Multiple nodal solutions for the Schrödinger-Poisson system with an asymptotically cubic term

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Abstract

This paper deals with the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{0.1}$$

where $\lambda > 0$ and $f(u)$ is a nonlinear term asymptotically cubic at the infinity. Taking advantage of the Miranda theorem and deformation lemma, we combine some new analytic techniques to prove that for each positive integer k, system (0.1) admits a radial nodal solution U_k^λ, which has exactly $k+1$ nodal domains and the corresponding energy is strictly increasing in k. Moreover, for any sequence $\{\lambda_n\} \to 0_+$ as $n \to \infty$, up to a subsequence, $U_k^{\lambda_n}$ converges to some $U_k^0 \in H_r^1(\mathbb{R}^3)$, which is a radial nodal solution with exactly $k+1$ nodal domains of (0.1) for $\lambda = 0$. These results give an affirmative answer to the open problem proposed in [Kim S, Seok J. Commun. Contemp. Math., 2012] for the Schrödinger-Poisson system with an asymptotically cubic term.

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1 Introduction

The following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \lambda\phi u = f(u) \quad \text{in } \mathbb{R}^3, \\ -\Delta\phi = u^2 \quad \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

has attracted much attention from the researchers, due to its importance in many physical applications [6, 14] and its difficulties and challenges in mathematical problems [11, 17]. As is well known, system (1.1) is equivalent to a single equation

$$-\Delta u + u + \lambda\phi_u u = f(u) \quad \text{in } \mathbb{R}^3, \tag{1.2}$$

which has a variational structure. The corresponding energy functional $I_\lambda : H^1(\mathbb{R}^3) \to \mathbb{R}$ is defined as

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)dx + \frac{\lambda}{4}\int_{\mathbb{R}^3}\phi_u u^2 dx - \int_{\mathbb{R}^3} F(u),$$

where $\phi_u(x) = \int_{\mathbb{R}^3}\frac{u^2(y)}{4\pi|x-y|}dy$ and $F(u) = \int_0^u f(\tau)d\tau$. Then its Gateaux derivative along the direction $v \in H^1(\mathbb{R}^3)$ is

$$I_\lambda(u)v = \int_{\mathbb{R}^3}(\nabla u\nabla v + uv)dx + \lambda\int_{\mathbb{R}^3}\phi_u uv - \int_{\mathbb{R}^3} f(u)v.$$

Moreover, the weak solutions of (1.1) can be found by the variational method as critical points of I_λ. In the last decades, various types of solutions of (1.1) have been found in the literature such as positive solutions, ground state solutions, multiple solutions, semiclassical states (see[1, 3, 4, 10, 17] and references therein).

Recently, many researchers start to focus on the existence of nodal solutions (also called sign-changing solutions) of (1.1). When $f(u) = |u|^{p-2}u$ with $p \in (4, 6)$, Wang and Zhou [20] proved that (1.1) admitted a least energy nodal solutions having two nodal domains by the Nehari manifold method and Brouwer degree theory. Later, Kim and Seok [11] and ianni [9] independently proved that (1.1) admits multiple nodal solutions with any prescribed numbers of nodes by the variational method and heat flow method, respectively. In these papers, the nonlinearity f is "super-cubic" at infinity, namely,

(F0) $\lim_{|u|\to\infty} \frac{F(u)}{u^4} = +\infty$.

Very recently, some efforts are taken to weaken the "super-cubic" condition **(F0)**. Li, Wang, Zhang [12] and Guo, Wang [7] proved that (1.1) admitted infinitely many sign-changing solutions under the assumption

(F0') $\lim_{|u|\to\infty} \frac{F(u)}{|u|^p} = +\infty$ with some $p \in (3, 6)$

by using the critical points theory of descending flow of invariant subsets and perturbation method. It is obvious that **(F0')** is weaker than **(F0)**. But unfortunately, these solutions in [7, 12] do not give any information on the numbers of nodal domains. For more related results about nodal solutions and details, one can refer to [2, 8, 19] and references therein.

To the best of our knowledge, the following

Open problem: can we construct nodal solutions with any prescribed number of nodal domains of (1.1) if **(F0)** is not satisfied?

proposed in [11] is still unsolved. This paper is devoted to answering this open problem. Note that Murcia and Siciliano [15] proved the existence of least energy nodal solution of (1.1), which has precisely two nodal domains. Similar as [15], we make the following assumptions that f satisfies

(F1) $f \in C(\mathbb{R}, \mathbb{R})$ and $f(t) = -f(-t)$ for $t \in \mathbb{R}$;

(F2) $\lim_{t\to 0} f(t)/t = 0$;

(F3) $\lim_{t\to\infty} f(t)/t^3 = 1$ and $f(t)/t^3 < 1\, for\, all\, t \in \mathbb{R}\backslash\{0\}$;

(F4) the function $t \mapsto f(t)/t^3$ is strictly increasing on $(0, \infty)$;

(F5) $\lim_{t\to\infty}[f(t)t - 4F(t)] = +\infty$.

Here assumption **(F3)** is called "asymptotically cubic condition" and **(F5)** is called "non-quadraticity condition". There are many functions satisfying **(F1)-(F5)** but not **(F0)**. For example, $f(u) = \frac{u^5}{1+u^2}$, $u \in \mathbb{R}$, which has a primitive $F(u) = \frac{u^4}{4} - \frac{u^2}{2} + \frac{1}{2}\ln(1+u^2)$.

Before stating our existence result, we introduce some preliminaries notations. For each positive integer k and $0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := +\infty$,, we denote by $\mathbf{r}_k = (r_1, \cdots, r_k) \in \mathbb{R}^k$ and define a Nehari type set

$$\begin{aligned} \mathcal{N}_k = \Big\{ u \in H_r^1(\mathbb{R}^3) : &\text{ there exists } \mathbf{r}_k \text{ such that } u_i \neq 0 \text{ in } B_i, \\ & I'_\lambda(u)u_i = 0,\ \forall 1 \leq i \leq k+1 \Big\}, \end{aligned} \tag{1.3}$$

where $B_1 = \{x \in \mathbb{R}^3 : |x| < r_1\}$, $B_i = \{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_1\}$, $u_i = u\chi_{B_i}$ and χ_{B_i} is the characteristic function on B_i. Similar as [5, 11], we consider the minimum level

$$c_k = \inf_{u \in \mathcal{N}_k} I_\lambda(u).$$

Our first existence result is as follows, which gives an affirmative answer to the above open problem.

Theorem 1.1. *Assume that **(F1)-(F5)** hold and $\lambda > 0$. Then for each positive integer k, problem* (1.1) *admits a radial least energy nodal solution U_k with precisely* $k+1$ *nodal domains such that $I_\lambda(U_k) = c_k$.*

One main difficulty in the proof of Theorem 1.1 is to obtain the nonempty of Nehari type set, because $\phi_u u$ is a 3−homogeneous term in the sense that

$$\phi_{tu}tu = t^3\phi_u u, \quad t \in \mathbb{R},$$

which competes with the asymptotically cubic term $f(u)$ sophisticatedly. Besides, compared with [11], all the techniques concerning the super-cubic case are no longer valid and hence some new ideas are necessary. We shall overcome them by construction method and some subtle analysis combined with the Miranda theorem.

Our another aim of this paper is to show that the energy of U_k is strictly increasing in k and the estimates of the energy $I_\lambda(U_k)$. Different from the super-cubic case, the asymptotically cubic term $f(u)$ and the nonlocal term $\lambda\phi_u u$ are in a more complicate competition, which makes the energy difficult to estimate. By taking advantage of the Miranda theorem and some subtle analysis, we have the following energy estimate.

We denote by $\mathcal{N} := \{u \in H_r^1(\mathbb{R}^3)\backslash\{0\} : I'_\lambda(u)u = 0\}$ the usual Nehari manifold, and by U_0 a positive ground state solution of (1.1), which is proved in [15].

Theorem 1.2. *Under the hypotheses of Theorem 1.1, the energy of U_k is strictly increasing in k. Namely,*

$$I_\lambda(U_{k+1}) > I_\lambda(U_k).$$

Moreover, $I_\lambda(U_k) > (k+1)I_\lambda(U_0)$.

Notice that U_k obtained in Theorem 1.1 depends on λ. To emphasize this dependence, we denote U_k by U_k^λ and then analyze the convergence properties of U_k^λ as $\lambda \to 0_+$.

Theorem 1.3. *Under the hypotheses of Theorem* (1.1)*, for any sequence $\{\lambda_n\}$ with $\lambda_n \to 0_+$ as $n \to \infty$, there exists a subsequence, still denoted by $\{\lambda_n\}$, such that $U_k^{\lambda_n}$ converges to U_k^0 strongly in $H_r^1(R^3)$ as $n \to \infty$, where U_k^0 is a radial least energy nodal solution among all the radial solutions having exactly $k+1$ nodal domains to the following equation*

$$-\Delta u + u = f(u). \tag{1.4}$$

Remark 1.4. *We point out that Theorems 1.1-1.3 are also true if the whole space $\mathbb{R}^3$ is replaced by an open ball $B_R(0) \subset \mathbb{R}^3$. Indeed, we only need to modify (ii) in Lemma 4.3 to (ii'): If $r_k \to R$, then $\psi(r_k) \to +\infty$. The remainder of the proof is similar with some necessary modifications.*

The contribution of this paper is twofold: on one hand, this paper gives an affirmative answer to the open problem addressed in [11]. On the other hand, by combining the Miranda theorem and some analytic techniques, we develop the gluing method to deal with the asymptotically cubic problem. Up to our knowledge, this paper is the first attempt to give the existence of nodal solutions with any prescribed number of nodes for an asymptotically cubic problem.

This paper is organized as follows. In Section 2, we give the variational framework of problem (1.1). In Section 3, we prove the nonempty of $\mathcal{N}_k$ and its related properties. In Section 4, nodal solutions of problem (1.1) will be constructed by the gluing method, and in Section 5, the energy comparison and asymptotic behaviors will be obtained.

2 Variational framework and preliminary results

In this section, we give the variational framework of (1.2) and some preliminary results. For each $k \in \mathbb{N}_+$, we define a set

$$\Gamma_k = \left\{ \mathbf{r}_k := (r_1, \cdots, r_k) \in (\mathbb{R}_{>0})^k : 0 =: r_0 < r_1 < \cdots < r_k < r_{k+1} := +\infty \right\} \tag{2.1}$$

and for each $\mathbf{r}_k \in \Gamma_k$, we denote

$$\begin{aligned} B_1^{\mathbf{r}_k} &:= \left\{ x \in \mathbb{R}^3 : |x| < r_1 \right\}, \\ B_i^{\mathbf{r}_k} &:= \left\{ x \in \mathbb{R}^3 : r_{i-1} < |x| < r_i \right\}, \quad i = 2, \cdots, k, \\ B_{k+1}^{\mathbf{r}_k} &:= \left\{ x \in \mathbb{R}^3 : |x| > r_k \right\}. \end{aligned}$$

Clearly, $B_1^{\mathbf{r}_k}$ is a ball, $B_2^{\mathbf{r}_k} \cdots, B_k^{\mathbf{r}_k}$ are annulus and $B_{k+1}^{\mathbf{r}_k}$ is the complement of a ball. Then we set a family of Hilbert spaces

$$H_i^{\mathbf{r}_k} := \left\{ u \in H_0^1(B_i^{\mathbf{r}_k}) : u(x) \text{ is radial} \right\}$$

with the norm $\|u\|_i = \left(\int_{B_i^{\mathbf{r}_k}} |\nabla u|^2 + u^2 \right)^{\frac{1}{2}}$ and let a product space

$$\mathcal{H}_k^{\mathbf{r}_k} = H_1^{\mathbf{r}_k} \times \cdots \times H_{k+1}^{\mathbf{r}_k}.$$

We introduce an auxiliary functional $E : \mathcal{H}_k^{\mathbf{r}_k} \to \mathbb{R}$ as

$$E(u_1, \cdots, u_{k+1}) = \sum_{i=1}^{k+1} \left(\frac{1}{2} \|u_i\|_i^2 + \frac{\lambda}{4} \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \int_{B_j^{\mathbf{r}_k}} \frac{|u_j(y)|^2 |u_i^2(x)|^2}{4\pi |x-y|} dx dy - \int_{B_i^{\mathbf{r}_k}} F(u_i) dx \right), \tag{2.2}$$

which is related to I_λ, because

$$E(u_1, \cdots, u_{k+1}) = I_\lambda\Big(\sum_{i=1}^{k+1} u_i\Big). \tag{2.3}$$

Then

$$\partial_{u_i} E(u_1, \cdots, u_{k+1}) u_i = \|u_i\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} f(u_i) u_i.$$

