Arbitrary stability for a Hopfield neural network problem with discrete and distributed delays

M. D. Kassim*
Department of Basic Engineering Sciences,
College of Engineering, Imam Abdulrahman Bin Faisal University,
P.O. Box 1982, Dammam 34151, Saudi Arabia
and
N.-E. Tatar†
King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Dhahran, 31261, Saudi Arabia

Abstract

We consider a Hopfield neural network system containing discrete as well as distributed delays. A stability result of arbitrary type is proved under weaker assumptions than the used ones so far. This result includes exponential and polynomial (or power type) stability as special cases. Our proof relies on a judicious choice of Lyapunov-type functionals and some appropriate manipulations.

Key Words and Phrases: Hopfield neural network system, Discrete delay, Distributed delay, exponentail stability, arbitrary stability, Lyapunov-type functional.

Mathematics Subject Classification: 34G20, 34C11, 92B20

1 Introduction

The problem of concern here is the following Hopfield neural network (HNN) system with two types of delays: discrete and distributed

$$\begin{cases} x_i'(t) = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t-\tau)) \\ \qquad\qquad + \sum_{j=1}^n d_{ij} \int_0^\infty k_j(s)\, f_j(x_j(t-s))\, ds + I_i, \ t > 0, \\ x_i(t) = \varphi_i(t), \ t \le 0, \end{cases} \tag{1}$$

*E-mail: mdkassim@iau.edu.sa
†E-mail: tatarn@kfupm.edu.sa

for $i = 1, 2, \ldots, n$, where

n	is the number of existing units
x_i	are the state of the neuron number i at the instant t
$c_i > 0$	are the rates of the passive delay
a_{ij}, b_{ij}, d_{ij}	denote the connection weight matrices
I_i	stand for the external inputs assumed constants
f_j	are the activation functions
k_j	are the delay feedback kernels
$\tau > 0$	is the discrete delay, and
φ_i	describes the history of the states

The activation functions in the discrete and distributed delays are in general different but we are considering them here equal just for simplicity.

The continuous deterministic HNN is a recurrent artificial neural network that is used in many applications to model the dynamics of systems with a large number of inputs and unknown parameters. The first model introduced in [8] had the form

$$x_i'(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)),\ t > 0,\ i = 1, 2, \ldots, n.$$

Hopfield [8] introduced this continuous deterministic model to describe the time evolution of the state of electronic devices with a large number of amplifiers in conjunction with feedback circuits made up of wires, resistors and capacitors. Such circuits have integrative time delays due to capacitance. Since then, HNN has been used to describe various systems that occur in engineering, biology, and economy [1-3,7,9-12,15,18,20,21,23].

Many complex processes with delays can be modeled as Hopfield neural network (HNN) systems with discrete and/or continuously distributed delays. Time retardation in electronic neural networks occur on account of the finite switching speed of amplifiers and can lead to instabilities in the form of oscillations [4-6,13,14,17,19,22,24-28].

Guo [6] analyzed the global asymptotic stability for (1) with piecewise continuous kernels. The global and local stability of the equilibrium states of (1) has been investigated under various conditions on the different coefficients, activation functions, and delays [4-6,13,14,17,19,22,24-28]. In addition, there is an interest in determining the speed of convergence to the equilibrium states. For this purpose, various exponential stability results have been established, see for example [19]. In all these papers the main condition for exponential asymptotic stability is $\int_0^\infty e^{\beta s} K(s) ds < \infty$ for some $\beta > 0$ in addition to the standard condition of the dominance of the damping on the other coefficients [13,16,17,19,22,24,28].

Yin and Fu [25] studied the μ-stability issue for a class of NNs (1) subject to impulses with a diagonal K and unbounded time-varying lags. They used a Lyapunov-Krasovskii functional to derive some conditions in the form of linear matrix inequalities. The μ-stability, roughly, means that the states converge asymptotically to equilibrium at the rate $1/\mu(t)$ in a certain norm. Cui *et* *al.* [4] extended (1) to a reaction-diffusion cellular NN. The delays there were unbounded and time-varying and the distributed delays were bounded. In both papers, the function $\mu(t)$ must satisfy the conditions

$$\frac{\mu'(t)}{\mu(t)} \leq \beta_1,\ \frac{\mu(t-\tau)}{\mu(t)} \geq \beta_2,\ \frac{\int_0^\infty k_j(s)\mu(t+s)\,ds}{\mu(t)} \leq \beta_3,\ t>0$$

where β_1, β_2 and β_3 are nonnegative scalars.

