The relativistic feature of Hydrogen-like atoms in the Heisenberg picture

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Abstract

The relativistic properties of Hydrogen-like atoms (HLAs) are here investigated in the Heisenberg picture for the first time. The relativistic vibrational Hamiltonian (RVH) H_{vib}^{rel} is first defined as a power series of harmonic oscillator Hamiltonian H_0 by using the relativistic energy eigenvalue E_{vib}^{rel}. By applying the first-order RVH (proportional to H_0) to the Heisenberg equation, a pair of coupled equations is turned out for the relativistic motion of the electron's position and linear momentum. A simple comparison of the first-order relativistic and nonrelativistic equations reveals this reality that the natural (fundamental) frequency of HLA (like entropy) is slowly raised by increasing the atomic number from $z = 20$. The second-order RVH (proportional to H_0^2) has then been implemented to determine an exact expression for the electron relativistic frequency in the different atomic energy levels. In general, the physical role of RVH H_{vib}^{rel} is fundamental because it not only specifies the temporal relativistic variations of position, velocity, and linear momentum of the oscillating electron, but also identifies the corresponding relativistic potential, kinetic, and mechanical energies. The results will finally be testified by demonstrating energy conservation.

1 Introduction

So far, much literature has been published about the special relativistic principles of Einstein[1-3] and their impacts on the physical and chemical quantities of Hydrogen-like atoms (HLAs) such as energy eigenvalue, Hamiltonian, angular frequency, and radius of the single bound electron, Enthalpy, and entropy.[4-9] We here demonstrate that the key relativistic parameter of Lorentz factor $\gamma = \left(1 - z^2\alpha^2/n^2\right)^{-1/2}$, in which z, α, and n are the respective atomic number, fine constant, and energy level, is responsible for all relativistic changes of physical and chemical quantities. For example, the relativistic angular frequency ω_n^{rel} is directly increased by the Lorentz factor γ, while the relativistic radius r_n^{rel} is reversely reduced by the Lorentz factor γ^{-1} so that the rotational speed of the electron remains unchanged over circular orbits of Bohr.[4, 7] The energy eigenvalue and Hamiltonian are simultaneously modified by the different relativistic perturbation terms which have been discussed by many authors in different ways.[10-12]

The initial aim of the present paper is to introduce a new method for calculating the relativistic vibrational Hamiltonian (RHV) H_{vib}^{rel} of HLAs. This method has recently been exploited to derive the nonrelativistic vibrational Hamiltonian (NRVH) $\hat{H}_{vib}$ as a power series of harmonic oscillator Hamiltonian H_0.[13] The well-known Langevin equation[14, 15] was formed by applying $\hat{H}_{vib}$ into the Heisenberg equation. In the end, it is demonstrated that the fluctuations in the position and linear momentum of the oscillating electron are responsible for the noise flux of spontaneous emission radiated from atoms with a Lorentzian profile.[13, 16]

The next aim is to extend the vibrational motion equations of HLAs from nonrelativistic to relativistic states. Accordingly, the relativistic energy eigenvalue E_{vib}^{rel} has been rendered by applying the special relativity principles to the kinetic and potential energies of the electron. The RVH H_{vib}^{rel} is then constructed by substituting the number operator $\hat{N}$ for the energy level number n in the relativistic energy eigenvalue E_{vib}^{rel} similar to the energy eigenvalue of a simple harmonic oscillator $E_n = (n+1/2)\hbar\omega_0$ which is directly converted to its Hamiltonian $\hat{H}_0 = (\hat{N}+1/2)\hbar\omega_0$.[13, 17] ω_0 is the natural frequency of harmonic oscillator whose definition for the nonrelativistic oscillation of electron has been introduced by the relation (10) 0f Ref [13] as $\omega_{vib}^{(1)} = \omega_0 = \hbar^{-1}\mu c^2(Z\alpha)^2$ ($\mu \approx m_e$).