If $(u_1,\cdots,u_{k+1})\in\mathcal{H}_k^{\mathbf{r}_k}$ is a critical point of E, then each component u_i satisfies the following system

$$\begin{cases} -\Delta u_i+u_i+\lambda\left(\displaystyle\int_{\mathbb{R}^3}\frac{|\sum_{j=1}^{k+1}u_j(y)|^2}{4\pi|x-y|}dy\right)u_i=f(u_i) \quad \text{in } B_i^{\mathbf{r}_k}, \quad i=1,\cdots,k+1,\\ u_i=0 \quad \text{on } \partial B_i^{\mathbf{r}_k}. \end{cases} \tag{2.4}$$

In order to search for the critical points of E with nonzero components, we consider the following infimum problem

$$\Psi(\mathbf{r}_k)=\inf_{(u_1,\cdots,u_{k+1})\in\mathcal{N}_k^{\mathbf{r}_k}}E(u_1,\cdots,u_{k+1}). \tag{2.5}$$

where

$$\mathcal{N}_k^{\mathbf{r}_k}:=\left\{(u_1,\cdots,u_{k+1})\in\mathcal{H}_k^{\mathbf{r}_k}:u_i\neq 0,\partial_{u_i}E(u_1,\cdots,u_{k+1})u_i=0,1\le i\le k+1\right\}. \tag{2.6}$$

Clearly, the nonempty of $\mathcal{N}_k^{\mathbf{r}_k}$ implies the nonempty of $\mathcal{N}_k$, which is defined in (1.3). Moreover,

$$\inf_{\mathbf{r}_k\in\Gamma_k}\Psi(\mathbf{r}_k)=\inf_{u\in\mathcal{N}_k}I_\lambda(u).$$

Let $u_+=\max\{u,0\},u_-=\max\{-u,0\}$ and

$$\mathcal{N}_{nod}=\{u\in H_r^1(\mathbb{R}^3)\backslash\{0\}:I_\lambda'(u)u_\pm=0,u_\pm\neq 0\}.$$

Since $\mathcal{N}_k\subset\mathcal{N}_{nod}$ and $\inf_{u\in\mathcal{N}_{nod}}I_\lambda(u)>0$ (see [15, Corollary 10]), it follows immediately that

$$\inf_{\mathbf{r}_k\in\Gamma_k}\Psi(\mathbf{r}_k)=c_k:=\inf_{u\in\mathcal{N}_k}I_\lambda(u)\ge\inf_{u\in\mathcal{N}_{nod}}I_\lambda(u)>0, \tag{2.7}$$

Next, we list a useful lemma, which plays an important role in the proofs of our main results.

Lemma 2.1. *([13, Miranda Theorem]) Let*

$$D=\{x:=(x_1,\cdots,x_n)\in\mathbb{R}^n:|x_i|<L \textit{ for all } 1\le i\le n\}.$$

Suppose that the mapping $H=(h_1,\cdots,h_n):\bar{D}\to\mathbb{R}^n$ is continuous on $\bar{D}$ satisfying

$$H(x)\neq\theta, \quad \forall\ x\in\partial D$$

and

(i) $\quad h_i(x_1, \cdots, x_{i-1}, -L, x_{i+1}, \cdots, x_n) \geq 0 \quad$ *for* $1 \leq i \leq n$,

(ii) $\quad h_i(x_1, \cdots, x_{i-1}, L, x_{i+1}, \cdots, x_n) \leq 0 \quad$ *for* $1 \leq i \leq n$,

where $\theta := (0, \cdots, 0)$*. Then* $H(x) = \theta$ *has a solution in* D*.*

3 Properties of the Nehari type set

In this section, we are devoted to proving the nonempty of $\mathcal{N}_k^{\mathbf{r}_k}$ and $\mathcal{N}_k$, and giving some properties of the Nehari type set.

First, we recall a useful result cited from [11]. Before stating it, for each $\mathbf{r}_k \in \Gamma_k$ and $p \in [4, 6)$, we define

$$E_p(u_1, \cdots, u_{k+1}) := \sum_{i=1}^{k+1} \left(\frac{1}{2} \|u_i\|_i^2 + \frac{\lambda}{4} \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \int_{B_j^{\mathbf{r}_k}} \frac{|u_j(y)|^2 |u_i^2(x)|^2}{4\pi |x-y|} dxdy - \int_{B_i^{\mathbf{r}_k}} |u_i|^p dx \right)$$

and a family of sets

$$M_{k,p}^{\mathbf{r}_k} := \left\{ (u_1, \cdots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k} : u_i \neq 0, \partial_{u_i} E_p(u_1, \cdots, u_{k+1}) u_i = 0,\ 1 \leq i \leq k+1 \right\}, \quad (3.1)$$

where

$$\partial_{u_i} E_p(u_1, \cdots, u_{k+1}) u_i = \|u_i\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} |u_i|^2 - \int_{B_i^{\mathbf{r}_k}} |u_i|^p. \quad (3.2)$$

Let $G_p^u : (\mathbb{R}_{\geq 0})^{k+1} \to \mathbb{R}$ be defined as

$$\begin{aligned} G_p^u(c_1, \cdots, c_{k+1}) &:= E_p(c_1 u_1, \cdots, c_{k+1} u_{k+1}) \\ &= \sum_{i=1}^{k+1} \left(\frac{1}{2} c_i^2 \|u_i\|_i^2 + \frac{\lambda c_i^2}{4} \sum_{j=1}^{k+1} c_j^2 \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} |u_i|^2 - \frac{c_i^p}{p} \int_{B_i^{\mathbf{r}_k}} |u_i|^p \right). \end{aligned} \quad (3.3)$$

Lemma 3.1. *([11, Lemmas 3.1 and 3.3]) Let* $p \in (4, 6)$*. Then for any* $\boldsymbol{r}_k \in \Gamma_k$ *and* $(u_1, \cdots, u_{k+1}) \in \mathcal{H}_k^{\boldsymbol{r}_k}$ *with all* $u_i \neq 0$*, there exists a unique maximum point* $(t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{>0})^{k+1}$ *of* G_p^u *in* $(\mathbb{R}_{\geq 0})^{k+1}$ *such that*

$$(t_1 u_1, \cdots, t_{k+1} u_{k+1}) \in M_{k,p}^{\boldsymbol{r}_k}.$$

By using Lemma 3.1, we prove the following result.

Lemma 3.2. *For each $\mathbf{r}_k \in \Gamma_k$, the set $M_{k,4}^{\mathbf{r}_k} \neq \emptyset$, which is defined in* (3.1) *with* $p = 4$.

Proof. For each $\mathbf{r}_k = (r_1, \cdots, r_{k+1}) \in \Gamma_k$, we take $(u_1, \cdots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k}$ with $u_i \neq 0$ such that $\min_i \left\{ \frac{\|\nabla u_i\|^2_{L^2(B_i^{\mathbf{r}_k})}}{\|u_i\|^4_{L^4(B_i^{\mathbf{r}_k})}} \right\} > 1$. Clearly, there exists $\delta_0 > 0$ such that

$$1 < \delta_0^2 < \min\{\|\nabla u_i\|^2_{L^2(B_i^{\mathbf{r}_k})} / \|u_i\|^4_{L^4(B_i^{\mathbf{r}_k})}, i = 1, \cdots, k+1\}. \tag{3.4}$$

Then we define a family of radial functions by

$$v_i^{\delta_0}(x) = \delta_0^2 u_i(r_{i-1} + \delta_0(|x| - r_{i-1})).$$

Note that the support set $supp(u_i) \subset B_i^{\mathbf{r}_k} := \{x \in \mathbb{R}^N : r_{i-1} < |x| < r_i\}$. Then by the choice of $\delta_0 > 1$ and some direct computations, we have

$$supp(v_i^{\delta_0}) \subset \{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_{i-1} + (r_i - r_{i-1})/\delta_0\} \subset B_i^{\mathbf{r}_k}$$

and

$$\begin{aligned} &\|v_i^{\delta_0}\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{\mathbb{R}^3} \phi_{v_j^{\delta_0}} |v_i^{\delta_0}|^2 - \int_{\mathbb{R}^3} |v_i^{\delta_0}|^4 \\ =&\delta_0^3 \|\nabla u_i\|^2_{L^2(B_i^{\mathbf{r}_k})} + \delta_0 \int_{B_i^{\mathbf{r}_k}} |u_i|^2 + \lambda \delta_0^3 \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \delta_0^5 \int_{B_i^{\mathbf{r}_k}} |u_i|^4. \end{aligned}$$

Let $g_i : [0, +\infty) \to \mathbb{R}$ be defined by

$$g_i(\delta) = \delta^3 \|\nabla u_i\|^2_{L^2(B_i^{\mathbf{r}_k})} + \delta \int_{B_i^{\mathbf{r}_k}} |u_i|^2 + \lambda \delta^3 \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \delta^5 \int_{B_i^{\mathbf{r}_k}} |u_i|^4.$$

Obviously, $g_i(\delta_0) \geq \delta_0^3 \|\nabla u_i\|^2_{L^2(B_i^{\mathbf{r}_k})} - \delta_0^5 \int_{B_i^{\mathbf{r}_k}} |u_i|^4 > 0$ due to (3.4), $g_i(0) = 0$ and $g_i(\delta) \to -\infty$ as $\delta \to +\infty$. Moreover, for each i, there is a unique zero point $\delta_i \in (\delta_0, +\infty)$ such that $g_i(\delta_i) = 0$ and $g_i > 0$ in $(0, \delta_i)$ and $g_i < 0$ in $(\delta_i, +\infty)$.

Let $\delta_{max} = \max\{\delta_1, \cdots, \delta_{k+1}\}$ and set

$$w_i(x) := v_i^{\delta_{\max}}(x) = \delta_{\max}^2 u_i(r_{i-1} + \delta_{\max}(|x| - r_{i-1})). \tag{3.5}$$

Then $g_i(\delta_{max}) \leq g_i(\delta_i) \leq 0$ and

$$supp(w_i) \subset \left\{x \in \mathbb{R}^3 : r_{i-1} < |x| < r_{i-1} + (r_i - r_{i-1})/\delta_{\max}\right\} \subset B_i^{\mathbf{r}_k}.$$

Hence

$$(w_1, \cdots, w_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k} \quad \text{and each } w_i \neq 0.$$

Now, we claim that there exists some $(t_{1,4}, \cdots, t_{k+1,4}) \in (\mathbb{R}_{>0})^{k+1}$ such that

$$(t_{1,4}w_1, \cdots, t_{k+1,4}w_{k+1}) \in M_{k,4}^{\mathbf{r}_k}. \tag{3.6}$$

Indeed, by Lemma 3.1, for each $p \in (4, 6)$, there exists a unique global maximum point $(t_{1,p}, \cdots, t_{k+1,p}) \in (\mathbb{R}_{>0})^{k+1}$ of G_p^w such that

$$t_{i,p}^2\|w_i\|_i^2 + \lambda \sum_{j=1}^{k+1} t_{i,p}^2 t_{j,p}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{w_j}|w_i|^2 - t_{i,p}^p \int_{B_i^{\mathbf{r}_k}} |w_i|^p = 0, \quad \forall i = 1, \cdots, k+1. \tag{3.7}$$

