GENERAL STABILITY FOR THE VISCOELASTIC WAVE EQUATION WITH NONLINEAR DAMPING AND NONLINEAR TIME-VARYING DELAY AND ACOUSTIC BOUNDARY CONDITIONS

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Abstract. In this paper, we are concerned with the energy decay rates for the viscoelastic wave equation with nonlinear damping and nonlinear time-varying delay in the boundary and acoustic boundary conditions. Here we consider with minimal condition on the relaxation function g, namely $g'(t) \leq -\mu(t)G(g(t))$, where G is an increasing and convex function near the origin and μ is a positive nonincreasing function. The decay rates of the energy depend on the functions μ, G and on the function F defined by f_0 which represents the growth at the origin of f_1.

Keywords: optimal decay; viscoelastic wave equation; nonlinear time-varying delay; acoustic boundary conditions

1. Introduction

In this paper, we are concerned with the energy decay rates for the viscoelastic wave equation with nonlinear damping and nonlinear time-varying delay in the boundary and acoustic boundary conditions

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds = 0, \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,t) = 0, \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.2}$$

$$\begin{aligned}&\frac{\partial u}{\partial \nu}(x,t) - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(x,s)ds + a_1 f_1(u_t(x,t)) + a_2 f_2(u_t(x,t-\tau(t)))\\ &\quad = w_t(x,t), \quad \text{on } \Gamma_1 \times (0,\infty),\end{aligned} \tag{1.3}$$

$$u_t(x,t) + h(x)w_t(x,t) + m(x)w(x,t) = 0, \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.4}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \tag{1.5}$$

$$u_t(x,t) = j_0(x,t), \quad \text{in } \Gamma_1 \times (-\tau(0),0), \tag{1.6}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \geq 1)$ with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, Γ_0 and Γ_1 are closed and disjoint, ν is the outward unit normal vector to Γ. $w(x,t)$ is the normal displacement into the domain of a point $x \in \Gamma_1$ at time t and $h, m : \Gamma_1 \to \mathbb{R}$ are functions that represent resistivity and spring constant per unit area, respectively, and are essential bounded, g represents the kernel of the memory term, $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are given functions, a_1, a_2 are real numbers with $a_1 > 0, a_2 \neq 0, \tau(t) > 0$ represents the time-varying delay and the initial data (u_0, u_1, j_0) belong to a suitable space. Boundary conditions

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(1.3) and (1.4) are called acoustic boundary conditions. (1.4) does not contain the second derivative w_{tt}, which physically means that the material of the surface is much lighter than a liquid flowing along it.

When $a_1 = a_2 = 0$, the model (1.1)-(1.5) are pertinent to noise control and suppression in practical applications. The noise propagates through some acoustic medium, for example, though air, in a room that is characterized by a bounded domain Ω and whose walls, floor and ceiling are described by the boundary conditions [1, 2]. Park and Park [3] studied the general decay for problem (1.1)-(1.5) under the conditions that $\int_0^\infty g(s)ds < \frac{1}{2}$ and $g'(t) \leq -\mu(t)g(t)$, for $t \geq 0$, where $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonincreasing differentiable function. Liu [4] improved the work of [3] to an arbitrary rate of decay with not necessarily of an exponential or polynomial one. Recently, Yoon et al. [5] generalized the work of [3, 4] to general decay rates without the assumption condition $\int_0^\infty g(s)ds < \frac{1}{2}$. The assumption on relaxation function g has been weakened compared to the conditions assumed in previous literature [3, 4].

Many phenomena depend on both the current state and past occurrences. There has been a notable increase in the research on the wave equation with delay effects, which frequently arise in various practical problems [6-8]. Kirane and Said-Houari [9] showed the global existence and asymptotic stability for the following viscoelastic wave equation with constant delay

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + a_1 u_t(x,t) + a_2 u_t(x,t-\tau) = 0,$$

where a_1 and a_2 are positive constants. Dai and Yang [10] proved the exponential decay results for the energy of the concerned problem in the case $a_1 = 0$ which solves an open problem proposed by Kirane and Said-Houari [9]. The viscoelastic wave equation involving time-varying delay instead of constant delay is studied by Liu [11]. Afterwards, systems with time-varying delay have been extensively considered by many authors (see [12-17] and references therein). Moreover, Benaissa et al. [18] investigated the global existence and energy decay of solutions for the following wave equation with a time-varying delay in the weakly nonlinear feedbacks

$$u_{tt}(x,t) - \Delta u(x,t) + a_1\sigma(t)f_1(u_t(x,t)) + a_2\sigma(t)f_2(u_t(x,t-\tau(t))) = 0,$$

where $a_1, a_2 > 0$ and σ, f_1, f_2 satisfy some conditions. This result extended the previous works [6, 8]. For the problem with nonlinear time-varying delay, we also refer [19, 20]. Motivated by these results, we study the general decay rates of solution for problem (1.1)-(1.6). We put a minimal and general assumption on relaxation function g, namely

$$g'(t) \leq -\mu(t)G(g(t)), \tag{1.7}$$

where μ is a positive nonincreasing function and G is linear or it is strictly increasing and strictly convex function near the origin. Also, our results obtained without imposing any restrictive growth assumption on the damping term. The decay rates of the energy depend on the functions μ, G and on the function F defined by f_0 which represents the growth at the origin of f_1. Recently, Al-Gharabli et al. [21] considered the general and optimal decay result for the viscoelastic equation with nonlinear boundary feedback. When relaxation function g satisfies the condition (1.7), the general decay of solution for the viscoelastic equation has been studied by several researchers(see [22, 23] and references therein).

2. Preliminary and statement of main results

Throughout this paper, we use the notation

$$V = \{u \in H^1(\Omega) : u = 0 \ \text{ on } \Gamma_0\}.$$

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^2(\Gamma_1)}$ by $\|\cdot\|$ and $\|\cdot\|_{\Gamma_1}$, respectively.

The Poincaré inequality hold in V, that is, there exist the smallest positive constants λ and λ_* such that

$$\|u\|^2 \leq \lambda\|\nabla u\|^2 \quad \text{and} \quad \|u\|_{\Gamma_1}^2 \leq \lambda_*\|\nabla u\|^2 \ \text{ for all } u \in V. \tag{2.1}$$

As in [5, 19, 21, 22, 23], we consider the following assumptions on g, f_1, f_2, τ, h and m.

(H1) $g : [0, \infty) \to (0, \infty)$ is a differentiable function satisfying

$$1 - \int_0^\infty g(s)ds = l > 0 \tag{2.2}$$

and there exists a C^1 function $G : (0, \infty) \to (0, \infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0, r_0]$, $r_0 \leq g(0)$, with $G(0) = G'(0) = 0$, such that

$$g'(t) \leq -\mu(t)G(g(t)), \quad \forall t \geq 0, \tag{2.3}$$

where μ is a positive nonincreasing differentiable function. G in (2.3) has been introduced for the first time in [24]. These are weaker conditions on G than those introduced in [24].

(H2) $f_1 : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function such that there exists a strictly increasing function $f_0 \in C^1(\mathbb{R}^+)$, with $f_0(0) = 0$, and positive constants c_1, c_2 and ε such that

$$f_0(|s|) \leq |f_1(s)| \leq f_0^{-1}(|s|) \ \text{ for all } |s| \leq \varepsilon, \tag{2.4}$$

$$c_1|s| \leq |f_1(s)| \leq c_2|s| \quad \text{for all } |s| \geq \varepsilon. \tag{2.5}$$

Moreover, we assume that the function F defined by $F(s) = \sqrt{s}f_0(\sqrt{s})$, is a strictly convex C^2 function on $(0, r_1]$, for some $r_1 > 0$, when f_0 is nonlinear.