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The last aim is to find the relativistic natural frequency ω_0^{ref} associated with the lowest energy level $n=1$ for the vibrational motion of the electron. This is achieved by placing the first-order RVH $\hat{H}_{rel}^{(1)}$ into the Heisenberg equation so that it is automatically turned out as $\omega_0^{rel} = \omega_0(1 + z^2\alpha^2/2)$. One thus expects that the relativistic vibrational frequency ω_n^{rel} always is larger than the nonrelativistic frequency ω_n in all higher circular orbits ($n>1$) due to the larger relativistic natural frequency $\omega_0^{rel} > \omega_0$. Finally, the results are confirmed by demonstrating energy conservation so that the sum of relativistic kinetic and potential energies of electron (mechanical energy) remains unchanged at all times.

2 The relativistic kinetic, potential, and vibrational energies

There are two different kinds of oscillation that simultaneously happen for the single electron of HLAs. The first one is concerned with the intrinsic spinal motion of the electron around its axis with an unknown driving force which is not tractable by the present theory. The second one is the relativistic motion of the electron over the stable orbits of Bohr which acts as a harmonic oscillator. The kinetic energy of electron thus obeys the special relativistic relation in the usual form[18, 19]

$$T_{rel} = m_e c^2 \left(1 + \frac{p_{rel}^2}{m_e^2 c^4}\right)^{1/2} - m_e c^2 = \frac{p_{rel}^2}{2m_e} + \frac{p_{rel}^4}{8m_e^3 c^2} + \cdots, \tag{1}$$

in which m_e is the rest mass of the electron and $p_{rel} = \gamma m_e v_n$ is the electron relativistic linear momentum over Bohr's circular orbits with the constant speed $v_n = ze^2/n\hbar = z\alpha c/n$ ($1/4\pi\varepsilon_0 = 1$) and the Lorentz factor $\gamma = \left(1 - z^2\alpha^2/n^2\right)^{-1/2}$.[4, 7, 12] As a result, the relativistic kinetic energy (1) can be approximated up to the fourth power $(z\alpha)^4/n^4$ after substituting the corresponding expansions γ^2 and γ^4 as

$$T_{rel} = \frac{1}{2}m_e c^2 \frac{(z\alpha)^2}{n^2} + \frac{3}{8}m_e c^2 \frac{(z\alpha)^4}{n^4}. \tag{2}$$

On the other side, the relativistic potential energy is assigned by the Coulomb potential as

$$V_{rel} = \frac{-ze^2}{r_n^{rel}}, \tag{3}$$

in which r_n^{rel} is the relativistic radius of Bohr's nth-orbit. The wave-particle nature of the electron is now used to determine the relativistic radius r_n^{rel}. We begin with the de Broglie relation in the relativistic form

$$\lambda_n^{rel} = \frac{h}{p_{rel}} = \frac{h}{\gamma m_e v_n} = \gamma^{-1}\lambda_n . \tag{4}$$

λ_n^{rel} is the relativistic wavelength of rotating electron which satisfies the standing-wave condition in the relativistic form

$$r_n^{rel} = \frac{n\lambda_n^{rel}}{2\pi} = \frac{n\hbar}{\gamma m_e v_n} = \gamma^{-1}r_n . \tag{5}$$

Therefore, the relativistic radius r_n^{rel} and wavelength λ_n^{rel} have commonly been contracted concerning their nonrelativistic values $r_n = n^2\hbar/(m_e z\alpha c)$ and $\lambda_n = nh/(m_e z\alpha c)$ by the reverse of the Lorentz factor $\gamma^{-1} = \left(1 - z^2\alpha^2/n^2\right)^{1/2}$. It is noteworthy to know that the relation (5) is in complete agreement with the relations (7) and (5) of Refs [4, 7], respectively, but disagrees with the corresponding relation (13) of Ref [20] in which the contraction factor is introduced as $2(\gamma+1)^{-1}$ rather γ^{-1}. Finally, the relativistic angular frequency ω_n^{rel} is turned out as

$$\omega_n^{rel} = \frac{v_n}{r_n^{rel}} = \gamma\,\omega_n , \tag{6}$$

in which the nonrelativistic angular frequency $\omega_n = \omega_0/n^3$ was already defined by the relation (18) of Ref [13] and its present relativistic value is raised by the Lorentz factor γ.