Zhang and Jin [26] established conditions for existence, uniqueness, and global asymptotic stability of the stationary state of HNN with fixed or distributed time delays. The results apply in case the interconnection matrices are symmetric and nonsymmetric. The activation functions are continuous and non-monotonic functions.

It is our objective here to derive sufficient conditions for stability with general rate including as a special case the exponential stability. Our results are obtained using new suitably selected functionals of Lyapunov-type in this theory and improve the existing results using completely different arguments. In view of the previous results we shall assume the existence of continuously differentiable solutions.

2 Preliminaries

In this part of the paper we shall present our assumptions, definitions, and useful lemmas.

We start with the presumptions

(B1) The delay kernel functions k_j are piecewise continuous nonnegative functions such that $\kappa_j := \int_0^\infty k_j(s)\,ds < \infty$.

(B2) The functions f_i are Lipschitz continuous on $\mathbb{R}$ with L_i, $i=1,2,\ldots,n$ as Lipschitz constants, that is

$$|f_i(x)-f_i(y)| \leq L_i|x-y|,\ \forall x,y \in \mathbb{R},\ i=1,2,\ldots,n.$$

(B3) The initial data $\varphi_i(t)$, $t \leq 0$ are continuous functions.

Definition 1: The point $x^* = (x_1^*,\ldots,x_n^*)^T$ is called an equilibrium point of problem (1) if for $i=1,2,\ldots,n$,

$$\begin{aligned} c_i x_i^* &= \textstyle\sum_{j=1}^n a_{ij} f_j(x_j^*) + \sum_{j=1}^n b_{ij} f_j(x_j^*) + \sum_{j=1}^n d_{ij} \int_0^\infty k_j(s)\, f_j(x_j^*)\,ds + I_i \\ &= \textstyle\sum_{j=1}^n \left[a_{ij} + b_{ij} + d_{ij}\int_0^\infty k_j(s)\ ds\right] f_j(x_j^*) + I_i,\ t>0. \end{aligned}$$

Definition 2: The equilibrium point x^* is said to be globally μ-stable if there exists a constant $A>0$ and a positive function $\mu(t)$ such that $\lim_{t\to\infty}\mu(t)=\infty$ and

$$\|x(t)-x^*\| \leq \frac{A}{\mu(t)},\ t>0$$

where $\|\cdot\|$ denotes any norm in R^n.

The existence of a unique equilibrium for this kind of problems has been shown for instance in [26,27] when the functions f_j are Lipschitz continuous. It has been proved also for 'Non-Lipschitz' continuous functions (see [5]).

These results apply for our case here. In fact, one can consult any result in Hopfield neural network theory even without (discrete and distributed) delays, as delays do not affect the proofs. As a matter of fact, they do not appear in the system satisfied by the equilibrium. However, there will be conditions on their coefficients.

3 General stability

This part is devoted to the study the stability of the equilibrium state x^* for (1). If we let

$$z(t) = x(t) - x^*,$$

then it is clear that the stability of x^* is equivalent to the stability of the zero state for the problem

$$\left\{\begin{array}{rcl} z_i'(t) & = & -c_i z_i(t) + \sum_{j=1}^n a_{ij} g_j(z_j(t)) + \sum_{j=1}^n b_{ij} g_j(z_j(t-\tau)) \\ & & + \sum_{j=1}^n d_{ij} \int_0^\infty k_j(s)\, g_j(z_j(t-s))\, ds,\ t>0,\ i=1,2,\ldots,n, \\ \multicolumn{3}{l}{z_i(t) = \psi_i(t) := \varphi_i(t) - x_i^*,\ t \le 0,\ i=1,2,\ldots,n,} \end{array}\right. \tag{2}$$

where

$$g_j(z_j(t)) = f_j(z_j(t) + x_j^*) - f_j(x_j^*),\ t \ge 0. \tag{3}$$

(B4) The initial data $\varphi_i(t)$ are such that $\psi_i \in L^2(-\infty, 0)$, $i = 1, 2, \ldots, n$.

To investigate the stability of the system (1), we employ the 'energy' functional

$$E(t) := \sum_{i=1}^n z_i^2(t),\ t \ge 0. \tag{4}$$

The first lemma is a straightforward consequence of **(B2)** and (3).

