On a study of Sobolev type fractional functional evolution equations

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Abstract

Sobolev type fractional functional evolution equations have many applications in the modeling of many physical processes. Therefore, we investigate fractional-order time-delay evolution equation of Sobolev type with multi-orders in a Banach space and introduce an analytical representation of a mild solution via a new delayed Mittag-Leffler type function which is generated by linear bounded operators. Furthermore, we derive an exact analytical representation of solutions for multi-dimensional fractional functional dynamical systems with nonpermutable and permutable matrices. We also study stability analysis of the given time-delay system in Ulam-Hyers sense with the help of Laplace transform.

Keywords: fractional functional evolution equation, Sobolev, delayed Mittag-Leffler type function, nonpermutable linear operators, Ulam-Hyers stability

1 Introduction

Multi-term FDEs have been studied due to their applications in modelling, and solved using various mathematical methods. Finding the solution to these equations is an interesting and challenging subject that attracted many scientists in the last decades. Up to now, various analytical and computational techniques have been investigated to find the solution of multi-term FDEs, of which we mention a few as follows. Luchko and several collaborators [15, 26, 27] used the method of operational calculus to solve multi-order FDEs with different types of fractional derivatives. In the realm of ordinary differential equations, Mahmudov and other collaborators [2, 33] have derived an analytical representation of solutions for special cases of fractional differential equations with multi-orders, namely: Langevin and Bagley-Torvik equations involving scalar coefficients and permutable matrices by using Laplace transform method and fractional analogue of variation of constants formula, respectively, while the other authors [36] have solved multi-term differential equations in Riemann-Liouville's sense with variable coefficients applying a new method to construct analytical solutions. Several results have been investigated on solving multi-dimensional time-delay deterministic and stochastic systems with permutable matrices [3, 16] in classical and fractional senses.

Khusainov et al. [20] have proposed a compact representation of a solution of the Cauchy problem for a linear inhomogeneous differential equation with non-singular matrix and pure delay via special matrix functions which are called delayed matrix cosine and sine. Diblik et al. [8] have studied a control problem for a system governed by delay oscillating equations that have considered in [20] and gave sufficient and necessary conditions for relative controllability from the point of rank criterion. In [22], Liang et al. have used a different approach to study controllability results for linear second-order delay differential system in terms of delay Gramian matrix involing delayed matrix sine polynomials. In addition, Diblik et al. [10], have represented an analytical solution of an inhomogeneous second-order differential equation with two constant delays by using matrix functions under the assumption that linear parts are given by permutable matrices. In [23], Liang and Wang have considered iterative learning control problem of an oscillation system

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with two constant delays that are studied in [10] by utilizing delayed matrix functions. In [24], Liang et al. have investigated the finite-time stability of linear delay differential equations via the delayed matrix cosine and sine of polynomial degrees and extended to the same issue of delay differential equation with nonlinearity by virtue of Gronwall's inequality approach. Particularly, time-delay systems with multi-delays has a potential to be more suitable for applications in engineering and science [9, 34, 39, 40, 41]. In [9], Diblik et al. have studied nonhomogeneous system of linear differential equations with multiple different delays and pairwise permutable matrices. In [39], Pospisil and Jaros have proposed a closed-form formula for a solution of system of nonhomogeneous linear differential equations with any finite number of constant delays and linear parts given by pair-wise permutable matrices by applying Laplace transform technique. Pospisil [40] has introduced analytical representation of solutions for linear differential equations with multiple constant delays without commutativity assumption on the matrix coefficients. Moreover, Pospisil [41] has analyzed asymptotic stability results for some nonlinear functional differential equations system with multiple time-varying delays and linear parts given by pair-wise permutable constant matrices via Gronwall's, Bihari's and Pinto's integral inequalities. Furthermore, Medved and Pospisil [34] have derived sufficient conditions for the asymptotic stability of nonlinear multi-delay differential equations using multi-delay exponential functions via Pinto's inequality.

Recently, Mahmudov [31] has introduced a fractional analogue of delayed matrices cosine and sine in the commutative case i.e., $AB = BA$ to solve the sequential Riemann-Liouville type linear time-delay systems whilst Liang et al. [25] have also obtained an explicit solution of the differential equation with pure delay and sequential Caputo type fractional derivative. However, there are a few papers involving non-permutable matrices which are recently studied fractional time-continuous [28] and discrete [29] systems with a constant delay using recursively defined matrix-equations, and also delayed linear difference equations [30] applying $\mathscr{Z}$-transform technique by Mahmudov.

Meanwhile, Sobolev type evolution equations and their fractional-order analogues have attracted a great deal of attention from applications' point of view and studied by several authors [4, 5, 7, 11, 32, 46, 47, 48] in recent decade. In [4], Balachandran and Dauer have derived sufficient conditions for controllability of partial functional differential systems of Sobolev type in a Banach space by using compact semigroups and Schauder's fixed point theorem. Moreover, Balachandran et al. [5] have considered existence results of solutions for nonlinear impulsive integrodifferential equations of Sobolev type with nonlocal conditions via Krasnoselkii's fixed point technique. In terms of fractional differential equations, Wang et al. [47] have investigated controllability results of Sobolev type fractional evolution equations in a seperable Banach space by using the theory of propagation family and contraction mapping principle. In addition, Feckan et al. [11] have presented controllability of fractional functional evolution sytems of Sobolev type with the help of new characteristic solution operators and well-known Schauder's fixed point approach. In addition, Mahmudov [32] have considered approximate controllability results for a class of fractional evolution equations of Sobolev type by using fixed point approach. In [48], Wang and Li have discussed stability analysis of fractional evolution equations of Sobolev type in Ulam-Hyers sense. In [7], Chang et al. have studied the asymptotic behaviour of resolvent operators of Sobolev type and their applications to the existence and uniqueness of mild solutions to fractional functional evolution equations in Banach spaces. Vijayakumar et al. [46] have presented approximate controllability results for Sobolev type time-delay differential systems of fractional-order in Hilbert spaces.

To the best of our knowledge, the fractional functional evolution equations of Sobolev type with non-permutable operators and two independent fractional orders of differentiation α and β which are assumed to be in the interval $(1, 2]$ and $(0, 1]$, respectively are an untreated topic in the present literature. Thus, motivated by the above research works, we consider the following Cauchy problem for fractional evolution equation of Sobolev type with orders $1 < \alpha \leq 2$ and $0 < \beta \leq 1$ on $\mathbb{J} := [-\tau, T]$:

$$\begin{cases} \left({}^C D^{\alpha}_{0+} Ey\right)(t) - A_0 \left({}^C D^{\beta}_{0+} y\right)(t) = B_0 y(t-\tau) + g(t), \quad t > 0, \quad \tau > 0, \\ Ey(t) = \varphi(t), \quad -\tau \leq t \leq 0, \\ Ey'(0) = \varphi'(0), \end{cases} \tag{1.1}$$

where ${}^C D^{\alpha}_{0+}$ and ${}^C D^{\beta}_{0+}$ Caputo fractional differential operators of orders $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively, with the lower limit zero, the operators $E : D(E) \subset X \to Y$, $A_0 : D(A_0) \subset X \to Y$ and

$B_0 : D(B_0) \subset X \to Y$ are linear, where X and Y are Banach spaces, $y(\cdot)$ is a X-valued function on $\mathbb{J}$, i.e., $y(\cdot) : \mathbb{J} \to X$, $\varphi(\cdot) \in \mathbb{C}(\mathbb{I}, Y)$ where $\mathbb{I} = [-\tau, 0]$, $\tau > 0$ is a fixed delay time, suppose $\varphi(\cdot)$ is continuously differentiable at initial data $t = 0$ and $\varphi(0), \varphi'(0) \in Y$. In addition, $g(\cdot) : \mathbb{L} \to Y$ is a continuous function, where $\mathbb{L} = [0, T]$ and $T = n\tau$, for a fixed $n \in \mathbb{N} = \{1, 2, 3, ...\}$. The domain $D(E)$ of E becomes a Banach space with respect to $\|y\|_{D(E)} = \|Ey\|_Y, y \in D(E)$.

The main idea is that under the hypotheses (H_1)-(H_4) we transform Sobolev type fractional multi-term functional evolution equation with linear operators (1.1) to fractional-order time-delay evolution equation with multi-orders and linear bounded operators (3.1). Secondly, we solve fractional functional evolution equation with nonpermutable linear bounded operators by using Laplace transform technique which is used as a necessary tool for solving and analyzing fractional-order differential equations and systems in [2, 6, 16, 19, 43]. Then we propose exact analytical representation of a mild solution of (3.1) and (1.1), respectively with the help of new defined Mittag-Leffler function which is expressed via linear bounded operators by removing the exponential boundedness of a forced term $g(\cdot)$ and $\left({}^C D^{\beta}_{0+}x\right)(\cdot)$ for $\beta \in (0, 1]$ (or $\left({}^C D^{\alpha}_{0+}x\right)(\cdot)$ for $\alpha \in (1, 2]$) in both cases: with nonpermutable and permutable linear operators $A, B \in \mathscr{B}(Y)$.

The structure of this paper contains a crucial improvement in the theory of Sobolev type fractional multi-term time-delay evolution equations and is organized as below. Section 2 is a preparatory section where we recall main definitions and results from fractional calculus, special functions and fractional differential equations. In Section 3, we establish a new delayed analogue of Mittag-Leffler type function which is generated by linear bounded operators via a double infinity series and investigate some necessary properties of this function which are accurate tools for testing the candidate solutions of fractional-order dynamical equations. Moreover, first, we introduce the sufficient conditions for exponential boundedness of (3.1) to guarantee the existence of Laplace integral transform of equation (3.1). Meanwhile, we tackle this strong condition and verify that the sufficient conditions can be omitted easily. In addition, we propose exact solutions for multi-term fractional delay dynamical systems with commutative and noncommutative matrices. Section 4 deals with an analytical representation of a mild solution to Sobolev type functional evolution equation with two independent fractional-orders and permutable linear bounded operators. In Section 6, we discuss our main contributions of this paper and future research work.

2 Preliminary concept

We embark on this section by briefly presenting some notations and definition fractional calculus and fractional differential equations [21, 37, 44] which are used throughout the paper.

Let $\mathbb{C}^2(\mathbb{J}, X) \coloneqq \{x(\cdot) \in \mathbb{C}(\mathbb{J}, X) : x'(\cdot), x''(\cdot) \in \mathbb{C}(\mathbb{J}, X)\}$ denote the Banach space of functions $x(t) \in X$ for $t \in \mathbb{J}$ equipped with a norm $\|x\|_{\mathbb{C}^2(\mathbb{J},X)} = \sum\limits_{i=0}^{2} \sup\limits_{t\in\mathbb{J}} \|x^{(i)}(t)\|$. The space of all bounded linear operators from X to Y is denoted by $\mathscr{B}(X, Y)$ and $\mathscr{B}(Y, Y)$ is written as $\mathscr{B}(Y)$.

Definition 2.1. [21, 37, 44] The fractional integral of order $\alpha > 0$ for a function $g \in ([0, \infty), \mathbb{R})$ is defined by

$$(I^{\alpha}_{0^+} g)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (t - s)^{\alpha-1} g(s)\, \mathrm{d}s, \quad t > 0, \tag{2.1}$$

where $\Gamma(\cdot)$ is the well-known Euler's gamma function.

Definition 2.2. [21, 37, 44] The Riemann-Liouville fractional derivative of order $n - 1 < \alpha \le n$, $n \in \mathbb{N}$ for a function $g \in ([0, \infty), \mathbb{R})$ is defined by

$$({}^{RL}D^{\alpha}_{0^+} g)(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{d}{dt}\right)^n \int\limits_0^t (t - s)^{n-\alpha-1} g(s)\, \mathrm{d}s, \quad t > 0, \tag{2.2}$$

where the function $g(\cdot)$ has absolutely continuous derivatives up to order n.

The following theorem and its corollary is regarding fractional analogue of the eminent Leibniz integral rule for general order $\alpha \in (n-1, n], n \in \mathbb{N}$ in Riemann-Liouville's sense which is more productive tool for the testing particular solution of inhomogeneous linear multi-order fractional differential equations with variable and constant coefficients is considered by Huseynov et al. [17].

Theorem 2.1. *Let the function $K : J \times J \to \mathbb{R}$ be such that the following assumptions are fulfilled:*

(a) For every fixed $t \in J$, the function $\hat{K}(t,s) = {}^{RL,t}D^{\alpha-1}_{s+}K(t,s)$ is measurable on J and integrable on J with respect to some $t^* \in J$;

(b) The partial derivative ${}^{RL,t}D^{\alpha}_{s+}K(t,s)$ exists for every interior point $(t,s) \in \hat{J} \times \hat{J}$;

(c) There exists a non-negative integrable function g such that $\left|{}^{RL,t}D^{\alpha}_{s+}K(t,s)\right| \leq g(s)$ for every interior point $(t,s) \in \hat{J} \times \hat{J}$;

(d) The derivative $\frac{d^{l-1}}{dt^{l-1}} \lim\limits_{s\to t-0} {}^{RL,t}D^{\alpha-l}_{s+}K(t,s)$, $l = 1, 2, \ldots, n$ exists for every interior point $(t,s) \in \hat{J} \times \hat{J}$.

Then, the following relation holds true for fractional derivative in Riemann-Liouville sense under Lebesgue integration for any $t \in \hat{J}$:

$$ {}^{RL}D^{\alpha}_{t_0+} \int\limits_{t_0}^{t} K(t,s)\mathrm{d}s = \sum_{l=1}^{n} \frac{d^{l-1}}{dt^{l-1}} \lim_{s\to t-0} {}^{RL,t}D^{\alpha-l}_{s+}K(t,s) + \int\limits_{t_0}^{t} {}^{RL,t}D^{\alpha}_{s+}K(t,s)\mathrm{d}s. \tag{2.3}$$

If we have $K(t,s) = f(t-s)g(s)$, $t_0 = 0$ and assumptions of Theorem 2.1 are fulfilled, then following equality holds true for convolution operator in Riemann-Liouville sense for any $n \in \mathbb{N}$:

$$\begin{aligned} {}^{RL}D^{\alpha}_{0+} \int\limits_{0}^{t} f(t-s)g(s)\mathrm{d}s = &\sum_{l=1}^{n} \lim_{s\to t-0} {}^{RL,t}D^{\alpha-l}_{s+} f(t-s) \frac{d^{l-1}}{dt^{l-1}} \lim_{s\to t-0} g(s) \\ &+ \int\limits_{0}^{t} {}^{RL,t}D^{\alpha}_{s+}f(t-s)g(s)\mathrm{d}s, \quad t > 0. \end{aligned} \tag{2.4}$$

where ${}^{RL,t}D^{\gamma}_{t_0+}K(t,s)$ is a partial Riemann-Liouville fractional differentiation operator of order $\gamma > 0$ [21] with respect to t of a function $K(t,s)$ of two variables with lower terminal t_0 and $J = [t_0, T]$, $\hat{J} = (t_0, T)$.

