A neutral fractional Halanay inequality and application to a Cohen-Grossberg neural network system

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Abstract

We extend the well-known Halanay inequality to the fractional order case in presence of distributed delays and delays of neutral type (in the fractional derivative). Both the discrete and distributed neutral delays are investigated. It is proved that solutions decay toward zero in a Mittag-Leffler manner under some rather general conditions. Some large classes of kernels and examples satisfying our assumptions are provided. We apply our findings to prove Mittag-Leffler stability for solutions of fractional neutral network systems of Cohen-Grossberg type.

Key Words and Phrases: Caputo fractional derivative, Cohen-Grossberg neural network system, Halanay inequality, Mittag-Leffler stability, neutral delay.

1 Introduction

In this work we extend an inequality from the integer order case to the fractional order case and taking into account discrete as well as distributed neutral delays. It is an inequality named after Halanay.

Lemma 1: Assume that $w(t)$ is a nonnegative solution of

$$w'(t) \leq -Aw(t) + B \sup_{t-\tau \leq s \leq t} w(s),\ t \geq a.$$

If $0 < B < A$, then there exist $M > 0$ and $\alpha > 0$ such that

$$w(t) \leq Me^{-\alpha(t-a)},\ t \geq a.$$

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It was used first by Halanay [10] to study the stability of the equation

$$v'(t) = -Av(t) + Bv(t-\tau),\ \tau > 0.$$

A generalization of this inequality to variable coefficients and variable (bounded and unbounded) delay exist in the literature [1] and [24,25]. In particular, they have been used in the investigation of some Volterra functional equations and also Hopfield Neural Network problems of the form

$$\begin{cases} x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau)) + I_i,\ t > 0, \\ x_i(t) = \phi_i(t),\ -\tau \le t \le 0, \end{cases}$$

with $i = 1, ..., n$, (see [15,26] and also [12] for another problem). These systems have been applied in many areas such as combinatorial optimization, cryptography, parallel computing, signal theory, image processing, biological, biomedical, medical (epidemiology), polymer composite and geology [14,16,17].

Furthermore, Halanay inequality has been generalized to the distributed delays case in [20]

$$w'(t) \le -A(t)w(t) + B(t)\int_0^\infty k(s)w(t-s)\,ds,\ t \ge 0.$$

The authors proved an exponential decay of solutions for kernels satisfying

$$\int_0^\infty e^{\beta s} k(s) ds < \infty$$

for some $\beta > 0$, provided that

$$B(t)\int_0^\infty k(s)\,ds \le A(t) - b,\ b > 0,\ t \in R.$$

See also [21].

In the present work, we extend the Halanay inequality from the integer (first order) case to the fractional order case. The justification of the use of fractional derivatives is traced back to [2,3]. The second feature is the consideration of delays of neutral type. That is, delays occurring in the highest derivative. Namely, we discuss the stability of the problem

$$\begin{cases} D_C^\alpha [w(t) - pw(t-\nu)] \le -aw(t) + \int_0^t k(t-s)w(s)\,ds,\ 0 < \alpha < 1,\ t, \nu, p > 0, \\ w(t) = \varphi(t),\ t \in [-\nu, 0]. \end{cases} \tag{1}$$

We find sufficient conditions on the delay kernel k ensuring the stability of the solutions. The obtained stability is of Mittag-Leffler type

$$w(t) \le AE_\alpha(-at^\alpha),\ t > 0.$$

Some examples of classes of functions satisfying our conditions are provided. Next, this result is applied to a problem arising in neural network theory. Namely, we discuss a Cohen-Grossberg neural network system [9] (which is a generalization of the corresponding Hopfield neural network system) which is of fractional order and with a delay of neutral type.

Fairly, there is a huge amount of work on the existence, asymptotic behavior and stability (finite time or not and with or without decay rates) for Cohen–Grossberg neural network systems. We report here only some of those treating the neutral delay case and/or the fractional case. For neutral Cohen-Grossberg systems of integer order type (first order) we refer the reader to [23,5,7]. The fractional case of Cohen-Grossberg systems with discrete delays has been studied in [13]. We mention here that the Halanay inequality has been already extended to the fractional case but only for discrete delays in [4,11]. We are not aware, however, of any work related to our problem (1).

The previous arguments in the literature which work in the integer order case are no longer valid in the fractional order case. For instance, the lack of the semi-group property for the Mittag-Leffler functions does not facilitate the study. The absence of an explicit appropriate estimation of the expression $E_\alpha(-a(t-\nu)^\alpha)/E_\alpha(-at^\alpha)$ is problematic for the convergence of the obtained series. Ideally, the rate would be of the form $E_\alpha(-at^\alpha)$ as is the case without the neutral delay but here this kind of delay brings some new challenges. In particular, we face difficulties estimating the delayed term $w(t-\nu)$ nearby ν. Passing rather to $t^{-\alpha}$ (using Mainardi's conjecture which says roughly that $E_\alpha(-at^\alpha)$ is between two functions of the form $t^{-\alpha}$) will not solve the problem completely.

The plan of the paper is as follows: Section 2 contains some preliminaries while Section 3 contains our main result on the fractional Halanay inequality with discrete neutral delay. In addition we give two possible classes of kernels in Section 4. Section 5 is devoted to a fractional Halanay inequality with distributed neutral delay. Section 6 contains a result on the problem studied in Section 3 but for solutions of arbitrary signs. An application to a Cohen-Grossberg system with delays of neutral type which arises in neural network theory is provided in Section 7. We conclude in Section 8.

2 Preliminaries

In this section we define the fractional derivative considered here and give some useful lemmas used in the proof of our results.

Definition 2: The Riemann-Liouville fractional integral of order $\alpha > 0$

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \ \alpha > 0$$

for any measurable function f provided that the right hand side exists. Here $\Gamma(\alpha)$ is the usual Gamma function.

Definition 3: The fractional derivative of order α in the sense of Caputo is defined by

$$D_C^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)ds, \ 0 < \alpha < 1$$

provided that the integral exists.

The one-parametric and two-parametric Mittag-Leffler functions are defined by

$$E_\alpha(z) := \sum\nolimits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \ \mathrm{Re}(\alpha) > 0,$$

and

$$E_{\alpha,\beta}(z) := \sum\nolimits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \ \mathrm{Re}(\alpha) > 0, \ \mathrm{Re}(\beta) > 0$$

resp. Notice that $E_{\alpha,1}(z) \equiv E_\alpha(z)$.