Henceforth, for each p, we denote the maximum element by

$$t_{i_p,p} = \max_{j \in \{1,\cdots,k+1\}} \{t_{j,p}\}$$

Firstly, we assert that $(t_{1,p}, \cdots, t_{k+1,p})$ is bounded for $p \to 4_+$. In fact, we argue it by contradiction. Suppose on the contrary that $t_{i_p,p} \to +\infty$ as $p \to 4_+$. Then it follows from (3.5) and (3.7) that

$$\begin{aligned}
0 &= t_{i_p,p}^{2-p}\|w_{i_p}\|_{i_p}^2 + \lambda \sum_{j=1}^{k+1} \frac{t_{j,p}^2}{t_{i_p,p}^2} t_{i_p,p}^{4-p} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i,p}^{\mathbf{r}_k}} |w_{i_p}|^p \\
&\leq t_{i_p,p}^{2-p}\|w_{i_p}\|_{i_p}^2 + \lambda \sum_{j=1}^{k+1} \int_{B_{i,p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i,p}^{\mathbf{r}_k}} |w_{i_p}|^p \\
&\to \lambda \sum_{j=1}^{k+1} \int_{B_{i,p}^{\mathbf{r}_k}} \phi_{w_j}|w_{i_p}|^2 - \int_{B_{i,p}^{\mathbf{r}_k}} |w_{i_p}|^4 \quad \text{as } p \to 4_+ \\
&= \lambda\delta_{max}^3 \sum_{j=1}^{k+1} \int_{B_{i,p}^{\mathbf{r}_k}} \phi_{u_j}u_{i_p}^2 - \delta_{max}^5 \int_{B_{i,p}^{\mathbf{r}_k}} |u_{i_p}|^4 \\
&= g_{i_p}(\delta_{max}) - \delta_{max}^3\|\nabla u_{i_p}\|_{L^2(B_{i,p}^{\mathbf{r}_k})}^2 - \delta_{max} \int_{B_{i,p}^{\mathbf{r}_k}} |u_{i_p}|^2 < 0,
\end{aligned} \tag{3.8}$$

which leads to a contradiction. Thus the assertion follows and $(t_{1,p}, \cdots, t_{k+1,p})$ is bounded for $p \to 4_+$. Then there is some $(t_{1,4}, \cdots, t_{k+1,4}) \in (\mathbb{R}_{\geq 0})^{k+1}$ such that

$$(t_{1,p}, \cdots, t_{k+1,p}) \to (t_{1,4}, \cdots, t_{k+1,4}) \text{ as } p \to 4_+.$$

By Lemma 3.1, $(t_{1,4}, \cdots, t_{k+1,4})$ is also a global maximum point of G_4^w defined in (3.3), because $(t_{1,p}, \cdots, t_{k+1,p})$ is the global maximum point of G_p^w, and

$$t_{i,4}^2\|w_i\|_i^2 + \lambda \sum_{j=1}^{k+1} t_{i,4}^2 t_{j,4}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{w_j} w_i^2 = t_{i,4}^4 \int_{B_i^{\mathbf{r}_k}} |w_i|^4. \tag{3.9}$$

Next, we prove $t_{i,4} > 0, \forall i = 1, \cdots, k+1$. Otherwise, we may suppose on the contrary that there is some $i_0 \in \{1, \cdots, k+1\}$ such that $t_{i_0,4} = 0$. Note that

$$\begin{aligned}
&G_4^w(t_{1,4}, \cdots, t_{i_0-1,4}, \mu, t_{i_0+1,4}, \cdots, t_{k+1,4}) \\
&= G_4^w(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4}) \\
&\quad + \frac{\mu^2}{2}\|w_{i_0}\|_{i_0}^2 + \frac{\lambda\mu^4}{4} \int_{B_{i_0}^{\mathbf{r}_k}} \phi_{w_{i_0}} w_{i_0}^2 + \frac{\lambda\mu^2}{4} \sum_{j \neq i_0} t_{j,4}^2 \int_{B_{i_0}^{\mathbf{r}_k}} \phi_{w_j} w_{i_0}^2 - \mu^4 \int_{B_{i_0}^{\mathbf{r}_k}} |w_{i_0}|^4 \\
&=: G_4^w(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4}) + \theta(\mu).
\end{aligned}$$

Clearly, $\theta(\mu) > 0$ if μ is sufficiently small. Then $(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4})$ is not a global maximum point of G_4^w in $(\mathbb{R}_{\geq 0})^{k+1}$, which contradicts with the fact that $(t_{1,4}, \cdots, t_{i_0-1,4}, 0, t_{i_0+1,4}, \cdots, t_{k+1,4})$ is the global maximum point of G_4^w in $(\mathbb{R}_{\geq 0})^{k+1}$. Thus $t_{i,4} > 0$ for all $i = 1, \cdots, k+1$.

Hence (3.6) follows from (3.9) and the fact $t_{i,4} > 0$. So the claim holds and $\mathcal{M}_{k,4}^{\mathbf{r}_k} \neq \emptyset$. The proof is completed. $\square$

With the aid of Lemma 3.2, we shall prove the nonempty set of $\mathcal{N}_k^{\mathbf{r}_k}$. To this end, we introduce $L_i : (\mathbb{R}_{\geq 0})^{k+1} \to \mathbb{R}$ by

$$\begin{aligned}
L_i(t_1, \cdots, t_{k+1}) &= \partial_{u_i} E(t_1 u_1, \cdots, t_{k+1} u_{k+1}) t_i u_i \\
&= t_i^2\|u_i\|_i^2 + \lambda t_i^2 \sum_{j=1}^{k+1} t_j^2 \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} f(t_i u_i) t_i u_i
\end{aligned} \tag{3.10}$$

and $F_i : (\mathbb{R}_{>0})^{k+1} \to \mathbb{R}$ by

$$\begin{aligned}
F_i(t_1, \cdots, t_{k+1}) &= \frac{\partial_{u_i} E(t_1 u_1, \cdots, t_{k+1} u_{k+1}) t_i u_i}{t_i^2 + t_i^4} \\
&= \frac{1}{1+t_i^2}\|u_i\|_i^2 + \frac{\lambda}{1+t_i^2} \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} t_j^2 \phi_{u_j} u_i^2 - \frac{t_i^2}{1+t_i^2} \int_{B_i^{\mathbf{r}_k}} \frac{f(t_i u_i) u_i}{t_i^3}.
\end{aligned} \tag{3.11}$$

It is easy to see that L_i and F_i are continuous functions.

Proposition 3.3. *For each $\mathbf{r}_k$, the set $\mathcal{N}_k^{\mathbf{r}_k} \neq \emptyset$, which is defined as in* (2.6).

Proof. By Lemma 3.2, we can take $(u_1, \cdots, u_{k+1}) \in M_{k,4}^{\mathbf{r}_k}$. Note from (3.11) that

$$F_i(T, \cdots, T) = \frac{1}{1+T^2}\|u_i\|_i^2 + \frac{T^2}{1+T^2}\left(\lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} \frac{f(Tu_i)u_i}{T^3}\right).$$

Then by **(F1)**, **(F4)** and (3.1), (3.2), we have

$$F_i(T, \cdots, T) \to \|u_i\|_i^2 > 0 \qquad \text{as } T \to 0,$$
$$F_i(T, \cdots, T) \to \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} |u_i|^4 < 0 \quad \text{as } T \to +\infty,$$

Thus there exists small $l > 0$ and large $L > 0$ such that for all i,

$$F_i(l, \cdots, l) > 0 \quad \text{and} \quad F_i(L, \cdots, L) < 0. \tag{3.12}$$

Let

$$D_{l,L} := \left\{(t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{>0})^{k+1} : l \leq t_i \leq L,\ \forall\, i = 1, \cdots, k+1\right\}.$$

Then we infer from (3.11) and (3.12) that for all $t_j \in [l, L]$,

$$F_i(t_1, \cdots, t_{i-1}, l, t_{i+1}, \cdots, t_{k+1}) \geq F_i(l, \cdots, l) > 0,$$
$$F_i(t_1, \cdots, t_{i-1}, L, t_{i+1} \cdots, t_{k+1}) \leq F_i(L, \cdots, L) < 0.$$

This together with the Miranda theorem (Lemma 2.1), yields that there is some $\bar{\mathbf{t}} := (\bar{t}_1, \cdots, \bar{t}_{k+1}) \in D_{l,L}$ such that

$$(F_1(\bar{\mathbf{t}}), \cdots, F_{k+1}(\bar{\mathbf{t}})) = \theta,$$

which implies $\partial_{u_i} E(\bar{t}_1 u_1, \cdots, \bar{t}_{k+1} u_{k+1})\bar{t}_i u_i = 0$ due to (3.11). Then

$$(\bar{t}_1 u_1, \cdots, \bar{t}_{k+1} u_{k+1}) \in \mathcal{N}_k^{\mathbf{r}_k}$$

and thus $\mathcal{N}_k^{\mathbf{r}_k}$ is not empty. The proof is completed. □

As a consequence of Proposition 3.3, the following result follows immediately.

Proposition 3.4. *The set $\mathcal{N}_k \neq \emptyset$, which is defined as in* (1.3).

Next, we list some properties of the Nehari type set $\mathcal{N}_k^{\mathbf{r}_k}$ below.

Lemma 3.5. *For each $\boldsymbol{r}_k \in \Gamma_k$, if $(u_1, \cdots, u_{k+1}) \in \mathcal{N}_k^{\boldsymbol{r}_k}$, then*

(i) $E(t_1u_1, \cdots, t_{k+1}u_{k+1}) < E(u_1, \cdots, u_{k+1})$, $\forall\, (t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{\geq 0})^{k+1} \backslash (1, \cdots, 1)$;

(ii) for any $r \in (0, 1)$ and $R \in (1, +\infty)$,

$$L_i(r, \cdots, r) > 0 \quad \textit{and} \quad L_i(R, \cdots, R) < 0,\ \forall i = 1, \cdots, k+1.$$

where L_i are defined in (3.10)*;*

(iii) there exists $\bar{\delta} > 0$ such that for all $(u_1, \cdots, u_{k+1}) \in \mathcal{N}_k^{\boldsymbol{r}_k}$,

$$\|u_i\|_i \geq \bar{\delta}, \qquad \forall i = 1, \cdots, k+1.$$

Proof. (i) For $(u_1, \cdots, u_{k+1}) \in \mathcal{N}_k^{\mathbf{r}_k}$, set

$$\xi(t) = (\frac{t^2}{2} - \frac{t^4}{4})\|u_i\|_i^2 + \int_{B_i^{\mathbf{r}_k}} \left(\frac{t^4}{4} f(u_i)u_i - F(tu_i) \right).$$

Then

$$\xi'(t) = t(1 - t^2)\|u_i\|_i^2 + t^3 \int_{B_i^{\mathbf{r}_k}} (\tfrac{f(u_i)}{u_i^3} - \tfrac{f(tu_i)}{t^3 u_i^3}) u_i^4 \text{ for any } t \in \mathbb{R}_+.$$

A direct computation gives that $\xi(t) < \xi(1)$ for any $t \in \mathbb{R}_+ \backslash \{1\}$, due to **(F4)**. If $(t_1, \cdots, t_{k+1}) \neq (1, \cdots, 1)$, it follows that

$$\begin{aligned}
&E(t_1u_1, \cdots, t_{k+1}u_{k+1}) \\
=&E(t_1u_1, \cdots, t_{k+1}u_{k+1}) - \sum_{i=1}^{k+1} \frac{t_i^4}{4} \partial_{u_i} E(u_1, \cdots, u_{k+1})u_i \\
=&\sum_{i=1}^{k+1} \left(\frac{1}{2} t_i^2 \|u_i\|^2 + \frac{\lambda t_i^2}{4} \sum_{j=1}^{k+1} t_j^2 \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} F(t_i u_i) \right) \\
&- \sum_{i=1}^{k+1} \frac{t_i^4}{4} \left(\|u_i\|^2 + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2 - \int_{B_i^{\mathbf{r}_k}} f(u_i)u_i \right) \\
=&\sum_{i=1}^{k+1} \left((\frac{t_i^2}{2} - \frac{t_i^4}{4})\|u_i\|^2 + \int_{B_i^{\mathbf{r}_k}} \left[\frac{t_i^4}{4} f(u_i)u_i - F(t_iu_i) \right] \right) + \lambda \sum_{\substack{i,j=1 \\ i \neq j}}^{k+1} \left(\frac{t_i^2 t_j^2 - t_i^4}{4} + \frac{t_i^2 t_j^2 - t_j^4}{4} \right) \int_{B_i^{\mathbf{r}_k}} \phi_{u_j} u_i^2
\end{aligned}$$

$$
\begin{aligned}
&< \sum_{i=1}^{k+1}\left(\frac{1}{4}\|u_i\|^2 + \int_{B_i^{\mathbf{r}_k}}\left[\frac{1}{4}f(u_i)u_i - F(u_i)\right]\right) - \frac{\lambda}{4}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1}(t_i^2 - t_j^2)^2\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2\\
&\le \sum_{i=1}^{k+1}\left(\frac{1}{4}\|u_i\|^2 + \int_{B_i^{\mathbf{r}_k}}\left[\frac{1}{4}f(u_i)u_i - F(u_i)\right]\right)\\
&= E(u_1,\cdots,u_{k+1}) - \sum_i \frac{1}{4}\partial_{u_i}E(u_1,\cdots,u_{k+1})u_i = E(u_1,\cdots,u_{k+1}).
\end{aligned}
$$

Thus (i) follows.