In the special cases, Riemann-Riouville type differentiation under integral sign holds for convolution operator [17]:

- If $\alpha \in (0, 1]$, then

$$\begin{aligned} {}^{RL}D^{\alpha}_{0+} \int\limits_{0}^{t} f(t-s)g(s)\mathrm{d}s = &\lim_{s\to t-0} {}^{RL,t}D^{\alpha-1}_{s+} f(t-s) \lim_{s\to t-0} g(s) \\ &+ \int\limits_{0}^{t} {}^{RL,t}D^{\alpha}_{s+}f(t-s)g(s)\mathrm{d}s, \quad t > 0; \end{aligned}$$

- If $\alpha \in (1, 2]$, then

$$\begin{aligned} {}^{RL}D^{\alpha}_{0+} \int\limits_{0}^{t} f(t-s)g(s)\mathrm{d}s = &\lim_{s\to t-0} {}^{RL,t}D^{\alpha-1}_{s+} f(t-s) \lim_{s\to t-0} g(s) \\ &+ \lim_{s\to t-0} {}^{RL,t}D^{\alpha-2}_{s+} f(t-s) \lim_{s\to t-0} g(s) + \int\limits_{0}^{t} {}^{RL,t}D^{\alpha}_{s+}f(t-s)g(s)\mathrm{d}s, \quad t > 0. \end{aligned}$$

Definition 2.3. [21, 37] The Caputo fractional derivative of order, $n-1<\alpha\leq n$, $n\in\mathbb{N}$ for a function $g\in([0,\infty),\mathbb{R})$ is defined by

$$({}^{C}D^{\alpha}_{0+}g)(t)=\frac{1}{\Gamma(n-\alpha)}\int\limits_{0}^{t}(t-s)^{n-\alpha-1}\left(\frac{d}{ds}\right)^{n}g(s)\,\mathrm{d}s,\quad t>0, \tag{2.5}$$

where the function $g(\cdot)$ has absolutely continuous derivatives up to order n.

Definition 2.4. [21, 37] The relationship between Caputo and Riemann-Liouville fractional differential operators of order $n-1<\alpha\leq n$, $n\in\mathbb{N}$ for a function $g\in([0,\infty),\mathbb{R})$ is defined by

$$({}^{C}D^{\alpha}_{0+}g)(t)={}^{RL}D^{\alpha}_{0+}\left(g(t)-\sum_{k=0}^{n-1}\frac{t^k}{k!}g^{(k)}(0)\right),\quad t>0, \tag{2.6}$$

where the function $g(\cdot)$ has absolutely continuous derivatives up to order n.

Remark 2.1. If $g(\cdot)$ is an abstract function with values in X, then the integrals which appear in Definition 2.1, 2.2, 2.3 and 2.4 are taken in Bochner's sense.

The Laplace transform of the Caputo's fractional differentiation operator [21] is defined by

$$\mathscr{L}\left\{({}^{C}D^{\alpha}_{0+}g)(t)\right\}(s)=s^{\alpha}G(s)-\sum_{k=1}^{n}s^{\alpha-k}g^{(k-1)}(0),\quad n-1<\alpha\leq n,\quad n\in\mathbb{N}, \tag{2.7}$$

where $G(s)=\mathscr{L}\left\{g(t)\right\}(s)$.

In the particular cases, the Laplace integral transform of the Caputo fractional derivative is:

- If $\alpha\in(0,1]$, then
$$\mathscr{L}\left\{\left({}^{C}D^{\alpha}_{0+}x\right)(t)\right\}(s)=s^{\alpha}X(s)-s^{\alpha-1}x(0);$$
- If $\alpha\in(1,2]$, then
$$\mathscr{L}\left\{\left({}^{C}D^{\alpha}_{0+}x\right)(t)\right\}(s)=s^{\alpha}X(s)-s^{\alpha-1}x(0)-s^{\alpha-2}x'(0),$$

where $X(s)=\mathscr{L}\left\{x(t)\right\}(s)$.

Lemma 2.1 ([51])**.** *Suppose that A is linear bounded operator defined on the Banach space X and assume that $\|A\|<1$. Then, $(I-A)^{-1}$ is linear bounded on X and*

$$(I-A)^{-1}=\sum_{k=0}^{\infty}A^k. \tag{2.8}$$

The following well-known generalized Gronwall inequality which plays an important role in the qualitative analysis of the solutions to fractional differential equations is stated and proved in [14, 50] for $\beta>0$. In particular case, if $\beta=1$, then the following relations hold true:

Theorem 2.2. *Suppose $a(t)$ is a nonnegative function locally integrable on $0\leq t<T$ (some $T\leq+\infty$), $b(t)$ is a nonnegative, nondecreasing continuous function defined on $0\leq t<T$, $|b(t)|\leq M$, (M is a positive constant) and suppose $u(t)$ is a nonnegative and locally integrable on $0\leq t<T$ with*

$$u(t)\leq a(t)+b(t)\int\limits_{0}^{t}u(s)ds,$$

on this interval; then

$$u(t)\leq a(t)+b(t)\int\limits_{0}^{t}\exp\left(b(t)(t-s)\right)a(s)ds,\quad 0\leq t<T.$$

Corollary 2.1. *Under the hypothesis of Theorem 2.2, let $a(t)$ be a nondecreasing function on $[0,T)$. Then*

$$u(t) \le a(t) \exp\left(b(t)t\right), \quad 0 \le t < T. \tag{2.9}$$

The Mittag-Leffler function is a natural generalization of the exponential function, first proposed as a single parameter function of one variable by using an infinite series [35]. Extensions to two or three parameters are well known and thoroughly studied in textbooks such as [13]. Extensions to two or several variables have been studied more recently [1, 12, 18, 45].

Definition 2.5 ([35])**.** The classical Mittag-Leffler function is defined by

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)}, \quad \alpha > 0, \quad t \in \mathbb{R}. \tag{2.10}$$

The two-parameter Mttag-Leffler function [49] is given by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)}, \quad \alpha > 0, \quad \beta \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{2.11}$$

The three-parameter Mittag-Leffler function [42] is determined by

$$E^{\gamma}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(k\alpha+\beta)} \frac{t^k}{k!}, \quad \alpha > 0, \quad \beta, \gamma \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{2.12}$$

where $(\gamma)_k$ is the Pochhammer symbol denoting $\frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$. These series are convergent, locally uniformly in t, provided the $\alpha > 0$ condition is satisfied. It is important to note that

$$E^1_{\alpha,\beta}(t) = E_{\alpha,\beta}(t), \quad E_{\alpha,1}(t) = E_\alpha(t), \quad E_1(t) = \exp(t).$$

Lemma 2.2 ([42])**.** *The Laplace transform of the three-parameter Mittag-Leffler function is given by*

$$\mathscr{L}\left\{t^{\beta-1} E^{\gamma}_{\alpha,\beta}(\lambda t^\alpha)\right\}(s) = s^{-\beta}\left(1 - \lambda s^{-\alpha}\right)^{-\gamma}, \tag{2.13}$$

where $\alpha > 0, \quad \beta, \gamma, \lambda \in \mathbb{R}$ and $Re(s) > 0$.

Definition 2.6. [12] A bivariate Mittag-Leffler type function which is a particular case of multivariate Mittag-Leffler function [26] is defined by

$$E_{\alpha,\beta,\gamma}(\lambda_1 x^\alpha, \lambda_2 y^\beta) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k+m}{m} \frac{\lambda_1^k \lambda_2^m x^{k\alpha} y^{m\beta}}{\Gamma(k\alpha + m\beta + \gamma)}, \quad \alpha, \beta > 0, \quad \gamma \in \mathbb{R}, \quad x, y \in \mathbb{R}. \tag{2.14}$$

Univariate version of bivariate Mittag-Leffler function (2.14) is defined by

$$E_{\alpha,\beta,\gamma}(\lambda_1 t^\alpha, \lambda_2 t^\beta) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k+m}{m} \frac{\lambda_1^k \lambda_2^m t^{k\alpha+m\beta}}{\Gamma(k\alpha + m\beta + \gamma)}, \quad \alpha, \beta > 0, \quad \gamma \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{2.15}$$

3 A representation of mild solution to (1.1) with non-permutable linear operators

In this section, we consider the Cauchy problem for fractional functional evolution equation of Sobolev type in a Banach space. Firstly, we introduce the following hypotheses on the linear operators A_0, B_0 and E:

(H_1): A_0 is a closed operator;

(H_2): B_0 is a bounded operator;

(H_3): $D(E) \subset D(A_0)$ and E is bijective;

(H_4): A linear operator $E^{-1}: Y \to D(E) \subset X$ is compact.

It is important to stress out that (H_4) implies E^{-1} is bounded. Furthermore, (H_4) also implies that E is closed since the fact: E^{-1} is closed and injective, then its inverse is also closed. It comes from the closed graph theorem, we acquire the boundedness of the linear operator $A \coloneqq A_0 E^{-1}: Y \to Y$. Furthermore, $B \coloneqq B_0 E^{-1}: Y \to Y$ is a linear bounded operator since E^{-1} and B_0 are bounded.

Obviously, the substitution $Ey(t) = x(t)$ is equivalent to $y(t) = E^{-1}x(t)$. The central idea is that applying the substitution $y(t) = E^{-1}x(t)$, under the hypotheses $(H_1) - (H_4)$, we transform the Sobolev type fractional-order functional evolution system (1.1) to the following multi-term evolution system with a constant delay and linear bounded operators $A, B \in \mathscr{B}(Y)$:

$$\begin{cases} \left({}^C D^{\alpha}_{0+}x\right)(t) - A\left({}^C D^{\beta}_{0+}x\right)(t) = Bx(t-\tau) + g(t), \quad \tau > 0, \quad t > 0, \\ x(t) = \varphi(t), \quad -\tau \le t \le 0, \\ x'(0) = \varphi'(0), \end{cases} \tag{3.1}$$

where $x(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$, $\varphi(\cdot) \in \mathbb{C}(\mathbb{I}, Y)$ and $\varphi(\cdot)$ is continuously differentiable at $t = 0$.

This signifies that a mild solution of the Cauchy problem for Sobolev type multi-term fractional functional evolution equation (1.1) is the multiplication of $E^{-1} \in \mathscr{B}(Y)$ and the solution of an initial value problem for fractional time-delay evolution equation with multi-orders and linear bounded operators (3.1).

Remark 3.1. Alternatively, we can modify the assumptions which are given above in a similar way:

(H_1'): A_0 is a bounded operator;

(H_2'): B_0 is a closed operator;

(H_3'): $D(E) \subset D(B_0)$ and E is bijective;

(H_4'): $E^{-1}: Y \to D(E) \subset X$ is compact.

It follows from the closed graph theorem $B \coloneqq B_0 E^{-1}: Y \to Y$ is a linear bounded operator. Furthermore, $A \coloneqq A_0 E^{-1}: Y \to Y$ is also a linear bounded operator since A_0 and E^{-1} are bounded. In conclusion, under the assumptions $(H_1')-(H_4')$, the Sobolev type fractional functional multi-term evolution equation with initial conditions (1.1) is converted to the fractional evolution system with a constant delay and linear bounded operators (3.1) by using the same transformation $y(t) = E^{-1}x(t)$.

To get an analytical representation of the mild solution of (3.1), first, we need to show that exponentially boundedness of $x(\cdot)$ and its Caputo derivatives $\left({}^C D^{\alpha}_{0+}x\right)(\cdot)$,$\left({}^C D^{\beta}_{0+}x\right)(\cdot)$ for $1 < \alpha \le 2$ and $0 < \beta \le 1$, respectively. To do this, we need to assume exponential boundedness for one of the given fractional differentiation operators and a forced term with the aid of following Theorem 3.1.

Theorem 3.1. *Assume* (3.1) *has a unique continuous solution $x(t)$, if $g(t)$ is continuous & exponentially bounded and $\left({}^C D^{\beta}_{0+}x\right)(t)$ for $0 < \beta \le 1$ is exponentially bounded on $[0, \infty)$, then $x(t)$ and its Caputo derivative $\left({}^C D^{\alpha}_{0+}x\right)(t)$ is exponentially bounded for $1 < \alpha \le 2$ on $[0, \infty)$ and, thus, their Laplace transforms exist.*

Proof. Since $g(t)$ and $\left({}^C D^{\beta}_{0+}x\right)(t)$ for $0 < \beta \le 1$ is exponentially bounded, there exists positive constants L, P, δ and sufficient large T such that $\|g(t)\| \le L\exp(\delta t)$ and $\|\left({}^C D^{\beta}_{0+}x\right)(t)\| \le P\exp(\delta t)$ for any $t \ge T$. It is clear that the system (3.1) is equivalent to the following Volterra fractional integral equation of second-kind:

$$\begin{aligned} x(t) &= \left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\varphi(0) + t\varphi'(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^t (t-r)^{\alpha-\beta-1}x(r)dr \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}[Bx(r-\tau) + g(r)]dr, \quad t > 0, \quad \tau > 0. \end{aligned} \tag{3.2}$$

This means that every solution of (3.2) is also a solution of (3.1) and vice versa. For $t \ge T$, (3.2) can be expressed as

$$x(t) = \left(1 - \frac{At^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\varphi(0) + t\varphi'(0) + \frac{A}{\Gamma(\alpha-\beta)}\int_0^T (t-r)^{\alpha-\beta-1}x(r)dr$$

$$+ \frac{1}{\Gamma(\alpha)}\int_0^T (t-r)^{\alpha-1}[Bx(r-\tau)+g(r)]dr + \frac{A}{\Gamma(\alpha-\beta)}\int_T^t (t-r)^{\alpha-\beta-1}x(r)dr$$

$$+ \frac{1}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}[Bx(r-\tau)+g(r)]dr, \quad t>0, \quad \tau>0.$$

In view of hypotheses of Theorem 3.1, the solution $x(t), (x(0)=\varphi(0), x'(0)=\varphi'(0))$ is unique and continuous on $[0,\infty)$, then $Ax(t)$ and $Bx(t)+g(t)$ are bounded on $[0,T]$, namely:

$$\exists M>0 \quad \text{s.t.} \quad \|Ax(t)\| \le M, \quad \forall t\in[0,T],$$

and

$$\exists N>0 \quad \text{s.t.} \quad \|Bx(t-\tau)+g(t)\| \le N, \quad \forall t\in[0,T], \quad \tau>0.$$

Let $z(t) = \max\{\|x(t+h)\| : h\in[-\tau,0]\}$. Then, we get

$$\|z(t)\| \le \left(1 + \frac{\|A\|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\|\varphi(0)\| + t\|\varphi'(0)\| + \frac{M}{\Gamma(\alpha-\beta)}\int_0^T (t-r)^{\alpha-\beta-1}dr$$

$$+ \frac{N}{\Gamma(\alpha)}\int_0^T (t-r)^{\alpha-1}dr + \frac{\|A\|}{\Gamma(\alpha-\beta)}\int_T^t (t-r)^{\alpha-\beta-1}\|z(r)\|dr$$

$$+ \frac{\|B\|}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}\|z(r)\|dr + \frac{1}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}\|g(r)\|dr.$$

Multiplying last inequality by $\exp(-\delta t)$ and note that

$$\exp(-\delta t) \le \exp(-\delta r), \quad r\in[T,t] \quad \text{and} \quad \exp(-\delta t) \le \exp(-\delta T), \quad \|g(t)\| \le L\exp(\delta t), \quad t\ge T.$$