Now by substituting the reduced radius r_n^{rel} from (5) into (3), a relativistic relation is derived for the relativistic potential energy correct to $(z\alpha)^4/n^4$ as

$$V_{rel} = \gamma\frac{-ze^2}{r_n} = -m_e c^2\frac{(z\alpha)^2}{n^2} - \frac{1}{2}m_e c^2\frac{(z\alpha)^4}{n^4}, \tag{7}$$

where the expansion $\gamma = 1 + (z\alpha)^2/2n^2 + \ldots$ has been used. It is also evident from (7) that the potential energy is increased by the Lorentz factor γ from the relativistic point of view. One can now render the relativistic energy eigenvalue (relativistic vibrational energy) by adding the relativistic kinetic energy (2) to the relativistic potential energy (7) as

$$E_{vib}^{rel} = -\frac{1}{2}m_e c^2\frac{(z\alpha)^2}{n^2} - \frac{1}{8}m_e c^2\frac{(z\alpha)^4}{n^4} = E_{vib} - \frac{1}{2m_e c^2}E_{vib}^2 , \tag{8}$$

which is in complete agreement with the corresponding relations (9) and (6) of Refs [4, 7], respectively. The first term E_{vib} is the nonrelativistic energy eigenvalue of HLAs and the second quadratic term (proportional to E_{vib}^2) is concerned with the perturbation that happens for HLAs due to the relativistic motion of the electron.

3 The relativistic vibrational Hamiltonian (RVH) $\hat{H}_{vib}^{rel}$

We have recently constructed the nonrelativistic vibrational Hamiltonian (NRVH) of HLAS in terms of that of a simple harmonic oscillator $\hat{H}_0$ by expanding the corresponding energy eigenvalue $E_{vib} = -0.5 m_e c^2 (Z\alpha)^2 / n^2$ around $n = 1$ in the form[13]

$$\hat{H}_{vib} = 0.5 \hbar \omega_0 \sum_{k=0}^{\infty} (-1)^{k+1} (k+1) \left(\frac{\hat{H}_0}{\hbar \omega_0} \right)^k , \qquad (9)$$

in which $\omega_0 = \hbar^{-1} \mu c^2 (Z\alpha)^2$ ($\mu \approx m_e$) is the nonrelativistic natural frequency associated with the lowest energy level $n = 1$, as it is evident in the relativistic frequency relation (6) with the corresponding value $\gamma = 1$.

The RVH $\hat{H}_{vib}^{rel}$ is now derived by substituting the NRVH $\hat{H}_{vib}$ for the corresponding energy eigenvalue E_{vib} in the relativistic vibrational energy eigenvalue (8) as

$$\hat{H}_{vib}^{rel} = \hat{H}_{vib} - \frac{1}{2m_e c^2} \hat{H}_{vib}^2 . \qquad (10)$$

Therefore, the NRVH $\hat{H}_{vib}$ is modified by the perturbation Hamiltonian $\hat{H}_{pertubation} = \frac{-1}{2m_e c^2} \hat{H}_{vib}^2$ due to the relativistic vibrational motion of the electron.

The most advantage of RVH $\hat{H}_{vib}^{rel}$ is to hand over the relativistic motion equation of the electron in the direction of an arbitrary axis such as x by using the well-known Heisenberg equation in the form[21]

$$\frac{d < \hat{x}(t) >_{rel}}{dt} = i \hbar^{-1} < [\hat{H}_{vib}^{rel} , \hat{x}(t)] > , \qquad (11)$$

in which $< \hat{x}(t) >_{rel}$ is the expectation (mean) value of electron vibrational position at the moment of t. Although the equation (11) is a first-order time-dependent differential equation, its general solution is rather difficult due to $\hat{H}_{vib}^{rel}$ which consists of the two infinite power series

$\hat{H}_{vib}$ and $\hat{H}_{vib}^2$ in terms of $\hat{H}_0$ according to the relations (9) and (10). However, this problem can be sorted out by solving the equation (11) step by step by implementing the first and second-order RVH $\hat{H}_{rel}^{(1)}$ and $\hat{H}_{rel}^{(2)}$ proportional to $\hat{H}_0$ and $\hat{H}_0^2$ in sections 4 and 5, respectively.