(ii) According to (3.10), we have

$$
L_i(T,\cdots,T) = T^2\|u_i\|_i^2 + \lambda T^4\sum_{j=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2 - T^4\int_{B_i^{\mathbf{r}_k}}\frac{f(Tu_i)}{T^3}u_i.
$$

Then by **(F4)** and $(u_1,\cdots,u_{k+1})\in\mathcal{N}_k^{\mathbf{r}_k}$, it follows that for any $r\in(0,1)$,

$$
\begin{aligned}
L_i(r,\cdots,r) &= r^2\|u_i\|_i^2 + r^4\left(\int_{B_i^{\mathbf{r}_k}}\left[\lambda\sum_{j=1}^{k+1}\phi_{u_j}u_i^2 - \frac{f(ru_i)}{r^3}u_i\right]\right)\\
&\ge r^2\|u_i\|_i^2 + r^4\left(\int_{B_i^{\mathbf{r}_k}}\left[\lambda\sum_{j=1}^{k+1}\phi_{u_j}u_i^2 - f(u_i)u_i\right]\right)\\
&= r^2\|u_i\|_i^2 - r^4\|u_i\|_i^2 > 0.
\end{aligned}
$$

Similarly, for any $R\in(1,+\infty)$, we have that

$$
L_i(R,\cdots,R) \le R^2\|u_i\|_i^2 - R^4\|u_i\|_i^2 < 0.
$$

Hence (ii) follows.

(iii) By **(F2)**,**(F3)** and the continuous embedding theorem $H_r^1(\mathbb{R}^3)\hookrightarrow L^s(\mathbb{R}^3)$ for any $s\in(2,6)$, it follows that

$$
\begin{aligned}
\|u_i\|_i^2 &\le \|u_i\|_i^2 + \lambda\sum_{j=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}\phi_{u_j}u_i^2 \quad = \int_{B_i^{\mathbf{r}_k}} f(u_i)u_i dx \le \frac{1}{2}\int_{B_i^{\mathbf{r}_k}}|u_i|^2dx + C\int_{B_i^{\mathbf{r}_k}}|u_i|^q\\
&\le \frac{1}{2}\|u_i\|_i^2 + C\|u_i\|_i^q,
\end{aligned}
$$

which implies $\|u_i\|_i \ge (\frac{1}{2C})^{\frac{1}{q-2}} =: \bar{\delta} > 0$. Thus (iii) follows and this lemma is proved. □

4 Existence of nodal solutions

With the aid of these lemmas in Section 3, we have the following result.

Proposition 4.1. *For $\boldsymbol{r}_k \in \Gamma_k$, if $(\bar{u}_1, \cdots, \bar{u}_{k+1}) \in \mathcal{N}_k^{\boldsymbol{r}_k}$ is a minimizer of $E|_{\mathcal{N}_k^{r_k}}$ such that*

$$E(\bar{u}_1, \cdots, \bar{u}_{k+1}) = \psi(\boldsymbol{r}_k) =: m,$$

then $(\bar{u}_1, \cdots, \bar{u}_{k+1})$ is a critical point of E in $\mathcal{H}_k^{\boldsymbol{r}_k}$.

Proof. We prove it by contradiction. Suppose on the contrary that

$$\|(\partial_{u_1} E, \cdots, \partial_{u_{k+1}} E)(\bar{u}_1, \cdots, \bar{u}_{k+1})\| \neq 0.$$

Then by the continuity of $\partial_{u_i} E$, there exist some $0 < \delta < \min\{\|u_i\|_i/6, \forall 1 \le i \le k+1\}$ and $\rho > 0$ such that

$$\|(\partial_{u_1} E, \cdots, \partial_{u_{k+1}} E)(u_1, \cdots, u_{k+1})\| \ge \rho, \quad \forall (u_1, \cdots, u_{k+1}) \in B_{3\delta}(\bar{u}_1, \cdots, \bar{u}_{k+1}).$$

Set

$$D = \left\{(t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{\ge 0})^{k+1} : |t_i - 1| < \frac{1}{2}, \forall i = 1, \cdots, k+1\right\}.$$

Then $\|(t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1}) - (\bar{u}_1, \cdots, \bar{u}_{k+1})\|_{\mathcal{H}_k^{\mathbf{r}_k}} > 3\delta$ on ∂D and by Lemma 3.5 (i), there holds

$$0 < \sigma := \sup_{\partial D} E(t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1}) < E(\bar{u}_1, \cdots, \bar{u}_{k+1}) = m. \tag{4.1}$$

Let $\varepsilon := \min\left\{\frac{m-\sigma}{2}, \frac{\delta\rho}{8}\right\}$ and $S := B_{3\delta}(\bar{u}_1, \cdots, \bar{u}_{k+1})$. According to the classical deformation lemma (see [21, Lemma 2.3]), there is a deformation $\eta := (\eta_1, \cdots, \eta_{k+1}) \in C([0,1] \times \mathcal{H}_k^{\mathbf{r}_k}, (\mathcal{H}_k^{\mathbf{r}_k})^{k+1})$ such that

(i) $\eta(1, u_1, \cdots, u_{k+1}) = (u_1, \cdots, u_{k+1})$, if $(u_1, \cdots, u_{k+1}) \notin E^{-1}(m - 2\varepsilon, m + 2\varepsilon) \bigcap S_{2\delta}$;

(ii) $\eta(1, E^{m+\varepsilon} \bigcap S) \subset E^{m-\varepsilon}$;

(iii) $E(\eta(1, (u_1, \cdots, u_{k+1}))) \le E(u_1, \cdots, u_{k+1})$ for any $u \in \mathcal{H}_k^{\mathbf{r}_k}$, where $E^c := \left\{(u_1, \cdots, u_{k+1}) \in \mathcal{H}_k^{\mathbf{r}_k} : E(u) \le c\right\}$. Then by (4.1) and (ii), (iii), it follows that

$$E(\eta(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1}))) \le m - \varepsilon, \qquad \text{if } \sum_i \|t_i\bar{u}_i - \bar{u}_i\|_i < \delta;$$
$$E(\eta(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1}))) \le E(t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1})) < m, \quad \text{if } \sum_i \|t_i\bar{u}_i - \bar{u}_i\|_i \ge \delta.$$

In any case, we conclude that

$$\sup_{(t_1,\cdots,t_{k+1})\in\bar{D}} E(\eta(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1}))) < m. \tag{4.2}$$

On the other hand, we have

$$\left\{\eta(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1})), t \in D\right\} \cap \mathcal{N}_k^{\mathbf{r}_k} \ne \emptyset. \tag{4.3}$$

In fact, for $(t_1, \cdots, t_{k+1}) \in \partial D$, it follows from (i) that

$$\eta(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1})) = (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1}).$$

By defining a vector map

$$\Phi = (\Phi_1, \cdots, \Phi_{k+1}) : D \to \mathbb{R}^{k+1}$$

with

$$\Phi_i(t_1, \cdots, t_{k+1}) = \partial_i E(\eta(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1})))\eta_i(1, (t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1})).$$

Note that on ∂D,

$$\Phi_i(t_1, \cdots, t_{k+1}) = \partial_i E(t_1\bar{u}_1, \cdots, t_{k+1}\bar{u}_{k+1})t_iu_i = L_i(t_1, \cdots, t_{k+1}).$$

where L_i is define in (3.10). Then by Lemma 3.5 (ii), it follows that for $|t_j - 1| < \frac{1}{2}$,

$$\Phi_i(t_1, \cdots, t_{i-1}, 1/2, t_{i+1}, \cdots, t_{k+1}) = L_i(t_1, \cdots, t_{i-1}, 1/2, t_{i+1}, \cdots, t_{k+1}) > 0,$$
$$\Phi_i(t_1, \cdots, t_{i-1}, 3/2, t_{i+1}, \cdots, t_{k+1}) = L_i(t_1, \cdots, t_{i-1}, 3/2, t_{i+1}, \cdots, t_{k+1}) < 0.$$

This together with Lemma 2.1, gives that there exists some $(\bar{t}_1, \cdots, \bar{t}_{k+1}) \in D$ such that $\Phi_i(\bar{t}_1, \cdots, \bar{t}_{k+1}) = \theta$, namely,

$$(\eta(1, (\bar{t}_1\bar{u}_1, \cdots, \bar{t}_{k+1}\bar{u}_{k+1})) \in \mathcal{N}_k^{\mathbf{r}_k}.$$

So $\sup_{\bar{D}} E(\eta(1,(t_1\bar{u}_1,\cdots,t_{k+1}\bar{u}_{k+1}))) \geq \inf_{\mathcal{N}_k^{\mathbf{r}_k}} E = m$ and thereby it leads to a contradiction with (4.2).

Thus $\|(\partial_{u_1}E,\cdots,\partial_{u_{k+1}}E)(\bar{u}_1,\cdots,\bar{u}_{k+1})\| = 0$ and $(\bar{u}_1,\cdots,\bar{u}_{k+1})$ is a critical point of E in $\mathcal{H}_k^{\mathbf{r}_k}$. The proof is finished. $\square$

The following result shows that there exists a minimizer of E on $\mathcal{N}_k^{\mathbf{r}_k}$. Then it combined with Proposition 4.1, gives that it is indeed a solution of system (2.4).

Lemma 4.2. *For each $\boldsymbol{r}_k \in \Gamma_k$, there exists a minimizer $(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}) \in \mathcal{N}_k^{\boldsymbol{r}_k}$ with components $(-1)^{i+1}u_i^{\boldsymbol{r}_k} > 0$ in B_i satisfying $E(u_1^{\boldsymbol{r}_k},\cdots,u_{k+1}^{\boldsymbol{r}_k}) = \Psi(\boldsymbol{r}_k)$ defined in* (2.5).

Proof. By Proposition 3.3 and the definition of Ψ, there exists a minimizing sequence $\{(u_1^n,\cdots,u_{k+1}^n)\}_n \subset \mathcal{N}_k^{\mathbf{r}_k}$ such that $E(u_1^n,\cdots,u_{k+1}^n) \to \Psi(\mathbf{r}_k) > 0$ as $n \to +\infty$.

We first observe from **(F2)** and **(F4)** that $G(t) := f(t)/t^3$ is increasing in $|t| > 0$ and thereby for any $|t| > 0$

$$tf(t) - 4F(t) = t^4G(t) - 4\int_0^t s^3G(s)ds = \int_0^t 4s^3(G(t)-G(s))ds > 0.$$

Thus

$$\frac{1}{4}f(t)t - F(t) \geq 0 \quad \text{for any } t \in \mathbb{R}. \tag{4.4}$$

So

$$\begin{aligned}\Psi(\mathbf{r}_k) + o(1) &= E(u_1^n,\cdots,u_{k+1}^n) - \frac{1}{4}\sum_i \partial_{u_i}E(u_1^n,\cdots,u_{k+1}^n)u_i^n \\ &= \sum_i \left(\frac{1}{4}\|u_i^n\|_i^2 + \frac{1}{4}\int_{B_i^{\mathbf{r}_k}} f(u_i^n)u_i^n - \int_{B_i^{\mathbf{r}_k}} F(u_i^n)\right) \geq \sum_i \frac{1}{4}\|u_i^n\|_i^2.\end{aligned}$$

Hence $\{(u_1^n,\cdots,u_{k+1}^n)\}_n$ is bounded in $\mathcal{H}_k^{\mathbf{r}_k}$ and there is some $(u_1^0,\cdots,u_{k+1}^0) \in \mathcal{H}_k^{\mathbf{r}_k}$ such that

$$(u_1^n,\cdots,u_{k+1}^n) \rightharpoonup (u_1^0,\cdots,u_{k+1}^0) \quad \text{in} \quad \mathcal{H}_k^{\mathbf{r}_k}$$

and $u_i^n \rightharpoonup u_i^0$ in $H_i^{\mathbf{r}_k}$. Then by **(F1)-(F4)** and the compact embedding theorem $H_i \hookrightarrow L^s$ for any $s \in (2,6)$, it follows that

$$\int_{B_i^{\mathbf{r}_k}} f(u_i^n)u_i^n \to \int_{B_i^{\mathbf{r}_k}} f(u_i^0)u_i^0 \text{ and } \int_{B_i^{\mathbf{r}_k}} F(u_i^n) \to \int_{B_i^{\mathbf{r}_k}} F(u_i^0) \quad \text{as } n \to +\infty. \tag{4.5}$$