Using the aforementioned inequalities, we attain

$$\|z(t)\|\exp(-\delta t) \le \|\varphi(0)\|\exp(-\delta t) + \frac{\|A\|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|\varphi(0)\|\exp(-\delta t)$$

$$+ t\|\varphi'(0)\|\exp(-\delta t) + \frac{M\exp(-\delta t)}{\Gamma(\alpha-\beta)}\int_0^T (t-r)^{\alpha-\beta-1}dr$$

$$+ \frac{N\exp(-\delta t)}{\Gamma(\alpha)}\int_0^T (t-r)^{\alpha-1}dr + \frac{\|A\|\exp(-\delta t)}{\Gamma(\alpha-\beta)}\int_T^t (t-r)^{\alpha-\beta-1}\|z(r)\|dr$$

$$+ \frac{\|B\|\exp(-\delta t)}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}\|z(r)\|dr + \frac{\exp(-\delta t)}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}\|g(r)\|dr$$

$$\le \|\varphi(0)\|\exp(-\delta T) + \frac{\|A\|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|\varphi(0)\|\exp(-\delta T) + t\|\varphi'(0)\|\exp(-\delta T)$$

$$+ \frac{M\exp(-\theta T)}{\Gamma(\alpha-\beta+1)}(t^{\alpha-\beta} - (t-T)^{\alpha-\beta}) + \frac{N\exp(-\delta T)}{\Gamma(\alpha+1)}(t^{\alpha} - (t-T)^{\alpha})$$

$$
\begin{aligned}
&+\frac{\|A\|}{\Gamma(\alpha-\beta)}\int_T^t (t-r)^{\alpha-\beta-1}\|z(r)\|\exp(-\delta r)dr\\
&+\frac{\|B\|}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}\|z(r)\|\exp(-\delta r)dr\\
&+\frac{L}{\Gamma(\alpha)}\int_T^t (t-r)^{\alpha-1}\exp(\delta(r-t))dr\\
&\le \|\varphi(0)\|\exp(-\delta T)+\frac{\|A\|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|\varphi(0)\|\exp(-\delta T)\\
&+t\|\varphi'(0)\|\exp(-\delta T)+\frac{M\exp(-\delta T)}{\Gamma(\alpha-\beta+1)}T^{\alpha-\beta}+\frac{N\exp(-\delta T)}{\Gamma(\alpha+1)}T^{\alpha}\\
&+\int_0^t\left(\frac{\|A\|(t-r)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{\|B\|(t-r)^{\alpha-1}}{\Gamma(\alpha)}\right)\|z(r)\|\exp(-\delta r)dr\\
&+\frac{L}{\Gamma(\alpha)}\int_0^t (t-r)^{\alpha-1}\exp(-\delta(t-r))dr\\
&\le \|\varphi(0)\|\exp(-\delta T)+\frac{\|A\|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|\varphi(0)\|\exp(-\delta T)\\
&+t\|\varphi'(0)\|\exp(-\delta T)+\frac{M\exp(-\delta T)}{\Gamma(\alpha-\beta+1)}T^{\alpha-\beta}+\frac{N\exp(-\delta T)}{\Gamma(\alpha+1)}T^{\alpha}\\
&+\left(\frac{\|A\|t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{\|B\|t^{\alpha-1}}{\Gamma(\alpha)}\right)\int_0^t \|z(r)\|\exp(-\delta r)dr+\frac{L}{\delta^{\alpha}},\quad t\ge T.
\end{aligned}
$$

Denote

$$
\begin{cases}
a(t)=\frac{\|A\|t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|\varphi(0)\|\exp(-\delta T)+t\|\varphi'(0)\|\exp(-\delta T)+\|\varphi(0)\|\exp(-\delta T)\\
\qquad +\frac{M\exp(-\delta T)}{\Gamma(\alpha-\beta+1)}T^{\alpha-\beta}+\frac{N\exp(-\delta T)}{\Gamma(\alpha+1)}T^{\alpha}+\frac{L}{\delta^{\alpha}},\\
b(t)=\frac{\|A\|t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+\frac{\|B\|t^{\alpha-1}}{\Gamma(\alpha)},\\
v(t)=\|z(t)\|\exp(-\delta t).
\end{cases}
$$

Thus, we attain

$$
v(t)\le a(t)+b(t)\int_0^t v(s)ds,\quad t\ge T. \tag{3.3}
$$

According to the Gronwall's inequality (2.9), we have

$$
v(t)\le a(t)\exp(tb(t))\le \exp(a(t)+tb(t)). \tag{3.4}
$$

Then, it yields from (3.4) that

$$
\|x(t)\|\le\|z(t)\|\le\exp(a(t)+tb(t)+\delta t),\quad t\ge T.
$$

Since $g(t)$ and $\left({}^{C}D^{\beta}_{0+}x\right)(t)$ for $\beta\in(0,1]$ are exponentially bounded on $[0,\infty)$, from equation (3.1), we acquire

$$
\|\left({}^{C}D^{\alpha}_{0+}x\right)(t)\|\le\|\left({}^{C}D^{\alpha}_{0+}z\right)(t)\|\le\|A\|\|\left({}^{C}D^{\beta}_{0+}z\right)(t)\|+\|B\|\|z(t)\|+\|g(t)\|
$$

$$\leq \|A\|P\exp(\delta t) + \|B\|\exp(a(t)+tb(t)+\delta t) + L\exp(\delta t)$$
$$\leq (\|A\|P + \|B\| + L)\exp(a(t)+tb(t)+\delta t), \quad t \geq T.$$

In other words, $\left({}^{C}D^{\alpha}_{0+}x\right)(t)$ is also exponentially bounded, the Laplace integral transforms of $x(t)$ and its Caputo derivatives $\left({}^{C}D^{\alpha}_{0+}x\right)(t)$, $\left({}^{C}D^{\beta}_{0+}x\right)(t)$ exist for $\alpha \in (1,2]$ and $\beta \in (0,1]$, respectively. The proof is complete. □

Alternatively, we can also use the following version of Theorem 3.1, for exponential boundedness of $x(\cdot)$ and its derivatives $\left({}^{C}D^{\alpha}_{0+}x\right)(\cdot)$,$\left({}^{C}D^{\beta}_{0+}x\right)(\cdot)$ of order $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively in Caputo's sense on $[0,\infty)$.

Theorem 3.2. *Assume* (3.1) *has a unique continuous solution $x(t)$, if $g(t)$ is continuous & exponentially bounded and $\left({}^{C}D^{\alpha}_{0+}x\right)(t)$ for $1 < \alpha \leq 2$ is exponentially bounded on $[0,\infty)$, then $x(t)$ and its Caputo derivative $\left({}^{C}D^{\beta}_{0+}x\right)(t)$ is exponentially bounded for $0 < \beta \leq 1$ on $[0,\infty)$ and, thus, their Laplace transforms exist.*

Proof. This proof is similar to the proof of Theorem 3.1. So, we omit it here. □

Definition 3.1. We define a new delayed Mittag-Leffler function $\mathscr{E}^{A,B,\tau}_{\alpha,\beta,\gamma}(\cdot) : \mathbb{R} \to Y$ generated by nonpermutable linear bounded operators $A, B \in \mathscr{B}(Y)$ for $\tau > 0$ as follows:

$$\mathscr{E}^{A,B,\tau}_{\alpha,\beta,\gamma}(t) := \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} Q^{A,B}_{k,m}\frac{(t-m\tau)^{k\alpha+m\beta+\gamma-1}}{\Gamma(k\alpha+m\beta+\gamma)}\mathcal{H}(t-m\tau), \quad \alpha,\beta > 0, \quad \gamma \in \mathbb{R}, \quad t \in \mathbb{R}, \tag{3.5}$$

where $\mathcal{H}(\cdot) : \mathbb{R} \to \mathbb{R}$ is a well-known Heaviside function which is determined by

$$\mathcal{H}(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

and a linear bounded operator $Q^{A,B}_{k,m} \in \mathscr{B}(Y)$, $k, m \in \mathbb{N}_0$ is defined by

$$Q^{A,B}_{k,m} := \sum_{l=0}^{k} A^{k-l}BQ^{A,B}_{l,m-1}, \quad k,m \in \mathbb{N}, \qquad Q^{A,B}_{k,0} := A^k, \quad k \in \mathbb{N}_0, \qquad Q^{A,B}_{0,m} := B^m, \quad m \in \mathbb{N}_0. \tag{3.6}$$

A linear bounded operator $Q^{A,B}_{k,m} \in \mathscr{B}(Y)$ is represented explicitly in Table 1.

Table 1: Explicit representation of $Q^{A,B}_{k,m}$ for $r, s \in \mathbb{N}_0$

$Q^{A,B}_{k,m}$	k=0	k=1	k=2	…	k=r
$m=0$	I	A	A^2	…	A^r
$m=1$	B	$AB+BA$	$A^2B+ABA+BA^2$	…	$A^rB+\ldots+BA^r$
$m=2$	B^2	$AB^2+BAB+B^2A$	$A^2B^2+ABAB+AB^2A$ $+BA^2B+BABA+B^2A^2$	…	$A^rB^2+\ldots+B^2A^r$
…	…	…	…	…	…
$m=s$	B^s	$AB^s+\ldots+B^sA$	$A^2B^s+\ldots+B^sA^2$	…	$A^rB^s+\ldots+B^sA^r$

From the above table, it can be easily seen that, in the case of commutativity $AB = BA$, we have $Q^{A,B}_{k,m} := \binom{k+m}{m}A^kB^m$, $k, m \in \mathbb{N}_0$.

If we consider a delayed Mittag-Leffler type function $\mathscr{E}^{A,B,\tau}_{\alpha,\beta,\gamma}(t)$ on $t \in \mathbb{J} = [-\tau, T]$, then we derive the piece-wise function as follows.

Definition 3.2. A new delayed Mittag-Leffler type function of three parameters $\mathscr{E}^{A,B,\tau}_{\alpha,\beta,\gamma} : \mathbb{J} \to Y$ for $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$ and $\tau > 0$ is defined by

$$\mathscr{E}^{A,B,\tau}_{\alpha,\beta,\gamma}(t) := \begin{cases} \Theta, & -\tau \leq t \leq 0, \\ \sum\limits_{k=0}^{\infty} \sum\limits_{m=0}^{n-1} Q^{A,B}_{k,m} \frac{(t-m\tau)^{k\alpha+m\beta+\gamma-1}}{\Gamma(k\alpha+m\beta+\gamma)}, & (n-1)\tau < t \leq n\tau, \quad n \in \mathbb{N}. \end{cases} \tag{3.7}$$

where Θ is a null operator.

Furthermore, the delayed Mittag-Leffler type function (3.7) can also be represented step by step on $\mathbb{J} = [-\tau, T] := [-\tau, 0] \cup (0, \tau] \cup (\tau, 2\tau] \cup \ldots \cup ((n-1)\tau, T]$, where $T = n\tau$ for a fixed $n \in \mathbb{N}$,

$$\mathscr{E}^{A,B,\tau}_{\alpha,\beta,\gamma}(t) = \begin{cases} \Theta, \quad -\tau \leq t \leq 0, \\ t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}), \quad 0 < t \leq \tau, \\ t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) + \sum\limits_{k=0}^{\infty} Q^{A,B}_{k,1} \frac{(t-\tau)^{k\alpha+\beta+\gamma-1}}{\Gamma(k\alpha+\beta+\gamma)}, \quad \tau < t \leq 2\tau, \\ t^{\gamma-1} E_{\alpha,\gamma}(At^{\alpha}) + \sum\limits_{k=0}^{\infty} Q^{A,B}_{k,1} \frac{(t-\tau)^{k\alpha+\beta+\gamma-1}}{\Gamma(k\alpha+\beta+\gamma)} + \\ + \sum\limits_{k=0}^{\infty} Q^{A,B}_{k,2} \frac{(t-2\tau)^{k\alpha+2\beta+\gamma-1}}{\Gamma(k\alpha+2\beta+\gamma)} + \cdots + \sum\limits_{k=0}^{\infty} Q^{A,B}_{k,n-1} \frac{(t-(n-1)\tau)^{k\alpha+(n-1)\beta+\gamma-1}}{\Gamma(k\alpha+(n-1)\beta+\gamma)}, \quad (n-1)\tau < t \leq n\tau. \end{cases} \tag{3.8}$$

Lemma 3.1. *A linear operator $Q^{A,B}_{k,m} \in \mathscr{B}(Y)$ for $k, m \in \mathbb{N}_0$ has the following properties:*

(*i*) *$Q^{A,B}_{k,m}$, $k, m \in \mathbb{N}$ generalizes classical Pascal's rule for linear operators $A, B \in \mathscr{B}(Y)$ as follows:*

$$Q^{A,B}_{k,m} = AQ^{A,B}_{k-1,m} + BQ^{A,B}_{k,m-1}, \quad k, m \in \mathbb{N}; \tag{3.9}$$

(*ii*) *If $AB = BA$, then we have*

$$Q^{A,B}_{k,m} = \binom{k+m}{m} A^k B^m, \quad k, m \in \mathbb{N}_0. \tag{3.10}$$

Proof. Lemma can be easily prove via a mathematical induction principle. Thus, we omit it here. □

According to the above lemma, a linear bounded operator $Q^{A,B}_{k,m}$ for $k, m \in \mathbb{N}$ satisfies the following Pascal's rule for permutable linear operators $A, B \in \mathscr{B}(Y)$ as below:

$$\binom{k+m}{m} A^k B^m = \binom{k+m-1}{m} A^{k-1} B^m + \binom{k+m-1}{m-1} A^k B^{m-1}, \quad k, m \in \mathbb{N}. \tag{3.11}$$

By using the property of $Q^{A,B}_{k,m} \in \mathscr{B}(Y)$ (3.10) we define the following delayed univariate version of a bivariate Mittag-Leffler function via permutable linear bounded operators which is a delayed analogue of (2.15).

Definition 3.3. We define a new delayed analogue of univariate form of a bivariate Mittag-Leffler type function $E^{A,B,\tau}_{\alpha,\beta,\gamma}(\cdot) : \mathbb{R} \to Y$ generated by permutable linear bounded operators $A, B \in \mathscr{B}(Y)$ for $\tau > 0$ as follows:

$$E^{A,B,\tau}_{\alpha,\beta,\gamma}(t) := \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \binom{k+m}{m} A^k B^m \frac{(t-m\tau)^{k\alpha+m\beta)+\gamma-1}}{\Gamma(k\alpha+m\beta+\gamma)} \mathcal{H}(t-m\tau), \quad \alpha, \beta > 0, \quad \gamma \in \mathbb{R}, \quad t \in \mathbb{R}. \tag{3.12}$$

If we consider a delayed Mittag-Leffler function (3.12) with permutable operators on $\mathbb{J} = [-\tau, T]$, then we derive the piece-wise function as follows.

Definition 3.4. A new delayed Mittag-Leffler type function of three parameters $E^{A,B,\tau}_{\alpha,\beta,\gamma} : \mathbb{J} \to Y$ generated by permutable linear operators $A, B \in \mathscr{B}(Y)$ for $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$, $\tau > 0$ is defined by

$$E^{A,B,\tau}_{\alpha,\beta,\gamma}(t) := \begin{cases} \Theta, & -\tau \leq t \leq 0, \\ \sum\limits_{k=0}^{\infty} \sum\limits_{m=0}^{n-1} \binom{k+m}{m} A^k B^m \frac{(t-m\tau)^{k\alpha+m\beta+\gamma-1}}{\Gamma(k\alpha+m\beta+\gamma)}, & (n-1)\tau < t \leq n\tau, \quad n \in \mathbb{N}. \end{cases} \tag{3.13}$$

In addition, (3.13) can be expressed via the following piece-wise function on $t \in \mathbb{J} := [-\tau, 0] \cup \bigcup_{i=1}^{n} ((i-1)\tau, i\tau]$ for a fixed $n \in \mathbb{N}$:

$$E^{A,B,\tau}_{\alpha,\beta,\gamma}(t) = \begin{cases} \Theta, \quad -\tau \leq t \leq 0, \\ t^{\gamma-1}E_{\alpha,\gamma}(At^{\alpha}), \quad 0 < t \leq \tau, \\ t^{\gamma-1}E_{\alpha,\gamma}(At^{\alpha}) + (t-\tau)^{\beta+\gamma-1}E^{2}_{\alpha,\beta+\gamma}(A(t-\tau)^{\alpha})B, \qquad \tau < t \leq 2\tau, \\ t^{\gamma-1}E_{\alpha,\gamma}(At^{\alpha}) + (t-\tau)^{\beta+\gamma-1}E^{2}_{\alpha,\beta+\gamma}(A(t-\tau)^{\alpha})B + \\ + \cdots + (t-(n-1)\tau)^{(n-1)\beta+\gamma-1}E^{n}_{\alpha,(n-1)\beta+\gamma}(A(t-(n-1)\tau)^{\alpha})B^{n-1}, \quad (n-1)\tau < t \leq n\tau. \end{cases} \tag{3.14}$$

The following lemma plays a crucial role for solving the Cauchy problem for functional fractional evolution equation with linear bounded operators (3.1). In general case, it holds true whenever $\alpha > 0$, $\alpha > \beta$, $\alpha > \gamma$.