Lemma 4: [19] For $\nu, \mu, b > 0$, we have

$$z^{1-\nu} \int_0^z (z-\xi)^{\nu-1} \xi^{\mu-1} e^{-b\xi} d\xi \leq b^{-\mu} \max\left(1, 2^{1-\nu}\right) \Gamma(\mu) \left(1 + \frac{\mu}{\nu}\right), \ z > 0.$$

For $\sigma, \gamma, \beta > 0$, we have [8]

$$I^{\sigma} t^{\gamma-1} E_{\alpha,\beta}(at^{\beta})(x) = x^{\sigma+\gamma-1} E_{\beta,\sigma+\gamma}(ax^{\beta}). \tag{2}$$

Mainardi's conjecture [18]**:** For all $t > 0$ and fixed α, $0 < \alpha < 1$, we have

$$\frac{1}{1 + a\Gamma(1-\alpha)t^{\alpha}} \leq E_{\alpha}(-at^{\alpha}) \leq \frac{1}{1 + a\Gamma(1+\alpha)^{-1}t^{\alpha}}, \ t \geq 0. \tag{3}$$

has been proved later in [6] and in [22].

3 Fractional Distributed Halanay Inequality with discrete neutral delay

We start by clarifying what we mean by Mittag-Leffler stability

Definition 5: If 0 is an equilibrium, then the solution $u(t)$ is said to be α-Mittag-Leffler stable ($0 < \alpha < 1$) if there exist two positive constants M and λ such that

$$\|u(t)\| \leq M E_{\alpha}(-\lambda t^{\alpha}), \ t > 0$$

for a certain norm $\|.\|$.

Theorem 6: Let $u(t)$ be a nonnegative solution of

$$\begin{cases} D_C^{\alpha}\left[u(t) - pu(t-\nu)\right] \leq -au(t) + \int_0^t k(t-s)u(s)\, ds, \ 0 < \alpha < 1, \ t, \nu, p > 0 \\ u(t) = \varphi(t) \geq 0, \ t \in [-\nu, 0]. \end{cases} \tag{4}$$

with $a > 0$ and k a nonnegative summable function. If $p > 0$ and k are such that the relation

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^{\alpha}) \left(\int_0^s E_{\alpha}(-a\sigma^{\alpha}) k(s-\sigma) d\sigma \right) ds \leq M E_{\alpha}(-at^{\alpha}), \ t > 0 \tag{5}$$

holds for some $M > 0$ with

$$M < 1 - \left(\frac{1}{a\nu^{\alpha}} + 2^{\alpha}\Gamma(1-\alpha) \right) p \tag{6}$$

and

$$\left(\frac{1}{a\nu^{\alpha}} + 2^{\alpha}\Gamma(1-\alpha) \right) p < 1, \tag{7}$$

then there exists a constant $C > 0$ such that

$$w(t) \leq C E_{\alpha}(-at^{\alpha}), \ t > 0.$$

Proof: We shall compare solutions of (4) with those of

$$\begin{cases} D_C^{\alpha}\left[w(t) - pw(t-\nu)\right] = -aw(t) + \int_0^t k(t-s)w(s)\, ds, \ 0 < \alpha < 1, \ t > 0 \\ w(t) = \varphi(t) \geq 0, \ t \in [-\nu, 0]. \end{cases} \tag{8}$$

Writing the equation in (8) in the form

$$D_C^{\alpha}\left[w(t)-pw(t-\nu)\right]=-a\left[w(t)-pw(t-\nu)\right]-apw(t-\nu)+\int_0^t k(t-s)w(s)\,ds,\ t>0$$

allows us to profit from the formulation

$$\begin{gathered}w(t)-pw(t-\nu)=E_\alpha(-at^\alpha)\left[\varphi(0)-p\varphi(-\nu)\right]\\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(-apw(s-\nu)+\int_0^s k(s-\sigma)w(\sigma)d\sigma\right)\,ds,\ t>0.\end{gathered}$$

As the solution is nonnegative (with nonnegative history), we find

$$\begin{gathered}w(t)\le\varphi(0)E_\alpha(-at^\alpha)+pw(t-\nu)\\ +\left[\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)w(\sigma)d\sigma\right)\,ds\right],\ t>0.\end{gathered}\tag{9}$$

Therefore

$$\begin{gathered}\tfrac{w(t)}{E_\alpha(-at^\alpha)}\le\varphi(0)+\tfrac{p}{E_\alpha(-at^\alpha)}w(t-\nu)\\ +\tfrac{1}{E_\alpha(-at^\alpha)}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)E_\alpha(-a\sigma^\alpha)\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}d\sigma\right)\,ds,\ t>0\end{gathered}$$

and

$$\begin{gathered}\tfrac{w(t)}{E_\alpha(-at^\alpha)}\le\varphi(0)+\tfrac{p}{E_\alpha(-at^\alpha)}w(t-\nu)\\ +\tfrac{1}{E_\alpha(-at^\alpha)}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)E_\alpha(-a\sigma^\alpha)d\sigma\right)\,ds\,\sup_{0\le\sigma\le t}\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}\\ \le\varphi(0)+\tfrac{p}{E_\alpha(-at^\alpha)}w(t-\nu)+M\sup_{0\le\sigma\le t}\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)},\ t>0.\end{gathered}$$

We shall use repeatedly the estimation

$$\begin{gathered}\tfrac{1}{E_\alpha(-at^\alpha)}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)w(\sigma)d\sigma\right)\,ds\\ \le\tfrac{1}{E_\alpha(-at^\alpha)}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)E_\alpha(-a\sigma^\alpha)\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}d\sigma\right)\,ds\\ \le M\sup_{0\le\sigma\le t}\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)},\ t>0\end{gathered}\tag{10}$$

and

$$\frac{w(t)}{E_\alpha(-at^\alpha)}\le\varphi(0)+\frac{p}{E_\alpha(-at^\alpha)}w(t-\nu)+M\sup_{0\le\sigma\le t}\frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)},\ t>0.\tag{11}$$

will be our first reference inequality.