Lemma 3: Let assumption **(B2)** hold. Then

$$2|z_i(t)\, g_j(z_j(t))| \le z_i^2(t) + L_j^2 z_j^2(t),\ t>0,\ i,j = 1,2,\ldots,n,$$

and

$$2|z_i(t)\, g_j(z_j(t-\tau))| \le z_i^2(t) + L_j^2 z_j^2(t-\tau),\ t>0,\ i,j = 1,2,\ldots,n.$$

Lemma 4: Let presumptions **(B1)-(B3)** hold. Then

$$\begin{array}{c} E'(t) \le \sum_{i=1}^n \left\{ -2c_i + \sum_{j=1}^n \left[a_{ij} + L_i^2 a_{ji} + b_{ij} + d_{ij} \right] \right\} z_i^2(t) \\ + \sum_{j=1}^n \lambda_{1j}\, z_j^2(t-\tau) + \sum_{j=1}^n \lambda_{2j} \int_0^\infty k_j(s)\, z_j^2(t-s)\, ds,\ t \ge 0, \end{array}$$

where

$$\lambda_{1j} = \left(\sum_{i=1}^n b_{ij} \right) L_j^2,\ \lambda_{2j} = \left(\sum_{i=1}^n d_{ij} \right) L_j^2 \kappa_j,\ j = 1,2,\ldots,n. \tag{5}$$

Proof: The differentiation of $E(t)$ in (4), along solutions of (2), yields for $t \geq 0$

$$\begin{array}{c} E'(t) = 2\sum_{i=1}^n \Big[-c_i z_i^2(t) + \sum_{j=1}^n a_{ij}\, z_i(t)\, g_j(z_j(t)) \\ + \sum_{j=1}^n b_{ij}\, z_i(t)\, g_j(z_j(t-\tau)) + \sum_{j=1}^n d_{ij}\, z_i(t)\, \int_0^\infty k_j(s)\, g_j(z_j(t-s))\, ds \Big]. \end{array}$$

By Lemma 4 we can write

$$\begin{array}{c} E'(t) \leq -2\sum_{i=1}^n c_i z_i^2(t) + \sum_{i,j=1}^n a_{ij}[z_i^2(t) + L_j^2\, z_j^2(t)] \\ + \sum_{i,j=1}^n b_{ij}[z_i^2(t) + L_j^2\, z_j^2(t-\tau)] \\ + \sum_{i,j=1}^n d_{ij} \left[z_i^2(t) + \left(\int_0^\infty k_j(s)\, L_j\, |z_j(t-s)|\, ds \right)^2 \right],\ t \geq 0. \end{array}$$

From Cauchy-Schwartz inequality we have the bound

$$\begin{array}{c} \left(\int_0^\infty k_j(s)\, L_j\, z_j(t-s)\, ds \right)^2 \leq \int_0^\infty k_j(s)\, ds\, \int_0^\infty k_j(s)\, L_j^2\, z_j^2(t-s)\, ds \\ \leq L_j^2 \kappa_j \int_0^\infty k_j(s)\, z_j^2(t-s)\, ds,\ t \geq 0. \end{array}$$

Consequently,

$$\begin{array}{l} E'(t) \leq \sum_{i=1}^n \left[-2c_i + \sum_{j=1}^n a_{ij} + L_i^2 \sum_{j=1}^n a_{ji} + \sum_{j=1}^n b_{ij} + \sum_{j=1}^n d_{ij} \right] \\ \times z_i^2(t) + \sum_{i,j=1}^n b_{ij}\, L_j^2\, z_j^2(t-\tau) + \sum_{i,j=1}^n d_{ij} L_j^2 \kappa_j \int_0^\infty k_j(s)\, z_j^2(t-s)\, ds \\ = \sum_{i=1}^n \left[-2c_i + \sum_{j=1}^n a_{ij} + L_i^2 \sum_{j=1}^n a_{ji} + \sum_{j=1}^n b_{ij} + \sum_{j=1}^n d_{ij} \right] z_i^2(t) \\ \qquad + \sum_{j=1}^n \left(\sum_{i=1}^n b_{ij} \right) L_j^2\, z_j^2(t-\tau) \\ \qquad + \sum_{j=1}^n \left(\sum_{i=1}^n d_{ij} \right) L_j^2 \kappa_j \int_0^\infty k_j(s)\, z_j^2(t-s)\, ds \\ = \sum_{i=1}^n \left[-2c_i + \sum_{j=1}^n a_{ij} + L_i^2 \sum_{j=1}^n a_{ji} + \sum_{j=1}^n b_{ij} + \sum_{j=1}^n d_{ij} \right] z_i^2(t) \\ \quad + \sum_{j=1}^n \lambda_{1j}\, z_j^2(t-\tau) + \sum_{j=1}^n \lambda_{2j} \int_0^\infty k_j(s)\, z_j^2(t-s)\, ds,\ t \geq 0. \end{array}$$