Lemma 3.2. *For $A, B \in \mathscr{B}(Y)$ which are satisfying $AB \neq BA$, we have:*

$$\begin{aligned} &\mathscr{L}^{-1}\left\{\frac{s^{\gamma}}{e^{ms\tau}s^{(m+1)\beta}}\left[(s^{\alpha-\beta}I - A)^{-1}B\right]^{m}(s^{\alpha-\beta}I - A)^{-1}\right\}(t) \\ &= \sum_{k=0}^{\infty}\frac{Q^{A,B}_{k,m}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\gamma)}(t-m\tau)^{k(\alpha-\beta)+m\alpha+\alpha-\gamma-1}\mathcal{H}(t-m\tau), \quad t \in \mathbb{R}, \quad m \in \mathbb{N}_0. \end{aligned} \tag{3.15}$$

Proof. To prove, we will use a mathematical induction principle with regard to $m \in \mathbb{N}_0$. Obviously, according to the relation (2.13), (3.15) is true for $m = 0$, which establishes the basis for induction:

$$\begin{aligned} &\mathscr{L}^{-1}\left\{s^{\gamma-\beta}(s^{\alpha-\beta}I - A)^{-1}\right\}(t) = t^{\alpha-\gamma-1}E^{1}_{\alpha-\beta,\alpha-\gamma}(At^{\alpha-\beta})\mathcal{H}(t) \\ &= t^{\alpha-\gamma-1}E_{\alpha-\beta,\alpha-\gamma}(At^{\alpha-\beta})\mathcal{H}(t) = \sum_{k=0}^{\infty}A^{k}\frac{t^{k(\alpha-\beta)+\alpha-\gamma-1}}{\Gamma(k(\alpha-\beta)+\alpha-\gamma)}\mathcal{H}(t) \\ &= \sum_{k=0}^{\infty}Q^{A,B}_{k,0}\frac{t^{k(\alpha-\beta)+\alpha-\gamma-1}}{\Gamma(k(\alpha-\beta)+\alpha-\gamma)}\mathcal{H}(t), \quad t \in \mathbb{R}, \quad \text{where} \quad Q^{A,B}_{k,0} := A^{k}, \quad k \in \mathbb{N}_0. \end{aligned} \tag{3.16}$$

For $m = 1$, we use the convolution property of Laplace integral transform and formula (3.16):

$$\begin{aligned} &\mathscr{L}^{-1}\left\{e^{-s\tau}s^{\gamma-2\beta}(s^{\alpha-\beta}I - A)^{-1}B(s^{\alpha-\beta}I - A)^{-1}\right\}(t) \\ =&\mathscr{L}^{-1}\left\{e^{-s\tau}s^{-\beta}(s^{\alpha-\beta}I - A)^{-1}B\right\}(t) * \mathscr{L}^{-1}\left\{s^{\gamma-\beta}(s^{\alpha-\beta}I - A)^{-1}\right\}(t) \\ =&(t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau)^{\alpha-\beta})\mathcal{H}(t-\tau)B * t^{\alpha-\gamma-1}E_{\alpha-\beta,\alpha-\gamma}(At^{\alpha-\beta})\mathcal{H}(t) \\ =&\int_{0}^{t}(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})\mathcal{H}(t-\tau-s)Bs^{\alpha-\gamma-1}E_{\alpha-\beta,\alpha-\gamma}(As^{\alpha-\beta})\mathcal{H}(s)\mathrm{d}s \\ =&\int_{0}^{t-\tau}(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})Bs^{\alpha-\gamma-1}E_{\alpha-\beta,\alpha-\gamma}(As^{\alpha-\beta})\mathrm{d}s \end{aligned} \tag{3.17}$$

Then interchanging the order of integration and summation in (3.17) which is permissible in accordance with the uniform convergence of the series (2.11), we attain:

$$\begin{aligned} &\mathscr{L}^{-1}\left\{e^{-s\tau}s^{\gamma-2\beta}(s^{\alpha-\beta}I - A)^{-1}B(s^{\alpha-\beta}I - A)^{-1}\right\}(t) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{A^{k}BA^{l}}{\Gamma(k(\alpha-\beta)+\alpha)\Gamma(l(\alpha-\beta)+\alpha-\gamma)}\int_{0}^{t-\tau}(t-\tau-s)^{k(\alpha-\beta)+\alpha-1}s^{l(\alpha-\beta)+\alpha-\gamma-1}\mathrm{d}s \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{A^{k}BA^{l}}{\Gamma(k(\alpha-\beta)+\alpha)\Gamma(l(\alpha-\beta)+\alpha-\gamma)}(t-\tau)^{(k+l)(\alpha-\beta)+2\alpha-\gamma-1}\mathcal{B}(k(\alpha-\beta)+\alpha, l(\alpha-\beta)+\alpha-\gamma) \end{aligned}$$

$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{A^kBA^l}{\Gamma((k+l)(\alpha-\beta)+2\alpha-\gamma)}(t-\tau)^{(k+l)(\alpha-\beta)+2\alpha-\gamma-1} \qquad (t>\tau)$$
$$=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{A^kBA^l}{\Gamma((k+l)(\alpha-\beta)+2\alpha-\gamma)}(t-\tau)^{(k+l)(\alpha-\beta)+2\alpha-\gamma-1}\mathcal{H}(t-\tau), \quad t\in\mathbb{R}, \tag{3.18}$$

where $\mathcal{B}(\cdot,\cdot)$ is a well-known beta function.

Applying the following eminent Cauchy product formula for absolutely convergent double infinity series

$$\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}a_kb_l=\sum_{k=0}^{\infty}\sum_{l=0}^{k}a_{k-l}b_l, \tag{3.19}$$

in (3.18), one can acquire that

$$\begin{aligned}
&\mathscr{L}^{-1}\left\{e^{-s\tau}s^{\gamma-2\beta}(s^{\alpha-\beta}I-A)^{-1}B(s^{\alpha-\beta}I-A)^{-1}\right\}(t)\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{A^{k-l}BA^l}{\Gamma(k(\alpha-\beta)+2\alpha-\gamma)}(t-\tau)^{k(\alpha-\beta)+2\alpha-\gamma-1}\mathcal{H}(t-\tau)\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{A^{k-l}BQ^{A,B}_{l,0}}{\Gamma(k(\alpha-\beta)+2\alpha-\gamma)}(t-\tau)^{k(\alpha-\beta)+2\alpha-\gamma-1}\mathcal{H}(t-\tau)\\
&=\sum_{k=0}^{\infty}\frac{Q^{A,B}_{k,1}}{\Gamma(k(\alpha-\beta)+\alpha+\alpha-\gamma)}(t-\tau)^{k(\alpha-\beta)+\alpha+\alpha-\gamma-1}\mathcal{H}(t-\tau), \quad t\in\mathbb{R},
\end{aligned} \tag{3.20}$$

where $\quad Q^{A,B}_{k,1}\coloneqq\sum\limits_{l=0}^{k}A^{k-l}BQ^{A,B}_{l,0}, \quad k\in\mathbb{N}_0.$

To verify the induction step, we assume that (3.15) holds true for $m=n$ where $n\in\mathbb{N}_0$:

$$\begin{aligned}
&\mathscr{L}^{-1}\left\{e^{-ns\tau}s^{\gamma-(n+1)\beta}\left[(s^{\alpha-\beta}I-A)^{-1}B\right]^n(s^{\alpha-\beta}I-A)^{-1}\right\}(t)\\
&=\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{A^{k-l}BQ^{A,B}_{l,n-1}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha-\gamma)}(t-n\tau)^{k(\alpha-\beta)+(n+1)\alpha-\gamma-1}\mathcal{H}(t-n\tau)\\
&=\sum_{k=0}^{\infty}\frac{Q^{A,B}_{k,n}}{\Gamma(k(\alpha-\beta)+n\alpha+\alpha-\gamma)}(t-n\tau)^{k(\alpha-\beta)+n\alpha+\alpha-\gamma-1}\mathcal{H}(t-n\tau), \quad t\in\mathbb{R},
\end{aligned} \tag{3.21}$$

where $\quad Q^{A,B}_{k,n}\coloneqq\sum\limits_{l=0}^{k}A^{k-l}BQ^{A,B}_{l,n-1}, \quad k\in\mathbb{N}_0.$

Then it yields that for $m=n+1$ as follows:

$$\begin{aligned}
&\mathscr{L}^{-1}\left\{e^{-(n+1)s\tau}s^{\gamma-(n+2)\beta}\left[(s^{\alpha-\beta}I-A)^{-1}B\right]^{n+1}(s^{\alpha-\beta}I-A)^{-1}\right\}(t)\\
&=\mathscr{L}^{-1}\left\{e^{-s\tau}s^{-\beta}(s^{\alpha-\beta}I-A)^{-1}B\right\}(t)*\mathscr{L}^{-1}\left\{e^{-ns\tau}s^{\gamma-(n+1)\beta}\left[(s^{\alpha-\beta}I-A)^{-1}B\right]^n(s^{\alpha-\beta}I-A)^{-1}\right\}(t)\\
&=(t-\tau)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau)^{\alpha-\beta})\mathcal{H}(t-\tau)B\\
&*\sum_{l=0}^{\infty}\frac{Q^{A,B}_{l,n}}{\Gamma(l(\alpha-\beta)+(n+1)\alpha-\gamma)}(t-n\tau)^{l(\alpha-\beta)+(n+1)\alpha-\gamma-1}\mathcal{H}(t-n\tau)\\
&=\int\limits_0^t(t-\tau-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta})\mathcal{H}(t-\tau-s)B\\
&\times\sum_{l=0}^{\infty}\frac{Q^{A,B}_{l,n}}{\Gamma(l(\alpha-\beta)+(n+1)\alpha-\gamma)}(s-n\tau)^{l(\alpha-\beta)+(n+1)\alpha-\gamma-1}\mathcal{H}(s-n\tau)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_{n\tau}^{t-\tau} (t-\tau-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-\tau-s)^{\alpha-\beta}) B \\
&\times \sum_{l=0}^{\infty} \frac{Q_{l,n}^{A,B}}{\Gamma(l(\alpha-\beta)+(n+1)\alpha-\gamma)} (s-n\tau)^{l(\alpha-\beta)+(n+1)\alpha-\gamma-1} \mathrm{d}s \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^k B Q_{l,n}^{A,B}}{\Gamma(k(\alpha-\beta)+\alpha)\Gamma(l(\alpha-\beta)+(n+1)\alpha-\gamma)} \int_{n\tau}^{t-\tau} (t-\tau-s)^{k(\alpha-\beta)+\alpha-1}(s-n\tau)^{l(\alpha-\beta)+(n+1)\alpha-\gamma-1}\mathrm{d}s \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^k B Q_{l,n}^{A,B}}{\Gamma(k(\alpha-\beta)+\alpha)\Gamma(l(\alpha-\beta)+(n+1)\alpha-\gamma)} (t-(n+1)\tau)^{(k+l)(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma-1}\mathcal{H}(t-(n+1)\tau) \\
&\times \mathcal{B}(k(\alpha-\beta)+\alpha, l(\alpha-\beta)+(n+1)\alpha-\gamma) \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{A^k B Q_{l,n}^{A,B}}{\Gamma((k+l)(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma)} (t-(n+1)\tau)^{(k+l)(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma-1}\mathcal{H}(t-(n+1)\tau) \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{k} \frac{A^{k-l} B Q_{l,n}^{A,B}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma)} (t-(n+1)\tau)^{k(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma-1}\mathcal{H}(t-(n+1)\tau) \\
&= \sum_{k=0}^{\infty} \frac{Q_{k,n+1}^{A,B}}{\Gamma(k(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma)} (t-(n+1)\tau)^{k(\alpha-\beta)+(n+1)\alpha+\alpha-\gamma-1}\mathcal{H}(t-(n+1)\tau), \quad t \in \mathbb{R}, \quad (3.22)
\end{aligned}
$$

$$
\text{where} \quad Q_{k,n+1}^{A,B} \coloneqq \sum_{l=0}^{k} A^{k-l} B Q_{l,n}^{A,B}, \quad k \in \mathbb{N}_0.
$$

Thus (3.22) holds true whenever (3.21) is true, and by the principle of mathematical induction, we conclude that the formula (3.15) holds true for all $m \in \mathbb{N}_0$. The proof is complete. ☐

Theorem 3.3. *Let $A, B \in \mathscr{B}(Y)$ with non-zero commutator, i.e., $[A, B] \coloneqq AB - BA \neq 0$. Assume that $g(\cdot) : \mathbb{J} \to Y$ and $\left({}^{C}D_{0+}^{\beta}x\right)(t)$ where $0 < \beta \leq 1$ (or $\left({}^{C}D_{0+}^{\alpha}x\right)(t)$ where $1 < \alpha \leq 2$) are exponentially bounded. A mild solution $x(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$ of the Cauchy problem (3.1) can be represented as*

$$
\begin{aligned}
x(t) &= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)} \right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0) \\
&+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} g(s)\mathrm{d}s \\
&= \left(I + \mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B \right)\varphi(0) + \mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\varphi'(0) \\
&+ \int_{-\tau}^{0} \mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t} \mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-s)g(s)\mathrm{d}s, \quad (n-1)\tau < t \leq n\tau. \quad (3.23)
\end{aligned}
$$

Proof. We recall that the existence of Laplace transform of $x(\cdot)$ and its Caputo derivatives ${}^{C}D_{0+}^{\alpha}x(\cdot)$ and ${}^{C}D_{0+}^{\beta}x(\cdot)$ for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively, is guaranteed by Theorem 3.1 or 3.2. Thus, to find the mild solution $x(t)$ of (3.1) satisfying the initial conditions $x(t) = \varphi(t)$ for $-\tau \leq t \leq 0$ and $x'(0) = \varphi'(0)$, we can use the Laplace integral transform. Taking the Laplace transform on both sides of equation (3.1) and using the following facts that

$$\mathscr{L}\left\{\left({}^C D^{\alpha}_{0+} x\right)(t)\right\}(s) = s^{\alpha} X(s) - s^{\alpha-1}\varphi(0) - s^{\alpha-2}\varphi'(0),$$
$$\mathscr{L}\left\{\left({}^C D^{\beta}_{0+} x\right)(t)\right\}(s) = s^{\beta} X(s) - s^{\beta-1}\varphi(0),$$

and for a delayed term, first, using substitution $t - \tau = \theta$, we obtain that

$$\begin{aligned}\mathscr{L}\left\{x(t-\tau)\right\}(s) &= \int_0^{\infty} e^{-st} x(t-\tau)\mathrm{d}t = e^{-s\tau}\int_{-\tau}^{\infty} e^{-s\theta}x(\theta)\mathrm{d}\theta \\ &= e^{-s\tau}\left(\int_{-\tau}^{0} e^{-s\theta}x(\theta)\mathrm{d}\theta + \int_0^{\infty} e^{-s\theta}x(\theta)\mathrm{d}\theta\right) \\ &= e^{-s\tau}X(s) + \int_{-\tau}^{0} e^{-s(\tau+\theta)}\varphi(\theta)\mathrm{d}\theta.\end{aligned}$$

Secondly, by making use of the substitution $\tau + \theta = r$, we acquire that

$$\begin{aligned}\mathscr{L}\left\{x(t-\tau)\right\}(s) &= e^{-s\tau}X(s) + \int_0^{\tau} e^{-sr}\varphi(r-\tau)\mathrm{d}r \\ &= e^{-s\tau}X(s) + \int_0^{\infty} e^{-st}\hat{\varphi}(t-\tau)\mathrm{d}t \\ &= e^{-s\tau}X(s) + \mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s),\end{aligned}$$

where $\hat{\varphi}(\cdot) : \mathbb{R} \to \mathbb{R}$ is the unit-step function defined as follows :

$$\hat{\varphi}(t) = \begin{cases}\varphi(t), & -\tau \leq t \leq 0, \\ 0, & t > 0.\end{cases} \tag{3.24}$$

Therefore, we derive that

$$\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)X(s) = s^{\alpha-1}\varphi(0) - As^{\beta-1}\varphi(0) + s^{\alpha-2}\varphi'(0) + B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s) + G(s), \tag{3.25}$$

where $X(s)$ and $G(s)$ represent the Laplace integral transforms of $x(t)$ and $g(t)$, respectively.