For $t\in[0,\nu]$, by the decreasingness of $E_\alpha(-at^\alpha)$, we have $E_\alpha(-at^\alpha)\ge E_\alpha(-a\nu^\alpha)$ and therefore

$$\frac{w(t)}{E_\alpha(-at^\alpha)}\le\left(1+\frac{p}{E_\alpha(-a\nu^\alpha)}\right)\sup_{-\nu\le\sigma\le0}\varphi(\sigma)+M\sup_{0\le\sigma\le t}\frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}$$

or

$$(1-M)\frac{w(t)}{E_\alpha(-at^\alpha)}\le\left(1+\frac{p}{E_\alpha(-a\nu^\alpha)}\right)\sup_{-\nu\le\sigma\le0}\varphi(\sigma).\tag{12}$$

For $t\in[\nu,2\nu]$, from our reference relation (11) and (12), we find

$$\begin{gathered}\tfrac{w(t)}{E_\alpha(-at^\alpha)}\le\sup_{-\nu\le\sigma\le0}\varphi(\sigma)+\tfrac{p}{1-M}\left(1+\tfrac{p}{E_\alpha(-a\nu^\alpha)}\right)\tfrac{E_\alpha(-a(t-\nu)^\alpha)}{E_\alpha(-at^\alpha)}\sup_{-\nu\le\sigma\le0}\varphi(\sigma)\\ +M\sup_{0\le\sigma\le t}\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}.\end{gathered}$$

Here, the expression $E_\alpha(-a(t-\nu)^\alpha)/E_\alpha(-at^\alpha)$ is dealt with as follows:

$$\frac{E_\alpha(-a(t-\nu)^\alpha)}{E_\alpha(-at^\alpha)} \leq \frac{1}{E_\alpha(-at^\alpha)} \leq \frac{1}{E_\alpha(-a\,(2\nu)^\alpha)} \leq 1 + a\Gamma(1-\alpha)\,(2\nu)^\alpha =: A. \tag{13}$$

Therefore

$$\frac{w(t)}{E_\alpha(-at^\alpha)} \leq \left[1 + \frac{A\,(E_\alpha(-a\nu^\alpha)+p)\,p}{(1-M)\,E_\alpha(-a\nu^\alpha)}\right] \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma) + M \sup_{0\leq\sigma\leq t}\frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}$$

or (passing to the sup)

$$(1-M)\,\frac{w(t)}{E_\alpha(-at^\alpha)} \leq \left[1 + \frac{AE_\alpha(-a\nu^\alpha)}{(1-M)\,E_\alpha(-a\nu^\alpha)}p + \frac{A}{(1-M)\,E_\alpha(-a\nu^\alpha)}p^2\right] \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma). \tag{14}$$

We will write (14) in the more convenient way

$$(1-M)\,\frac{w(t)}{E_\alpha(-at^\alpha)} \leq \frac{A}{E_\alpha(-a\nu^\alpha)}\left[1 + \frac{p}{1-M} + \left(\frac{p}{1-M}\right)^2\right] \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma). \tag{15}$$

For $t \in [2\nu, 3\nu]$, in view of the estimations $\frac{t^\alpha}{(t-\nu)^\alpha} \leq 2^\alpha$ and the relations (3), we infer the new estimation for $E_\alpha(-a(t-\nu)^\alpha)/E_\alpha(-at^\alpha)$

$$\begin{aligned} \tfrac{E_\alpha(-a(t-\nu)^\alpha)}{E_\alpha(-at^\alpha)} &\leq \tfrac{1+a\Gamma(1-\alpha)t^\alpha}{1+a\Gamma(1+\alpha)^{-1}(t-\nu)^\alpha} \leq \tfrac{\Gamma(1+\alpha)}{a\nu^\alpha} + \tfrac{\Gamma(1+\alpha)\Gamma(1-\alpha)t^\alpha}{(t-\nu)^\alpha} \\ &\leq \Gamma(1+\alpha)\left[\tfrac{1}{a\nu^\alpha} + 2^\alpha\Gamma(1-\alpha)\right],\ t \geq 2\nu. \end{aligned} \tag{16}$$

Notice that, as $\Gamma(1+\alpha)$ is very close to (and below) 1, we may ignore it.

Back to our reference inequality (11), taking into account (15) and (16), we may write

$$\begin{aligned} \tfrac{w(t)}{E_\alpha(-at^\alpha)} \leq \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma) + \tfrac{p}{1-M}\tfrac{E_\alpha(-a(t-\nu)^\alpha)}{E_\alpha(-at^\alpha)}\tfrac{A}{E_\alpha(-a\nu^\alpha)}\left[1 + \tfrac{p}{1-M} + \left(\tfrac{p}{1-M}\right)^2\right]\sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma) \\ +M\sup_{0\leq\sigma\leq t}\tfrac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)} \end{aligned}$$

or

$$(1-M)\,\frac{w(t)}{E_\alpha(-at^\alpha)} \leq \left\{1 + \frac{p}{1-M}\frac{AV}{E_\alpha(-a\nu^\alpha)}\left[1 + \frac{p}{1-M} + \left(\frac{p}{1-M}\right)^2\right]\right\} \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma) \tag{17}$$

where

$$V := \frac{1}{a\nu^\alpha} + 2^\alpha\Gamma(1-\alpha).$$

As $\frac{AV}{E_\alpha(-a\nu^\alpha)} > 1$, we may rewrite (17) in the form

$$(1-M)\,\frac{w(t)}{E_\alpha(-at^\alpha)} \leq \frac{A}{E_\alpha(-a\nu^\alpha)}\left\{1 + \frac{pV}{1-M} + \left(\frac{pV}{1-M}\right)^2 + \left(\frac{p}{1-M}\right)^3\right\} \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma).$$

Let us prove the following claim.

Claim: For $t \in [(n-1)\nu, n\nu]$,

$$(1-M)\,\frac{w(t)}{E_\alpha(-at^\alpha)} \leq \frac{A}{E_\alpha(-a\nu^\alpha)}\sum_{k=0}^{n}\left(\frac{pV}{1-M}\right)^k \sup_{-\nu\leq\sigma\leq 0}\varphi(\sigma).$$

This claim will be proved by recurrence. Clearly, from the above findings, it holds for $n=1$, 2 and 3.