(H3) $f_2 : \mathbb{R} \to \mathbb{R}$ is an odd nondecreasing C^1 function such that there exist positive constants c_3, c_4 and c_5 satisfy

$$|f_2'(s)| \leq c_3, \quad c_4 s f_2(s) \leq F_2(s) \leq c_5 s f_1(s), \quad \text{for} \ \ s \in \mathbb{R}, \tag{2.6}$$

where $F_2(t) = \int_0^t f_2(s)ds$.

(H4) For the time-varying delay, we assume that $\tau \in W^{2,\infty}([0,T])$ for $T > 0$ and there exist positive constants τ_1, τ_2 and τ_3 satisfy

$$0 < \tau_1 \leq \tau(t) \leq \tau_2 \ \text{ and } \tau'(t) \leq \tau_3 < 1 \ \text{ for all } t > 0. \tag{2.7}$$

Moreover, for $c_4\tau_3 < 1$, we assume that a_1 and a_2 satisfy

$$0 < |a_2| < \frac{c_4(1-\tau_3)}{c_5(1-c_4\tau_3)} a_1. \tag{2.8}$$

(H5) We assume that $h, m \in C(\Gamma_1)$ and $h(x) > 0$ and $m(x) > 0$ for all $x \in \Gamma_1$. This assumption implies that there exist positive constants h_i and $m_i (i = 1, 2)$ such that

$$h_1 \leq h(x) \leq h_2, \quad m_1 \leq m(x) \leq m_2 \ \text{ for all } \ x \in \Gamma_1. \tag{2.9}$$

Remark 2.1. ([23]) 1. By (H1), we obtain $\lim\limits_{t\to+\infty} g(t) = 0$. Then there exists $t_0 \geq 0$ large enough such that

$$g(t_0) = r_0 \ \Rightarrow \ g(t) \leq r_0, \quad \forall t \geq t_0. \tag{2.10}$$

As g and μ are positive nonincreasing continuous functions and G is a positive continuous function then

$$c_6 \leq \mu(t)G(g(t)) \leq c_7, \quad \forall t \in [0, t_0],$$

for some positive constants c_6 and c_7. From (2.3), we obtain

$$g'(t) \leq -\mu(t)G(g(t)) \leq -\frac{c_6}{g(0)} g(0) \leq -c_8 g(t), \quad \forall t \in [0, t_0], \tag{2.11}$$

where $c_8 = \frac{c_6}{g(0)}$ is a positive constant.

2. If G is a strictly increasing and strictly convex C^2 function on $(0, r_0]$, with $G(0) = G'(0) = 0$, then it has an extension $\overline{G}$, which is strictly increasing and strictly convex C^2 function on $(0, \infty)$. The same remark can be established for $\overline{F}$.

We recall the well-known Jensen's inequality which will be used essentially to establish our main result. If ϕ is a convex function on $[a, b]$, $p : \Omega \to [a, b]$ and k are integrable functions on Ω, $k(x) \geq 0$ and $\int_\Omega k(x)dx = k_0 > 0$, then Jensen's inequality states that

$$\phi\left[\frac{1}{k_0}\int_\Omega p(x)k(x)dx\right] \leq \frac{1}{k_0}\int_\Omega \phi[p(x)]k(x)dx. \tag{2.12}$$

Let H^* be the conjugate of the convex function H defined by $H^*(s) = \sup\limits_{t\geq 0}(st - H(t))$, then

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0. \tag{2.13}$$

Moreover, due to the argument given in [25], it holds that

$$H^*(s) = s(H')^{-1}(s) - H\big((H')^{-1}(s)\big), \quad \forall s \geq 0. \tag{2.14}$$

As in [6, 8], we introduce the following new function

$$z(x, \kappa, t) = u_t(x, t - \kappa\tau(t)), \quad \text{for } (x, \kappa, t) \in \Gamma_1 \times (0, 1) \times (0, \infty).$$

Then, problem (1.1)-(1.6) is equivalent to

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds = 0, \text{ in } \Omega \times (0,\infty), \tag{2.15}$$

$$\tau(t)z_t(x,\kappa,t) + (1-\kappa\tau'(t))z_\kappa(x,\kappa,t) = 0, \text{ in } \Gamma_1 \times (0,1) \times (0,\infty), \tag{2.16}$$

$$u(x,t) = 0, \text{ in } \Gamma_0 \times (0,\infty), \tag{2.17}$$

$$\frac{\partial u}{\partial \nu}(x,t) - \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(x,s)ds + a_1 f_1(u_t(x,t)) + a_2 f_2(z(x,1,t)) = w_t(x,t), \text{ on } \Gamma_1 \times (0,\infty), \tag{2.18}$$

$$u_t(x,t) + h(x)w_t(x,t) + m(x)w(x,t) = 0, \text{ on } \Gamma_1 \times (0,\infty), \tag{2.19}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \text{ in } \Omega, \tag{2.20}$$

$$z(x,\kappa,0) = j_0(x, -\kappa\tau(0)), \text{ in } \Gamma_1 \times (0,1). \tag{2.21}$$

We state the global existence result, which can be established by the arguments of [18, 26].

Theorem 2.1. Let initial data $(u_0,u_1) \in (V \cap H^2(\Omega)) \times V$ and $j_0 \in L^2(\Gamma_1 \times (0,1))$. Suppose that (H1)-(H5) hold. Then, for any $T > 0$, there exists a unique pair of functions (u,w,z) which is a solution to problem (2.15)-(2.21) in the class

$$u \in L^\infty(0,T; V \cap H^2(\Omega)), \quad u_t \in L^\infty(0,T;V), \quad u_{tt} \in L^\infty(0,T;L^2(\Omega)),$$
$$z \in L^\infty(0,T;L^2(\Gamma_1 \times (0,1))), \quad w, w_t \in L^2(0,\infty; L^2(\Gamma_1)).$$

Now, we introduce the energy

$$\begin{aligned} E(t) = {} & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \frac{1}{2}(g \circ \nabla u)(t) + \frac{1}{2}\int_{\Gamma_1} m(x)w^2(t)d\Gamma \\ & + \frac{\zeta\tau(t)}{2}\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma, \end{aligned} \tag{2.22}$$

where $(g \circ \nabla u)(t) = \int_0^t g(t-s)\|\nabla u(t) - \nabla u(s)\|^2 ds$ and

$$\frac{2|a_2|(1-c_4)}{c_4(1-\tau_3)} < \zeta < \frac{2(a_1 - |a_2|c_5)}{c_5}. \tag{2.23}$$

To show the main results of this paper, we need the following lemma.

Lemma 2.1. Let (H3) and (H4) hold. Then, there exist positive constants γ_0 and γ_1 satisfying

$$\begin{aligned} E'(t) \le {} & \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - \int_{\Gamma_1} h(x)w_t^2(t)d\Gamma \\ & -\gamma_0 \int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma - \gamma_1 \int_{\Gamma_1} f_2(z(x,1,t))z(x,1,t)d\Gamma. \end{aligned} \tag{2.24}$$

Proof. Multiplying in (2.15) by $u_t(t)$, integrating over Ω and using Green's formula, (2.18) and (2.19), we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left[\|u_t(t)\|^2 + \left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + (g \circ \nabla u)(t) + \int_{\Gamma_1} m(x)w^2(t)d\Gamma\right] + \int_{\Gamma_1} h(x)w_t^2(t)d\Gamma \\ & = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2 - a_1\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma - a_2\int_{\Gamma_1} f_2(z(x,1,t))u_t(t)d\Gamma, \end{aligned} \tag{2.25}$$

where we used the relation

$$
\begin{aligned}
&-\int_\Omega \nabla u_t(t)\int_0^t g(t-s)\nabla u(s)dsdx\\
&=\frac{d}{dt}\left[\frac{1}{2}(g\circ\nabla u)(t)-\frac{1}{2}\int_0^t g(s)ds\|\nabla u(t)\|^2\right]-\frac{1}{2}(g'\circ\nabla u)(t)+\frac{1}{2}g(t)\|\nabla u(t)\|^2.
\end{aligned}
$$