Thus, after solving the above equation with respect to the $X(s)$, we get

$$\begin{aligned}X(s) &= s^{\alpha-1}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}\varphi(0) + s^{\alpha-2}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}\varphi'(0) \\ &\quad - s^{\beta-1}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}A\varphi(0) + \left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s) \\ &\quad + \left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}G(s) = s^{-1}\varphi(0) + s^{-1}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}Be^{-s\tau}\varphi(0) \\ &\quad + s^{\alpha-2}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}\varphi'(0) + \left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s) \\ &\quad + \left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}G(s).\end{aligned}$$

For nonpermutable linear operators $A, B \in \mathscr{B}(Y)$ and sufficiently large s such that

$$\|\left(s^{\alpha-\beta}I - A\right)^{-1}Bs^{-\beta}e^{-s\tau}\| < 1,$$

a linear bounded operator $s^{\alpha}I - As^{\beta} - Be^{-s\tau}$ is invertible and it holds that

$$\begin{aligned}
\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1} &= \left(s^{\beta}\left[s^{\alpha-\beta}I - A - Bs^{-\beta}e^{-s\tau}\right]\right)^{-1} \\
&= \left(s^{\beta}(s^{\alpha-\beta}I - A)\left[I - (s^{\alpha-\beta}I - A)^{-1}Bs^{-\beta}e^{-s\tau}\right]\right)^{-1} \\
&= \left(s^{\beta}\left[I - \left(s^{\alpha-\beta}I - A\right)^{-1}Bs^{-\beta}e^{-s\tau}\right]\right)^{-1}\left(s^{\alpha-\beta}I - A\right)^{-1} \\
&= \left[I - (s^{\alpha-\beta}I - A)^{-1}Bs^{-\beta}e^{-s\tau}\right]^{-1}s^{-\beta}\left(s^{\alpha-\beta}I - A\right)^{-1} \\
&= \sum_{m=0}^{\infty}\frac{1}{e^{ms\tau}s^{\beta m}}\left[\left(s^{\alpha-\beta}I - A\right)^{-1}B\right]^{m}s^{-\beta}\left(s^{\alpha-\beta}I - A\right)^{-1} \\
&= \sum_{m=0}^{\infty}\frac{1}{e^{ms\tau}s^{(m+1)\beta}}\left[\left(s^{\alpha-\beta}I - A\right)^{-1}B\right]^{m}\left(s^{\alpha-\beta}I - A\right)^{-1}
\end{aligned}$$

Then, by taking inverse Laplace transform, we have

$$\begin{aligned}
x(t) &= \mathscr{L}^{-1}\left\{s^{-1}\right\}(t)\varphi(0) + \mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{s^{-1}}{e^{(m+1)s\tau}s^{(m+1)\beta}}\left[\left(s^{\alpha-\beta}I - A\right)^{-1}B\right]^{m}\left(s^{\alpha-\beta}I - A\right)^{-1}\right\}(t)B\varphi(0) \\
&+ \mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{s^{\alpha-2}}{e^{ms\tau}s^{(m+1)\beta}}\left[\left(s^{\alpha-\beta}I - A\right)^{-1}B\right]^{m}\left(s^{\alpha-\beta}I - A\right)^{-1}\right\}(t)\varphi'(0) \\
&+ \mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{1}{e^{ms\tau}s^{(m+1)\beta}}\left[\left(s^{\alpha-\beta}I - A\right)^{-1}B\right]^{m}\left(s^{\alpha-\beta}I - A\right)^{-1}B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s)\right\}(t) \\
&+ \mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{1}{e^{ms\tau}s^{(m+1)\beta}}\left[\left(s^{\alpha-\beta}I - A\right)^{-1}B\right]^{m}\left(s^{\alpha-\beta}I - A\right)^{-1}G(s)\right\}(t). \qquad (3.26)
\end{aligned}$$

Therefore, in accordance with Lemma 3.2, in general case, we attain the following result for $t \in \mathbb{R}_{+}$:

$$\begin{aligned}
x(t) &= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\mathcal{H}(t-(m+1)\tau)\right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}Q_{k,m}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\mathcal{H}(t-m\tau)\varphi'(0) \\
&+ \int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\mathcal{H}(t-(m+1)\tau-s)\varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}Q_{k,m}^{A,B}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\mathcal{H}(t-m\tau-s)g(s)\mathrm{d}s \\
&:= \left(I + \mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B\right)\varphi(0) + \mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\varphi'(0) \\
&+ \int_{-\tau}^{0}\mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t}\mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-s)g(s)\mathrm{d}s, \quad t > 0, \qquad (3.27)
\end{aligned}$$

where we have used the formula (3.24) and substitution $s - \tau = r$ and acquired that

$$\begin{aligned}
&\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}Q_{k,m}^{A,B}B\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\mathcal{H}(t-m\tau-s)\hat{\varphi}(s-\tau)\mathrm{d}s \\
&= \int_{0}^{\tau}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}Q_{k,m}^{A,B}B\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\mathcal{H}(t-m\tau-s)\varphi(s-\tau)\mathrm{d}s
\end{aligned}$$

$$= \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau - r)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} \mathcal{H}(t-(m+1)\tau - r)\varphi(r)\mathrm{d}r$$

$$= \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau - s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} \mathcal{H}(t-(m+1)\tau - s)\varphi(s)\mathrm{d}s.$$

□

Thus, the analytical representation of a mild solution of (3.1) can be expressed with respect to $t \in (-(n-1)\tau, n\tau]$, $n \in \mathbb{N}$ as follows:

$$\begin{aligned}
x(t) &= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)} \right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)} \varphi'(0) \\
&+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau - s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} \varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau - s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} g(s)\mathrm{d}s \\
&\coloneqq \left(I + \mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B \right)\varphi(0) + \mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\varphi'(0) \\
&+ \int_{-\tau}^{0} \mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t} \mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-s)g(s)\mathrm{d}s, \quad (n-1)\tau < t \leq n\tau. \qquad (3.28)
\end{aligned}$$

The following lemma will be of significance for the results of next theorem.

Lemma 3.3. *For arbitrary $t \in ((n-1)\tau, n\tau]$ for fixed $n \in \mathbb{N}$, $\tau > 0$ and any parameters $\alpha, \beta, \gamma \in \mathbb{R}$ satisfying $\alpha, \beta > 0$ and $\gamma - 1 > \lfloor \nu \rfloor$, we have:*

$$ {}^{C}D_{0+}^{\nu} \left\{ \mathscr{E}_{\alpha,\beta,\gamma}^{A,B,\tau}(s) \right\}(t) = \mathscr{E}_{\alpha,\beta,\gamma-\nu}^{A,B,\tau}(t), \quad n-1 < \nu \leq n, \quad n \in \mathbb{N}.$$

Proof. We have [37]:

$$\frac{d^n}{\mathrm{d}t^n}\left(t^{\xi}\right) = \frac{\Gamma(\xi+1)}{\Gamma(\xi-n+1)} t^{\xi-n}, \quad n \in \mathbb{N}, \quad \xi \in \mathbb{R}. \qquad (3.29)$$

Since $(t-m\tau)^{k\alpha+m\beta+\gamma-1} = (t-m\tau)^{k\alpha+m\beta+\gamma-1}\mathcal{H}(t-m\tau)$, $t > m\tau$ for $m = 0, 1, \ldots, n-1$ and given the condition $\gamma - 1 > \lfloor \nu \rfloor$, we can attain that

$$\begin{aligned}
&{}^{C}D_{0+}^{\nu} \left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(s-m\tau)^{k\alpha+m\beta+\gamma-1}}{\Gamma(k\alpha+m\beta+\gamma)} \right)(t) \\
&= {}^{C}D_{0+}^{\nu} \left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(s-m\tau)^{k\alpha+m\beta+\gamma-1}\mathcal{H}(s-m\tau)}{\Gamma(k\alpha+m\beta+\gamma)} \right)(t) \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B}\, {}^{C}D_{0+}^{\nu} \left(\frac{(s-m\tau)^{k\alpha+m\beta+\gamma-1}\mathcal{H}(s-m\tau)}{\Gamma(k\alpha+m\beta+\gamma)} \right)(t) \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{1}{\Gamma(n-\nu)} \int_{0}^{t} (t-s)^{n-\nu-1} \frac{d^n}{\mathrm{d}s^n} \left(\frac{(s-m\tau)^{k\alpha+m\beta+\gamma-1}\mathcal{H}(s-m\tau)}{\Gamma(k\alpha+m\beta+\gamma)} \right) \mathrm{d}s
\end{aligned}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} \frac{(s-m\tau)^{k\alpha+m\beta+\gamma-n-1}\mathcal{H}(s-m\tau)}{\Gamma(k\alpha+m\beta+\gamma-n)} \mathrm{d}s$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{1}{\Gamma(n-\nu)} \int_{m\tau}^t (t-s)^{n-\nu-1} \frac{(s-m\tau)^{k\alpha+m\beta+\gamma-n-1}}{\Gamma(k\alpha+m\beta+\gamma-n)} \mathrm{d}s$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k\alpha+m\beta+\gamma-\nu-1}}{\Gamma(n-\nu)\Gamma(k\alpha+m\beta+\gamma-n)} \mathcal{B}\left(n-\nu, k\alpha+m\beta+\gamma-n\right)$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k\alpha+m\beta+\gamma-\nu-1}}{\Gamma(k\alpha+m\beta+\gamma-\nu)} = \mathscr{E}_{\alpha,\beta,\gamma-\nu}^{A,B,\tau}(t), \quad (n-1)\tau < t \leq n\tau.$$

This completes the proof. □

It should be stressed out that the assumption on the exponential boundedness of the function $g(\cdot)$ and $\left({}^C D_{0+}^{\beta} x\right)(\cdot)$ where $0<\beta\leq 1$ (alternatively, $\left({}^C D_{0+}^{\alpha} x\right)(\cdot)$ for $1<\alpha\leq 2$) can be omitted.

Theorem 3.4. *Let $A, B \in \mathscr{B}(Y)$ with non-zero commutator, i.e., $[A,B] \coloneqq AB - BA \neq 0$. A mild solution $x(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$ of the Cauchy problem* (3.1) *can be represented as*

$$\begin{aligned}
x(t) &= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0) \\
&+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} g(s)\mathrm{d}s \\
&\coloneqq \left(I + \mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B\right)\varphi(0) + \mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\varphi'(0) \\
&+ \int_{-\tau}^{0} \mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t} \mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-s)g(s)\mathrm{d}s, \quad (n-1)\tau < t \leq n\tau.
\end{aligned} \tag{3.30}$$

Proof. For making use of verification by substitution, we apply superposition principle for the initial value problem of linear inhomogeneous multi-order fractional evolution equation (3.1). For this, firstly let us consider the following homogeneous system with inhomogeneous initial conditions:

$$\begin{cases}
\left({}^C D_{0+}^{\alpha} x\right)(t) - A\left({}^C D_{0+}^{\beta} x\right)(t) - Bx(t-\tau) = 0, \quad \tau > 0, \quad t > 0, \\
x(t) = \varphi(t), \quad -\tau \leq t \leq 0, \\
x'(0) = \varphi'(0),
\end{cases} \tag{3.31}$$

has a mild solution

$$\begin{aligned}
x(t) &= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} B \frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s\\
&=\left(I+\mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B\right)\varphi(0)+\mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\varphi'(0)\\
&+\int_{-\tau}^{0}\mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s,\quad (n-1)\tau<t\leq n\tau. \qquad (3.32)
\end{aligned}
$$

With the help of verification by substitution and the property of $Q_{k,m}^{A,B}$ (3.9), we confirm that (3.32) is a mild solution of linear homogeneous fractional functional evolution equation (3.31):

$$
\begin{aligned}
\left({}^{C}D_{0+}^{\alpha}x\right)(t)&={}^{C}D_{0+}^{\alpha}\left(I+\mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B\right)\varphi(0)+{}^{C}D_{0+}^{\alpha}\left(\mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\right)\varphi'(0)\\
&+{}^{C}D_{0+}^{\alpha}\left(\int_{-\tau}^{0}\mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s\right)\\
&={}^{C}D_{0+}^{\alpha}\left(I+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0)\\
&+{}^{C}D_{0+}^{\alpha}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\right)\varphi'(0)\\
&+{}^{C}D_{0+}^{\alpha}\left(\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s\right).
\end{aligned}
$$

In this case, we first apply the property of $Q_{k,m}^{A,B}$ (3.9) before Caputo differentiation the first and second terms above, in accordance with the following formula [37]:

$$
{}^{C}D_{0+}^{\nu}\left(\frac{t^{\eta}}{\Gamma(\eta+1)}\right)=\begin{cases}\frac{t^{\eta-\nu}}{\Gamma(\eta-\nu+1)}, & \eta>\lfloor\nu\rfloor,\\ 0, & \eta=0,1,2,\ldots,\lfloor\nu\rfloor,\\ \text{undefined}, & \text{otherwise}.\end{cases} \qquad (3.33)
$$

Then, we have

$$
\begin{aligned}
\left({}^{C}D_{0+}^{\alpha}x\right)(t)&={}^{C}D_{0+}^{\alpha}\Big[I+B\frac{(t-\tau)^{\alpha}}{\Gamma(\alpha+1)}+\sum_{k=1}^{\infty}\sum_{m=0}^{n-1}AQ_{k-1,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\\
&+\sum_{k=0}^{\infty}\sum_{m=1}^{n}BQ_{k,m-1}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\Big]\varphi(0)\\
&+{}^{C}D_{0+}^{\alpha}\Big[tI+\sum_{k=1}^{\infty}\sum_{m=0}^{n-1}AQ_{k-1,m}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\\
&+\sum_{k=0}^{\infty}\sum_{m=1}^{n}BQ_{k,m-1}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\Big]\varphi'(0)\\
&+{}^{C}D_{0+}^{\alpha}\Big(\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s\Big)\\
&=B\varphi(0)+\sum_{k=1}^{\infty}\sum_{m=0}^{n-1}AQ_{k-1,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+1)}\varphi(0)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=0}^{\infty}\sum_{m=1}^{n} BQ_{k,m-1}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+1)}\varphi(0)\\
&+\sum_{k=1}^{\infty}\sum_{m=0}^{n-1} AQ_{k-1,m}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1-\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+2-\alpha)}\varphi'(0)\\
&+\sum_{k=0}^{\infty}\sum_{m=1}^{n} BQ_{k,m-1}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1-\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+2-\alpha)}\varphi'(0)\\
&+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha)}\varphi(s)\mathrm{d}s.
\end{aligned}
$$