Assume that the assertion is true for n *i.e.* on $[(n-1)\nu, n\nu]$. Let $t\in[n\nu,(n+1)\nu]$, then our reference inequality (11) and the fact that $t-\nu\in[(n-1)\nu,n\nu]$, yield

$$\begin{array}{c}\frac{w(t)}{E_\alpha(-at^\alpha)}\le \sup_{-\nu\le\sigma\le 0}\varphi(\sigma)+\frac{pE_\alpha(-a(t-\nu)^\alpha)}{(1-M)E_\alpha(-at^\alpha)}\frac{A}{E_\alpha(-a\nu^\alpha)}\sum_{k=0}^{n}\left(\frac{pV}{1-M}\right)^k\sup_{-\nu\le\sigma\le 0}\varphi(\sigma)\\ +M\sup_{0\le\sigma\le t}\frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}\end{array}$$

and by (16)

$$\begin{array}{c}(1-M)\,\frac{w(t)}{E_\alpha(-at^\alpha)}\le\left[1+\frac{pV}{1-M}\sum_{k=0}^{n}\left(\frac{pV}{1-M}\right)^k\right]\sup_{-\nu\le\sigma\le 0}\varphi(\sigma)\\ \le\frac{A}{E_\alpha(-a\nu^\alpha)}\left[1+\sum_{k=1}^{n+1}\left(\frac{pV}{1-M}\right)^k\right]\sup_{-\nu\le\sigma\le 0}\varphi(\sigma)=\frac{A}{E_\alpha(-a\nu^\alpha)}\sum_{k=0}^{n+1}\left(\frac{pV}{1-M}\right)^k\sup_{-\nu\le\sigma\le 0}\varphi(\sigma).\end{array}$$

The claim is proved. Therefore

$$w(t)\le\left[\frac{A}{(1-M)\,E_\alpha(-a\nu^\alpha)}\sum_{k=0}^{\infty}\left(\frac{pV}{1-M}\right)^k\sup_{-\nu\le\sigma\le 0}\varphi(\sigma)\right]E_\alpha(-at^\alpha),\ t>0. \tag{18}$$

Our assumptions (6) and (7) guarantee the convergence of the series in (18). The proof is complete.

4 Examples

Here we give two classes of functions for which our assumptions in the theorem apply.

First class:

We consider the set of all kernels k satisfying

$$\int_0^s E_\alpha(-a\sigma^\alpha)k(s-\sigma)d\sigma\le C_1 s^{\nu-1},\ s\ge 0,\ C_1,\nu>0. \tag{19}$$

The class of functions $k(t)\le C_2t^{-\alpha}e^{-bt}$ fulfills this condition for some b and C_2 to be determined. Indeed, as

$$E_\alpha(-at^\alpha)\le\frac{1}{1+\frac{at^\alpha}{\Gamma(1+\alpha)}}=\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)+at^\alpha}\le\frac{\Gamma(1+\alpha)}{at^\alpha},\ t>0 \tag{20}$$

we see that

$$\begin{array}{c}\int_0^s E_\alpha(-a(s-\sigma)^\alpha)k\,(\sigma)\,d\sigma\le\frac{C_2\Gamma(1+\alpha)}{a}\int_0^s(s-\sigma)^{-\alpha}\sigma^{-\alpha}e^{-b\sigma}d\sigma\\ \le\frac{2^{\alpha+1}C_2\Gamma(1+\alpha)}{a}b^{\alpha-1}\Gamma(1-\alpha)s^{-\alpha},\ s>0.\end{array}$$

Therefore (19) holds with

$$C_1:=\frac{2^{\alpha+1}C_2\Gamma(1+\alpha)}{a}b^{\alpha-1}\Gamma(1-\alpha),\ \nu:=1-\alpha.$$

Using the formula (2), we get

$$\begin{array}{c}\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s E_\alpha(-a\sigma^\alpha)k(s-\sigma)d\sigma\right)\,ds\\ \le C_1\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)s^{-\alpha}\,ds\\ \le C_1\Gamma(1-\alpha)E_{\alpha,1}(-at^\alpha).\end{array} \tag{21}$$

The assumption (6) holds if we choose C_1 (or C_2 for the specific example) such that

$$C_1\Gamma(1-\alpha) < 1 - \left(\frac{1}{a\nu^\alpha} + 2^\alpha\Gamma(1-\alpha)\right) p.$$

Second class:

Assume that $k(t) \leq C_3 t^{\alpha-1}E_{\alpha,\alpha}(-bt^\alpha)$ for some $b > 0$ and $C_3 > 0$ to be determined. Applying twice formula (2) and (20), we obtain

$$\begin{array}{l} C_3 \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,\alpha}(-b(s-\sigma)^\alpha)E_\alpha(-a\sigma^\alpha)d\sigma\right) ds \\ \leq \frac{C_3\Gamma(1+\alpha)}{a}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s (s-\sigma)^{\alpha-1}E_{\alpha,\alpha}(-b(s-\sigma)^\alpha)\sigma^{-\alpha}d\sigma\right) ds \\ \qquad \leq \frac{C_3\Gamma(1+\alpha)\Gamma(1-\alpha)}{a}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)E_{\alpha,1}(-bs^\alpha)\, ds \\ \qquad \leq C_3\frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{a}\frac{\Gamma(1+\alpha)}{b}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)s^{-\alpha}\, ds \\ \qquad\qquad \leq \frac{C_3\Gamma^2(1+\alpha)\Gamma^2(1-\alpha)}{ab}E_{\alpha,1}(-at^\alpha)\,. \end{array} \tag{22}$$

It appears that $M = \frac{C_3\Gamma^2(1+\alpha)\Gamma^2(1-\alpha)}{ab}$ and we impose the condition on C_3 or/and b so that the assumption on M in the theorem is fulfilled.

5 Fractional Distributed Halanay Inequality with distributed neutral delay

In this section we treat the case where the neutral delay is of distributed type. Namely, we consider the inequality

$$\left\{\begin{array}{l} D_C^\alpha\left[u(t) - p\int_0^t g(t-s)u(s)\, ds\right] \leq -au(t) + \int_0^t k(t-s)u(s)\, ds,\ 0<\alpha<1,\ t,\nu,p>0 \\ u(t) = u_0 \geq 0,\ t\in[-\nu,0] \end{array}\right. \tag{23}$$

which we will compare to

$$\left\{\begin{array}{l} D_C^\alpha\left[w(t) - p\int_0^t g(t-s)w(s)\, ds\right] = -aw(t) + \int_0^t k(t-s)w(s)\, ds,\ 0<\alpha<1,\ t,\nu,p>0 \\ w(t) = w_0 = u_0 \geq 0,\ t\in[-\nu,0]. \end{array}\right. \tag{24}$$

Here, the solutions are supposed to be nonnegative and g is a continuous function which we will determine below.