Theorem 5: Let assumptions **(B1)-(B4)** hold. If

$$\sum_{j=1}^n \left[a_{ij} + b_{ij} + d_{ij} + L_i^2 \left(a_{ji} + b_{ji} + \kappa_i^2 d_{ji} \right) \right] < 2c_i,\ i = 1, 2, \ldots, n,$$

then $E(t)$ is uniformly bounded.

Proof: Consider the functionals

$$V_1(t) := \sum_{j=1}^n \lambda_{1j} \int_{t-\tau}^t z_j^2(s) ds,\ t \geq 0, \tag{6}$$

and

$$\begin{array}{c} V_2(t) := \sum_{j=1}^\infty \lambda_{2j} \int_{-\infty}^t \left(\int_t^\infty k_j(\sigma - s)\, d\sigma \right)\, z_j^2(s)\, ds \\ = \sum_{j=1}^\infty \lambda_{2j} \int_0^\infty k_j(s) \int_{t-s}^t z_j^2(\sigma)\, d\sigma\, ds,\ t \geq 0. \end{array} \tag{7}$$

Note that

$$V_1(0) = \sum_{j=1}^n \lambda_{1j} \int_{-\tau}^0 z_j^2(s)\, ds = \sum_{j=1}^n \lambda_{1j} \int_{-\tau}^0 \psi_j^2(s)\, ds < \infty,$$

and

$$V_2(0) = \sum_{j=1}^{\infty} \lambda_{2j} \int_0^{\infty} k_j(s) \int_{-s}^{0} \psi_j^2(\sigma)\, d\sigma\, ds < \infty.$$

Moreover,

$$V_1'(t) = \sum_{j=1}^{n} \lambda_{1j} \left[z_j^2(t) - z_j^2(t-\tau) \right], \ t \geq 0, \tag{8}$$

and

$$\begin{aligned} V_2'(t) &= \textstyle\sum_{j=1}^{n} \lambda_{2j} \left(\int_t^{\infty} k_j(\sigma - t)\, d\sigma \right) z_j^2(t) - \sum_{j=1}^{n} \lambda_{2j} \int_{-\infty}^{t} k_j(t-s)\, z_j^2(s)\, ds \\ &= \textstyle\sum_{j=1}^{n} \lambda_{2j} \left(\int_0^{\infty} k_j(s)\, ds \right) z_j^2(t) - \sum_{j=1}^{n} \lambda_{2j} \int_0^{\infty} k_j(s)\, z_j^2(t-s)\, ds \\ &= \textstyle\sum_{j=1}^{n} \lambda_{2j}\kappa_j\, z_j^2(t) - \sum_{j=1}^{n} \lambda_{2j} \int_0^{\infty} k_j(s)\, z_j^2(t-s)\, ds, \ t \geq 0. \end{aligned} \tag{9}$$

Let

$$\mathcal{E}(t) = E(t) + V_1(t) + V_2(t), \ t \geq 0. \tag{10}$$

Then, $\mathcal{E}(0) < \infty$ and

$$\begin{aligned} &\mathcal{E}'(t) = E'(t) + V_1'(t) + V_2'(t) \\ &\leq \textstyle\sum_{i=1}^{n} \left[-2c_i + \sum_{j=1}^{n} a_{ij} + L_i^2 \sum_{j=1}^{n} a_{ji} + \sum_{j=1}^{n} b_{ij} + \sum_{j=1}^{n} d_{ij} \right] z_i^2(t) \\ &\quad + \textstyle\sum_{j=1}^{n} \lambda_{1j}\, z_j^2(t-\tau) + \sum_{j=1}^{n} \lambda_{2j} \int_0^{\infty} k_j(s)\, z_j^2(t-s)\, ds \\ &\quad + \textstyle\sum_{j=1}^{n} \lambda_{1j} \left[z_j^2(t) - z_j^2(t-\tau) \right] + \sum_{j=1}^{n} \lambda_{2j}\kappa_j\, z_j^2(t) \\ &\quad - \textstyle\sum_{j=1}^{n} \lambda_{2j} \int_0^{\infty} k_j(s)\, z_j^2(t-s)\, ds \end{aligned}$$

or

$$\begin{aligned} \mathcal{E}'(t) &\leq \textstyle\sum_{i=1}^{n} \left[-2c_i + \sum_{j=1}^{n} \left(a_{ij} + L_i^2 a_{ji} + b_{ij} + d_{ij} \right) \right] z_i^2(t) \\ &\quad + \textstyle\sum_{i=1}^{n} \lambda_{1i} z_i^2(t) + \sum_{i=1}^{n} \lambda_{2i}\kappa_i\, z_i^2(t), \ t \geq 0. \end{aligned}$$

