upsilon_1 x)(t) := \Im_{\varphi_1, x_1}(t) + \Im_{\varphi_2, x_1}(t) + \Im_{\varphi_3, x_1}(t) + \Im_{I_i, x_1}(t),$$

obviously, $(\Upsilon_1 x)(\cdot) \in AA(\mathbb{T}, L^p(\mathbb{H}))$. Similar to the prove of Step 2 in Theorem 4.1, it is not difficult to investigate the operator Υ_1 is a contraction mapping under (17). Furthermore, x_1^* is the unique p-mean piecewise almost automorphic mild solution of Eq.(18).

Example 4.1. Consider the nonlinear stochastic impulsive equations on time scales

$$\begin{cases} \frac{\partial \zeta(t,x)}{\Delta_1 t} = \frac{\partial^2 \zeta(t,x)}{\Delta_2 x^2} + F_1(t, \zeta(t,x)) + \int_{-\infty}^{t} B(t-u)F_2(u, \zeta(u,x))\Delta_1 u \\ \qquad + \int_{-\infty}^{t} C(t-u)F_3(u, \zeta(u,x))\frac{\partial W(u)}{\Delta_1 t}, \quad (t,x) \in \mathbb{T} \times [0,\pi]_{\mathbb{T}}, \ t \neq t_i \\ \Delta_2 \zeta(t_i, x) = d\left(\cos i + \sin\sqrt{5}i\right)\zeta(t_i, x), \quad i \in \mathbb{Z}, \ x \in [0,\pi]_{\mathbb{T}}, \\ \zeta(t,0) = \zeta(t,\pi) = 0, \ t \in \mathbb{T}, \end{cases} \tag{19}$$

where $t_i = i + \frac{1}{16}\left|\cos(i+1) - \sin\sqrt{3}t\right|$ for $i \in \mathbb{Z}$ and

$$F_i(t, \zeta) = d(\sin 2\pi t + \sin\sqrt{2}t)\sin\zeta + \frac{1}{10}e^{-|t|}\zeta, \ \ i = 1, 2, 3.$$

Define $A = \frac{\partial^2}{\Delta_2 r^2}$, it is not difficult to deduce that the evolution family $\{T(t,s)\}_{-\infty<s\leq t<+\infty}$ satisfies

$$\|T(t,s)\| \leq e_{\ominus\frac{1}{2}}(t,s) \text{ for } t \geq s.$$

Let $x(t) = \zeta(t, \cdot)$ and

$$\rho(t) = q(t) = \begin{cases} t+1, & \text{for } t > 0, \\ e^{-t^2}, & \text{for } t \leq 0, \end{cases}$$

then Eq.(19) can be formulated in abstract form as Eq.(1) and the assumptions (H_1)-(H_3), $(\bar{H}_2)$ hold, where $\gamma(t) = \frac{1}{10}e^{-|t|}$, $\Phi(h) = h$, $L = 2d$, $\phi_0 = \inf_{i\in\mathbb{Z}}(t_{i+1} - t_i) > \frac{17}{20}$, $\bar{\mu} = \frac{4}{3}$, $K = 1$, $\delta = \frac{1}{2}$. In addition, let $\|B\|_{L^1(0,+\infty)} = 0.2$ and $\|C\|_{L^2(0,+\infty)} = 0.1$, by choosing sufficiently small positive constant d, from Theorem 4.1 and Theorem 4.2, it follows (19) admits a unique p-mean doubly weighted piecewise pseudo almost automorphic mild solution.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