Let us rewrite it as

$$\left\{\begin{array}{l} D_C^\alpha\left[w(t) - p\int_0^t g(t-s)w(s)\, ds\right] = -a\left[w(t) - p\int_0^t g(t-s)w(s)\, ds\right] \\ -ap\int_0^t g(t-s)w(s)\, ds + \int_0^t k(t-s)w(s)\, ds,\ 0<\alpha<1,\ t,\nu,p>0 \\ w(t) = w_0 \geq 0,\ t\in[-\nu,0]. \end{array}\right.$$

Therefore

$$\begin{array}{c} w(t) - p\int_0^t g(t-s)w(s)\, ds = E_\alpha(-at^\alpha)w_0 \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(-ap\int_0^s g(s-\sigma)w(\sigma)\, d\sigma + \int_0^s k(s-\sigma)w(\sigma)d\sigma\right) ds \end{array}$$

and

$$\begin{array}{c} w(t) \leq E_\alpha(-at^\alpha)w_0 + p\int_0^t g(t-s)w(s)\, ds \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)w(\sigma)d\sigma\right) ds,\ t>0. \end{array} \tag{25}$$

Dividing both sides of (25) by $E_\alpha(-at^\alpha)$, we find

$$\frac{w(t)}{E_\alpha(-at^\alpha)} \le w_0 + \frac{p}{E_\alpha(-at^\alpha)} \int_0^t g(t-s)w(s)\,ds$$
$$+\frac{1}{E_\alpha(-at^\alpha)} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^\alpha) \left(\int_0^s k(s-\sigma)E_\alpha(-a\sigma^\alpha)d\sigma\right) ds \sup_{0\le\sigma\le t} \frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)}$$

or

$$\frac{w(t)}{E_\alpha(-at^\alpha)} \le w_0 + \frac{p}{E_\alpha(-at^\alpha)} \int_0^t g(t-s)E_\alpha(-as^\alpha)\frac{w(s)}{E_\alpha(-as^\alpha)}\,ds + M \sup_{0\le\sigma\le t} \frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)},\ t>0.$$

We assume that

$$\frac{p}{E_\alpha(-at^\alpha)} \int_0^t g(t-s)E_\alpha(-as^\alpha)\,ds \le M^*$$

for some $M^* > 0$. Then

$$\frac{w(t)}{E_\alpha(-at^\alpha)} \le w_0 + (M^*+M) \sup_{0\le\sigma\le t} \frac{w(\sigma)}{E_\alpha(-a\sigma^\alpha)},\ t>0.$$

and

$$w(t) \le \frac{w_0}{1-M^*-M} E_\alpha(-at^\alpha),\ t>0$$

in case

$$M^*+M<1.$$

Example: If k is as in the previous section and g is such that

$$g(t) \le C_4 t^{\alpha-1} E_{\alpha,\alpha}(-ct^\alpha)$$

for some $C_4, c > a$, then

$$\int_0^t g(t-s)E_\alpha(-as^\alpha)\,ds \le C_4 \frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{a} E_{\alpha,1}(-at^\alpha),\ t>0.$$

A value for M^* would be

$$M^* = \frac{C_4 p \Gamma(1+\alpha)\Gamma(1-\alpha)}{a}.$$

Therefore, we have proved

Theorem 7: Let $u(t)$ be a nonnegative solution of (23) with $a,p>0$ and k, g are nonnegative continuous functions such that

$$\int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^\alpha) \left(\int_0^s E_\alpha(-a\sigma^\alpha)k(s-\sigma)d\sigma\right) ds \le M E_\alpha(-at^\alpha),\ t>0,$$

$$p\int_0^t g(t-s)E_\alpha(-as^\alpha)\,ds \le M^* E_\alpha(-at^\alpha),\ t>0,$$

hold for some $M, M^* > 0$ with

$$M^*+M<1.$$

Then, there exists a constant $C>0$ such that

$$w(t) \le C E_\alpha(-at^\alpha),\ t>0.$$

6 The case of solutions of arbitrary signs

Before we go to the applications, we notice that in all previous Halanay inequality results (including the above one), it is assumed that solutions are nonnegative. This assumption is enough in applications, for instance, to neural network systems (without neutral delays). Indeed, to prove stability of the equilibrium solution, we shift this issue to the stability of zero by a change of functions. Then, we consider the absolute value of the solutions (or other appropriate norms). In the present case, the situation is different. In presence of a neutral delay, working with the norm of the difference operator $(u(t)-pu(t-\nu))$ or with the norm of the solution and move the neutral delay term to the right hand side, both are problematic. Instead, we shall discuss how to prove directly the stability of the inequality for arbitrary sign solutions. This task also is not easy because now the delays will be inside convolution integrals. The estimations in this case are more involved and tedious.

Back to

$$\begin{cases} D_C^\alpha\left[u(t)-pu(t-\nu)\right] \leq -au(t)+\int_0^t k(t-s)u(s)\,ds,\ 0<\alpha<1,\ t,\nu,p>0 \\ u(t)=\varphi(t)\geq 0,\ t\in[-\nu,0] \end{cases}$$

with $|\varphi(s)| \leq w_0 E_\alpha(-a(s+\nu)^\alpha)$ for $s\in[-\nu,0]$, $w_0>0$. To fix ideas, let us assume $0<p<1$ and consider the formulation

$$\begin{gathered} w(t)-pw(t-\nu)=E_\alpha(-at^\alpha)\left[\varphi(0)-p\varphi(-\nu)\right] \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(-apw(s-\nu)+\int_0^s k(s-\sigma)w(\sigma)d\sigma\right)\,ds,\ t>0. \end{gathered}$$

Then

$$\begin{gathered} |w(t)| \leq 2w_0E_\alpha(-at^\alpha)+p\,|w(t-\nu)| \\ +ap\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\,|w(s-\nu)|\,ds \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\,ds,\ t>0. \end{gathered} \tag{26}$$