Again by making use of an important property of $Q_{k,m}^{A,B}$ (3.9) and relation (3.33) one can attain that

$$
\begin{aligned}
\left({}^{C}D_{0+}^{\alpha}x\right)(t) &= B\varphi(0)+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ_{k,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha-\beta}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta+1)}\varphi(0)\\
&+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ_{k,m}^{A,B}B\frac{(t-(m+2)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\varphi(0)\\
&+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ_{k,m}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1-\beta}}{\Gamma(k(\alpha-\beta)+m\alpha+2-\beta)}\varphi'(0)\\
&+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ_{k,m}^{A,B}\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0)\\
&+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta)}\varphi(s)\mathrm{d}s\\
&+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ_{k,m}^{A,B}B\frac{(t-(m+2)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s, \quad (n-1)\tau<t\leq n\tau.
\end{aligned}
$$

Then, the Caputo fractional differentiation of $x(t)$ (3.32) of order $0<\beta\leq 1$ is as follows:

$$
\begin{aligned}
\left({}^{C}D_{0+}^{\beta}x\right)(t) &= {}^{C}D_{0+}^{\beta}\left(I+\mathscr{E}_{\alpha-\beta,\alpha,\alpha+1}^{A,B,\tau}(t-\tau)B\right)\varphi(0)+{}^{C}D_{0+}^{\beta}\left(\mathscr{E}_{\alpha-\beta,\alpha,2}^{A,B,\tau}(t)\right)\varphi'(0)\\
&+{}^{C}D_{0+}^{\beta}\left(\int_{-\tau}^{0}\mathscr{E}_{\alpha-\beta,\alpha,\alpha}^{A,B,\tau}(t-\tau-s)B\varphi(s)\mathrm{d}s\right)\\
&= {}^{C}D_{0+}^{\beta}\left(I+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\beta+\alpha}}{\Gamma(k(\alpha-\beta)+m\beta+\alpha+1)}\right)\varphi(0)\\
&+{}^{C}D_{0+}^{\beta}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} Q_{k,m}^{A,B}\frac{(t-m\tau)^{k(\alpha-\beta)+m\beta+1}}{\Gamma(k(\alpha-\beta)+m\beta+2)}\right)\varphi'(0)\\
&+{}^{C}D_{0+}^{\beta}\Big(\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s\Big)\\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q_{k,m}^{A,B}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\beta+\alpha-\beta}}{\Gamma(k(\alpha-\beta)+m\beta+\alpha-\beta+1)}\varphi(0)
\end{aligned}
$$

$$+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau)^{k(\alpha-\beta)+m\beta+1-\beta}}{\Gamma(k(\alpha-\beta)+m\beta+2-\beta)}\varphi'(0)$$
$$+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta)}\varphi(s)\mathrm{d}s, \quad (n-1)\tau<t\leq n\tau.$$

Finally, taking a linear combination of above results, we acquire the desired result:

$$\begin{aligned}\left({}^{C}D^{\alpha}_{0+}x\right)(t)-A\left({}^{C}D^{\beta}_{0+}x\right)(t) &= B\varphi(0)+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ^{A,B}_{k,m}B\frac{t^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\varphi(0)\\ &+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ^{A,B}_{k,m}\frac{t^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0)\\ &+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ^{A,B}_{k,m}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s := Bx(t-\tau).\end{aligned}$$

Next, we consider the following linear inhomogeneous fractional evolution equation for $t\in\mathbb{J}$:

$$\left({}^{C}D^{\alpha}_{0+}x\right)(t)-A\left({}^{C}D^{\beta}_{0+}x\right)(t)-Bx(t-\tau)=g(t), \quad t>0, \quad \tau>0, \tag{3.34}$$

with zero initial conditions

$$x(t)=0, \quad x'(0)=0, \quad -\tau\leq t\leq 0,$$

has an integral representation of a mild solution which is a particular solution of (3.1):

$$\bar{x}(t)=\int_{0}^{t}\mathscr{E}^{A,B}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\mathrm{d}s=\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}g(s)\mathrm{d}s, \quad (n-1)\tau<t\leq n\tau.$$

In accordance with fractional analogue of variation of constants formula any particular mild solution of inhomogeneous differential equation of fractional-order (3.34) should be looked for in the form of

$$\bar{x}(t)=\int_{0}^{t}\mathscr{E}^{A,B}_{\alpha-\beta,\alpha,\alpha}(t-s)f(s)\mathrm{d}s=\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s, \quad (n-1)\tau<t\leq n\tau, \tag{3.35}$$

where $f(s)$ is unknown function for $s\in[0,t]$ with $\bar{x}(0)=\bar{x}'(0)=0$.

Because of this homogeneous initial values $\bar{x}(0)=\bar{x}'(0)=0$, it follows that in this case, for any given order either in $(1,2]$ and $(0,1]$, the Riemann–Liouville and Caputo type fractional differentiation operators are equal in accordance with (2.6). Therefore, in the work below we will apply Riemann–Liouville derivative instead of Caputo one to verify the mild solution of evolution equation with two independent fractional-orders.

Applying the property of a linear operator $Q^{A,B}_{k,m}$ (3.9) and having Caputo differentiation of order $1<\alpha\leq 2$ of $\bar{x}(t)$, we obtain:

$$\begin{aligned}&\left({}^{C}D^{\alpha}_{0+}\bar{x}\right)(t)=\left({}^{RL}D^{\alpha}_{0+}\bar{x}\right)(t)\\ &={}^{RL}D^{\alpha}_{0+}\Big[\int_{0}^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s)\mathrm{d}s+\int_{0}^{t}\sum_{k=1}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k-1,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s\\ &+\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=1}^{n} BQ^{A,B}_{k,m-1}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s\Big]\end{aligned}$$

$$= \left({}^{RL}D^{\alpha}_{0+}(I^{\alpha}_{0+}f)\right)(t) + {}^{RL}D^{\alpha}_{0+}\Big[\int\limits_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha-\beta)}f(s)\mathrm{d}s\Big]$$

$$+ {}^{RL}D^{\alpha}_{0+}\Big[\int\limits_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ^{A,B}_{k,m}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha)}f(s)\mathrm{d}s\Big].$$

By making use of the fractional Leibniz integral rules (2.4) in Riemann-Liouville's sense for the second and third terms of the above expression, we get

$$\left({}^{C}D^{\alpha}_{0+}\bar{x}\right)(t) = \left({}^{RL}D^{\alpha}_{0+}\bar{x}\right)(t)$$

$$= f(t) + \lim_{s\to t-0} {}^{RL,t}D^{\alpha-1}_{0+}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha-\beta)}\right)\lim_{s\to t-0} f(s)$$

$$+ \lim_{s\to t-0} {}^{RL,t}D^{\alpha-2}_{0+}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha-\beta)}\right)\frac{d}{dt}\lim_{s\to t-0} f(s)$$

$$+ \int\limits_0^t {}^{RL,t}D^{\alpha}_{0+}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha-\beta)}f(s)\mathrm{d}s$$

$$+ \lim_{s\to t-0} {}^{RL,t}D^{\alpha-1}_{0+}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha)}\right)\lim_{s\to t-0} f(s)$$

$$+ \lim_{s\to t-0} {}^{RL,t}D^{\alpha-2}_{0+}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha)}\right)\frac{d}{dt}\lim_{s\to t-0} f(s)$$

$$+ \int\limits_0^t {}^{RL,t}D^{\alpha}_{0+}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+2\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+2\alpha)}f(s)\mathrm{d}s$$

$$= f(t) + \lim_{s\to t-0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta+1)}\lim_{s\to t-0} f(s)$$

$$+ \lim_{s\to t-0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta+1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta+2)}\frac{d}{dt}\lim_{s\to t-0} f(s)$$

$$+ \int\limits_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta)}f(s)\mathrm{d}s$$

$$+ \lim_{s\to t-0} {}^{RL,t}D^{\alpha-1}_{0+}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\lim_{s\to t-0} f(s)$$

$$+ \lim_{s\to t-0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\lim_{s\to t-0}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+2)}\frac{d}{dt}\lim_{s\to t-0} f(s)$$

$$+ \int\limits_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s$$

$$= f(t) + \int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} AQ^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta)}f(s)\mathrm{d}s$$

$$+ \int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ^{A,B}_{k,m}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s$$

Then, Caputo fractional derivative of $\bar{x}(t)$ of order $0 \leq \beta \leq 1$ is

$$\begin{aligned}
&\left({}^{C}D^{\beta}_{0+}\bar{x}\right)(t) = \left({}^{RL}D^{\beta}_{0+}\bar{x}\right)(t) \\
&= {}^{RL}D^{\beta}_{0+}\Big[\int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s\Big] \\
&= \lim_{s\to t-0} {}^{RL,t}D^{\beta-1}_{0+}\left(\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\right)\lim_{s\to t-0} f(s) \\
&+ \int_0^t {}^{RL}D^{\beta}_{0+}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s \\
&= \lim_{s\to t-0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta+1)}\lim_{s\to t-0} f(s) \\
&+ \int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta)}f(s)\mathrm{d}s \\
&= \int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-\beta-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\beta)}f(s)\mathrm{d}s.
\end{aligned}$$

Thus, linear combinations of above results yield that

$$\begin{aligned}
&\left({}^{C}D^{\alpha}_{0+}\bar{x}\right)(t) - A\left({}^{C}D^{\beta}_{0+}\bar{x}\right)(t) \\
&= f(t) + \int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} BQ^{A,B}_{k,m}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}f(s)\mathrm{d}s \\
&= f(t) + B\bar{x}(t-\tau) = g(t) + B\bar{x}(t-\tau), \quad (n-1)\tau < t \leq n\tau, \quad \tau > 0.
\end{aligned}$$

Therefore, $f(t) = g(t)$, $(n-1)\tau < t \leq n\tau$ which confirms the desired verification. The proof is complete. $\square$

Then it follows that by using the substitution $y(t) = E^{-1}x(t)$, we can acquire a mild solution of (1.1) as below.

Theorem 3.5. *Let $A, B \in \mathscr{B}(Y)$ with non-zero commutator, i.e., $[A,B] := AB - BA \neq 0$. A mild solution $y(\cdot) \in \mathbb{C}^2(\mathbb{J}, X)$ of the Cauchy problem (1.1) can be represented as*

$$\begin{aligned}
y(t) &= \left(E^{-1} + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} E^{-1}Q^{A,B}_{k,m}B\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E^{-1}Q^{A,B}_{k,m}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0) \\
&+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} E^{-1}Q^{A,B}_{k,m}B\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s \\
&+ \int_0^t \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} E^{-1}Q^{A,B}_{k,m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}g(s)\mathrm{d}s \\
&:= \left(E^{-1} + t^{\alpha}E^{-1}\mathscr{E}^{A,B}_{\alpha-\beta,\alpha,\alpha+1}(t-\tau)B\right)\varphi(0) + tE^{-1}\mathscr{E}^{A,B}_{\alpha-\beta,\alpha,2}(t)\varphi'(0)
\end{aligned}$$

$$+\int_{-\tau}^{0} E^{-1}\mathscr{E}^{A,B}_{\alpha-\beta,\alpha,\alpha}(t-\tau-s)B\varphi(s)\mathrm{d}s+\int_{0}^{t} E^{-1}\mathscr{E}^{A,B}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\mathrm{d}s, \quad (n-1)\tau < t \leq n\tau. \tag{3.36}$$

Remark 3.2. Let $\alpha = 2, \beta = 1$. Then, a mild solution $y(\cdot) \in \mathbb{C}^2(\mathbb{J}, X)$ of the Cauchy problem for the following second-order functional evolution equation

$$\begin{cases} \left(Ey''\right)(t) - A_0 y'(t) - B_0 y(t-\tau) = g(t), \quad t > 0, \quad \tau > 0, \\ y(t) = \varphi(t), \quad -\tau \leq t \leq 0, \\ y'(0) = \varphi'(0), \end{cases} \tag{3.37}$$

can be determined by means of a new delayed Mittag-Leffler type function as follows

$$\begin{aligned} y(t) &= \left(E^{-1} + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} E^{-1} Q^{A,B}_{k,m} B \frac{(t-(m+1)\tau)^{k+2m+2}}{(k+2m+2)!}\right)\varphi(0) \\ &+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} E^{-1} Q^{A,B}_{k,m} \frac{(t-m\tau)^{k+2m+1}}{(k+2m+1)!}\varphi'(0) \\ &+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} E^{-1} Q^{A,B}_{k,m} B \frac{(t-(m+1)\tau-s)^{k+2m+1}}{(k+2m+1)!}\varphi(s)\mathrm{d}s \\ &+ \int_{0}^{t} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} E^{-1} Q^{A,B}_{k,m} \frac{(t-m\tau-s)^{k+2m+1}}{(k+2m+1)!} g(s)\mathrm{d}s \\ &:= \left(E^{-1} + t^{\alpha} E^{-1}\mathscr{E}^{A,B}_{1,2,3}(t-\tau)B\right)\varphi(0) + tE^{-1}\mathscr{E}^{A,B}_{1,2,2}(t)\varphi'(0) \\ &+ \int_{-\tau}^{0} E^{-1}\mathscr{E}^{A,B}_{1,2,2}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t} E^{-1}\mathscr{E}^{A,B}_{1,2,2}(t-s)g(s)\mathrm{d}s, \quad (n-1)\tau < t \leq n\tau. \end{aligned} \tag{3.38}$$

Remark 3.3. In particular case, we consider the following initial value problem for multi-dimensional multi-term fractional time-delay differential equation with noncommutative matrices

$$\begin{cases} \left({}^{C}D^{\alpha}_{0+}y\right)(t) - A_0\left({}^{C}D^{\beta}_{0+}y\right)(t) - B_0 y(t-\tau) = g(t), \quad t > 0, \quad \tau > 0, \\ y(t) = \varphi(t), \quad -\tau \leq t \leq 0, \\ y'(0) = \varphi'(0), \end{cases} \tag{3.39}$$

where ${}^{C}D^{\alpha}_{0+}$ and ${}^{C}D^{\beta}_{0+}$ Caputo fractional derivatives of orders $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively, with the lower limit zero. $E = I \in \mathbb{R}^{n\times n}$ is an identity matrix, the matrices $A_0, B_0 \in \mathbb{R}^{n\times n}$ are nonpermutable i.e., $AB \neq BA$, $y(\cdot) \in \mathbb{R}^n$ is a vector-valued function on $\mathbb{J}$, i.e., $y(\cdot) : \mathbb{J} \to \mathbb{R}^n$ and $\varphi \in \mathbb{C}(\mathbb{J}, \mathbb{R}^n)$ and $\varphi'(\cdot)$ is continuously differentiable at $t = 0$. In addition, a forced term $g(\cdot) : \mathbb{L} \to \mathbb{R}^n$ is a continuous function.