From (2.22) and (2.25), we obtain

$$
\begin{aligned}
E'(t)&=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|^2-\int_{\Gamma_1}h(x)w_t^2(t)d\Gamma\\
&-a_1\int_{\Gamma_1}f_1(u_t(t))u_t(t)d\Gamma-a_2\int_{\Gamma_1}f_2(z(x,1,t))u_t(t)d\Gamma\\
&+\frac{\zeta\tau'(t)}{2}\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma+\frac{\zeta\tau(t)}{2}\int_{\Gamma_1}\int_0^1 f_2(z(x,\kappa,t))z_t(x,\kappa,t)d\kappa d\Gamma, \qquad (2.26)
\end{aligned}
$$

where $F_2(t)=\int_0^t f_2(s)ds$. We multiply in (2.16) by $f_2(z(x,\kappa,t))$ and integrate over $\Gamma_1\times(0,1)$ to obtain

$$
\begin{aligned}
&\frac{\zeta\tau(t)}{2}\int_{\Gamma_1}\int_0^1 f_2(z(x,\kappa,t))z_t(x,\kappa,t)d\kappa d\Gamma\\
&=-\frac{\zeta}{2}\int_{\Gamma_1}\left[(1-\tau'(t))F_2(z(x,1,t))-F_2(z(x,0,t))+\int_0^1\tau'(t)F_2(z(x,\kappa,t))d\kappa\right]d\Gamma.
\end{aligned}
$$

Applying this to (2.26) and noting that $z(x,0,t)=u_t(x,t)$, it follows that

$$
\begin{aligned}
E'(t)&=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|^2-\int_{\Gamma_1}h(x)w_t^2(t)d\Gamma-a_1\int_{\Gamma_1}f_1(u_t(t))u_t(t)d\Gamma\\
&-a_2\int_{\Gamma_1}f_2(z(x,1,t))u_t(t)d\Gamma-\frac{\zeta}{2}\int_{\Gamma_1}\left[(1-\tau'(t))F_2(z(x,1,t))-F_2(u_t(x,t))\right]d\Gamma. \qquad (2.27)
\end{aligned}
$$

From (2.6) and (2.7), we get

$$
\begin{aligned}
&-\frac{\zeta}{2}\int_{\Gamma_1}\left[(1-\tau'(t))F_2(z(x,1,t))-F_2(u_t(x,t))\right]d\Gamma\\
&\le-\frac{\zeta c_4}{2}(1-\tau_3)\int_{\Gamma_1}f_2(z(x,1,t))z(x,1,t)d\Gamma+\frac{\zeta c_5}{2}\int_{\Gamma_1}f_1(u_t(t))u_t(t)d\Gamma. \qquad (2.28)
\end{aligned}
$$

Substituting (2.28) into (2.27), we obtain

$$
\begin{aligned}
E'(t)&\le\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|^2-\int_{\Gamma_1}h(x)w_t^2(t)d\Gamma-\Big(a_1-\frac{\zeta c_5}{2}\Big)\int_{\Gamma_1}f_1(u_t(t))u_t(t)d\Gamma\\
&-a_2\int_{\Gamma_1}f_2(z(x,1,t))u_t(t)d\Gamma-\frac{\zeta c_4}{2}(1-\tau_3)\int_{\Gamma_1}f_2(z(x,1,t))z(x,1,t)d\Gamma. \qquad (2.29)
\end{aligned}
$$

The definition of F_2 and (2.14) give

$$
F_2^*(s)=sf_2^{-1}(s)-F_2(f_2^{-1}(s)),\quad \text{for } s\ge 0. \qquad (2.30)
$$

Hence, using (2.6), (2.13) and (2.30) with $s=f_2(z(x,1,t))$ and $t=u_t(t)$, we get(see details in [20])

$$
\begin{aligned}
&-a_2\int_{\Gamma_1}f_2(z(x,1,t))u_t(t)d\Gamma\\
&\le|a_2|\int_{\Gamma_1}\Big(f_2(z(x,1,t))z(x,1,t)-F_2(z(x,1,t))+F_2(u_t(t))\Big)d\Gamma\\
&\le|a_2|\Big((1-c_4)\int_{\Gamma_1}f_2(z(x,1,t))z(x,1,t)d\Gamma+c_5\int_{\Gamma_1}f_1(u_t(t))u_t(t)d\Gamma\Big). \qquad (2.31)
\end{aligned}
$$

By using (2.29) and (2.31) and selecting ζ satisfying (2.23), we obtain the desired inequality (2.24) where $\gamma_0 = a_1 - \frac{\zeta c_5}{2} - |a_2|c_5 > 0$ and $\gamma_1 = \frac{\zeta c_4}{2}(1-\tau_3) - |a_2|(1-c_4) > 0$. $\square$

Our main results are the following.

Theorem 2.2. Assume that (H1)-(H5) hold and f_0 is linear. Then there exist positive constants k_1, k_2, k_3 and k_4 such that the energy functional satisfies, for all $t \geq t_0$,

$$E(t) \leq k_2 e^{-k_1 \int_{t_0}^t \mu(s)ds}, \quad \text{if } G \text{ is linear}, \tag{2.32}$$

$$E(t) \leq k_4 G_1^{-1}\Big(k_3 \int_{t_0}^t \mu(s)ds\Big), \quad \text{if } G \text{ is nonlinear}, \tag{2.33}$$

where $G_1(t) = \int_t^{r_0} \frac{1}{sG'(s)}ds$ is strictly decreasing and convex on $(0, r_0]$.

Theorem 2.3. Assume that (H1)-(H5) hold and f_0 is nonlinear. Then there exist positive constants $\alpha_1, \alpha_2, \alpha_3$ and α_4 such that the energy functional satisfies

$$E(t) \leq \alpha_2 F_1^{-1}\Big(\alpha_1 \int_{t_0}^t \mu(s)ds\Big), \quad \forall t \geq t_0, \text{ if } G \text{ is linear}, \tag{2.34}$$

where $F_1(t) = \int_t^{r_1} \frac{1}{sF'(s)}ds$ and

$$E(t) \leq \alpha_4 (t-t_0) K_1^{-1}\Big(\frac{\alpha_3}{(t-t_0)\int_{t_1}^t \mu(s)ds}\Big), \quad \forall t \geq t_1, \text{ if } G \text{ is nonlinear}, \tag{2.35}$$

where $K_1(t) = tK'(\varepsilon_2 t),\ 0 < \varepsilon_2 < r_2 = \min\{r_0, r_1\}$ and $K = \big(\overline{G}^{-1} + \overline{F}^{-1}\big)^{-1}$.

3. Technical Lemmas

In this section, we prove the following lemmas to obtain the general decay rates of the solution for problem (2.15)-(2.21).

Lemma 3.1. Under the assumption (H1), the functional Φ_1 defined by

$$\Phi_1(t) = \int_\Omega u(t)u_t(t)dx + \int_{\Gamma_1} u(t)w(t)d\Gamma + \frac{1}{2}\int_{\Gamma_1} h(x)w^2(t)d\Gamma$$

satisfies

$$\begin{aligned}\Phi_1'(t) \leq{}& \|u_t(t)\|^2 - \frac{l}{2}\|\nabla u(t)\|^2 + \frac{2C(\xi)}{l}(i\circ\nabla u)(t) + \frac{8\lambda_*}{l}\|w_t(t)\|^2_{\Gamma_1}\\ &+\frac{a_1 a_3}{l}\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \frac{|a_2|a_3}{l}\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma,\end{aligned} \tag{3.1}$$

for any $0 < \xi < 1$, where

$$C(\xi) = \int_0^\infty \frac{g^2(s)}{i(s)}ds \quad \text{and} \quad i(t) = \xi g(t) - g'(t). \tag{3.2}$$