For $t\in[0,\nu]$,

$$\begin{gathered} \tfrac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 3w_0+\tfrac{apw_0}{E_\alpha(-at^\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)E_\alpha(-as^\alpha)\,ds \\ +M\sup_{0\leq\sigma\leq t}\tfrac{|w(\sigma)|}{E_\alpha(-a\sigma^\alpha)} \end{gathered}$$

where M is as in (5). Again, as

$$\begin{gathered} \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)E_\alpha(-as^\alpha)\,ds \\ \leq \tfrac{\Gamma(1+\alpha)}{a}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)s^{-\alpha}\,ds \\ \leq \tfrac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{a}E_{\alpha,1}(-at^\alpha) \end{gathered} \tag{27}$$

we may write

$$\tfrac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p+M\sup_{0\leq\sigma\leq t}\tfrac{|w(\sigma)|}{E_\alpha(-a\sigma^\alpha)}$$

or

$$(1-M)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p. \tag{28}$$

In case $t\in[\nu,2\nu]$, we first notice that

$$\begin{aligned} |w(t-\nu)| &\leq \tfrac{3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}\tfrac{E_\alpha(-a(t-\nu)^\alpha)}{E_\alpha(-at^\alpha)}E_\alpha(-at^\alpha) \\ &\leq A\tfrac{3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}E_\alpha(-as^\alpha) \end{aligned}$$

where A is as in (13). Using the fact that

$$w_0 \le A\frac{3w_0 + w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}$$

and the relations (26) and (28), we get

$$\begin{array}{c} |w(t)| \le 2w_0 E_\alpha(-at^\alpha) + Ap\frac{3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}E_\alpha(-at^\alpha) \\ +apA\frac{3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\, E_\alpha(-as^\alpha)ds \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\, ds. \end{array}$$

Next, we apply (27), to obtain

$$\begin{array}{c} |w(t)| \le 2w_0 E_\alpha(-at^\alpha) + Ap\frac{3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}E_\alpha(-at^\alpha) \\ +apA\frac{3w_0+w_0\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M}\frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{a}E_\alpha(-at^\alpha) \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\, ds \end{array}$$

or

$$\begin{array}{c} (1-M)\,\frac{|w(t)|}{E_\alpha(-at^\alpha)} \le 2w_0 + Apw_0\left[1+\Gamma(1+\alpha)\Gamma(1-\alpha)\right]\frac{3+\Gamma(1+\alpha)\Gamma(1-\alpha)p}{1-M} \\ \le 2w_0 + \frac{3w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}p + \frac{w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^2}{1-M}p^2. \end{array} \tag{29}$$

For $t \in [2\nu, 3\nu]$, according to (16),

$$\frac{E_\alpha(-a(t-\nu)^\alpha)}{E_\alpha(-at^\alpha)} \le \frac{1}{a\nu^\alpha} + 2^\alpha\Gamma(1-\alpha) =: V > 1$$

and therefore

$$\begin{array}{c} |w(t)| \le 2w_0 E_\alpha(-at^\alpha) + \frac{pV}{1-M}\left[2w_0 + \frac{3w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}p + \frac{w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^2}{1-M}p^2\right]E_\alpha(-at^\alpha) \\ +pV\frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{1-M}\left[2w_0 + \frac{3w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}p + \frac{w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^2}{1-M}p^2\right]E_\alpha(-at^\alpha) \\ +\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\, ds. \end{array}$$

So

$$\begin{array}{c} (1-M)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \le 2w_0 + \frac{pV}{1-M}\left[2w_0 + \frac{3w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}p + \frac{w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^2}{1-M}p^2\right] \\ +pV\frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{1-M}\left[2w_0 + \frac{3w_0A[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}p + \frac{Aw_0[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^2}{1-M}p^2\right] \end{array}$$

or

$$\begin{array}{c} (1-M)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \le 2w_0 + 2w_0\frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M} + 3w_0A\frac{p^2V[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^2}{(1-M)^2} \\ +w_0A\frac{V[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]^3}{(1-M)^2}p^3. \end{array} \tag{30}$$

Writing (30) in the form

$$\begin{array}{c} (1-M)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \le 2w_0 + 2w_0\frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M} + 3w_0A\left(\frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^2 \\ +w_0A\left(\frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^3 \\ \le 3w_0A\left\{1 + \frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M} + \left(\frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^2 + \left(\frac{pV[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^3\right\} \end{array} \tag{31}$$

allows us to make the claim

Claim: On $[(n-1)\nu, n\nu]$, we have

$$\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 3A\sum\nolimits_{k=0}^{n}\left(\frac{Vp\left[1+\Gamma(1+\alpha)\Gamma(1-\alpha)\right]}{1-M}\right)^k.$$

By (28), (29) and (31), the claim is true for $n=1,2$ and 3. Let $t\in[n\nu,(n+1)\nu]$, then from (26)

$$\begin{gathered}|w(t)| \leq 2w_0E_\alpha(-at^\alpha) + 3ApV\tfrac{w_0}{1-M}\sum\nolimits_{k=0}^{n}\left(\tfrac{Vp[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^k E_\alpha(-at^\alpha)\\ +3Ap\tfrac{w_0V}{1-M}\Gamma(1+\alpha)\Gamma(1-\alpha)\sum\nolimits_{k=0}^{n}\left(\tfrac{Vp[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^k E_\alpha(-at^\alpha)\\ +\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s k(s-\sigma)\,|w(\sigma)|\,d\sigma\right)\,ds\end{gathered}$$

or

$$\begin{gathered}\left(\tfrac{1-M}{w_0}\right)\tfrac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 2+\tfrac{3ApV}{1-M}\sum\nolimits_{k=0}^{n}\left(\tfrac{Vp[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^k\\ +\tfrac{3ApV}{1-M}\Gamma(1+\alpha)\Gamma(1-\alpha)\sum\nolimits_{k=0}^{n}\left(\tfrac{Vp[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^k.\end{gathered}$$

Then

$$\begin{gathered}\left(\tfrac{1-M}{w_0}\right)\tfrac{|w(t)|}{E_\alpha(-at^\alpha)}\\ \leq 3A\left\{1+[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]\tfrac{pV}{1-M}\sum\nolimits_{k=0}^{n}\left(\tfrac{Vp[1+\Gamma(1+\alpha)\Gamma(1-\alpha)]}{1-M}\right)^k\right\}\end{gathered}$$

i.e.