The exact analytical representation of solution $y(\cdot) \in \mathbb{C}^2(\mathbb{J}, \mathbb{R}^n)$ of (3.39) can be expressed by

$$\begin{aligned} y(t) &:= \left(1 + \mathscr{E}^{A_0,B_0}_{\alpha-\beta,\alpha,\alpha+1}(t-\tau)B_0\right)\varphi(0) + \mathscr{E}^{A_0,B_0}_{\alpha-\beta,\alpha,2}(t)\varphi'(0) \\ &+ \int_{-\tau}^{0} \mathscr{E}^{A_0,B_0}_{\alpha-\beta,\alpha,\alpha}(t-\tau-s)B_0\varphi(s)\mathrm{d}s + \int_{0}^{t} \mathscr{E}^{A_0,B_0}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\mathrm{d}s \\ &= \left(1 + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} Q^{A_0,B_0}_{k,m} B_0 \frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0) \end{aligned}$$

$$
+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A_0,B_0}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0)
$$

$$
+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A_0,B_0}B_0\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s
$$

$$
+\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}Q_{k,m}^{A_0,B_0}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}g(s)\mathrm{d}s,\quad (n-1)\tau<t\leq n\tau. \tag{3.40}
$$

4 A representation of solutions of (1.1) with permutable linear operators

To get an analytical representation of a mild solution of (3.1) with permutable linear operators i.e., $AB = BA$, first, we need to prove auxiliary lemma for making use of Laplace integral transform according to the Theorem 3.1. Moreover, the scalar analogue of following theorem has been considered by Ahmadova and Mahmudov for fractional delay-free Langevin equations with constant coefficients in [2]. In general, the following theorem is true for $\alpha > 0$, $\alpha > \beta$ and $\alpha > \gamma$.

Theorem 4.1. *Let $m \in \mathbb{N}_0$ and $Re(s) > 0$. For $A, B \in \mathscr{B}(Y)$ with $[A, B] = AB - BA = 0$, we have:*

$$
\begin{aligned}
\mathscr{L}^{-1}\Big\{\frac{s^{\gamma}B^{m}e^{-ms\tau}}{(s^{\alpha}I-As^{\beta})^{m+1}}\Big\}(t) &= (t-m\tau)^{m\alpha+\alpha+\gamma-1}\sum_{k=0}^{\infty}\binom{k+m}{m}\frac{A^{k}B^{m}(t-m\tau)^{k(\alpha-\beta)}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\gamma)}\\
&= (t-m\tau)^{m\alpha+\alpha-\gamma-1}E_{\alpha-\beta,m\alpha+\alpha-\gamma}^{m+1}(A(t-m\tau)^{\alpha-\beta})B^{m}.
\end{aligned}
$$

Proof. By using the Taylor series representation of $\frac{1}{(1-t)^{m+1}}$, $m \in \mathbb{N}_0$ of the form

$$
\frac{1}{(1-t)^{m+1}}=\sum_{k=0}^{\infty}\binom{k+m}{m}t^{k},\quad |t|<1,
$$

we achieve that

$$
\begin{aligned}
\frac{s^{\gamma}B^{m}e^{-ms\tau}}{(s^{\alpha}I-As^{\beta})^{m+1}} = \frac{s^{\gamma}B^{m}e^{-ms\tau}}{(s^{\alpha}I)^{m+1}}\frac{1}{(1-\frac{A}{s^{\alpha-\beta}})^{m+1}} &= \frac{s^{\gamma}B^{m}e^{-ms\tau}}{s^{(m+1)\alpha}}\sum_{k=0}^{\infty}\binom{k+m}{m}\Big(\frac{A}{s^{\alpha-\beta}}\Big)^{k}\\
&=\sum_{k=0}^{\infty}\binom{k+m}{m}\frac{A^{k}B^{m}e^{-ms\tau}}{s^{(m+1)\alpha+k(\alpha-\beta)-\gamma}}.
\end{aligned}
$$

By using the inverse Laplace integral formula for the above function, we get the desired result:

$$
\begin{aligned}
\mathscr{L}^{-1}\Big\{\frac{s^{\gamma}B^{m}e^{-ms\tau}}{(s^{\alpha}I-As^{\beta})^{m+1}}\Big\}(t) &= \sum_{k=0}^{\infty}A^{k}B^{m}\binom{k+m}{m}\mathscr{L}^{-1}\Big\{\frac{e^{-ms\tau}}{s^{k(\alpha-\beta)+(m+1)\alpha-\gamma}}\Big\}(t)\\
&=\sum_{k=0}^{\infty}A^{k}B^{m}\binom{k+m}{m}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+\alpha-\gamma-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha-\gamma)}\\
&=(t-m\tau)^{m\alpha+\alpha-\gamma-1}E_{\alpha-\beta,m\alpha+\alpha-\gamma}^{m+1}(A(t-m\tau)^{\alpha-\beta})B^{m}.
\end{aligned}
$$

We have required an extra condition on s such that

$$
s^{\alpha-\beta} > \|A\|,
$$

for proper convergence of the series. But, this condition can be removed at the end of calculation since analytic continuation of both sides, to give the desired result for all $s \in \mathbb{C}$ which is satisfying $Re(s) > 0$. □

Then, we acquire analytical representation of mild solution for multi-term fractional functional evolution equation with permutable linear bounded operators via the following theorem.

Theorem 4.2. *Let $A, B \in \mathscr{B}(Y)$ with zero commutator, i.e., $[A, B] := AB - BA = 0$. Assume that $g(\cdot) : \mathbb{J} \to X$ and $\left({}^C D^{\beta}_{0+} x\right)(\cdot)$ for $0 < \beta \leq 1$ are exponentially bounded. A mild solution $x(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$ of the Cauchy problem* (3.1) *can be represented by means of delayed analogue of bivariate Mittag-Leffler type functions* (2.14) *as follows*

$$
\begin{aligned}
x(t) &= \left(I + BE^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha+1}(t-\tau)\right)\varphi(0) + E^{A,B,\tau}_{\alpha-\beta,\alpha,2}(t)\varphi'(0) \\
&+ \int_{-\tau}^{0} E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t} E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\mathrm{d}s \\
&= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^k B^{m+1}\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^k B^{m}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0) \\
&+ \int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^k B^{m+1}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^k B^{m}\frac{(t-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}g(s)\mathrm{d}s. \qquad (4.1)
\end{aligned}
$$

Proof. We recall that the existence of Laplace transform of $x(\cdot)$ and its Caputo derivatives $\left({}^C D^{\alpha}_{0+} x\right)(\cdot)$ and $\left({}^C D^{\beta}_{0+} x\right)(\cdot)$ for $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, respectively, is guaranteed by Theorem 3.1. Thus, to find the mild solution $x(t)$ of (1.1) with permutable linear operators, i.e., $AB = BA$, we can use the Laplace transform technique. Taking the Laplace transform technique on both sides of equation (3.1) and solving the equation with respect to the $X(s)$, we get Thus, after solving the above equation with respect to the $X(s)$, we get

$$
\begin{aligned}
X(s) &= s^{-1}\varphi(0) + s^{-1}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}Be^{-s\tau}\varphi(0) \\
&+ s^{\alpha-2}\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}\varphi'(0) + \left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s) \\
&+ \left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1}G(s).
\end{aligned}
$$

For nonpermutable linear operators $A, B \in \mathscr{B}(Y)$ and sufficiently large s such that

$$
\|I - \left(s^{\alpha}I - As^{\beta}\right)^{-1}Be^{-s\tau}\| < 1,
$$

a linear bounded operator $s^{\alpha}I - As^{\beta} - Be^{-s\tau}$ is invertible and it holds that

$$
\begin{aligned}
\left(s^{\alpha}I - As^{\beta} - Be^{-s\tau}\right)^{-1} &= \left(s^{\alpha}I - As^{\beta}\right)^{-1}\left(I - \left(s^{\alpha}I - As^{\beta}\right)^{-1}Be^{-s\tau}\right)^{-1} \\
&= \left(s^{\alpha}I - As^{\beta}\right)^{-1}\sum_{m=0}^{\infty}\left(s^{\alpha}I - As^{\beta}\right)^{-m}B^m e^{-ms\tau} \\
&= \sum_{m=0}^{\infty}\frac{B^m e^{-ms\tau}}{\left(s^{\alpha}I - As^{\beta}\right)^{(m+1)}}.
\end{aligned}
$$

Then taking inverse Laplace transform, we have

$$
x(t) = \mathscr{L}^{-1}\left\{s^{-1}\right\}(t)\varphi(0) + \mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{s^{-1}B^{m+1}e^{-(m+1)s\tau}}{\left(s^{\alpha}I - As^{\beta}\right)^{(m+1)}}\right\}(t)B\varphi(0)
$$

$$
\begin{aligned}
&+\mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{s^{\alpha-2}B^{m}e^{-ms\tau}}{(s^{\alpha}I-As^{\beta})^{(m+1)}}\right\}(t)\varphi'(0)\\
&+\mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{B^{m+1}e^{-(m+1)s\tau}}{(s^{\alpha}I-As^{\beta})^{(m+1)}}\varphi(s)\right\}(t)\\
&+\mathscr{L}^{-1}\left\{\sum_{m=0}^{\infty}\frac{B^{m}e^{-ms\tau}}{(s^{\alpha}I-As^{\beta})^{(m+1)}}G(s)\right\}(t). \qquad (4.2)
\end{aligned}
$$

Therefore, in accordance with Theorem 4.1, we acquire on $t\in\mathbb{J}=[-\tau,T]$:

$$
\begin{aligned}
x(t)&=\left\{I+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{k}\frac{A^{k}B^{m+1}(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right\}\varphi(0)\\
&+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{k}\frac{A^{k}B^{m}(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0)\\
&+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{k}\frac{A^{k}B^{m+1}(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s\\
&+\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{k}\frac{A^{k}B^{m}(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}g(s)\mathrm{d}s\\
&\coloneqq\left(I+E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha+1}(t-\tau)\right)\varphi(0)+E^{A,B,\tau}_{\alpha-\beta,\alpha,2}(t)\varphi'(0)\\
&+\int_{-\tau}^{0}E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha}(t-\tau-s)B\varphi(s)\mathrm{d}s+\int_{0}^{t}E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\mathrm{d}s,\quad (n-1)\tau<t\leq n\tau. \qquad (4.3)
\end{aligned}
$$

□

Remark 4.1. The analytical mild solution for the initial value problem for (3.1) can be attained from the property of $Q^{A,B}_{k,m}$ (3.10) for linear bounded operators $A,B\in\mathscr{B}(Y)$ satisfying $AB=BA$ where

$$
Q^{A,B}_{k,m}=\binom{k+m}{m}A^{k}B^{m},\quad k,m\in\mathbb{N}_0.
$$

It should be emphasized that the assumption on the exponential boundedness of the function $g(\cdot)$ and $\left({}^{C}D^{\beta}_{0+}x\right)(\cdot)$ for $0<\beta\leq1$ ($\left({}^{C}D^{\alpha}_{0+}x\right)(\cdot)$ for $1<\alpha\leq2$) can be omitted for the case of permutable linear bounded operators, too.

Theorem 4.3. *Let $A,B\in\mathscr{B}(Y)$ with zero commutator, i.e., $[A,B]\coloneqq AB-BA=0$. A mild solution $x(\cdot)\in\mathbb{C}^2(\mathbb{J},Y)$ of the Cauchy problem* (3.1) *can be expressed as*

$$
\begin{aligned}
x(t)&=\left(I+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^{k}B^{m+1}\frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)}\right)\varphi(0)\\
&+\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^{k}B^{m}\frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)}\varphi'(0)\\
&+\int_{-\tau}^{0}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^{k}B^{m+1}\frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}\varphi(s)\mathrm{d}s\\
&+\int_{0}^{t}\sum_{k=0}^{\infty}\sum_{m=0}^{n-1}\binom{k+m}{m}A^{k}B^{m}\frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}g(s)\mathrm{d}s,\quad (n-1)\tau<t\leq n\tau. \qquad (4.4)
\end{aligned}
$$

Proof. For linear homogeneous and inhomogeneous cases, by using the following Pascal identity for binomial coefficients:

$$\binom{k+m}{m} = \binom{k+m-1}{m} + \binom{k+m-1}{m-1}, \quad k,m \in \mathbb{N},$$

the formula (3.33) and fractional Leibniz integral rules (2.4), it can be easily shown that (4.4) is a mild solution of the Cauchy problem for (3.1) with permutable linear bounded operators. Moreover, similar case have considered by Mahmudov et al. for multi-dimensional delay-free Bagley-Torvik equations with permutable matrices in [33]. ☐

Theorem 4.4. *Let $A, B \in \mathscr{B}(Y)$ with zero commutator, i.e., $[A,B] := AB - BA = 0$. A mild solution $y(\cdot) \in \mathbb{C}^2(\mathbb{J}, X)$ of the Cauchy problem (1.1) can be determined as below*

$$\begin{aligned}
y(t) &= \left\{ E^{-1} + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{k} \frac{E^{-1}A^k B^{m+1}(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)} \right\} \varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{k} \frac{E^{-1}A^k B^{m}(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)} \varphi'(0) \\
&+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{k} \frac{E^{-1}A^k B^{m+1}(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} \varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{k} \frac{E^{-1}A^k B^{m}(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} g(s)\mathrm{d}s \\
&:= \left(E^{-1} + E^{-1}E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha+1}(t-\tau) \right) \varphi(0) + E^{-1}E^{A,B,\tau}_{\alpha-\beta,\alpha,2}(t)\varphi'(0) \\
&+ \int_{-\tau}^{0} E^{-1}E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha}(t-\tau-s)B\varphi(s)\mathrm{d}s + \int_{0}^{t} E^{-1}E^{A,B,\tau}_{\alpha-\beta,\alpha,\alpha}(t-s)g(s)\mathrm{d}s \quad (n-1)\tau < t \leq n\tau. \qquad (4.5)
\end{aligned}$$

Remark 4.2. In a special case, the exact analytical representation of solution $y(\cdot) \in \mathbb{C}^2(\mathbb{J}, \mathbb{R}^n)$ of Cauchy problem for multi-dimensional fractional functional differential equation with multi-orders and permutable matrices $A_0, B_0 \in \mathbb{R}^{n\times n}$ i.e., $A_0B_0 = B_0A_0$ (3.39) can be represented by

$$\begin{aligned}
y(t) &= \left(I + \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{m} A_0^k B_0^{m+1} \frac{(t-(m+1)\tau)^{k(\alpha-\beta)+m\alpha+\alpha}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha+1)} \right) \varphi(0) \\
&+ \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{m} A_0^k B_0^{m} \frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+1}}{\Gamma(k(\alpha-\beta)+m\alpha+2)} \varphi'(0) \\
&+ \int_{-\tau}^{0} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{m} A_0^k B_0^{m+1} \frac{(t-(m+1)\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} \varphi(s)\mathrm{d}s \\
&+ \int_{0}^{t} \sum_{k=0}^{\infty}\sum_{m=0}^{n-1} \binom{k+m}{m} A_0^k B_0^{m} \frac{(t-m\tau-s)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} g(s)\mathrm{d}s, \quad (n-1)\tau < t \leq n\tau. \qquad (4.6)
\end{aligned}$$

5 Stability analysis

In this section, we investigate stability results of fractional functional evolution equation of Sobolev type with linear bounded operators $A, B \in \mathscr{B}(Y)$ in Ulam-Hyers sense, without loss of generality, by considering nonpermutable case : $AB \neq BA$. In general, Ulam-Hyers stability results are studied via fixed point approach in [2, 48]. Unlike otherss, we consider stability analysis results for Sobolev type fractional multi-order functional evolution equation via Laplace transform technique.