This together with Lemma 3.5(iii) and $(u_1^n, \cdots, u_{k+1}^n) \in \mathcal{N}_k^{\mathbf{r}_k}$, gives that

$$\bar{\delta}^2 \leq \liminf_{n\to\infty} \|u_i^n\|_i^2 \leq \liminf_{n\to\infty} \int_{B_i^{\mathbf{r}_k}} f(u_i^n)u_i^n = \int_{B_i^{\mathbf{r}_k}} f(u_i^0)u_i^0.$$

So

$$u_i^0 \neq 0 \text{ for all } i = 1, \cdots, k+1.$$

Next, we prove that $(u_1^n, \cdots, u_{k+1}^n) \to (u_1^0, \cdots, u_{k+1}^0)$ in $\mathcal{H}_k^{\mathbf{r}_k}$. In fact, suppose by contradiction that there is some $i_0 \in \{1, \cdots, k+1\}$ such that $\|u_{i_0}^0\|_{i_0} < \liminf\limits_{n\to\infty} \|u_{i_0}^n\|_{i_0}$. Then by (3.10), (4.5) and $\int_{B_{i_0}^{\mathbf{r}_k}} \phi_{u_j^n}(u_i^n)^2 \to \int_{B_{i_0}^{\mathbf{r}_k}} \phi_{u_j^0}(u_i^0)^2$, we have that

$$\begin{aligned}
L_{i_0}^{u^0}(1, \cdots, 1) &:= \|u_{i_0}^0\|_{i_0}^2 + \lambda \sum_{j=1}^{k+1} \int_{B_{i_0}^{\mathbf{r}_k}} \phi_{u_j^0}|u_{i_0}^0|^2 - \int_{B_{i_0}^{\mathbf{r}_k}} f(u_{i_0}^0)u_{i_0}^0 \\
&< \liminf_n \left(\|u_{i_0}^n\|_{i_0}^2 + \lambda \sum_{j=1}^{k+1} \int_{B_{i_0}^{\mathbf{r}_k}} \phi_{u_j^n}|u_{i_0}^n|^2 - \int_{B_{i_0}^{\mathbf{r}_k}} f(u_{i_0}^n)u_{i_0}^n \right) = 0, \\
L_i^{u^0}(1, \cdots, 1) &:= \|u_i^0\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j^0}|u_i^0|^2 - \int_{B_i^{\mathbf{r}_k}} f(u_i^0)u_i^0 \\
&\leq \liminf_n \left(\|u_i^n\|_i^2 + \lambda \sum_{j=1}^{k+1} \int_{B_i^{\mathbf{r}_k}} \phi_{u_j^n}|u_i^n|^2 - \int_{B_i^{\mathbf{r}_k}} f(u_i^n)u_i^n \right) = 0, \quad \text{for } i \neq i_0,
\end{aligned} \tag{4.6}$$

Furthermore, observe from Lemma 3.5 (ii) that for small $\delta > 0$,

$$L_i^{u^0}(\delta, \cdots, \delta) > 0, \quad \forall i = 1, \cdots, k+1. \tag{4.7}$$

Then we deduce from (3.10), (4.6) and (4.7) that for all i,

$$\begin{aligned}
L_i^{u^0}(t_1, \cdots, t_{i-1}, \delta, t_{i+1}, \cdots, t_{k+1}) > 0, \\
L_i^{u^0}(t_1, \cdots, t_{i-1}, 1, t_{i+1}, \cdots, t_{k+1}) \leq 0.
\end{aligned}$$

By Lemma 2.1, there is some $(\bar{t}_1, \cdots, \bar{t}_{k+1}) \in \left\{x \in \mathbb{R}^{k+1} : \delta \leq x_i \leq 1\right\} \setminus \{(1, \cdots, 1)\}$ such that

$$(L_1^{u^0}(\bar{t}_1, \cdots, \bar{t}_{k+1}), \cdots, L_{k+1}^{u^0}(\bar{t}_1, \cdots, \bar{t}_{k+1})) = \theta,$$

which yields

$$(\bar{t}_1 u_1^0, \cdots, \bar{t}_{k+1} u_{k+1}^0) \in \mathcal{N}_k^{\mathbf{r}_k}.$$

This combined with (2.5), (4.5) and Lemma 3.5(i), gives that

$$
\begin{aligned}
\Psi(\mathbf{r}_k) &\leq E(\bar{t}_1 u_1^0, \cdots, \bar{t}_{k+1} u_{k+1}^0) \\
&= \sum_{i=1}^{k+1} \left(\frac{\bar{t}_i^2}{2} \|u_i^0\|_i^2 + \frac{\lambda}{4} \sum_{j=1}^{k+1} \bar{t}_i^2 \bar{t}_j^2 \int_{\mathbb{R}^3} \phi_{u_j^0} (u_i^0)^2 - \int_{\mathbb{R}^3} F(\bar{t}_i u_i^0) \right) \\
&< \liminf_n \sum_{i=1}^{k+1} \left(\frac{\bar{t}_i^2}{2} \|u_i^n\|_i^2 + \frac{\lambda}{4} \sum_{j=1}^{k+1} \bar{t}_i^2 \bar{t}_j^2 \int_{\mathbb{R}^3} \phi_{u_j^n} (u_i^n)^2 - \int_{\mathbb{R}^3} F(\bar{t}_i u_i^n) \right) \\
&\leq \liminf_n E(\bar{t}_1 u_1^n, \cdots, \bar{t}_{k+1} u_{k+1}^n) \\
&\leq \liminf_n E(u_1^n, \cdots, u_{k+1}^n) = \Psi(\mathbf{r}_k),
\end{aligned}
$$

which leads to a contradiction. Thus $(u_1^n, \cdots, u_{k+1}^n) \to (u_1^0, \cdots, u_{k+1}^0)$ in $\mathcal{H}_k^{\mathbf{r}_k}$ strongly and thereby $(u_1^0, \cdots, u_{k+1}^0) \in \mathcal{N}_k^{\mathbf{r}_k}$ is a minimizer of $E|_{\mathcal{N}_k^{\mathbf{r}_k}}$ such that

$$
E(u_1^0, \cdots, u_{k+1}^0) = \Psi(\mathbf{r}_k).
$$

In addition, it is direct to verify that

$$
(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) := (|u_1^0|, (-1)|u_2^0|, \cdots, (-1)^k |u_{k+1}^0|) \in \mathcal{N}_k^{\mathbf{r}_k},
$$

and $E(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) = \Psi(\mathbf{r}_k)$. Then $(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k})$ is also a minimizer of $E|_{\mathcal{N}_k^{\mathbf{r}_k}}$ and satisfies system (2.4). By the standard elliptic regularity theory, each $u_i^{\mathbf{r}_k} \in C^2(B_i^{\mathbf{r}_k})$ and thus by the strong maximum principle, $(-1)^i u_{i+1}^{\mathbf{r}_k} > 0$ in $B_i^{\mathbf{r}_k}$. Therefore, $(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k})$ is the desired solution and the proof is completed. □

Next, we show that there exists $\bar{\mathbf{r}}_k \in \Gamma_k$ such that $\Psi(\bar{\mathbf{r}}_k) = \inf_{\mathbf{r}_k \in \Gamma_k} \Psi(\mathbf{r}_k)$. Then we use it to prove that the sum $\sum_{i=1}^{k+1} u_i^{\bar{\mathbf{r}}_k}$ is a nodal solution of (1.1) that has exactly $k+1$ nodal domains.

Since not all functions in $H_r^1(\mathbb{R}^3)$ can be projected on the Nehari type set $\mathcal{N}_k$ here, it brings some difficulties in the proof of existence of minimum point $\bar{\mathbf{r}}_k$. We shall overcome them by introducing the Miranda theorem and subtle analysis. This is a novel point.

Lemma 4.3. *For $k \in \mathbb{N}_+$ and $\boldsymbol{r}_k = (r_1, \cdots, r_{k+1}) \in \Gamma_k$,*

(i) if $r_i - r_{i-1} \to 0$ for some $i \in \{1, \cdots, k\}$, then $\Psi(\boldsymbol{r}_k) \to +\infty$;

(ii) if $r_k \to +\infty$, then $\Psi(r_k) \to +\infty$;

(iii) Ψ is continuous in $\mathbf{r}_k$.

Particularly, there exists a minimum point $\bar{\mathbf{r}}_k := (\bar{r}_1, \cdots, \bar{r}_{k+1}) \in \Gamma_k$ of Ψ.

Proof. According to Lemma 4.2, for each $\mathbf{r}_k \in \Gamma_k$, there exists a minimizer $(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) \in \mathcal{N}_k^{\mathbf{r}_k}$ such that

$$E(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) = \Psi(\mathbf{r}_k).$$

Then by (4.4),

$$\begin{aligned}
\Psi(\mathbf{r}_k) = E(u_1, \cdots, u_{k+1}) &= E(u_1, \cdots, u_{k+1}) - \frac{1}{4}\sum_{i=1}^{k+1} I_\lambda'(u_1, \cdots, u_{k+1})u_i \\
&= \frac{1}{4}\sum_{i=1}^{k+1} \|u_i\|_i^2 + \frac{1}{4}\sum_{i=1}^{k+1}\int_{B_i^{\mathbf{r}_k}} (f(u_i)u_i - 4F(u_i)) \qquad (4.8)\\
&\geq \frac{1}{4}\sum_{i=1}^{k+1} \|u_i\|_i^2 \geq \frac{1}{4}\|u_i\|_i^2.
\end{aligned}$$

(i) By **(F2)** and **(F3)**, we get

$$\|u_i^{\mathbf{r}_k}\|_i^2 \leq \int_{B_i^{\mathbf{r}_k}} f(u_i^{\mathbf{r}_k})u_i^{\mathbf{r}_k} \leq \int_{B_i^{\mathbf{r}_k}} (\frac{1}{2}|u_i^{\mathbf{r}_k}|^2 + C|u_i^{\mathbf{r}_k}|^4),$$

which, together with the Hölder inequality and Sobolev inequality, yields that

$$\frac{1}{2}\|u_i^{\mathbf{r}_k}\|_i^2 \leq C\int_{B_i^{\mathbf{r}_k}} |u_i^{\mathbf{r}_k}|^4 \leq C\left(\int_{B_i^{\mathbf{r}_k}} |u_i^{\mathbf{r}_k}|^6\right)^{\frac{2}{3}} |B_i^{\mathbf{r}_k}|^{\frac{1}{3}} \leq C\|u_i^{\mathbf{r}_k}\|_i^4 |B_i^{\mathbf{r}_k}|^{\frac{1}{3}},$$

where $C > 0$ and $|B_i^{\mathbf{r}_k}|$ denotes the volume of $B_i^{\mathbf{r}_k}$. Thus $\|u_i^{\mathbf{r}_k}\|_i \to +\infty$ if $r_i - r_{i-1} \to 0$. This combined with (4.8), gives that

$$\Psi(\mathbf{r}_k) \to +\infty, \quad \text{if } r_i - r_{i-1} \to 0.$$

So (i) follows.

(ii) Recall the Strauss inequality [18] that for all $u \in H_r^1(\mathbb{R}^3)$, there exists $A > 0$ such that

$$|u(x)| \leq A\frac{\|u\|}{|x|} \quad \text{a.e. in } \mathbb{R}^3$$

Then by (2.4), it follows that

$$
\begin{aligned}
\|u_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 &\leq \int_{B_{k+1}} f(u_{k+1}^{\mathbf{r}_k})u_{k+1}^{\mathbf{r}_k} \\
&\leq \int_{B_{k+1}} \frac{1}{2}|u_{k+1}^{\mathbf{r}_k}|^2 + C|u_{k+1}^{\mathbf{r}_k}|^4 \\
&\leq \frac{1}{2}\|u_{k+1}^{\mathbf{r}_k}\|_{L^2}^2 + CA|r_k|^{-2}\|u_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 \int_{B_{k+1}} |u_{k+1}^{\mathbf{r}_k}|^2 \\
&\leq \frac{1}{2}\|u_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 + CAr_k^{-2}\|u_{k+1}^{\mathbf{r}_k}\|_{k+1}^4,
\end{aligned}
$$

which implies that $\|u_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 \geq \frac{r_k^2}{2CA}$. Then $\|u_{k+1}^{\mathbf{r}_k}\|_{k+1}^2 \to +\infty$ if $r_k \to +\infty$, and (ii) follows immediately due to (4.8).