$$\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 3A\left\{1+\sum\nolimits_{k=0}^{n}\left(\frac{Vp\left[1+\Gamma(1+\alpha)\Gamma(1-\alpha)\right]}{1-M}\right)^{k+1}\right\}.$$

Thus

$$\left(\frac{1-M}{w_0}\right)\frac{|w(t)|}{E_\alpha(-at^\alpha)} \leq 3A\sum\nolimits_{k=0}^{n+1}\left(\frac{Vp\left[1+\Gamma(1+\alpha)\Gamma(1-\alpha)\right]}{1-M}\right)^k$$

and the claim is proved. The series is convergent if

$$\frac{1+\Gamma(1+\alpha)\Gamma(1-\alpha)}{1-M}Vp<1.$$

We just proved the following result

Theorem 8: Let $u(t)$ be a solution of

$$\begin{cases} D_C^\alpha\left[u(t)-pu(t-\nu)\right] \leq -au(t)+\int_0^t k(t-s)u(s)\,ds,\ 0<\alpha<1,\ t,\nu,p>0\\ u(t)=\varphi(t),\ t\in[-\nu,0].\end{cases}$$

with $|\varphi(t)|\leq E_\alpha(-a(t+\nu)^\alpha)$, $t\in[-\nu,0]$, $a>0$, $p>0$ and k is a nonnegative function such that

$$\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)\left(\int_0^s E_\alpha(-a\sigma^\alpha)k(s-\sigma)d\sigma\right)ds \leq ME_\alpha(-at^\alpha),\ t>0$$

holds for some M such that

$$M<1-\left[1+\Gamma(1+\alpha)\Gamma(1-\alpha)\right]Vp$$

with

$$\left[1+\Gamma(1+\alpha)\Gamma(1-\alpha)\right]Vp<1.$$

Then, there exists a constant $C>0$ such that

$$|w(t)|\leq CE_\alpha(-at^\alpha),\ t>0.$$

7 Application in neural network theory

An important component of artificial intelligence is neural network systems. They are used extensively nowadays to solve complex problems in different areas. Here we apply our findings to neural network systems of Cohen-Grossberg type. Namely, we consider the problems

$$\begin{cases} D_C^\alpha \left[x_i(t) - p x_i(t-\nu)\right] = -h_i\left(x_i\left(t\right)\right) \left[g_i\left(x_i\left(t\right)\right) - \sum\limits_{j=1}^n a_{ij} f_j\left(x_j\left(t\right)\right) - \sum\limits_{j=1}^n b_{ij} l_j\left(x_j\left(t-\tau\right)\right) \right. \\ \left. - \sum\limits_{j=1}^n d_{ij} \int_0^\infty k_j\left(s\right) \varphi_j\left(x_j\left(t-s\right)\right) ds - I_i \right], \quad t, p, \nu > 0, \\ x_i\left(t\right) = x_{i0}\left(t\right), \; t \in [-\nu, 0], \; i = 1, 2, ..., n, \end{cases}$$

and

$$\begin{cases} D_C^\alpha \left[x_i(t) - p\int_0^t \psi_i(t-s) x_i(s) ds\right] = -h_i\left(x_i\left(t\right)\right) \left[g_i\left(x_i\left(t\right)\right) - \sum\limits_{j=1}^n a_{ij} f_j\left(x_j\left(t\right)\right) \right. \\ \left. - \sum\limits_{j=1}^n b_{ij} l_j\left(x_j\left(t-\tau\right)\right) - \sum\limits_{j=1}^n d_{ij} \int_0^\infty k_j\left(s\right) \varphi_j\left(x_j\left(t-s\right)\right) ds - I_i \right], \quad t, p > 0, \\ x_i\left(0\right) = x_{i0}\left(t\right), \; t \leq 0, \; i = 1, 2, ..., n, \end{cases}$$

where n is the number of neurons in the network, $x_{i.}\left(t\right)$ corresponds to the state of the ith neuron at time t; h_i represents an amplification function; g_i is an appropriately behaved function; I_i denotes external input to the $i$the neuron, a_{ij}, b_{ij}, d_{ij} denote the connection strengths of the jth neuron on the ith neuron, respectively; f_j, l_j, φ_j denote the activation functions, ψ_i are the neutral delay kernels, τ corresponds to the transmission delay, ν is the neutral delay, k_j denotes the delay kernel function, ϕ_i is the pre-history of the ith state. This is a quite general Cohen-Grossberg neural network (CGNNs) system which involves discrete and continuously distributed delays. Of course we may also consider different discrete delays or even variable discrete delays. We refrained to go to more general systems to avoid distracting the attention of the reader from the main contribution here. In fact we will consider the simpler problem

$$\begin{cases} D_C^\alpha \left[x_i(t) - p x_i(t-\nu)\right] = -h_i\left(x_i\left(t\right)\right) \left[g_i\left(x_i\left(t\right)\right) - \sum\limits_{j=1}^n d_{ij} \int_0^\infty k_j\left(s\right) f_j\left(x_j\left(t-s\right)\right) ds - I_i \right], \\ x_i\left(t\right) = x_{i0}\left(t\right), \; t \in [-\nu, 0], \; i = 1, 2, ..., n. \end{cases} \tag{32}$$

for $t, p, \nu > 0$.

The following assumptions are somewhat standard

(A1) The delay kernel functions k_j are piecewise continuous nonnegative functions such that $\kappa_j = \int_0^\infty k_j\left(s\right) ds < \infty$, $j = 1, 2, ..., n$.

(A2) The functions f_i are Lipschitz continuous on $\mathbb{R}$ with Lipschitz constants L_i, $i = 1, 2, ..., n$, that is

$$\left|f_i\left(x\right) - f_i\left(y\right)\right| \leq L_i \left|x - y\right|, \; \forall x, y \in \mathbb{R}, \; i = 1, 2, ..., n.$$

(A3) Each function h_i is positive, continuous and there exist constants $\underline{\alpha}_i$ and $\overline{\alpha}_i$, such that

$$0 < \underline{\alpha}_i \leq h_i(u) \leq \overline{\alpha}_i, \; i = 1, 2, ..., n, \text{ for all } u \in \mathbb{R}.$$

(A4) The functions g_i are differentiable with derivatives bounded by G.

For simplicity, we will assume zero history, i.e. $x_{i0}(t) \equiv 0$, $t \leq 0$.