Proof. Using equation (2.15), (2.17)-(2.19) and utilizing (2.2) and Young's inequality, we obtain

$$\begin{aligned}\Phi_1'(t) ={}& \|u_t(t)\|^2 - \Big(1 - \int_0^t g(s)ds\Big)\|\nabla u(t)\|^2 + \int_0^t g(t-s)(\nabla u(s) - \nabla u(t), \nabla u(t))ds\\ &-a_1\int_{\Gamma_1} f_1(u_t(t))u(t)d\Gamma - a_2\int_{\Gamma_1} f_2(z(x,1,t))u(t)d\Gamma + 2\int_{\Gamma_1} u(t)w_t(t)d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma\end{aligned}$$

$$\leq \|u_t(t)\|^2 - \frac{7l}{8}\|\nabla u(t)\|^2 + \frac{2}{l}\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2 dx$$
$$-a_1\int_{\Gamma_1} f_1(u_t(t))u(t)d\Gamma - a_2\int_{\Gamma_1} f_2(z(x,1,t))u(t)d\Gamma + 2\int_{\Gamma_1} u(t)w_t(t)d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma.$$

Using Cauchy-Schwarz inequality and (3.2), we have(see [23, 27])

$$\int_\Omega \Big(\int_0^t g(t-s)|\nabla u(s) - \nabla u(t)|ds\Big)^2 dx \leq \Big(\int_0^t \frac{g^2(s)}{i(s)}ds\Big)(i\circ\nabla u)(t) \leq C(\xi)(i\circ\nabla u)(t). \tag{3.3}$$

Applying Young's inequality and (2.1), we obtain for $\eta > 0$,

$$\Big| - a_1\int_{\Gamma_1} f_1(u_t(t))u(t)d\Gamma\Big| \leq \eta a_1\lambda_*\|\nabla u(t)\|^2 + \frac{a_1}{4\eta}\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma, \tag{3.4}$$

$$\Big| - a_2\int_{\Gamma_1} f_2(z(x,1,t))u(t)d\Gamma\Big| \leq \eta|a_2|\lambda_*\|\nabla u(t)\|^2 + \frac{|a_2|}{4\eta}\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma, \tag{3.5}$$

and

$$2\int_{\Gamma_1} u(t)w_t(t)d\Gamma \leq \frac{l}{8}\|\nabla u(t)\|^2 + \frac{8\lambda_*}{l}\|w_t(t)\|^2_{\Gamma_1}. \tag{3.6}$$

Combining estimates (3.3)-(3.6), we see that

$$\Phi_1'(t) \leq \|u_t(t)\|^2 - (\frac{3l}{4} - \eta a_1\lambda_* - \eta|a_2|\lambda_*)\|\nabla u(t)\|^2 + \frac{2C(\xi)}{l}(i\circ\nabla u)(t) + \frac{8\lambda_*}{l}\|w_t(t)\|^2_{\Gamma_1}$$
$$+\frac{a_1}{4\eta}\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \frac{|a_2|}{4\eta}\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma - \int_{\Gamma_1} m(x)w^2(t)d\Gamma.$$

Setting $a_3 = (a_1 + |a_2|)\lambda_*$ and choosing $\eta = \frac{l}{4a_3}$ leads to (3.1). □

Lemma 3.2. Under the assumption (H1), the functional Φ_2 defined by

$$\Phi_2(t) = -\int_\Omega u_t(t)\int_0^t g(t-s)(u(t) - u(s))dsdx$$

satisfies

$$\Phi_2'(t) \leq -\Big(\int_0^t g(s)ds - \delta\Big)\|u_t(t)\|^2 + \delta\|\nabla u(t)\|^2 + \frac{C_1(1 + C(\xi))}{\delta}(i\circ\nabla u)(t) + \delta\lambda_*\|w_t(t)\|^2_{\Gamma_1}$$
$$+\delta a_1\lambda_*\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \delta|a_2|\lambda_*\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma, \tag{3.7}$$

for any $0 < \delta < 1$.

Proof. Using equation (2.15), (2.17) and (2.18), we get

$$\Phi_2'(t) = \Big(1 - \int_0^t g(s)ds\Big)\int_\Omega \nabla u\cdot\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))dsdx$$
$$+\int_\Omega \Big(\int_0^t g(t-s)(\nabla u(t) - \nabla u(s))ds\Big)^2 dx - \int_{\Gamma_1} w_t(t)\int_0^t g(t-s)(u(t) - u(s))dsd\Gamma$$
$$+a_1\int_{\Gamma_1} f_1(u_t(t))\int_0^t g(t-s)(u(t) - u(s))dsd\Gamma + a_2\int_{\Gamma_1} f_2(z(x,1,t))\int_0^t g(t-s)(u(t) - u(s))dsd\Gamma$$
$$-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t) - u(s))dsdx - \Big(\int_0^t g(s)ds\Big)\|u_t(t)\|^2$$
$$= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 - \Big(\int_0^t g(s)ds\Big)\|u_t(t)\|^2.$$

By Young's inequality, (2.1) and (3.3), we obtain for $\delta > 0$,

$$\begin{aligned}
&I_1 \le \delta\|\nabla u(t)\|^2 + \frac{C(\xi)}{4\delta}(i\circ\nabla u)(t),\\
&I_2 \le C(\xi)(i\circ\nabla u)(t),\\
&|I_3| \le \delta\lambda_*\|w_t(t)\|^2_{\Gamma_1} + \frac{C(\xi)}{4\delta}(i\circ\nabla u)(t),\\
&|I_4| \le \delta a_1\lambda_*\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \frac{a_1C(\xi)}{4\delta}(i\circ\nabla u)(t),\\
&|I_5| \le \delta|a_2|\lambda_*\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma + \frac{|a_2|C(\xi)}{4\delta}(i\circ\nabla u)(t).
\end{aligned}$$

Using Young's inequality, (2.1), (2.2), (3.2) and (3.3), we see that

$$\begin{aligned}
I_6 &= \int_\Omega u_t(t)\int_0^t i(t-s)(u(t)-u(s))dsdx - \int_\Omega u_t(t)\int_0^t \xi g(t-s)(u(t)-u(s))dsdx\\
&\le \delta\|u_t(t)\|^2 + \frac{1}{2\delta}\int_\Omega\Big(\int_0^t i(t-s)|u(s)-u(t)|ds\Big)^2dx + \frac{\xi^2}{2\delta}\int_\Omega\Big(\int_0^t g(t-s)|u(t)-u(s)|ds\Big)^2dx\\
&\le \delta\|u_t(t)\|^2 + \frac{\lambda(g(0)+\xi)}{2\delta}(i\circ\nabla u)(t) + \frac{\lambda\xi^2C(\xi)}{2\delta}(i\circ\nabla u)(t).
\end{aligned}$$

Combining all above estimates and taking $C_1 = \max\{\frac{\lambda(g(0)+\xi)}{2},\ \delta + \frac{1+\lambda\xi^2}{2} + \frac{a_1+|a_2|}{4}\}$, the desired inequality (3.7) is established. □

Lemma 3.3. Under the assumptions (H3) and (H4), the functional Φ_3 defined by

$$\Phi_3(t) = \tau(t)\int_{\Gamma_1}\int_0^1 e^{-\kappa\tau(t)}F_2(z(x,\kappa,t))d\kappa d\Gamma$$

satisfies

$$\begin{aligned}
\Phi_3'(t) \le& -e^{-\tau_2}\tau(t)\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma - c_4(1-\tau_3)e^{-\tau_2}\int_{\Gamma_1} f_2(z(x,1,t))z(x,1,t)d\Gamma\\
&+c_5\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma.
\end{aligned}\tag{3.8}$$