(iii) We take $\{\mathbf{r}_k^n\} \subset \Gamma_k$ such that $\mathbf{r}_k^n \to \mathbf{r}_k \in \Gamma_k$, and denote their minimizers by $\mathbf{u}_k^{\mathbf{r}_k}$ and $\mathbf{u}_k^{\mathbf{r}_k^n}$, respectively. In the following, we show that

$$
\Psi(\mathbf{r}_k) \geq \limsup_{n\to\infty} \Psi(\mathbf{r}_k^n) \quad \text{and} \quad \liminf_{n\to\infty} \Psi(\mathbf{r}_k^n) \geq \Psi(\mathbf{r}_k).
$$

We divide the proof into two steps.

Step 1: Prove $\Psi(\mathbf{r}_k) \geq \limsup_{n\to\infty} \Psi(\mathbf{r}_k^n)$. To this end, we take $0 < r_0 < 1, R_0 > 1$ and set

$$
D_{r_0}^{R_0} := \left\{(t_1, \cdots, t_{k+1}) \in (\mathbb{R}_{>0})^{k+1} : r_0 \leq t_i \leq R_0,\ \forall i = 1, \cdots, k+1\right\},
$$

and for each n, we define

$$
\begin{aligned}
&v_i^{\mathbf{r}_k^n}(r) = t_i^n u_i^{\mathbf{r}_k}\left(\frac{r_i - r_{i-1}}{r_i^n - r_{i-1}^n}(r - r_{i-1}^n) + r_{i-1}\right), \quad \forall i = 1, \cdots, k, \\
&v_{k+1}^{\mathbf{r}_k^n}(r) = t_{k+1}^n u_{k+1}^{\mathbf{r}_k}\left(\frac{r_k}{r_k^n} r\right).
\end{aligned}
$$

Firstly, we claim that for large n,

$$
\text{there exists } (t_1^n, \cdots, t_{k+1}^n) \in D_{r_0}^{R_0} \text{ such that } (v_1^{\mathbf{r}_k^n}, \cdots, v_{k+1}^{\mathbf{r}_k^n}) \in \mathcal{N}_k^{\mathbf{r}_k^n}. \tag{4.9}
$$

In fact, some direct computations give that

$$\|v_i^{\mathbf{r}_k^n}\|_i^2 = (t_i^n)^2\|u_i^{\mathbf{r}_k}\|_i^2 + o(1),$$

$$\int_{B_i^{\mathbf{r}_k^n}}\int_{B_j^{\mathbf{r}_k^n}} \frac{|v_i^{\mathbf{r}_k^n}(x)|^2|v_j^{\mathbf{r}_k^n}(y)|^2}{4\pi|x-y|}dxdy = (t_i^n)^2(t_j^n)^2\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}} \frac{|u_i^{\mathbf{r}_k}(x)|^2|u_j^{\mathbf{r}_k}(y)|^2}{4\pi|x-y|}dxdy + o(1),$$

$$\int_{B_i^{\mathbf{r}_k^n}} f(v_i^{\mathbf{r}_k^n})v_i^{\mathbf{r}_k^n} = \int_{B_i^{\mathbf{r}_k}} f(t_i^n u_i^{\mathbf{r}_k})t_i^n u_i^{\mathbf{r}_k} + o(1).$$

With these equalities, we denote by

$$\begin{aligned} &L_i^{\mathbf{u}^{\mathbf{r}_k}}(t_1^n,\cdots,t_{k+1}^n)\\ &:= (t_i^n)^2\|u_i^{\mathbf{r}_k}\|_i^2 + \lambda\sum_{j=1}^{k+1}(t_i^n)^2(t_j^n)^2\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}} \frac{|u_i^{\mathbf{r}_k}(x)|^2|u_j^{\mathbf{r}_k}(y)|^2}{4\pi|x-y|}dxdy - \int_{B_i^{\mathbf{r}_k}} f(t_i^n u_i^{\mathbf{r}_k})t_i^n u_i^{\mathbf{r}_k} \end{aligned}$$

and

$$\begin{aligned} &h_i^n(t_1^n,\cdots,t_{k+1}^n) := \|v_i^{\mathbf{r}_k^n}\|_i^2 + \lambda\int_{B_i^{\mathbf{r}_k^n}}\int_{B_j^{\mathbf{r}_k^n}} \frac{|v_i^{\mathbf{r}_k^n}(x)|^2|v_j^{\mathbf{r}_k^n}(y)|^2}{4\pi|x-y|}dxdy - \int_{B_i^{\mathbf{r}_k^n}} f(v_i^{\mathbf{r}_k^n})v_i^{\mathbf{r}_k^n}\\ &= (t_i^n)^2\|u_i^{\mathbf{r}_k}\|_i^2 + \lambda\sum_{j=1}^{k+1}(t_i^n)^2(t_j^n)^2\int_{B_i^{\mathbf{r}_k}}\int_{B_j^{\mathbf{r}_k}} \frac{|u_i^{\mathbf{r}_k}(x)|^2|u_j^{\mathbf{r}_k}(y)|^2}{4\pi|x-y|}dxdy - \int_{B_i^{\mathbf{r}_k}} f(t_i^n u_i^{\mathbf{r}_k})t_i^n u_i^{\mathbf{r}_k} + o(1)\\ &= L_i^{\mathbf{u}^{\mathbf{r}_k}}(t_1^n,\cdots,t_{k+1}^n) + o(1). \end{aligned} \tag{4.10}$$

Then

$$L_i^{\mathbf{u}^{\mathbf{r}_k}}(1,\cdots,1) = 0 \text{ for all } i$$

and $\mathbf{u}_k^{\mathbf{r}_k} \in \mathcal{N}_k^{\mathbf{r}_k}$. Moreover, by Lemma 3.5 (ii), it follows that

$$L_i^{\mathbf{u}^{\mathbf{r}_k}}(t_1,\cdots,t_{i-1},r_0,t_{i+1},\cdots,t_{k+1}) > 0,$$
$$L_i^{\mathbf{u}^{\mathbf{r}_k}}(t_1,\cdots,t_{i-1},R_0,t_{k+1},\cdots,t_{k+1}) < 0.$$

This together with (4.10), implies that there exists $N_0 > 0$ depending on r_0 and R_0 such that for any $n \geq N_0$ and $r_0 \leq t_j \leq R_0$,

$$h_i^n(t_1,\cdots,t_{i-1},r_0,t_{i+1},\cdots,t_{k+1}) > 0,$$
$$h_i^n(t_1,\cdots,t_{i-1},R_0,t_{k+1},\cdots,t_{k+1}) < 0.$$

By Lemma 2.1, there exists some $(t_1^n, \cdots, t_{k+1}^n) \in D_{r_0}^{R_0}$ such that

$$h_i^n(t_1^n, \cdots, t_{k+1}^n) = 0 \quad \text{for all} \quad i = 1, \cdots, k+1,$$

which means that $(v_1^{\mathbf{r}_k^n}, \cdots, v_{k+1}^{\mathbf{r}_k^n}) \in \mathcal{N}_k^{\mathbf{r}_k^n}$. Thus the claim (4.9) follows.

Next, we prove that

$$\lim_{n\to\infty}(t_1^n, \cdots, t_{k+1}^n) = (1, \cdots, 1). \tag{4.11}$$

In fact, in view of the claim (4.9), we have

$$E^{\mathbf{r}_k^n}(v_1^{\mathbf{r}_k^n}, \cdots, v_{k+1}^{\mathbf{r}_k^n}) \geq E^{\mathbf{r}_k^n}(u_1^{\mathbf{r}_k^n}, \cdots, u_{k+1}^{\mathbf{r}_k^n}) = \Psi(\mathbf{r}_k^n). \tag{4.12}$$

and

$$(b_1, \cdots, b_{k+1}) := \limsup_{n\to\infty}(t_1^n, \cdots, t_{k+1}^n) \in D_{r_0}^{R_0}.$$

Then by (4.10), it follows that

$$(b_1 u_1^{\mathbf{r}_k}, \cdots, b_{k+1} u_{k+1}^{\mathbf{r}_k}) \in \mathcal{N}_k^{\mathbf{r}_k}.$$

Thus by Lemma 3.5 (i),

$$E(b_1 u_1^{\mathbf{r}_k}, \cdots, b_{k+1} u_{k+1}^{\mathbf{r}_k}) \leq E(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}).$$

On the other hand,

$$E(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) \leq E(b_1 u_1^{\mathbf{r}_k}, \cdots, b_1 u_{k+1}^{\mathbf{r}_k})$$

due to $(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) \in \mathcal{N}_k^{\mathbf{r}_k}$ and Lemma 3.5 (i). Thus we conclude that $(b_1, \cdots, b_{k+1}) = (1, \cdots, 1)$ and (4.11) follows.

Finally, by (4.9), (4.11) and (4.12), we see that

$$\Psi(\mathbf{r}_k) = E^{\mathbf{r}_k}(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) = \limsup_{n\to\infty} E^{\mathbf{r}_k^n}(v_1^{\mathbf{r}_k^n}, \cdots, v_{k+1}^{\mathbf{r}_k^n}) \geq \limsup_{n\to\infty} \Psi(\mathbf{r}_k^n).$$

So $\Psi(\mathbf{r}_k) \geq \limsup_n \Psi(\mathbf{r}_k^n)$.

Step 2: Prove $\Psi(\mathbf{r}_k) \leq \liminf_{n\to\infty} \Psi(\mathbf{r}_k^n)$.

Indeed, similar as the former case, we define $v_i^{\mathbf{r}_k^n} : [r_{i-1}, r_i] \to \mathbb{R}$ by

$$v_i^{\mathbf{r}_k}(r) = b_i^n w_i^{\mathbf{r}_k^n}\left(\frac{r_i^n - r_{i-1}^n}{r_i - r_{i-1}}(r - r_{i-1}) + r_{i-1}^n\right), \quad i = 1, \cdots, k.$$

$$v_{k+1}^{\mathbf{r}_k}(r) = b_{k+1}^n w_{k+1}^{\mathbf{r}_k^n}\left(\frac{r_k^n}{r_k} r\right),$$

and there exists $(b_1^n, \cdots, b_{k+1}^n) \in D_{r_0}^{R_0}$ such that $(v_1^{\mathbf{r}_k^n}, \cdots, v_{k+1}^{\mathbf{r}_k^n}) \in \mathcal{N}_k^{\mathbf{r}_k}$ and $b_i^n \to 1$ as $n \to +\infty$ for all i. Then

$$\begin{aligned}\Psi(\mathbf{r}_k) &= E^{\mathbf{r}_k}(u_1^{\mathbf{r}_k}, \cdots, u_{k+1}^{\mathbf{r}_k}) \leq \liminf_{n\to\infty} E^{\mathbf{r}_k^n}(v_1^{\mathbf{r}_k^n}, \cdots, v_{k+1}^{\mathbf{r}_k^n}) = \liminf_{n\to\infty} E^{\mathbf{r}_k^n}(u_1^{\mathbf{r}_k^n}, \cdots, u_{k+1}^{\mathbf{r}_k^n}) \\ &= \liminf_{n\to\infty} \Psi(\mathbf{r}_k^n).\end{aligned}$$

Thus (iii) follows.

Therefore, by (i)-(iii), there is a minimum point $\bar{\mathbf{r}}_k = (\bar{r}_1, \cdots, \bar{r}_{k+1}) \in \Gamma_k$ of Ψ. The proof is finished. □

Now, we start to prove Theorem 1.1. Precisely, by using the deformation lemma and Miranda theorem, we shall prove that the solution $(w_1^{\bar{\mathbf{r}}_k}, \cdots, w_{k+1}^{\bar{\mathbf{r}}_k})$ of the system (2.4), corresponding to $\bar{\mathbf{r}}_k$ found in Lemma 4.3, is the desired element that can be used to construct a nodal solution of (1.1) with exactly $k+1$ nodal domains.