4 The first-order RVH $\hat{H}_{rel}^{(1)}$ and natural (fundamental) frequency ω_0^{ref}

The first-order RVH $\hat{H}_{rel}^{(1)}$ is obtained by applying the power series $\hat{H}_{vib}$ from (9) to (10) and choosing terms proportional to $\hat{H}_0$ as

$$\hat{H}_{rel}^{(1)} = 0.5\hbar\omega_0\left[-1 + \frac{2\hat{H}_0}{\hbar\omega_0} + \frac{\hbar\omega_0}{4m_e c^2}\left(-1 + \frac{4\hat{H}_0}{\hbar\omega_0}\right)\right]. \tag{12}$$

The motion equations for the mean value of the relativistic position and linear momentum operators $<\hat{x}(t)>_{rel}$ and $<\hat{p}(t)>_{rel}$ are now derived by substituting the first-order RVH (12) into the Heisenberg equation in the forms

$$\frac{d<\hat{x}(t)>_{rel}}{dt} = i\hbar^{-1}<[\hat{H}_{rel}^{(1)}, \hat{x}(t)]> = \frac{1}{m_e}\left(1 + \frac{\hbar\omega_0}{2m_e c^2}\right)<\hat{p}(t)>_{rel} \tag{13}$$

and

$$\frac{d<\hat{p}(t)>_{rel}}{dt} = i\hbar^{-1}<[\hat{H}_{vib}^{(1)}, \hat{p}(t)]> = -m_e\omega_0^2\left(1 + \frac{\hbar\omega_0}{2m_e c^2}\right)<\hat{x}(t)>_{rel}, \tag{14}$$

where the commutation relations $[\hat{H}_0, \hat{x}] = -i\hbar m_e^{-1}\hat{p}$ and $[\hat{H}_0, \hat{p}] = i\hbar m_e\omega_0^2\hat{x}$ have been used.[22] The relativistic equation of motion for the variable $<\hat{x}(t)>_{rel}$ will ultimately be extracted after substituting (14) into the derivative of equation (13) as

$$\frac{d^2<\hat{x}(t)>_{rel}}{dt^2} + \omega_{vib}^{(1)2}<\hat{x}(t)>_{rel} = 0, \tag{15}$$

in which the first-order relativistic vibrational frequency $\omega_{vib}^{(1)}$ is the same as the relativistic natural frequency ω_0^{ref} in the form

$$\omega_{vib}^{(1)} = \omega_0^{ref} = \omega_0\left(1 + \frac{z^2\alpha^2}{2}\right). \tag{16}$$

On the other hand, if one expands the Lorentz factor γ in the general relativistic frequency relation (6) as

$$\gamma = \left(1 - z^2\alpha^2/n^2\right)^{-1/2} = 1 + \frac{1}{2}\frac{z^2\alpha^2}{n^2} + \frac{3}{8}\frac{z^4\alpha^4}{n^4} + \ldots, \tag{17}$$