Proof. Using the equation (2.16), integration by parts, (2.6) and (2.7), we obtain(see [19])

$$\begin{aligned}
\Phi_3'(t) =& \tau'(t)\int_{\Gamma_1}\int_0^1 e^{-\kappa\tau(t)}F_2(z(x,\kappa,t))d\kappa d\Gamma - \tau(t)\int_{\Gamma_1}\int_0^1 \kappa\tau'(t)e^{-\kappa\tau(t)}F_2(z(x,\kappa,t))d\kappa d\Gamma\\
&-\int_{\Gamma_1}\int_0^1 e^{-\kappa\tau(t)}(1-\kappa\tau'(t))\frac{d}{d\kappa}F_2(z(x,\kappa,t))d\kappa d\Gamma\\
&= -\Phi_3(t) - e^{-\tau(t)}\int_{\Gamma_1}(1-\tau'(t))F_2(z(x,1,t))d\Gamma + \int_{\Gamma_1}F_2(u_t(x,t))d\Gamma\\
&\le -e^{-\tau_2}\tau(t)\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma - c_4(1-\tau_3)e^{-\tau_2}\int_{\Gamma_1} f_2(z(x,1,t))z(x,1,t)d\Gamma\\
&+c_5\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma.
\end{aligned}$$

□

Lemma 3.4. ([23]) Under the assumption (H1), the functional Φ_4 defined by

$$\Phi_4(t) = \int_\Omega \int_0^t G_2(t-s)|\nabla u(s)|^2 dsdx,$$

satisfies

$$\Phi_4'(t) \leq 3(1-l)\|\nabla u(t)\|^2 - \frac{1}{2}(g \circ \nabla u)(t), \tag{3.9}$$

where $G_2(t) = \int_t^\infty g(s)ds$.

Next, let us define the perturbed modified energy by

$$L(t) = NE(t) + N_1\Phi_1(t) + N_2\Phi_2(t) + \Phi_3(t) + b_1E(t), \tag{3.10}$$

where N, N_1, N_2 and b_1 are some positive constants.

As in [3, 19], for $N > 0$ large enough, there exist positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t).$$

Lemma 3.5. Assume that (H1), (H3)-(H5) hold. Then, there exist positive constants β_3, β_4 and β_5 such that

$$L'(t) \leq -\beta_3 E(t) + \beta_4 \int_{t_0}^t g(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds + \beta_5 \int_{\Gamma_1} f_1^2(u_t(t))d\Gamma, \quad \forall t \geq t_0, \tag{3.11}$$

where t_0 was introduced in (2.10).

Proof. Let $g_0 = \int_0^{t_0} g(s)ds$. Using the fact that $i(t) = \xi g(t) - g'(t)$ and combining (2.24), (3.1), (3.7), (3.8) and (3.10), we get, for all $t \geq t_0$,

$$\begin{aligned}
L'(t) \leq{}& \frac{\xi N}{2}(g \circ \nabla u)(t) - \Big(\frac{lN_1}{2} - \delta N_2\Big)\|\nabla u(t)\|^2 - \Big(g_0 N_2 - \delta N_2 - N_1\Big)\|u_t(t)\|^2 \\
&-\Big(\frac{N}{2} - \frac{2C(\xi)N_1}{l} - \frac{C_1(1+C(\xi))N_2}{\delta}\Big)(i \circ \nabla u)(t) - N_1 \int_{\Gamma_1} m(x)w^2(t)d\Gamma + b_1E'(t) \\
&-\Big(h_1 N - \frac{8\lambda_* N_1}{l} - \delta\lambda_* N_2\Big)\|w_t(t)\|^2_{\Gamma_1} - e^{-\tau_2}\tau(t)\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma \\
&-\big(\gamma_0 N - c_5\big)\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma - \Big(\gamma_1 N + c_4(1-\tau_3)e^{-\tau_2}\Big)\int_{\Gamma_1} f_2(z(x,1,t))z(x,1,t)d\Gamma \\
&+\Big(\frac{a_1a_3N_1}{l} + \delta a_1\lambda_* N_2\Big)\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + \Big(\frac{|a_2|a_3N_1}{l} + \delta|a_2|\lambda_* N_2\Big)\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma.
\end{aligned} \tag{3.12}$$

From (2.6), we find that

$$\int_{\Gamma_1} f_2^2(z(x,1,t))d\Gamma \leq c_3 \int_{\Gamma_1} f_2(z(x,1,t))z(x,1,t)d\Gamma. \tag{3.13}$$

Applying (3.13) to (3.12) and taking $\delta = \frac{l}{4N_2}$, we get, for all $t \geq t_0$,

$$\begin{aligned}
L'(t) \leq{}& \frac{\xi N}{2}(g \circ \nabla u)(t) - \Big(\frac{lN_1}{2} - \frac{l}{4}\Big)\|\nabla u(t)\|^2 - \Big(g_0 N_2 - N_1 - \frac{l}{4}\Big)\|u_t(t)\|^2 \\
&-\Big(\frac{N}{2} - \frac{4C_1N_2^2}{l} - C(\xi)\Big[\frac{2N_1}{l} + \frac{4C_1N_2^2}{l}\Big]\Big)(i \circ \nabla u)(t) - N_1\int_{\Gamma_1} m(x)w^2(t)d\Gamma \\
&-\Big(h_1 N - \frac{8\lambda_* N_1}{l} - \frac{l\lambda_*}{4}\Big)\|w_t(t)\|^2_{\Gamma_1} - e^{-\tau_2}\tau(t)\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma
\end{aligned}$$

$$-\big(\gamma_0 N - c_5\big)\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma + \left(\frac{a_1 a_3 N_1}{l} + \frac{a_1 l\lambda_*}{4}\right)\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + b_1 E'(t)$$
$$-\Big(\gamma_1 N + c_4(1-\tau_3)e^{-\tau_2} - \frac{|a_2|a_3c_3N_1}{l} - \frac{|a_2|c_3 l\lambda_*}{4}\Big)\int_{\Gamma_1} f_2(z(x,1,t))z(x,1,t)d\Gamma.$$

We choose N_1 large enough so that

$$\frac{lN_1}{2} - \frac{l}{4} > 4(1-l),$$

then N_2 large enough so that

$$g_0 N_2 - N_1 - \frac{l}{4} > 1.$$

Using the fact that $\frac{\xi g^2(s)}{i(s)} < g(s)$ and the Lebesgue dominated convergence theorem, we deduce that

$$\xi C(\xi) = \int_0^\infty \frac{\xi g^2(s)}{i(s)} ds \to 0 \text{ as } \xi \to 0.$$

Hence, there is $0 < \xi_0 < 1$ such that if $\xi < \xi_0$, then

$$\xi C(\xi)\big[\frac{2N_1}{l} + \frac{4C_1N_2^2}{l}\big] < \frac{1}{8}.$$

Finally, selecting $\xi = \frac{1}{2N}$ and choosing N large enough so that

$$N > \max\Big\{\frac{16C_1N_2^2}{l},\ \frac{1}{h_1}\Big(\frac{8\lambda_* N_1}{l} + \frac{l\lambda_*}{4}\Big),\ \frac{c_5}{\gamma_0},\ \frac{1}{\gamma_1}\Big(\frac{|a_2|a_3c_3N_1}{l} + \frac{|a_2|c_3l\lambda_*}{4} - c_4(1-\tau_3)e^{-\tau_2}\Big)\Big\},$$

we obtain

$$L'(t) \le -\|u_t(t)\|^2 - 4(1-l)\|\nabla u(t)\|^2 + \frac{1}{4}(g\circ\nabla u)(t) - N_1\int_{\Gamma_1} m(x)w^2(t)d\Gamma$$
$$-e^{-\tau_2}\tau(t)\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma + \beta_5\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma + b_1E'(t), \quad \forall t \ge t_0, \tag{3.14}$$

where $\beta_5 = \frac{a_1a_3N_1}{l} + \frac{a_1l\lambda_*}{4}$. Using (2.11) and (2.24), we find that, for any $t \ge t_0$,

$$\int_0^{t_0} g(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le -\frac{1}{c_8}\int_0^{t_0} g'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le -\frac{2}{c_8}E'(t). \tag{3.15}$$

Combining (2.22), (3.14) and (3.15) and taking a suitable choice of b_1, we obtain the estimate (3.11). □

Lemma 3.6. ([21]) Assume that (H2) holds and $\max\{r_1, f_0(r_1)\} < \varepsilon$, where ε was introduced in (2.4). Then there exist positive constants C_2, C_3 and C_4 such that

$$\int_{\Gamma_1} f_1^2(u_t(t))d\Gamma \le \begin{cases} C_2\displaystyle\int_{\Gamma_1} f_1(u_t(t))u_t(t)d\Gamma, & \text{if } f_0 \text{ is linear,} \\ C_3F^{-1}(\chi(t)) - C_3E'(t), & \text{if } f_0 \text{ is nonlinear,} \end{cases} \tag{3.16}$$

where

$$\chi(t) = \frac{1}{|\Gamma_{11}|}\int_{\Gamma_{11}} f_1(u_t(t))u_t(t)d\Gamma \le -C_4E'(t), \tag{3.17}$$

$\Gamma_{11} = \{x \in \Gamma_1 : |u_t(t)| \le \varepsilon_1\}$ and $0 < \varepsilon_1 = \min\{r_1, f_0(r_1)\}$.