This may be rewritten simply as

$$\begin{aligned} \mathcal{E}'(t) &\leq \textstyle\sum_{i=1}^{n} \left\{ -2c_i + \sum_{j=1}^{n} \left[a_{ij} + L_i^2 a_{ji} + b_{ij} + d_{ij} + L_i^2 b_{ji} + L_i^2 \kappa_i^2 d_{ji} \right] \right\} z_i^2(t) \\ &= \textstyle\sum_{i=1}^{n} \left\{ -2c_i + \sum_{j=1}^{n} \left[a_{ij} + b_{ij} + d_{ij} + L_i^2 \left(a_{ji} + b_{ji} + \kappa_i^2 d_{ji} \right) \right] \right\} z_i^2(t), \ t \geq 0. \end{aligned} \tag{11}$$

From the condition stated in the theorem and (11) we see that $\mathcal{E}'(t) \leq 0$, $t \geq 0$. Therefore,

$$E(t) \leq \mathcal{E}(t) \leq \mathcal{E}(0), \ t \geq 0.$$

The proof is complete.

We now specify our main condition on the kernels

(B5) There are nonnegative continuous functions $\eta_j(t)$ such that $\lim_{t\to\infty} \eta(t) := \lim_{t\to\infty} \min_{1\leq j\leq n} \eta_j(t) = \bar{\eta}$ and

$$k_j(t-s) \geq \eta_j(t) \int_t^{\infty} k_j(\sigma - s)\, d\sigma, \ j = 1, 2, \ldots, n, \ 0 \leq s \leq t.$$

Theorem 6: Let assumptions **(B1)-(B5)** hold and

$$2c_i > \sum_{j=1}^{n} \left\{ a_{ij} + b_{ij} + d_{ij} + L_i^2 \left[a_{ji} + (1+\varepsilon) b_{ji} \ + 2\kappa_i^2 d_{ji} \right] \right\},\ i = 1, 2, \ldots, n,$$

for some $\varepsilon > 0$. Then, if $\lim_{t\to\infty} \eta(t) = \bar{\eta} = 0$ we have

$$E(t) \leq C_1\, e^{-C_2 \int_0^t \eta(s) ds},\ t \geq 0$$

and

$$E(t) \leq C_3 e^{-C_4 t},\ t \geq 0$$

in case $0 < \bar{\eta} \leq \infty$, for some positive constants C_i, $i = 1, 2, 3, 4$.

Remark: If $\eta(t) = \frac{\mu'(t)}{\mu(t)}$ for some differentiable function $\mu(t)$, then we obtain

$$E(t) \leq \frac{A}{|\mu(t)|^{\sigma}},\ t \geq 0$$

for some positive constants A and σ.

Proof (of Theorem 6): For $0 < \delta < 1/2$, consider the functional

$$\tilde{\mathcal{E}}(t) := E(t) + V_3(t) + \frac{1}{1-\delta} V_2(t),\ t \geq 0, \tag{12}$$

where

$$V_3(t) := e^{-\beta t} \sum_{j=1}^{n} \lambda_{1j} \int_{t-\tau}^{t} e^{\beta(s+\tau)}\, z_j^2(s)\, ds,\ t \geq 0,\ \beta > 0,$$

λ_{1j} as in (5), and V_2 as in (7). Here β is selected so small that $e^{\beta\tau} \leq 1 + \varepsilon$ (ε is in the statement of the theorem).