then the relativistic natural frequency (16) will expectedly be revisited by applying the lowest vibrational level $n = 1$ to the relativistic general relation (6) and ignoring the negligible terms of powers higher than $z^2\alpha^2$ ($z^4\alpha^4 \ll z^2\alpha^2$). The variations ratio of relativistic to nonrelativistic natural frequencies ω_0^{ref}/ω_0 versus the atomic number z is demonstrated in Fig. 1.
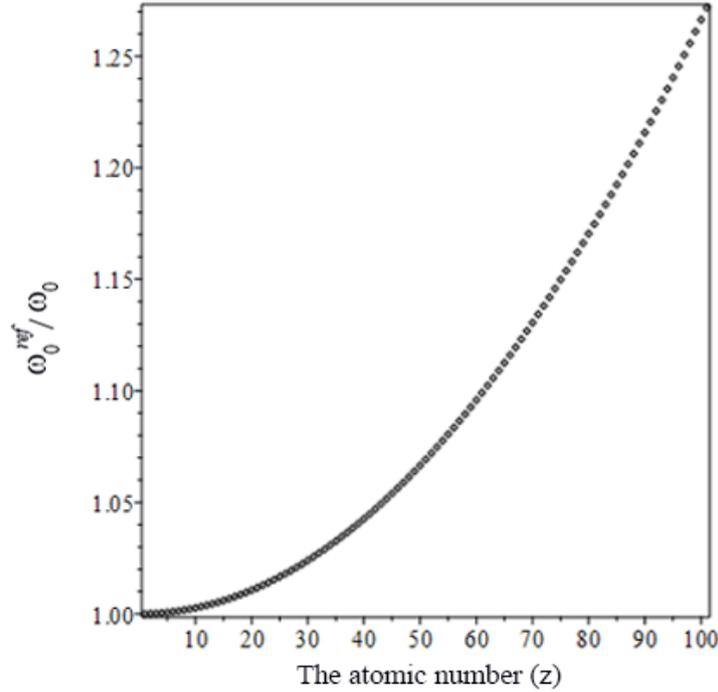
Fig. 1. The gradual increment of the normalized relativistic natural frequency ω_0^{ref}/ω_0 with respect to the atomic number z is shown in the range $1 \leq z \leq 100$ (Fermium).

According to the relation (16), the slope ($\alpha^2 z$) of the ratio ω_0^{ref}/ω_0 is slowly raised by increasing the atomic number z due to the negligible value of coefficient α^2 ($\alpha^2 \ll 1$). This gradual increment is evident in Fig. 1 so that the normalized relativistic natural frequency ω_0^{ref}/ω_0 is raised from 1 to 1.01 for the lighter elements with the atomic numbers in the range $1 \leq z \leq 20$ and suddenly raised from 1.01 to 1.27 for the heavier elements with the atomic numbers in the range $20 < z \leq 100$. Interestingly, similar behavior has been observed for the linear entropy of HLAS in Fig. 4 of Refs [9] and [22] where the ratio of relativistic to nonrelativistic estimates of the linear entropy $S_r^{R/NR}$ (red curve) is negligibly increased from 1 in the range $1 \leq z \leq 20$, but considerably raised from almost 1 to 3.5 in the range $20 < z \leq 100$

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. As a result, the physical quantities of HLAs have been sensibly affected by increasing the atomic number z, starting from $z=20$, due to the relativistic motion of a single electron.