Next, we define $\rho(t)$ by

$$\rho(t) := -\int_{t_0}^t g'(s)\int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dxds \le -2E'(t). \tag{3.18}$$

Lemma 3.7. Assume that (H1) and (H2) hold and G is nonlinear. Then, the solution of (2.15)-(2.21) satisfies the estimates

$$\int_{t_0}^{t} g(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2dxds \leq \begin{cases} \dfrac{1}{\theta}\,\overline{G}^{-1}\left(\dfrac{\theta\rho(t)}{\mu(t)}\right), \ \forall t\geq t_0, \ \text{if } f_0 \text{ is linear}, \\ \dfrac{t-t_0}{\theta}\,\overline{G}^{-1}\left(\dfrac{\theta\rho(t)}{(t-t_0)\mu(t)}\right), \ \forall t> t_0, \ \text{if } f_0 \text{ is nonlinear}, \end{cases} \tag{3.19}$$

where $\theta\in(0,1)$ and $\overline{G}$ is an extension of G.

Proof. First, we prove the estimate (3.19) when f_0 is linear. We introduce the functional

$$\mathcal{L}(t)=L(t)+\Phi_4(t),$$

which is nonnegative. From (3.9) and (3.14), we see that, for all $t\geq t_0$,

$$\mathcal{L}'(t)\leq -\|u_t(t)\|^2-(1-l)\|\nabla u(t)\|^2-\frac{1}{4}(g\circ\nabla u)(t)-N_1\int_{\Gamma_1}m(x)w^2(t)d\Gamma$$
$$-e^{-\tau_2}\tau(t)\int_{\Gamma_1}\int_0^1 F_2(z(x,\kappa,t))d\kappa d\Gamma+\beta_5\int_{\Gamma_1}f_1^2(u_t(t))d\Gamma+b_1E'(t).$$

Applying (2.22), (2.24) and (3.16) and selecting a suitable choice of b_1, we have

$$\mathcal{L}'(t)\leq -d_1E(t),$$

where d_1 is some positive constant. This implies that

$$\int_0^\infty E(s)ds<\infty. \tag{3.20}$$

For $0<\theta<1$, we define $I(t)$ by

$$I(t):=\theta\int_{t_0}^{t}\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2dxds.$$

By (3.20), θ is taken so small that, for all $t\geq t_0$,

$$I(t)<1. \tag{3.21}$$

Since G is strictly convex on $(0,r_0]$, then

$$G(qy)\leq qG(y), \tag{3.22}$$

where $0\leq q\leq 1$ and $y\in(0,r_0]$. Using the fact that μ is a positive nonincreasing function and applying (2.3), (3.21), (3.22) and Jensen's inequality (2.12), we find that(see details in [21, 23])

$$\begin{aligned}\rho(t) &\geq \frac{\mu(t)}{\theta I(t)}\int_{t_0}^{t} I(t)G(g(s))\int_{\Omega}\theta|\nabla u(t)-\nabla u(t-s)|^2dxds\\ &\geq \frac{\mu(t)}{\theta I(t)}\int_{t_0}^{t} G(I(t)g(s))\int_{\Omega}\theta|\nabla u(t)-\nabla u(t-s)|^2dxds\\ &\geq \frac{\mu(t)}{\theta}\,\overline{G}\left(\theta\int_{t_0}^{t} g(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2dxds\right).\end{aligned} \tag{3.23}$$

Since $\overline{G}$ is strictly increasing, we obtain

$$\int_{t_0}^{t} g(s)\int_{\Omega}|\nabla u(t)-\nabla u(t-s)|^2dxds\leq \frac{1}{\theta}\overline{G}^{-1}\left(\frac{\theta\rho(t)}{\mu(t)}\right).$$

Now, we show the estimate (3.19) when f_0 is nonlinear. Since we cannot guarantee (3.20), we define the following function

$$\delta(t) := \frac{\theta}{t-t_0}\int_{t_0}^{t}\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2 dxds,\ \forall t>t_0.$$

Using the fact that $E'(t)\le 0$ and (2.22), we have

$$\delta(t) \le \frac{2\theta}{t-t_0}\int_{t_0}^{t}(||\nabla u(t)||^2+||\nabla u(t-s)||^2)ds \le \frac{8\theta E(0)}{l}.$$

Choosing θ small enough so that, for all $t>t_0$,

$$\delta(t)\le 1. \tag{3.24}$$

Similar to (3.23), using (2.3), (3.22), (3.24) and Jensen's inequality (2.12), we obtain

$$\begin{aligned}
\rho(t) &= \frac{t-t_0}{\theta\delta(t)}\int_{t_0}^{t}\delta(t)(-g'(s))\int_\Omega \frac{\theta}{t-t_0}|\nabla u(t)-\nabla u(t-s)|^2 dxds\\
&\ge \frac{(t-t_0)\mu(t)}{\theta\delta(t)}\int_{t_0}^{t}G(\delta(t)g(s))\int_\Omega \frac{\theta}{t-t_0}|\nabla u(t)-\nabla u(t-s)|^2 dxds\\
&\ge \frac{(t-t_0)\mu(t)}{\theta}\ \overline{G}\bigg(\frac{\theta}{t-t_0}\int_{t_0}^{t}g(s)\int_\Omega |\nabla u(t)-\nabla u(t-s)|^2 dxds\bigg).
\end{aligned}$$

This implies that

$$\int_{t_0}^{t}g(s)\int_\Omega|\nabla u(t)-\nabla u(t-s)|^2dxds \le \frac{t-t_0}{\theta}\ \overline{G}^{-1}\bigg(\frac{\theta\rho(t)}{(t-t_0)\mu(t)}\bigg).$$

□

4. Proof of Theorem 2.2.

In this section, we prove the main result of our work. Now, we consider the following two cases.