Proof of Theorem 1.1. According to Proposition 4.1, Lemma 4.2 and Lemma 4.3, for each $k \in \mathbb{N}_+$, there exists $\bar{\mathbf{r}}_k \in \Gamma_k$ and a corresponding critical point $(u_1^{\bar{\mathbf{r}}_k}, \cdots, u_{k+1}^{\bar{\mathbf{r}}_k}) \in \mathcal{N}_k^{\bar{\mathbf{r}}_k}$ of E with $(-1)^{i-1}u_i^{\bar{\mathbf{r}}_k} > 0$ in $B_i^{\bar{\mathbf{r}}_k}$ such that

$$E(u_1^{\bar{\mathbf{r}}_k}, \cdots, u_{k+1}^{\bar{\mathbf{r}}_k}) = \Psi(\bar{\mathbf{r}}_k) = \inf_{\mathbf{r}_k \in \Gamma_k} \Psi(\mathbf{r}_k) = c_k.$$

Then $(u_1^{\bar{\mathbf{r}}_k}, \cdots, u_{k+1}^{\bar{\mathbf{r}}_k})$ satisfies (2.4). Let

$$U_k := \sum_{i=1}^{k+1} \bar{u}_i^{\bar{\mathbf{r}}_k}.$$

Obviously, U_k changes sign exactly k times and $I_\lambda(U_k) = c_k$, due to (2.3) and (2.7).

We further show that U_k is indeed a solution of (1.1). Indeed, if NOT, by the principle of symmetric criticality (see [16]), we may suppose on the contrary that there is a radial function $\phi \in C_0^\infty(\mathbb{R}^3) \bigcap H_r^1(\mathbb{R}^3)$ such that

$$I_\lambda'(U_k)\phi = -2.$$

For the sake of convenience, we denote by $\mathfrak{s} := (s_1, \cdots, s_{k+1})$ and define a continuous function $g : (\mathbb{R}_{>0})^{k+1} \times \mathbb{R} \to H_r^1(\mathbb{R}^3)$ by

$$g(\mathfrak{s}, \varepsilon) = \sum_{i=1}^{k+1} s_i u_i^{\bar{\mathbf{r}}_k} + \tau\phi,$$

Note that $g(\mathbf{1},0) = U_k$ changes sign k times, where $\mathbf{1} := (1,\cdots,1) \in (\mathbb{R}_{>0})^{k+1}$. Then there exists a small $\tau_0 \in (0,1)$ such that for any $\tau \in [0,\tau_0]$ and s_i satisfying $|s_i - 1| < \tau_0$, the function $g(\mathfrak{s},\varepsilon)$ changes sign exactly k times with k nodes $0 < \rho_1(\mathfrak{s},\varepsilon) < \cdots < \rho_k(\mathfrak{s},\varepsilon) < +\infty$, and

$$I'_\lambda(g(\mathfrak{s},\varepsilon))\phi < -1. \tag{4.13}$$

Set

$$D_{\tau_0} := \{\mathfrak{s} \in (\mathbb{R}_{>0})^{k+1} : |s_i - 1| < \tau_0, \forall i,\cdots,k+1\}.$$

We take a radial cut-off function $\eta \in C^\infty(D_{\tau_0},[0,1])$ by

$$\eta(\mathfrak{s}) = \begin{cases} 1, & \text{if } \mathfrak{s} \in D_{\frac{\tau_0}{4}}, \\ 0, & \text{if } \mathfrak{s} \notin D_{\frac{\tau_0}{2}}, \\ \in (0,1), & \quad \mathit{others}, \end{cases} \tag{4.14}$$

and define another function $\bar{g} : D_{\tau_0} \to H^1(\mathbb{R}^3)$ by

$$\bar{g}(\mathfrak{s}) = \sum_{i=1}^{k+1} s_i u_i^{\bar{\mathbf{r}}_k} + \tau_0 \eta(\mathfrak{s})\phi.$$

Then $\bar{g} \in C(D_{\tau_0}, H_r^1(\mathbb{R}^3))$ and for any $\mathfrak{s} \in D_{\tau_0}$, $\bar{g}(\mathfrak{s})$ changes sign k times with k-nodes $0 < \bar{\rho}_1(\mathfrak{s}) < \cdots < \bar{\rho}_k(\mathfrak{s}) < +\infty$. Moreover, $\bar{\rho}_i(\mathfrak{s})$ is continuous about $\mathfrak{s}$.

Now, we assert that there is $\bar{\mathfrak{s}} \in D_{\tau_0}$ such that

$$\bar{g}(\bar{s}) \in \mathcal{N}_k, \tag{4.15}$$

where $\mathcal{N}_k$ is define in (1.3). In fact, we denote by $\langle\cdot,\cdot\rangle$ the dual product between $H_r^1(\mathbb{R}^3)$ and its dual space $(H_r^1(\mathbb{R}^3))^*$. Let

$$V_i(\mathfrak{s}) := \langle I'_\lambda(\bar{g}(\mathfrak{s}), (\bar{g}(\mathfrak{s}))_i\rangle,$$

where $(\bar{g}(\mathfrak{s}))_i$ is the constraint of $g(\mathfrak{s})$ on $\{x \in \mathbb{R}^3 : \bar{\rho}_{i-1}(\mathfrak{s}) < |x| \le \bar{\rho}_i(\mathfrak{s})\}$. Then by the definition of η, it follows that for any $\mathfrak{s} \in \partial D_{\tau_0}$,

$$\begin{aligned} &\bar{g}(\mathfrak{s}) = \sum_{i=1}^{k+1} s_i w_i^{\bar{\mathbf{r}}_k}, \qquad (\bar{g}(\mathfrak{s}))_i = s_i w_i^{\bar{\mathbf{r}}_k} \quad \text{and} \\ &V_i(\mathfrak{s}) = s_i^2 \|u_i^{\bar{\mathbf{r}}_k}\|_i^2 + \sum_{j=1}^{k+1} s_i^2 s_j^2 \int_{B_j^{\mathbf{r}_k}} \phi_{u_i^{\bar{\mathbf{r}}_k}} |u_j^{\bar{\mathbf{r}}_k}|^2 - \int_{B_j^{\mathbf{r}_k}} f(s_i u_i^{\bar{\mathbf{r}}_k}) s_i u_i^{\bar{\mathbf{r}}_k}. \end{aligned} \tag{4.16}$$

Clearly, $V_i(1,\cdots,1)=0$ and

$$V_i(1-\tau_0,\cdots,1-\tau_0)>0,$$
$$V_i(1+\tau_0,\cdots,1+\tau_0)<0.$$

This together with Lemma 3.5 (ii), implies that

$$\begin{cases} V_i(s_1,\cdots,s_{i-1},1-\tau_0,s_{i+1},\cdots,s_{k+1})>0, & \forall 1-\tau_0\le s_j\le 1+\tau_0,\\ V_i(s_1,\cdots,s_{i-1},1+\tau_0,s_{i+1},\cdots,s_{k+1})<0, & \forall 1-\tau_0\le s_j\le 1+\tau_0. \end{cases}$$

According to Lemma 2.1, there exists some $\bar{\mathfrak{s}}\in D_{\tau_0}$ such that $V_i(\bar{\mathfrak{s}})=0$ for all i, namely, $\bar{g}(\bar{\mathfrak{s}})\in\mathcal{N}_k$. Thus the assertion follows.

According to the assertion (4.15), we deduce that

$$I_\lambda(\bar{g}(\bar{\mathfrak{s}}))\ge\inf_{\mathbf{r}_k\in\Gamma_k}\Psi(\mathbf{r}_k)=c_k. \tag{4.17}$$

However, on the other hand, note that

$$I_\lambda(\bar{g}(\bar{\mathfrak{s}}))=I_\lambda(\sum_{i=1}^{k+1}\bar{s}_iw_i^{\bar{\mathbf{r}}_k})+\int_0^1\langle I'_\lambda(\sum\bar{s}_iw_i^{\bar{\mathbf{r}}_k}+\theta\tau_0\eta(\bar{s})\phi),\tau_0\eta(\bar{\mathfrak{s}})\phi\rangle d\theta.$$

If $\bar{\mathfrak{s}}\in D_{\tau_0/2}$, then by (4.13), (4.14) and Lemma 3.5 (i),

$$I_\lambda(\bar{g}(\bar{\mathfrak{s}}))<I_\lambda(\sum_{i=1}^{k+1}\bar{s}_iw_i^{\bar{\mathbf{r}}_k})\le I_\lambda(\sum_{i=1}^{k+1}u_i^{\bar{\mathbf{r}}_k})=c_k.$$

If $\bar{\mathfrak{s}}\notin D_{\tau_0/2}$, then $\eta(\bar{s})=0$ and by (4.14) and Lemma 3.5(i),

$$I_\lambda(\bar{g}(\bar{\mathfrak{s}}))=I_\lambda(\sum_{i=1}^{k+1}\bar{s}_iw_i^{\bar{\mathbf{r}}_k})<I_\lambda(\sum_{i=1}^{k+1}u_i^{\bar{\mathbf{r}}_k})=c_k.$$

In any cases, there always holds

$$I_\lambda(\bar{g}(\bar{\mathfrak{s}}))<c_k,$$

which leads to a contradiction with (4.17). So U_k is a solution of (1.1).

In summary, $U_k=\sum_{i=1}^{k+1}u_i^{\bar{\mathbf{r}}_k}$ has exactly $k+1$ nodal domains and is a nodal solution of (1.1) satisfying $I_\lambda(U_k)=c_k$. The proof is completed. □

5 Energy comparison and the asymptotic behaviors

By Theorem 1.1, we have showed that for each $k \in \mathbb{N}_+$, the problem (1.1) admits a radial nodal solution U_k having exactly $k+1$ nodal domains. In this section, we are going to prove further that the energy $I_\lambda(U_k)$ is strictly increasing in k and $I_\lambda(U_k) > (k+1)I_\lambda(U_0)$.

Proof of Theorem 1.2:

By Lemmas 4.2 and 4.3, for each $k \in \mathbb{N}_+$, there exist $\bar{\mathbf{r}}_k = (\bar{r}_k, \cdots, \bar{r}_k) \in \Gamma_k$ and $\mathbf{u}_k^{\bar{\mathbf{r}}_k} := (u_1^{\bar{\mathbf{r}}_k}, \cdots, u_{k+1}^{\bar{\mathbf{r}}_k}) \in \mathcal{N}_k^{\bar{\mathbf{r}}_k}$ satisfying (2.4) such that

$$\psi(\bar{\mathbf{r}}_k) = \inf_{\mathbf{r}_k \in \Gamma_k} \psi(\mathbf{r}_k),$$

where $\psi(\mathbf{r}_k)$ is defined in (2.5). Moreover, by the proof of Theorem (1.1), $U_k := \sum_{i=1}^{k+1} u_i^{\bar{\mathbf{r}}_k}$ is a radial nodal solution of (1.1), which changes sign exactly k times. Similarly, there exist $\tilde{\mathbf{r}}_{k+1} = (\tilde{r}_1, \cdots, \tilde{r}_{k+1}) \in \Gamma_{k+1}$ and $\mathbf{u}_{k+1}^{\tilde{\mathbf{r}}_{k+1}} = (u_1^{\tilde{\mathbf{r}}_{k+1}}, \cdots, u_{k+2}^{\tilde{\mathbf{r}}_{k+1}}) \in \mathcal{N}_{k+1}^{\tilde{\mathbf{r}}_{k+1}}$ such that $U_{k+1} = \sum_{i=1}^{k+2} u_i^{\tilde{\mathbf{r}}_{k+1}}$ is a radial nodal solution of (1.1) changing sign exactly $k+1$ times.