Definition 9: The point $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$ is called an equilibrium if, for $i = 1, 2, ..., n$, x^* is a solution of the system

$$\begin{aligned} g_i(x_i^*) &= \sum_{j=1}^{n} a_{ij} f_j\left(x_j^*\right) + \sum_{j=1}^{n} d_{ij} \int_0^\infty k_j(s) f_j\left(x_j^*\right) ds + I_i \\ &= \sum_{j=1}^{n} (a_{ij} + d_{ij}\kappa_j) f_j\left(x_j^*\right) + I_i, \ t > 0. \end{aligned}$$

The existence and uniqueness of an equilibrium is guaranteed and has been proved in the previous works. We first shift the equilibrium to zero by the substitution $x(t) = u(t) + x^*$. We obtain

$$\begin{cases} D_C^\alpha \left[u_i(t) - pu_i(t-\nu)\right] = -h_i\left(u_i(t) + x_i^*\right) \left[g_i\left(u_i(t) + x_i^*\right) \right. \\ \qquad \left. - \sum_{j=1}^{n} d_{ij} \int_0^t k_j(s) f_j\left(u_j(t-s) + x_j^*\right) ds - I_i \right], \quad t > 0, \ i = 1, 2, ..., n, \\ u_i(t) = \psi_i(t) := \phi_i(t) - x_i^*, \ t \in [-\nu, 0], \ i = 1, 2, ..., n, \end{cases}$$

or

$$\begin{cases} D_C^\alpha \left[u_i(t) - pu_i(t-\nu)\right] = -H_i\left(u_i(t)\right) \left[G_i\left(u_i(t)\right) \right. \\ \left. - \sum_{j=1}^{n} d_{ij} \int_0^t k_j(s) F_j\left(u_j(t-s)\right) ds \right], \quad t > 0, \ i = 1, 2, ..., n, \\ u_i(t) = \psi_i(t) := \phi_i(t) - x_i^*, \ t \in [-\nu, 0], \ i = 1, 2, ..., n \end{cases}$$

where

$$\begin{gathered} H_i\left(u_i(t)\right) = h_i\left(u_i(t) + x_i^*\right), \ G_i\left(u_i(t)\right) = g_i\left(u_i(t) + x_i^*\right) - g_i(x^*) \\ F_i\left(u_i(t)\right) = f_i\left(u_i(t) + x_i^*\right) - f_i\left(x_i^*\right), \quad t > 0, \ i = 1, 2, ..., n. \end{gathered}$$

Next, by the mean value theorem, we see that

$$\begin{aligned} & D_C^\alpha \left|u_i(t) - pu_i(t-\nu)\right| \leq sgn\left[u_i(t) - pu_i(t-\nu)\right] D_C^\alpha \left[u_i(t) - pu_i(t-\nu)\right] \\ & = -H_i\left(u_i(t)\right) sgn\left[u_i(t) - pu_i(t-\nu)\right] \left[g_i'\left(\bar{x}_i(t)\right) u_i(t) - \sum_{j=1}^{n} d_{ij} \int_0^\infty k_j(s) F_j\left(u_j(t-s)\right) ds\right] \end{aligned}$$

and adding and subtracting $pg_i'\left(\bar{x}_i(t)\right) u_i(t-\nu)$, we get

$$\begin{aligned} D_C^\alpha \left|u_i(t) - pu_i(t-\nu)\right| &\leq -H_i\left(u_i(t)\right) sgn\left[u_i(t) - pu_i(t-\nu)\right] \left[g_i'\left(\bar{x}_i(t)\right)\left[u_i(t) - pu_i(t-\nu)\right] \right. \\ & \left. + pg_i'\left(\bar{x}_i(t)\right) u_i(t-\nu) - \sum_{j=1}^{n} d_{ij} \int_0^\infty k_j(s) F_j\left(u_j(t-s)\right) ds \right], \quad t > 0, \ i = 1, 2, ..., n, \end{aligned}$$

or

$$\begin{aligned} D_C^\alpha \left|u_i(t) - pu_i(t-\nu)\right| &\leq -H_i\left(u_i(t)\right) G\left|u_i(t) - pu_i(t-\nu)\right| + pGH_i\left(u_i(t)\right)\left|u_i(t-\nu)\right| \\ & + H_i\left(u_i(t)\right) \sum_{j=1}^{n} d_{ij} \int_0^\infty k_j(s) L_j \left|u_j(t-s)\right| ds, \quad t > 0, \ i = 1, 2, ..., n. \end{aligned}$$

Therefore

$$\begin{aligned} D_C^\alpha \left|u_i(t) - pu_i(t-\nu)\right| &\leq -G\underline{\alpha}_i \left|u_i(t) - pu_i(t-\nu)\right| + pG\overline{\alpha}_i \left|u_i(t-\nu)\right| \\ & + \overline{\alpha}_i \sum_{j=1}^{n} L_j d_{ij} \int_0^\infty k_j(s) \left|u_j(t-s)\right| ds, \quad t > 0, \ i = 1, 2, ..., n. \end{aligned}$$

Finally we pass to the identity in w_i and write the formulation

$$|w_i(t) - pw_i(t-\nu)| = E_\alpha(-G\underline{\alpha}_i t^\alpha)\,|\varphi_i(0) - p\varphi_i(-\nu)| + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-a(t-s)^\alpha)$$
$$\times \left(pG\overline{\alpha}_i\,|w_i(t-\nu)| + \overline{\alpha}_i \sum_{j=1}^{n} L_j d_{ij} \int_0^\infty k_j(s)\,|w_j(t-s)|\,ds \right) ds, \quad t > 0,\ i = 1, 2, ..., n.$$

Now we can apply the previous result to obtain the Mittag-Leffler stability of this problem.

8 Conclusion

We have studied a general Halanay inequality of fractional order with distributed delays and involving delays of neutral type. General sufficient conditions have been found guaranteeing the Mittag-Leffler stability of the solutions and some examples were given. It seems that this is the best rate we can obtain as is the case in previous fractional problems. Finally, an application to a problem arising in neural networks theory has been presented. It is clear from our argument that these results may be extended to more general situations such variable delays and also for problems involving other terms. It is also important to mention that the conditions on the different parameters involved in the systems may be considerably improved as we managed no efforts in choosing the best estimations and bounds. In this regards, it would be nice to look for possibly optimal bounds on delay coefficient p and the kernel k.

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