Case 1: $G(t)$ is linear. Multiplying (3.11) by the positive nonincreasing function $\mu(t)$ and using (2.3), (2.24) and (3.16), we get

$$\begin{aligned}
&\mu(t)L'(t) \le -\beta_3\mu(t)E(t)+\beta_4\int_{t_0}^{t}\mu(s)g(s)\int_\Omega|\nabla u(t)-\nabla u(t-s)|^2dxds+\beta_5\mu(t)\int_{\Gamma_1}f_1^2(u_t(t))d\Gamma\\
&\le -\beta_3\mu(t)E(t)-\beta_4\int_{t_0}^{t}g'(s)\int_\Omega|\nabla u(t)-\nabla u(t-s)|^2dxds+\beta_5C_2\mu(0)\int_{\Gamma_1}f_1(u_t(t))u_t(t)d\Gamma\\
&\le -\beta_3\mu(t)E(t)-C_5E'(t),
\end{aligned}$$

where $C_5 = 2\beta_4+\frac{\beta_5C_2\mu(0)}{\gamma_0}$ is a positive constant. From $\mu(t)$ is nonincreasing, we have

$$(\mu L + C_5E)'(t)\le -\beta_3\mu(t)E(t),\quad \forall t\ge t_0.$$

Since $\mu(t)L(t)+C_5E(t)\sim E(t)$, for some positive constants k_1 and k_2, we obtain

$$E(t)\le k_2e^{-k_1\int_{t_0}^{t}\mu(s)ds}.$$

Case 2: $G(t)$ is nonlinear. This case is obtained through the ideas presented in [23] as follows. Using (2.24), (3.11), (3.16) and (3.19), we obtain

$$L'(t) \le -\beta_3 E(t) + \frac{\beta_4}{\theta}\overline{G}^{-1}\left(\frac{\theta\rho(t)}{\mu(t)}\right) - \frac{\beta_5 C_2}{\gamma_0}E'(t), \quad \forall t \ge t_0. \tag{4.1}$$

Let $L_1(t) = L(t) + \frac{\beta_5 C_2}{\gamma_0}E(t) \sim E(t)$, then (4.1) becomes

$$L_1'(t) \le -\beta_3 E(t) + \frac{\beta_4}{\theta}\overline{G}^{-1}\left(\frac{\theta\rho(t)}{\mu(t)}\right), \quad \forall t \ge t_0. \tag{4.2}$$

For $0 < \varepsilon_0 < r_0$, using (4.2) and the fact that $E' \le 0, \overline{G}' > 0$ and $\overline{G}'' > 0$, we find that the functional L_2, defined by

$$L_2(t) := \overline{G}'\left(\varepsilon_0 \frac{E(t)}{E(0)}\right)L_1(t) \sim E(t)$$

satisfies

$$L_2'(t) \le -\beta_3 E(t)\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\beta_4}{\theta}\overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)\overline{G}^{-1}\left(\frac{\theta\rho(t)}{\mu(t)}\right), \quad \forall t \ge t_0. \tag{4.3}$$

With $s = \overline{G}'\left(\varepsilon_0\frac{E(t)}{E(0)}\right)$ and $t = \overline{G}^{-1}\left(\frac{\theta\rho(t)}{\mu(t)}\right)$, using (2.13), (2.14) and (4.3), we get

$$L_2'(t) \le -\beta_3 E(t)G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\varepsilon_0\beta_4}{\theta}\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) + \frac{\beta_4\rho(t)}{\mu(t)},$$

where, we have used that $\varepsilon_0\frac{E(t)}{E(0)} < r_0$ and $\overline{G}' = G'$ on $(0, r_0]$. Multiplying this by $\mu(t)$ and using (3.18), we obtain

$$\mu(t)L_2'(t) \le -\Big(\beta_3 E(0) - \frac{\varepsilon_0\beta_4}{\theta}\Big)\frac{\mu(t)E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) - 2\beta_4 E'(t).$$

By defining $L_3(t) = \mu(t)L_2(t) + 2\beta_4 E(t)$, we see that, for some positive constants γ_2 and γ_3,

$$\gamma_2 L_3(t) \le E(t) \le \gamma_3 L_3(t). \tag{4.4}$$

With a suitable choice of ε_0, we get, for some positive constant d_2,

$$L_3'(t) \le -d_2\mu(t)\frac{E(t)}{E(0)}G'\left(\varepsilon_0\frac{E(t)}{E(0)}\right) = -d_2\mu(t)G_2\left(\frac{E(t)}{E(0)}\right), \quad \forall t \ge t_0, \tag{4.5}$$

where $G_2(t) = tG'(\varepsilon_0 t)$. Using the strict convexity of G on $(0, r_0]$ and $G_2'(t) = G'(\varepsilon_0 t) + \varepsilon_0 tG''(\varepsilon_0 t)$, we see that $G_2(t), G_2'(t) > 0$ on $(0, 1]$. Finally, defining

$$Q(t) = \frac{\gamma_2 L_3(t)}{E(0)}$$

and using (4.4), we have

$$Q(t) \le \frac{E(t)}{E(0)} \le 1 \ \text{ and } Q(t) \sim E(t). \tag{4.6}$$

From (4.5), (4.6) and the fact that $G_2'(t) > 0$ on $(0, 1]$, we arrive at

$$Q'(t) \le -k_3\mu(t)G_2(Q(t)), \quad \forall t \ge t_0,$$

where $k_3 = \frac{d_2\gamma_2}{E(0)}$ is a positive constant. Integrating this over (t_0, t) and using variable transformation, we find that(see details in [23])

$$\int_t^{t_0}\frac{\varepsilon_0 Q'(s)}{\varepsilon_0 Q(s)G'(\varepsilon_0 Q(s))}ds \ge k_3\int_{t_0}^t \mu(s)ds \Longrightarrow \int_{\varepsilon_0 Q(t)}^{\varepsilon_0 Q(t_0)}\frac{1}{sG'(s)}ds \ge k_3\int_{t_0}^t \mu(s)ds.$$

Since $\varepsilon_0 < r_0$ and $Q(t) \le 1$, for all $t \ge t_0$, we have

$$G_1(\varepsilon_0 Q(t)) = \int_{\varepsilon_0 Q(t)}^{r_0} \frac{1}{sG'(s)} ds \ge k_3 \int_{t_0}^{t} \mu(s)ds \Longrightarrow Q(t) \le \frac{1}{\varepsilon_0} G_1^{-1}\Big(k_3 \int_{t_0}^{t} \mu(s)ds\Big), \tag{4.7}$$

where $G_1(t) = \int_t^{r_0} \frac{1}{sG'(s)} ds$. Here, we have used the fact that G_1 is strictly decreasing function on $(0, r_0]$. Therefore, using (4.6) and (4.7), the estimate (2.33) is established.

5. Proof of Theorem 2.3

Case 1: $G(t)$ is linear. Multiplying (3.11) by the positive nonincreasing function $\mu(t)$ and using (2.3), (2.24) and (3.16), we get

$$\mu(t)L'(t) \le -\beta_3\mu(t)E(t) + \beta_5 C_3 \mu(t) F^{-1}(\chi(t)) - C_6 E'(t), \tag{5.1}$$

where $C_6 = 2\beta_4 + \beta_5 C_3 \mu(0)$ is a positive constant. Since $\mu(t)$ is nonincreasing, (5.1) becomes

$$F_3'(t) \le -\beta_3\mu(t)E(t) + \beta_5 C_3 \mu(t) F^{-1}(\chi(t)), \quad \forall t \ge t_0, \tag{5.2}$$

where $F_3(t) = \mu(t)L(t) + C_6 E(t) \sim E(t)$. For $0 < \varepsilon_1 < r_1$, using (5.2) and the fact that $E' \le 0, F' > 0$ and $F'' > 0$ on $(0, r_1]$, the functional F_4 defined by

$$F_4(t) := F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) F_3(t) \sim E(t)$$

satisfies

$$F_4'(t) \le -\beta_3\mu(t)E(t)F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) + \beta_5 C_3 \mu(t) F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) F^{-1}(\chi(t)).$$

As (2.13) and (2.14) with $s = F'\big(\varepsilon_1 \frac{E(t)}{E(0)}\big)$ and $t = F^{-1}(\chi(t))$, using (3.17), we obtain that

$$\begin{aligned} F_4'(t) &\le -\beta_3\mu(t)E(t)F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) + \varepsilon_1\beta_5 C_3 \frac{\mu(t)E(t)}{E(0)} F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) + \beta_5 C_3 \mu(0)\chi(t) \\ &\le -\big(\beta_3 E(0) - \varepsilon_1 \beta_5 C_3\big) \frac{\mu(t)E(t)}{E(0)} F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) - \beta_5 C_3 C_4 \mu(0) E'(t), \quad \forall t \ge t_0. \end{aligned}$$