In the following, we denote by

$$\hat{\mathbf{r}}_k = (\tilde{r}_2, \cdots, \tilde{r}_{k+1}) \text{ and } \hat{\mathbf{u}}_k := (u_2^{\tilde{\mathbf{r}}_{k+1}}, \cdots, u_{k+2}^{\tilde{\mathbf{r}}_{k+1}}),$$

where $u_2^{\tilde{\mathbf{r}}_{k+1}}$ is regarded as a function defined in $B_{\tilde{r}_2}(0)$ but it vanishes in $B_{\tilde{r}_1}(0)$. Observe that $(s_1 u_2^{\tilde{\mathbf{r}}_{k+1}}, \cdots, s_{k+1} u_{k+2}^{\tilde{\mathbf{r}}_{k+1}}) \in \mathcal{N}_k^{\hat{\mathbf{r}}_k}$ if and only if

$$\begin{aligned} 0 &= s_{i-1}^2 \|u_i^{\tilde{\mathbf{r}}_{k+1}}\|_i^2 + \sum_{j=2}^{k+2} s_{j-1}^2 s_{i-1}^2 \int_{B_i^{\tilde{\mathbf{r}}_{k+1}}} \phi_{u_j^{\tilde{\mathbf{r}}_{k+1}}} |u_i^{\tilde{\mathbf{r}}_{k+1}}|^2 - \int_{B_i^{\tilde{\mathbf{r}}_{k+1}}} f(s_{i-1} u_i^{\tilde{\mathbf{r}}_{k+1}}) s_{i-1} u_i^{\tilde{\mathbf{r}}_{k+1}} \\ &=: L_{i-1}^{\hat{\mathbf{u}}_k}(s_1, \cdots, s_{k+1}). \end{aligned}$$

Obviously, $L_{i-1}^{\hat{\mathbf{u}}_k}(\delta, \cdots, \delta) > 0$ for small $\delta \in (0,1)$ and

$$L_{i-1}^{\hat{\mathbf{u}}_k}(1, \cdots, 1) < 0, \quad \forall i = 2, \cdots, k+2,$$

because

$$\|u_i^{\tilde{\mathbf{r}}_{k+1}}\|_i^2 + \sum_{j=1}^{k+2} \int_{B_i^{\tilde{\mathbf{r}}_{k+1}}} \phi_{u_j^{\tilde{\mathbf{r}}_{k+1}}} |u_i^{\tilde{\mathbf{r}}_{k+1}}|^2 - \int_{B_i^{\tilde{\mathbf{r}}_{k+1}}} f(u_i^{\tilde{\mathbf{r}}_{k+1}}) u_i^{\tilde{\mathbf{r}}_{k+1}} = 0.$$

Then by the definition of $L_i^{\hat{\mathbf{u}}_k}$, there hold

$$\begin{aligned} &L_{i-1}^{\hat{\mathbf{u}}_k}(s_1,\cdots,s_{i-2},\delta,s_i,\cdots,s_{k+1})>0, \quad \forall s_j\in[\delta,1],\\ &L_{i-1}^{\hat{\mathbf{u}}_k}(s_1,\cdots,s_{i-2},1,s_i,\cdots,s_{k+1})<0, \quad \forall s_j\in[\delta,1]. \end{aligned} \tag{5.1}$$

Let $D_\delta^1:=\{(s_1,\cdots,s_{k+1})\in(\mathbb{R}_{>0})^{k+1}:\delta<s_j<1\}$. By Lemma 2.1, it follows from (5.1) that there is $\tilde{\mathbf{s}}:=(\tilde{s}_1,\cdots,\tilde{s}_{k+1})\in D_\delta^1$ such that

$$(L_1^{\hat{\mathbf{u}}_k}(\tilde{\mathbf{s}}),\cdots,L_{k+1}^{\hat{\mathbf{u}}_k}(\tilde{\mathbf{s}}))=\theta,$$

namely,

$$(\tilde{s}_1u_2^{\bar{\mathbf{r}}_{k+1}},\cdots,\tilde{s}_{k+1}u_{k+2}^{\bar{\mathbf{r}}_{k+1}})\in\mathcal{N}_k^{\hat{\mathbf{r}}_k}.$$

Hence

$$E(\tilde{s}_1u_2^{\bar{\mathbf{r}}_{k+1}},\cdots,\tilde{s}_{k+1}u_{k+2}^{\bar{\mathbf{r}}_{k+1}})\geq\inf_{\mathbf{r}_k\in\Gamma_k}\psi(\mathbf{r}_k)=E(u_1^{\bar{\mathbf{r}}_{k+1}},\cdots,u_{k+1}^{\bar{\mathbf{r}}_{k+1}})=I_\lambda(U_k).$$

Since Lemma 3.5(i) gives

$$I_\lambda(U_{k+1})=E(u_1^{\bar{\mathbf{r}}_{k+1}},\cdots,u_{k+2}^{\bar{\mathbf{r}}_{k+1}})>E(0,\tilde{s}_1u_2^{\bar{\mathbf{r}}_{k+1}},\cdots,\tilde{s}_{k+1}u_{k+2}^{\bar{\mathbf{r}}_{k+1}}),$$

it yields that

$$I_\lambda(U_{k+1})>I_\lambda(U_k).$$

Now, we turn to prove $I_\lambda(U_k)>(k+1)I_\lambda(U_0)$. In fact, since $I'_\lambda(U_k)u_i^{\bar{\mathbf{r}}_k}=0$, a direct computation yields that for any i,

$$\|u_i^{\bar{\mathbf{r}}_k}\|_i^2+\int_{\mathbb{R}^3}\phi_{u_i^{\bar{\mathbf{r}}_k}}|u_i^{\bar{\mathbf{r}}_k}|^2-\int_{\mathbb{R}^3}f(u_i^{\bar{\mathbf{r}}_k})u_i^{\bar{\mathbf{r}}_k}<0.$$

Note that there exists small $\hat{\delta}>0$ such that for all i,

$$\hat{\delta}^2\|u_i^{\bar{\mathbf{r}}_k}\|_i^2+\hat{\delta}^4\int_{\mathbb{R}^3}\phi_{u_i^{\bar{\mathbf{r}}_k}}|u_i^{\bar{\mathbf{r}}_k}|^2-\int_{\mathbb{R}^3}f(\hat{\delta}u_i^{\bar{\mathbf{r}}_k})\hat{\delta}u_i^{\bar{\mathbf{r}}_k}>0.$$

Then by the mean value theorem of continuous function, for each i, there exists $\hat{\delta}_i\in(\hat{\delta},1)$ such that

$$\hat{\delta}_i^2\|u_i^{\bar{\mathbf{r}}_k}\|_i^2+\hat{\delta}_i^4\int_{\mathbb{R}^3}\phi_{u_i^{\bar{\mathbf{r}}_k}}|u_i^{\bar{\mathbf{r}}_k}|^2-\int_{\mathbb{R}^3}f(\hat{\delta}_iw_i^{\bar{\mathbf{r}}_k})\hat{\delta}_iw_i^{\bar{\mathbf{r}}_k}=0,$$

which gives $\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k} \in \mathcal{N}$. Hence

$$I_\lambda(\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k}) \geq I_\lambda(U_0),$$

where U_0 is the ground state solution of (1.1). Then

$$\begin{aligned}
(k+1)I_\lambda(U_0) &\leq \sum_{i=1}^{k+1} I_\lambda(\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k}) = \sum_{i=1}^{k+1} \left(I_\lambda(\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k}) - \frac{1}{4} I'_\lambda(\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k}) \hat{\delta}_i w_i^{\bar{\mathbf{r}}_k} \right) \\
&= \sum_{i=1}^{k+1} \left[\frac{1}{4} \hat{\delta}_i^2 \|u_i^{\bar{\mathbf{r}}_k}\|_i^2 + \frac{1}{4} \int_{B_i^{\bar{\mathbf{r}}_k}} \left(f(\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k}) \hat{\delta}_i w_i^{\bar{\mathbf{r}}_k} - 4F(\hat{\delta}_i w_i^{\bar{\mathbf{r}}_k}) \right) \right] \\
&< \sum_{i=1}^{k+1} \left[\frac{1}{4} \|u_i^{\bar{\mathbf{r}}_k}\|^2 + \frac{1}{4} \int_{B_i^{\bar{\mathbf{r}}_k}} \left(f(u_i^{\bar{\mathbf{r}}_k}) u_i^{\bar{\mathbf{r}}_k} - 4F(u_i^{\bar{\mathbf{r}}_k}) \right) \right] \\
&= I_\lambda(\sum_{i=1}^{k+1} u_i^{\bar{\mathbf{r}}_k}) - \frac{1}{4} I'_\lambda(\sum_{i=1}^{k+1} u_i^{\bar{\mathbf{r}}_k}) u_i^{\bar{\mathbf{r}}_k} \\
&= I_\lambda(\sum_{i=1}^{k+1} u_i^{\bar{\mathbf{r}}_k}) = I_\lambda(U_k).
\end{aligned}$$

This completes the proof. □

Now, we are in a position to prove Theorem 1.3. In the sequel, we regard λ as a parameter and analyze the asymptotic behaviors of U_k^λ as $\lambda \to 0_+$.

Proof of Theorem 1.3:

Suppose that $\tilde{\mathbf{v}}_k = \sum_{i=1}^{k+1} v_i$ is a least energy nodal solution of (1.4) with exactly $k+1$ nodal domains. Obviously, $(a_{1,n}v_1, \cdots, a_{k+1,n}v_{k+1}) \in \mathcal{N}_{k,\lambda_n}^{\mathbf{r}_k}$ if and only if

$$g_i^n(a_{1,n}, \cdots, a_{k+1,n}) := a_{i,n}^2 \|v_i\|_i^2 + \sum_{j=1}^{k+1} \lambda_n a_{i,n}^2 a_{j,n}^2 \int_{B_i^{\mathbf{r}_k}} \phi_{v_j} v_i^2 - \int_{B_i^{\mathbf{r}_k}} f(a_{i,n} v_i) a_{i,n} v_i = 0. \tag{5.2}$$

Note from $I'_0(\tilde{\mathbf{v}}_k) v_i = 0$ that

$$g_i^n(1, \cdots, 1) := \|v_i\|_i^2 + \sum_{j=1}^{k+1} \lambda_n \int_{B_i^{\mathbf{r}_k}} \phi_{v_j} v_i^2 - \int_{B_i^{\mathbf{r}_k}} f(v_i) v_i > \|v_i\|_i^2 - \int_{B_i^{\mathbf{r}_k}} f(v_i) v_i = 0, \tag{5.3}$$

and from $(f4)$ that for any $L > 1$,

$$
\begin{aligned}
L^2\|v_i\|_i^2 - \int_{B_i^{\mathbf{r}_k}} f(Lv_i)Lv_i &= L^2\left(\|v_i\|_i^2 - \int_{B_i^{\mathbf{r}_k}} \frac{f(Lv_i)}{L^3}L^2 v_i\right) \\
&= L^2\int_{B_i^{\mathbf{r}_k}}\left(f(v_i)v_i - \frac{f(Lv_i)}{L^3}L^2v_i\right) \\
&< L^2\int_{B_i^{\mathbf{r}_k}}\left(f(v_i)v_i - f(v_i)L^2v_i\right) < 0.
\end{aligned}
$$

Then for sufficiently small λ_n,

$$
g_i^n(L,\cdots,L) = L^2\|v_i\|_i^2 + \lambda_n L^4\sum_{j=1}^{k+1}\int_{B_i^{\mathbf{r}_k}}\phi_{v_j}v_i^2 - \int_{B_i^{\mathbf{r}_k}} f(Lv_i)Lv_i < 0. \tag{5.4}
$$

By taking $L = 1+\frac{1}{n}$ and $\lambda_n > 0$ small enough such that (5.4) holds, it follows from Lemma 2.1 and (5.3), (5.4) that there is

$$
(a_{1,n},\cdots,a_{k+1,n}) \in D_1^{1+\frac{1}{n}} := \left\{(s_1,\cdots,s_{k+1}) \in (\mathbb{R}_{>0})^{k+1} : 1 \le s_i \le 1+\frac{1}{n}\right\}
$$

such that (5.2) holds. Moreover, $(a_{1,n},\cdots,a_{k+1,n}) \to (1,\cdots,1)$ as $n \to +\infty$. Thus

$$
\begin{aligned}
I_0(\tilde{\mathbf{v}}_k) \le I_0(U_k^0) &= \lim_{\lambda_n\to 0} I_{\lambda_n}(U_k^{\lambda_n}) = \lim_{n\to\infty} E_{\lambda_n}(w_1^{\lambda_n},\cdots,w_{k+1}^{\lambda_n}) \\
&\le \lim_{n\to\infty} E_{\lambda_n}(a_{1,n}v_1,\cdots,a_{k+1,n}v_{k+1}) = E_0(v_1,\cdots,v_{k+1}) = I_0(\sum_{i=1}^{k+1} v_i) = I(\tilde{\mathbf{v}}_k).
\end{aligned} \tag{5.5}
$$

There U_k^0 is a least energy nodal solution of (1.1) among the nodal solutions having exactly $k+1$ nodal domains. The proof is completed. □

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