References

1. Liu Z, Sun K. Almost automorphic solutions for stochastic differential equations driven by Lévy noise. J. Funct. Anal. 2014; 266(3):1115-1149.
2. Dhama S, Abbas S. Square mean almost automorphic solution of stochastic evolution equations with impulses on time scales. Differ. Equ. Appl. 2018; 10(4):449-469.
3. Diop M, Ezzinbi K, Mbaye M. Measure theory and square-mean pseudo almost periodic and automorphic process: Application to stochastic evolution equations. Bull. Malays. Math. Sci. Soc. 2018; 40(1):287-310.
4. Zhu H, Chu J, Zhang W. Mean-square almost automorphic solutions for stochastic differential equations with hyperbolicity. Discrete Contin. Dyn. Syst. 2018; 38:1935-1953.
5. Zhu P. Poisson μ-pseudo almost automorphic stochastic processes and its applications to nonlinear Sobolev-type SDEs with Markov switching. Math Meth Appl Sci. 2021; 44(2):1688-1712.
6. Wang YW, Yang W, Xiao JW, et al. Impulsive multisynchronization of coupled multistable neural networks with time-varying delay, IEEE Transactions on Neural Networks and Learning Systems, 2017; 28(7):1560-1571.
7. Ren Y, Cheng X, Sakthivel R. Impulsive neutral stochastic functional integro-differential equations with infinite delay driven by fBm. Appl. Math. Comput. 2014; 247:205-212.
8. Hilger S. Analysis on measure chains-a unified approach to continuous and discrete calculus. Results Math. 1990; 18(1-2):18-56.
9. Bohner M, Peterson A. Advances in Dynamic Equations on Time Scales. Boston, USA: Birkhäuser; 2003.
10. Guseinov GS. Integration on time scales. J. Math. Anal. Appl. 2003; 285(1):107-127.
11. Li Y, Wang C. Almost periodic functions on time scales and applications. Discrete Dyn. Nat. Soc. 2011; 2011: 1-20
12. Lizama C, Mesquita JG. Almost automorphic solutions of dynamic equations on time scales. J. Funct. Anal. 2013; 265(10):2267-2311.
13. Bohner M, Sanyal S. The stochastic dynamic exponential and geometric Brownian motion on isolated time scales. Commun. Math. Anal. 2010; 8(3):120-135.
14. Rafeeq S, Kalsoom H, Hussain S, et al. Delay dynamic double integral inequalities on time scales with applications. Adv. Differ Equ. 2020; 40:1-32.
15. Lu XD, Li HT, Wang CK, et al. Stability analysis of positive switched impulsive systems with delay on time scales. Internat. J. Robust Nonlinear Control. 2020; 30(16):6879-6890.
16. Wang C, Agarwal RP. Weighted piecewise pseudo almost automorphic functions with applications to abstract impulsive ∇-dynamic equations on time scales. Adv. Difference Equ. 2014; 153:1-29.
17. Bochner S. Curvature and Betti numbers in real and complex vector bundles. Universitá di Torino. Rendiconti del Seminario Natematico. 1955; 15:225-253.
18. Diagana T. Doubly-weighted pseudo-almost periodic functions. Afr. Diaspora J. Math. 2011; 12(1):121-136.
19. Blot J, Mophou GM, N'Guérékata GM, et al. Weighted pseudo almost automorphic functions and applications to abstract differential equations. Nonlinear Anal Theory Method Appl. 2009; 71(3):903-909.
20. Yang QG, Zhu P. Stepanov-like doubly weighted pseudo almost automorphic processes and its application to Sobolev-type stochastic differential equations driven by G-Brownian motion. Math Meth Appl Sci. 2017; 40(18):6602-6622.
21. Singh V, Pandey DN. Doubly weighted pseudo almost automorphic solutions for two-term fractinal order differential equations. J. Nonlinear Evol. Equ. Appl. 2019; 2018:39-55.
22. Ji DS, Zhang CY. Translation invariance of weighted pseudo almost periodic functions and related problems. J Math Anal Appl. 2012; 391(2):350-362.

23. Zheng ZM, Ding HS. On completeness of the space of weighted pseudo almost automorphic functions. J Funct Anal. 2015; 268(10):3211-3218.
24. Dhama S, Abbas S. Square mean almost automorphic solution of stochastic evolution equations with impulses on time scales. Differ. Equ. Appl. 2018; 10(4):449-469.
25. Chen Z, Lin W. Square-mean weighted pseudo almost automorphic solutions for non-autonomous stochastic evolution equations. J Math Pures Appl. 2013; 100(4):476-504.
26. Diop MA, Ezzinbi K, Lo M. Exponential stability for some stochastic neutral partial functional integrodifferential equations with delays and Poisson jumps. Semigroup Forum 2014; 88:595-609.
27. Li KX. Weighted pseudo almost automorphic solutions for nonautonomous SPDEs driven by Lévy noise. J. Math. Anal. Appl. 2015; 427:686-721.
28. Yang QG, Zhu P. Doubly-weighted pseudo almost automorphic solutions for nonlinear stochastic differential equations driven by Lévy noise. Stochastics Int J Prob Stochastic Proc. 2018;90(5):701-719.
29. Smart D. Fixed Point Theorems. Cambridge, UK: Cambridge University Press; 1980.