By direct differentiation we have

$$V_2'(t) = -\beta\, V_3(t) + e^{\beta\tau} \sum_{j=1}^{n} \lambda_{1j}\, z_j^2(t) - \sum_{j=1}^{n} \lambda_{1j}\, z_j^2(t-\tau),\ t \geq 0. \tag{13}$$

Next, we estimate $V_2'(t)$ in light of our new assumption **(B5)** on the kernels. Clearly, for $t \geq 0$

$$\begin{aligned} V_2'(t) &= \textstyle\sum_{j=1}^{n} \lambda_{2j} \kappa_j\, z_j^2(t) - \sum_{j=1}^{n} \lambda_{2j} \int_{-\infty}^{t} k_j(t-s)\, z_j^2(s)\, ds \\ &= \textstyle\sum_{j=1}^{n} \lambda_{2j} \kappa_j\, z_j^2(t) - \delta \sum_{j=1}^{n} \lambda_{2j} \int_{-\infty}^{t} k_j(t-s)\, z_j^2(s)\, ds \\ &\quad \textstyle - (1-\delta) \sum_{j=1}^{n} \lambda_{2j} \int_{-\infty}^{t} k_j(t-s)\, z_j^2(s)\, ds \\ &\leq \textstyle\sum_{j=1}^{n} \lambda_{2j}\, \kappa_j\, z_j^2(t) - \delta \sum_{j=1}^{n} \lambda_{2j}\, \eta_j(t) \int_{-\infty}^{t} \left(\int_t^{\infty} k_j(\sigma - s)\, d\sigma \right) z_j^2(s)\, ds \\ &\quad \textstyle - (1-\delta) \sum_{j=1}^{n} \lambda_{2j} \int_{-\infty}^{t} k_j(t-s)\, z_j^2(s)\, ds \\ &\leq \textstyle\sum_{j=1}^{n} \lambda_{2j}\, \kappa_j\, z_j^2(t) - \delta\, \eta(t)\, V_3(t) - (1-\delta) \sum_{j=1}^{n} \lambda_{2j} \int_{-\infty}^{t} k_j(t-s)\, z_j^2(s)\, ds. \end{aligned} \tag{14}$$

Taking into account (12)-(14), the differentiation along solutions of (2) yields

$$\begin{gathered}\tilde{\mathcal{E}}'(t) \le \sum_{i=1}^n \left[-2c_i + \sum_{j=1}^n \left(a_{ij} + L_i^2 a_{ji} + b_{ij} + d_{ij}\right)\right] z_i^2(t) \\ + \sum_{j=1}^n \lambda_{1j}\, z_j^2(t-\tau) + \sum_{j=1}^n \lambda_{2j} \int_0^\infty k_j(s)\, z_j^2(t-s)\, ds + e^{\beta\tau} \sum_{j=1}^n \lambda_{1j}\, z_j^2(t) \\ -\beta\, V_2(t) - \sum_{j=1}^n \lambda_{1j}\, z_j^2(t-\tau) + \tfrac{1}{1-\delta} \left\{\sum_{j=1}^n \lambda_{2j}\, \kappa_j\, z_j^2(t) - \delta\, \eta(t)\, V_3(t)\right\} \\ - \sum_{j=1}^n \lambda_{2j} \int_{-\infty}^t k_j(t-s)\, z_j^2(s)\, ds,\ t \ge 0\end{gathered}$$

or

$$\begin{gathered}\tilde{\mathcal{E}}'(t) \le \sum_{i=1}^n \left[-2c_i + \sum_{j=1}^n \left(a_{ij} + L_i^2 a_{ji} + b_{ij} + d_{ij}\right)\right] z_i^2(t) \\ + \sum_{j=1}^n \left[e^{\beta\tau}\lambda_{1j} + \tfrac{\lambda_{2j}\,\kappa_j}{1-\delta}\right] z_j^2(t) - \beta\, V_3(t) - \tfrac{\delta}{1-\delta}\, \eta(t)\, V_2(t),\ t \ge 0.\end{gathered}$$

In view of (5), we find

$$\begin{gathered}\tilde{\mathcal{E}}'(t) \le \sum_{i=1}^n \left[-2c_i + \sum_{j=1}^n \left(a_{ij} + L_i^2 a_{ji} + b_{ij} + d_{ij}\right)\right] z_i^2(t) \\ + \sum_{j=1}^n \left[e^{\beta\tau} \left(\sum_{i=1}^n b_{ij}\right) L_j^2 + \tfrac{\kappa_j}{1-\delta} \left(\sum_{i=1}^n d_{ij}\right) L_j^2 \kappa_j\right] z_j^2(t) \\ -\beta\, V_3(t) - \tfrac{\delta}{1-\delta}\, \eta(t)\, V_2(t),\ t \ge 0\end{gathered}$$