Definition 5.1. System (3.1) is stable in Ulam-Hyers sense on $\mathbb{J} = [-\tau, T]$ if there exists $\Lambda > 0$ such that for any $\varepsilon > 0$, a function $x^*(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$ satisfying the following inequality:

$$\| \left({}^C D_{0+}^{\alpha} x^* \right)(t) - A \left({}^C D_{0+}^{\beta} x^* \right)(t) - Bx^*(t-\tau) - g(t) \| \leq \varepsilon, \tag{5.1}$$

with the initial conditions

$$\begin{aligned} &x^*(t) = \varphi(t), \quad -\tau \leq t \leq 0, \\ &x^{*\prime}(0) = \varphi'(0), \end{aligned} \tag{5.2}$$

there exists a mild solution $x(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$ of (3.1) such that

$$\|x^*(t) - x(t)\| \leq \Lambda\varepsilon, \quad t \in \mathbb{J}. \tag{5.3}$$

Theorem 5.1. *Let $g(\cdot) : \mathbb{L} \to Y$ be a continuous function. Then, the system* (3.1) *is stable in Ulam-Hyers sense on $\mathbb{J} = [-\tau, T]$.*

Proof. Let $x^*(\cdot) \in \mathbb{C}^2(\mathbb{J}, Y)$ satisfy the inequality (5.1) with initial conditions (5.2). Putting

$$h(t) = \left({}^C D_{0+}^{\alpha} x^* \right)(t) - A \left({}^C D_{0+}^{\beta} x^* \right)(t) - Bx^*(t-\tau) - g(t), \quad t \in \mathbb{J}. \tag{5.4}$$

It follows from Definition 5.1 that $\|h(t)\| \leq \varepsilon$. We apply Laplace integral transform to both sides of equation (5.1) and using (3.25), then we acquire

$$H(s) = \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right) X^*(s) - s^{\alpha-1}\varphi(0) + As^{\beta-1}\varphi(0) - s^{\alpha-2}\varphi'(0) - B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s) - G(s), \tag{5.5}$$

where $H(s)$ and $X^*(s)$ are Laplace transforms of $h(t)$ and $x^*(t)$, respectively.

Then, after solving (5.5) with respect to $X^*(s)$, we attain

$$\begin{aligned} X^*(s) = &-H(s) \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1} + s^{\alpha-1} \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1} \varphi(0) \\ &- As^{\beta-1} \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1} \varphi(0) + s^{\alpha-2} \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1} \varphi'(0) \\ &+ B\mathscr{L}\left\{\hat{\varphi}(t-\tau)\right\}(s) \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1} + G(s) \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1}. \end{aligned}$$

In accordance with Theorem we obtain a mild solution of (3.1) via Laplace transform technique. Thus, since linearity property of Laplace transform, we derive

$$\mathscr{L}\left\{x(t) - x^*(t)\right\}(s) = H(s) \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1}.$$

Let $L(s) = \left(s^{\alpha} I - As^{\beta} - Be^{-s\tau} \right)^{-1}$. By using convolution property of Laplace transform and applying inverse Laplace integral formula, we obtain

$$x(t) - x^*(t) = \mathscr{L}^{-1}\left\{H(s)L(s)\right\} = h(t) * l(t)$$

where $l(t) = \sum\limits_{k=0}^{\infty} \sum\limits_{m=0}^{n-1} Q_{k,m}^{A,B} \frac{(t-m\tau)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)}$.

Suppose for $\tau > 0$,

$$\begin{aligned} \Lambda &:= \max\left\{ \int_0^t l(s)\mathrm{d}s, t \in ((n-1)\tau, n\tau] \right\} \\ &= \max\left\{ \int_0^t \sum_{k=0}^{\infty} \sum_{m=0}^{n-1} \|Q_{k,m}^{A,B}\| \frac{(s-m\tau)^{k(\alpha-\beta)+m\alpha+\alpha-1}}{\Gamma(k(\alpha-\beta)+m\alpha+\alpha)} \mathrm{d}s, t \in ((n-1)\tau, n\tau] \right\}. \end{aligned}$$

Therefore, for any $t \in ((n-1)\tau, n\tau]$, we attain a desired result:

$$\|x(t) - x^*(t)\| \leq \| \int_0^t h(t-s)l(s)\mathrm{d}s \| \leq \varepsilon \| \int_0^t l(s)\mathrm{d}s \| \leq \Lambda\varepsilon.$$

Thus, (3.1) is Ulam-Hyers stable on $\mathbb{J} = [-\tau, T]$. The proof is complete. $\square$

This implies that the functional evolution system of Sobolev type (1.1) is also Ulam-Hyers stable on $\mathbb{J} = [-\tau, T]$.

Remark 5.1. It should be pointed out that stability analysis in Ulam-Hyers sense can also be discussed for permutable case $AB = BA$ by using the property of $Q_{k,m}^{A,B}$ (4.1) with binomial coefficients. In this case, the only change happens for $l(t)$ that is expressed in terms of Mittag-Leffler type functions which are generated by linear operators $A, B \in \mathscr{B}(Y)$.

6 Conclusions

In this research work, we first convert Sobolev type fractional time-delay evolution equation with multi-orders (1.1) to multi-term fractional functional evolution equation with linear bounded operators (3.1). Secondly, we give the sufficient conditions to guarantee the rationality of solving multi-term fractional evolution equation with constant delay and linear bounded operators by the Laplace transform method. Then, we solve linear inhomogeneous time-delay evolution equations with nonpermutable & permutable linear bounded operators $A, B \in \mathscr{B}(Y)$ by making use of Laplace transform. Next, we propose an exact analytical representation of a mild solution to (3.1) and (1.1) in terms of newly defined delayed Mittag-Leffler type function which is generated by linear bounded operators by removing the strong condition that is an exponential boundedness of a forced term and one of fractional orders with the help of analytical methods, namely: verification by substitution and fractional analogue of variation of constants formulas. Moreover, we investigate stability results for fractional-order functional evolution equation in Ulam-Hyers sense without fixed point approach.

The main contributions of this paper are as follows:

- we introduce a new delayed Mittag-Leffler type function which is generated by linear bounded operators $A, B \in \mathscr{B}(Y)$ via a double infinite series ;
- we propose the property of $Q_{k,m}^{A,B}$ with nonpermutable linear operators $A, B \in \mathscr{B}(Y)$ which is a generalization of well-known Pascal's rule for binomial coefficients.
- we acquire the analytical representation of a mild solution for linear Sobolev type multi-term fractional functional evolution system with nonpermutable and permutable linear operators;
- we derive the exact analytical representation of multi-dimensional time-delay system with two independent orders and nonpermutable & permutable matrices.
- we investigate Ulam-Hyers type stability results for time-delay evolution equation of Sobolev type with the help of Laplace transform;

The possible directions for future work in which to extend the results of this paper is looking at Sobolev type fractional impulsive evolution equations with multi-orders and investigate asymptotic stability and approximate controllability results for Sobolev type multi-term fractional differential equations (1.1).

References

[1] A. Ahmadova, I.T. Huseynov, A. Fernandez, N.I. Mahmudov, Mittag-Leffler functions used to solve multi-order systems of fractional differential equations, Commun. Nonlinear Sci. Numer. Simulat., 105735, 2021, https://doi.org/10.1016/j.cnsns.2021.105735.

[2] A. Ahmadova, N.I. Mahmudov, Langevin differential equations with general fractional orders and their applications to electric circuit theory, J. Comput. Appl. Math., 2020, https://doi.org/10.1016/j.cam.2020.113299.

[3] A. Ahmadova, N.I. Mahmudov, Asymptotic behavior of solutions to fractional stochastic multi-term differential equation systems, Math. Meth. Appl. Sci., accepted, 2021.

[4] K. Balachandran, J.P. Dauer, Controllability of functional differential systems of Sobolev type in Banach spaces, Kybernetika, 34(3), 1998, 349-357.

[5] K. Balachandran, S. Kiruthika, J.J. Trujillo, On fractional impulsive equations of Sobolev type with nonlocal condition in Banach spaces, Comp. Math. Appl., 62, 2011, 1157-1165.

[6] E. Bazhlekova, Properties of the fundamental and the impulse-response solutions of multi-term fractional differential equations, Complex Analysis and Applications'13 (Proc. Intern. Conf., Sofia), Bulg. Acad. Sci. Sofia, 2013, 55–64.

[7] Y-K. Chang, R. Ponce, S. Rueda, Fractional differential equations of Sobolev type with sectorial operators, Semigroup Forum, 99, 2019, 591-606.

[8] J. Diblik, D. Ya. Khusainov, J. Lukacova, M. Ruzickova, Control of oscillating systems with a single delay, Adv. Difference Equ., 2010, 108218.

[9] J. Diblik, M. Feckan, M. Pospisil, Representation of a solution of the Cauchy problem for an oscillating system with multiple delays and pairwise permutable matrices, Abstract and Applied Analysis, 2013, https://doi.org/10.1155/2013/931493.

[10] J. Diblik, M. Feckan, M. Pospisil, Representation of a solution of the Cauchy problem for an oscillating system with two delays and permutable matrices, Ukrainian Mathematical Journal, 65(1), 2013, 58–69.

[11] M. Feckan, J. Wang, Y. Zhou, Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators, J. Optim. Theory Appl., 156, 2013, 79-95.

[12] A. Fernandez, C. Kürt, M.A. Özarslan, A naturally emerging bivariate Mittag-Leffler function and associated fractional-calculus operators, Comp. Appl. Math., 39, 2020, 200.

[13] R. Gorenflo, A.A. Kilbas, F. Mainardi, S.V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications, Springer-Verlag, Berlin, 2014.

[14] D. Henry, Geometric theory of semilinear parabolic equations, Lecture notes in Mathematics, Springer-Vellag, New York/Berlin, 840, 1981.

[15] R. Hilfer, Y. Luchko, Z. Tomovski, Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives, Fract. Calc. Appl. Anal., 12, 2009, 299–318.

[16] I.T. Huseynov, N.I. Mahmudov, Delayed analogue of three-parameter Mittag-Leffler functions and their applications to Caputo type fractional time-delay differential equations, Math. Meth. Appl. Sci., 2020, 1-25, https://doi.org/10.1002/mma.6761.

[17] I.T. Huseynov, A. Ahmadova, N.I. Mahmudov, Fractional Leibniz integral rules for Riemann-Liouville and Caputo fractional derivatives and their applications, arXiv:2012.11360.

[18] I.T. Huseynov, A. Ahmadova, A. Fernandez, N.I. Mahmudov, Explicit analytic solutions of incommensurate fractional differential equation systems, Appl. Math. Comput., 390, 2021, https://doi.org/10.1016/j.amc.2020.125590.

[19] L. Kexue, P. Jigen, Laplace transform and fractional differential equations, Appl. Math. Lett., 24, 2011, 2019-2023.

[20] D.Ya. Khusainov, J. Diblik, M. Ruzickova, J. Lukacova, Representation of a solution of the Cauchy problem for an oscillating system with pure delay, Nonlinear Oscill. 11, 2008, 276–285, https://doi.org/10.1007/s11072-008-0030-8.

[21] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and applications of fractional differential equations, Elsevier Sceince B.V. 204, 2006.

[22] Ch. Liang, J. Wang, D. O'Regan, Controllability of nonlinear delay oscillating systems, Electron. J. Qual. Theory Differ. Equ., 47, 2017, 1–18.

[23] Ch. Liang, J. Wang, Analysis of iterative learning control for an oscillating control system with two delays, Transactions of the Institute of Measurement and Control, 40(6), 2018, 1757-1765.

[24] Ch. Liang, W. Wei, J. Wang, Stability of delay differential equations via delayed matrix sine and cosine of polynomial degrees, Adv. Differ. Equ., 131, 2017, https://doi.org/10.1186/s13662-017-1188-0.

[25] Ch. Liang, J.R. Wang, D. O'Regan, Representation of a solution for a fractional linear system with pure delay, Appl. Math. Lett., 77, 2018, 72-78.

[26] Y.F. Luchko, R. Gorenflo, An operational method for solving fractional differential equations with Caputo derivatives, Acta Math. Vietnam., 24, 1999, 207-233.

[27] Y.F. Luchko, S. Yakubovich, An operational method for solving some classes of integro-differential equations, Differ. Equ., 30, 1994, 247–256.

[28] N.I. Mahmudov, Delayed perturbation of Mittag-Leffler functions their applications to fractional linear delay differential equations, Math. Methods Appl. Sci., 2018, 1-9, http://dx.doi.org/10.1002/mma.5446.

[29] N.I. Mahmudov, Representation of solutions of discrete linear delay systems with non-permutable matrices, Appl. Math. Lett., 85, 2018, 8-14.

[30] N.I. Mahmudov, Delayed linear difference equations : the method of $\mathscr{Z}$-transform, Electron. J. Qual. Theory Differ. Equ., 53, 2020, 1-12, https://doi.org/10.14232/ejqtde.2020.1.53.

[31] N.I. Mahmudov, A novel fractional delayed matrix cosine and sine, Appl. Math. Lett., 92, 2019, 41-48.

[32] N.I. Mahmudov, Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces, Abst. Appl. Anal., 2013, https://dx.doi.org/10.1155/2013/502839.

[33] N.I. Mahmudov, I.T. Huseynov, N.A. Aliev, F.A. Aliev, Analytical approach to a class of Bagley-Torvik equations, TWMS J. Pure Appl. Math., 11, 2020, 238-258.

[34] M. Medved, M. Pospisil, Sufficient conditions for the asymptotic stability of nonlinear multidelay differential equations with linear parts defined by pairwise permutable matrices, Nonlinear Analysis, 75, 2012, 3348-3363.

[35] M.G. Mittag-Leffler, Sopra la funzione $E_\alpha(x)$, Rend. R. Acc. Lincei., 13, 1904, 3–5.

[36] S. Pak, H.K. Choi, K.R. Sin, Analytical solutions of linear inhomogeneous fractional differential equation with continuous variable coefficients, Advan. Differ. Equ., 256, 2019.

[37] I. Podlubny, Fractional differential equations; Academic Press, San Diego, 1999.

[38] M. Pospisil, L. Skripkova, Representation of solutions of neutral differential equations with delay and linear parts defined by pairwise permutable matrices, Miskolc Mathematical Notes, 16(1), 2015, 423-438.

[39] M. Pospisil, F. Jaros, On the representation of solutions of delayed differential equations via Laplace transform, Electron. J. Qual. Theory Differ. Equ., 117, 2016, 1–13.

[40] M. Pospisil, Representation of solutions of systems of linear differential equations with multiple delays and nonpermutable variable coefficients, Mathematical Modelling and Analysis, 25(2), 303-322, 2020, https://doi.org/10.3846/mma.2020.11194.

[41] M. Pospisil, Representation and stability of solutions of systems of functional differential equations with multiple delays, Electron. J. Qual. Theory Differ. Equ., 54, 2012, 1–30.

[42] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 19, 1971, 7-15.

[43] J. Sabatier, M. Moze, C. Farges, LMI stability for fractional order systems, Comp. Math. Appl., 59, 2010, 1594-1609.

[44] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, New York, 1993.

[45] R.K. Saxena, S.L. Kalla, R. Saxena, Multivariate analogue of generalised Mittag-Leffler function. Integr. Transf. and Spec. F. 22, 2011, 533-548.

[46] V. Vijayakumar, R. Udhayakumar, Y. Zhou, N. Sakthivel, Approximate controllability results for Sobolev type delay differential system of fractional order without uniqueness, Numer. Methods Partial Differential Eq., 2020, 1-20, https://doi.org/10.1002/num.22642.

[47] J. Wang, M. Feckan, Y. Zhou, Controllability of Sobolev type fractional evolution systems, Dynamics of PDE, 11(1), 2014, 71-87.

[48] J. Wang, X. Li, A uniform method to Ulam-Hyers stability for some linear fractional equations, Mediterr. J. Math., 13, 2016, 625-635.

[49] A. Wiman, Über den Fundamentalsatz der Theorie der Funkntionen $E_\alpha(x)$, Acta Math., 29, 1905, 191–201.

[50] H. Ye, J. Gao, J. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, J. Math. Anal. Appl., 328(2), 2007, 1075-1081.

[51] K. Yosida, Functional Analysis, Springer-Verlag, 1968.