Let $F_5(t) = F_4(t) + \beta_5 C_3 C_4 \mu(0) E(t)$, then which satisfies, for positive constants γ_4 and γ_5,

$$\gamma_4 F_5(t) \le E(t) \le \gamma_5 F_5(t). \tag{5.3}$$

Consequently, with a suitable choice of ε_1, we have, for some positive constant d_3,

$$F_5'(t) \le -d_3\mu(t)\frac{E(t)}{E(0)} F'\Big(\varepsilon_1 \frac{E(t)}{E(0)}\Big) = -d_3 \mu(t) F_0\Big(\frac{E(t)}{E(0)}\Big), \quad \forall t \ge t_0, \tag{5.4}$$

where $F_0(t) = tF'(\varepsilon_1 t)$. From the strict convexity of F on $(0, r_1]$, we obtain $F_0(t), F_0'(t) > 0$ on $(0, 1]$. Let

$$J(t) = \frac{\gamma_4 F_5(t)}{E(0)},$$

and from (5.3) and (5.4), we get

$$J(t) \le \frac{E(t)}{E(0)} \le 1 \quad \text{and} \quad J'(t) \le -\alpha_1 \mu(t) F_0(J(t)), \quad \forall t \ge t_0,$$

where $\alpha_1 = \frac{d_3\gamma_4}{E(0)}$ is a positive constant. Then, similar to (4.7), the integration over (t_0, t) and variable transformation yield

$$J(t) \leq \frac{1}{\varepsilon_1} F_1^{-1}\Big(\alpha_1 \int_{t_0}^{t} \mu(s)ds\Big), \tag{5.5}$$

where $F_1(t) = \int_t^{r_1} \frac{1}{sF'(s)}ds$, which is strictly decreasing function on $(0, r_1]$. Combining (5.3) and (5.5), the estimate (2.34) is proved.

Case 2: $G(t)$ is nonlinear. This case is obtained by the ideas presented in [21] as follows. Using (3.11), (3.16) and (3.19), we obtain

$$L'(t) \leq -\beta_3 E(t) + \frac{\beta_4(t-t_0)}{\theta}\overline{G}^{-1}\left(\frac{\theta\rho(t)}{(t-t_0)\mu(t)}\right) + \beta_5 C_3 F^{-1}(\chi(t)) - \beta_5 C_3 E'(t), \quad \forall t > t_0. \tag{5.6}$$

Since $\lim\limits_{t\to\infty} \dfrac{1}{t-t_0} = 0$, there exists $t_1 > t_0$ such that

$$\frac{1}{t-t_0} < 1, \quad \forall t \geq t_1. \tag{5.7}$$

Using the strictly increasing and strictly convex function of $\overline{F}$ and (3.22) with $q = \frac{1}{t-t_0}$, we see that

$$\overline{F}^{-1}(\chi(t)) \leq (t-t_0)\overline{F}^{-1}\left(\frac{\chi(t)}{t-t_0}\right), \quad \forall t \geq t_1. \tag{5.8}$$

Combining (5.6) and (5.8), we arrive at

$$R_1'(t) \leq -\beta_3 E(t) + \frac{\beta_4(t-t_0)}{\theta}\overline{G}^{-1}\left(\frac{\theta\rho(t)}{(t-t_0)\mu(t)}\right) + \beta_5 C_3 (t-t_0)\overline{F}^{-1}\left(\frac{\chi(t)}{t-t_0}\right), \quad \forall t \geq t_1, \tag{5.9}$$

where $R_1(t) = L(t) + \beta_5 C_3 E(t) \sim E(t)$. Let

$$r_2 = \min\{r_0, r_1\}, \quad \varphi(t) = \max\left\{\frac{\theta\rho(t)}{(t-t_0)\mu(t)}, \frac{\chi(t)}{t-t_0}\right\} \text{ and } K = \left(\overline{G}^{-1} + \overline{F}^{-1}\right)^{-1}, \quad \forall t \geq t_1. \tag{5.10}$$

So, (5.9) reduces to

$$R_1'(t) \leq -\beta_3 E(t) + C_7(t-t_0)K^{-1}(\varphi(t)), \quad \forall t \geq t_1, \tag{5.11}$$

where $C_7 = \max\{\frac{\beta_4}{\theta}, \ \beta_5 C_3\}$. The strictly increasing and strictly convex properties of $\overline{G}$ and $\overline{F}$ imply that

$$K' = \frac{\overline{G}'\overline{F}'}{\overline{G}' + \overline{F}'} > 0 \ \text{ and } \ K'' = \frac{\overline{G}''(\overline{F}')^2 + (\overline{G}')^2\overline{F}''}{(\overline{G}' + \overline{F}')^2} > 0, \tag{5.12}$$

on $(0, r_2]$. Now, for $0 < \varepsilon_2 < r_2$, using (5.7), we see that $\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)} < r_2$. Defining

$$R_2(t) = K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)R_1(t), \quad \forall t \geq t_1,$$

and using (5.11) and (5.12), we find that

$$\begin{aligned} R_2'(t) &= \left(-\frac{\varepsilon_2}{(t-t_0)^2}\frac{E(t)}{E(0)} + \frac{\varepsilon_2}{t-t_0}\frac{E'(t)}{E(0)}\right)K''\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)R_1(t) + K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)R_1'(t) \\ &\leq -\beta_3 E(t) K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right) + C_7(t-t_0)K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)K^{-1}(\varphi(t)), \quad \forall t \geq t_1. \end{aligned} \tag{5.13}$$

Using (2.13) and (2.14) with $s = K'\big(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\big)$ and $t = K^{-1}(\varphi(t))$ and applying (5.13), we get

$$R_2'(t) \leq -\beta_3 E(t) K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right) + \varepsilon_2 C_7 \frac{E(t)}{E(0)} K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right) + C_7(t-t_0)\varphi(t). \tag{5.14}$$

From (3.17), (3.18) and (5.10), we obtain

$$(t-t_0)\mu(t)\varphi(t) \le -C_8 E'(t), \tag{5.15}$$

where $C_8 = \min\{2\theta,\ C_4\mu(0)\}$. Multiplying (5.14) by the positive nonincreasing function $\mu(t)$ and using (5.15), we have

$$R_3'(t) \le -\Big(\beta_3 E(0) - \varepsilon_2 C_7\Big)\frac{\mu(t)E(t)}{E(0)} K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right), \quad \forall t \ge t_1,$$

where $R_3(t) = \mu(t)R_2(t) + C_7C_8E(t) \sim E(t)$. For a suitable choice of ε_2, we find that

$$R_3'(t) \le -d_4\frac{\mu(t)E(t)}{E(0)} K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right), \quad \forall t \ge t_1, \tag{5.16}$$

where d_4 is a positive constant. An integration of (5.16) yields

$$\frac{d_4}{E(0)}\int_{t_1}^{t} E(s)K'\left(\frac{\varepsilon_2}{s-t_0}\frac{E(s)}{E(0)}\right)\mu(s)ds \le \int_t^{t_1} R_3'(s)ds \le R_3(t_1).$$

Using (5.12) and the non-increasing property of E, we see that the map $t \to E(t)K'\big(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\big)$ is non-increasing and consequently, we obtain

$$d_4\frac{E(t)}{E(0)}K'\left(\frac{\varepsilon_2}{t-t_0}\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}\mu(s)ds \le R_3(t_1), \quad \forall t \ge t_1. \tag{5.17}$$

Multiplying (5.17) by $\frac{1}{t-t_0}$, we get

$$d_4K_1\left(\frac{1}{t-t_0}\frac{E(t)}{E(0)}\right)\int_{t_1}^{t}\mu(s)ds \le \frac{R_3(t_1)}{t-t_0}, \quad \forall t \ge t_1,$$

where $K_1(s) = sK'(\varepsilon_2 s)$ which is strictly increasing. Therefore, we deduce that

$$E(t) \le \alpha_4(t-t_0)K_1^{-1}\left(\frac{\alpha_3}{(t-t_0)\int_{t_1}^{t}\mu(s)ds}\right), \quad \forall t \ge t_1,$$

where α_3 and α_4 are positive constants. This completes the proof.

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Conflict of interest statement

This work does not have any conflicts of interest.

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