or

$$\begin{aligned}\tilde{\mathcal{E}}'(t) &\le \sum_{i=1}^n \left\{-2c_i + \sum_{j=1}^n \left[a_{ij} + b_{ij} + d_{ij} + L_i^2 \left(a_{ji} + e^{\beta\tau} b_{ji} + \tfrac{\kappa_i^2}{1-\delta} d_{ji}\right)\right]\right\} \\ &\quad \times z_i^2(t) - \beta\, V_3(t) - \tfrac{\delta}{1-\delta}\, \eta(t)\, V_2(t) \\ &\le -\alpha E(t) - \beta\, V_3(t) - \tfrac{\delta}{1-\delta}\, \eta(t)\, V_2(t),\ t \ge 0\end{aligned} \tag{15}$$

where

$$\alpha = \min_{1 \le i \le n} \left\{2c_i - \sum_{j=1}^n \left[a_{ij} + b_{ij} + d_{ij} + L_i^2 \left(a_{ji} + e^{\beta\tau} b_{ji} + \frac{\kappa_i^2}{1-\delta} d_{ji}\right)\right]\right\}.$$

From the hypotheses we have $\alpha > 0$.

We discuss two cases:

Case 1: $\lim_{t\to\infty} \eta(t) = 0$

Let $t^* > 0$ be large enough so that

$$\eta(t) \le \frac{1}{\delta} \min\{\alpha, \beta\},\ t \ge t^*. \tag{16}$$

Therefore

$$\begin{aligned}\tilde{\mathcal{E}}'(t) &\le -\alpha E(t) - \beta\, V_3(t) - \tfrac{\delta}{1-\delta}\, \eta(t)\, V_2(t) \\ &\le -\delta\eta(t) E(t) - \delta\eta(t)\, V_3(t) - \tfrac{\delta}{1-\delta}\, \eta(t)\, V_2(t) \\ &\le -\delta\eta(t)\tilde{\mathcal{E}}(t),\ t \ge t^*.\end{aligned}$$

This implies that

$$\tilde{\mathcal{E}}(t) \le \tilde{\mathcal{E}}(t^*)\, e^{-\delta \int_{t^*}^t \eta(s) ds},\ t \ge t^*.$$

By continuity and Theorem 5, we may derive a similar estimate on $[0, t^*]$.

Case 2: $0 < \bar{\eta} \le \infty$

In this case

$$\exists\, t_* > 0 \quad s.t. \quad \eta(t) \ge \frac{\bar{\eta}}{2},\ \forall t \ge t_*. \tag{17}$$

In case $\bar{\eta} = +\infty$, we consider any positive constant ξ, $\eta(t) \ge \xi$.

In view of (15) and (17), we see that

$$\tilde{\mathcal{E}}'(t) \le -\alpha E(t) - \beta V_3(t) - \frac{\delta}{1-\delta}\frac{\bar{\eta}}{2} V_2(t) \le -\gamma \tilde{\mathcal{E}}(t),\ t \ge t_*,$$

where

$$\gamma = \min\left\{\alpha, \beta, \frac{\delta\bar{\eta}}{2}\right\} > 0.$$

Therefore,

$$\tilde{\mathcal{E}}(t) \le \tilde{\mathcal{E}}(t_*)\, e^{-\gamma(t-t_*)},\ t \ge t_*.$$

A continuity argument and Theorem 5 gives a similar estimates on $[0, t_*]$. The proof is complete.

References

[1] *A. Bouzerdoum, T. R. Pattison*: Neural network for quadratic optimization with bound constraints. IEEE Trans. Neural Netw. **4** (3) (1993), 293–304.

[2] *L. O. Chua: T. Roska*, Stability of a class of nonreciprocal cellular neural networks. IEEE Trans. Circuits Syst. I **37** (1990), 1520–1527.

[3] *B. Crespi*: Storage capacity of non-monotonic neurons. Neural Network. **12** (1999), 1377–1389.

[4] *H. Cui, J. Guo, J. Feng, T. Wang*: Global Mu-stability of impulsive reaction–diffusion neural networks with unbounded time-varying delays and bounded continuously distributed delays. Neurocomputing. **157** (2015), 1–10.

[5] *C. Feng, R. Plamondon*: On the stability analysis of delayed neural network systems. Neural Network. **14** (2001), 1181–1188.

[6] *Y. Guo*: Global asymptotic stability analysis for integro-differential systems modeling neural networks with delays. Z. Angew. Math. Phys. **61** (2010), 971–978.

[7] *J. J. Hopfield*: Neural networks and physical systems with emergent collective computational abilities. Proc. Natl. Acad. Sci. **79** (1982), 2554–2558.

[8] *J. J. Hopfield*: Neurons with graded response have collective computational properties like those of two-state neurons. Proc. Natl. Acad. Sci. U S A. **81** (8) (1984), 3088–3092.

[9] *J. J. Hopfield, D. W. Tank*: Computing with neural circuits: a model. Science **233** (1986), 625–633.

[10] *J. I. Inoue*: Retrieval phase diagrams of non-monotonic Hopfield networks. J. Phys. A: Math. Gen. **29** (1996), 4815–4826.

[11] *M. P. Kennedy, L. O. Chua*: Neural networks for non-linear programming. IEEE Trans. Circ. Syst. I, Fundam. Theory Appl. **35** (1998), 554–562.

[12] *B. Kosko: Neural Network and Fuzzy System - A Dynamical System Approach to Machine Intelligence*, New Delhi: Prentice-Hall of India (1991).

[13] *B. Liu, W. Luc, T. Chen*: New criterion of asymptotic stability for delay systems with time-varying structures and delays. Neural Networks. **54** (2014), 103–111.

[14] *T. T. Loan, D. A. Tuan*: Global exponential stability of a class of neural networks with unbounded delays. Ukrainian Math. J. **60** (1) (2008), 1633–1649.

[15] *S. Mohamad*: Exponential stability in Hopfield-type neural networks with impulses. Chaos, Solitons & Fractals. **32** (2007), 456–467.

[16] *S. Mohamed and K. Gopalsamy*: Continuous and discrete Halanay-type inequalities., Bull. Austral. Math. Soc. **61** (2000), 371–385.

[17] *S. Mohamad, K. Gopalsamy and H. Akca*: Exponential stability of artificial neural networks with distributed delays and large impulses. Nonl. Anal.: Real World Appl. **9** (2008), 872–888.

[18] *H. Qiao, J.G. Peng, Z. Xu*: Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. IEEE Trans. Neural Netw. **12** (2001), 360–370.

[19] *Q. Song, Z. Zhao*: Global dissipativity of neural networks with both variable and unbounded delays. Chaos, Solitons and Fractals. **25** (2005), 393–401.

[20] *S. I. Sudharsanan, M. K. Sundareshan*: Exponential stability and a systematic synthesis of a neural network for quadratic minimization. Neural Networks. **4** (1991), 599–613.

[21] *P. van Driessche, X. Zou*: Global attractivity in delayed Hopfield neural network models. SIAM J. Appl. Math. **58** (6) (1998), 1878–1890.

[22] *Y. Wang, W. Xiong, Q. Zhou, B. Xiao, Y. Yu*: Global exponential stability of cellular neural networks with continuously distributed delays and impulses. Physics Letters A **350** (2006), 89–95.

[23] *H. Yanai, S. Ammari*: Auto-associative memory with two stage dynamics of non-monotonic neurons. IEEE Trans. Neural Netw. **7** (1996), 803–815.

[24] *L. Yin, Y. Chen, Y. Zhao*: Global exponential stability for a class of neural networks with continuously distributed delays. Adv. Dyn. Syst. Appl. **4** (2) (2009), 221–229.

[25] *L. Yin, X. Fu:* Mu-stability of impulsive neural networks with unbounded time-varying delays and continuously distributed delays. Adv. Difference Eqs. **2011** (2011), 1–12.

[26] *J. Zhang, X. Jin*: Global stability analysis in delayed hopfield neural network models. Neural Networks. **13** (7), (2000), 745–753.

[27] *J. Zhang, Y. Suda, T. Iwasa*: Absolutely exponential stability of a class of neural networks with unbounded delay. Neural Networks. **17** (3) (2004), 391–397.

[28] *J. Zhou, S. Li, Z. Yang*: Global exponential stability of Hopfield neural networks with distributed delays. Appl. Math. Model. **33** (2009), 1513–1520.

Authors' addresses: Mohammed D Kassim, College of Engineering, Imam Abdulrahman Bin Faisal University, Department of Basic Sciences and Humanities, P.O. Box 1982, Dammam 34151, Saudi Arabia, e-mail: mdkassim@iau.edu.sa.

Nasser-eddine Tatar, King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran, 31261 Saudi Arabia, e-mail: tatarn@kfupm.edu.sa.