Finally, one may ask why the atomic number $z=20$ is a critical point for the simultaneous relativistic changes of both physical and chemical quantities. The answer should be related to the Lorentz factor γ as a key parameter of special relativity. Firstly, it should be noticed that the role of the Lorentz factor $\gamma = \left(1 - v^2/c^2\right)^{-1/2} = \left(1 - z^2\alpha^2/n^2\right)^{-1/2}$ will be disappeared in all relativistic relations (1) to (17) by tending the light speed c to infinity ($c \to \infty$) because the relativistic terms $v/c = z\alpha/n$ ($\alpha = e^2/\hbar c$) tends to zero, and consequently γ approaches to unity ($\gamma \to 1$). Secondly, it is evident from (17) that the numerical value of γ (v/c) is directly depending on the atomic number z as a chemical characteristic that distinguishes the elements of the single-electron group (HLAs) from each other. The variations of the Lorentz factor γ versus the atomic number z are illustrated in Fig. 2 for the different energy levels $n=1$, 2, and 3.
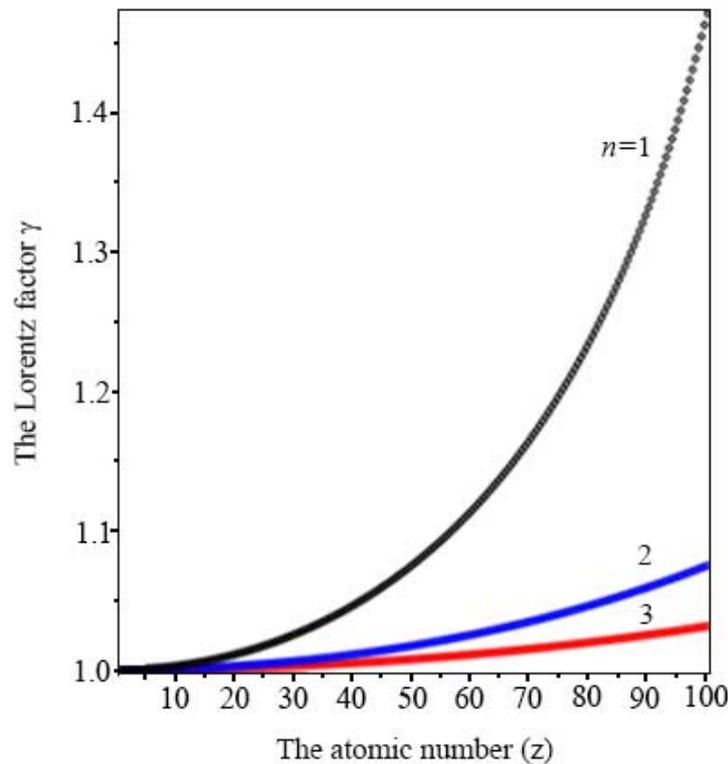

Fig. 2. The variations of the Lorentz factor γ versus the atomic number z are shown in the range $1 \le z \le 100$ (Fermium). The early point of gradual increment is associated with the atomic number $z=20$ and $n=1$ in agreement with Fig. 2, Fig. 1 of Ref [9], and Figs. 1 and 4 of Ref [22].

5 The second-order RVH $\hat{H}_{rel}^{(2)}$ and energy conservation

The second-order RVH of HLAs is assigned by substituting NRVH (9) to RVH (10) and choosing the terms correct to H_0^2 as

$$\hat{H}_{rel}^{(2)} = 0.5\hbar\omega_0\left[-1+\frac{2\hat{H}_0}{\hbar\omega_0}-\frac{3\hat{H}_0^2}{\hbar^2\omega_0^2}+\frac{\hbar\omega_0}{4m_e c^2}\left(-1+\frac{4\hat{H}_0}{\hbar\omega_0}-\frac{10\hat{H}_0^2}{\hbar^2\omega_0^2}\right)\right]. \tag{18}$$

The motion equations of relativistic variables $<\hat{x}>_{rel}$ and $<\hat{p}>_{rel}$ have similarly been rendered by substituting $\hat{H}_{rel}^{(2)}$ for $\hat{H}_{rel}^{(1)}$ into the Heisenberg equations (13) and (14) as

$$\frac{d<\hat{x}(t)>_{rel}}{dt}=\frac{1}{m_e}\left(1-\frac{3E_n^0}{\hbar\omega_0}+\frac{\hbar\omega_0-5E_n^0}{2m_e c^2}\right)<\hat{p}(t)>_{rel}-\frac{1}{2}i\omega_0\left(3+\frac{5\hbar\omega_0}{2m_e c^2}\right)<\hat{x}(t)>_{rel} \tag{19}$$

and

$$\frac{d<\hat{p}(t)>_{rel}}{dt}=-m_e\omega_0^2\left(1-\frac{3E_n^0}{\hbar\omega_0}+\frac{\hbar\omega_0-5E_n^0}{2m_e c^2}\right)<\hat{x}(t)>_{rel}-\frac{1}{2}i\omega_0\left(3+\frac{5\hbar\omega_0}{2m_e c^2}\right)<\hat{p}(t)>_{rel}. \tag{20}$$

The motion equation for the variable $<\hat{x}>_{rel}$ will finally be achieved by the respective substitution $<\hat{p}(t)>_{rel}$ and $d<\hat{p}(t)>_{rel}/dt$ from (19) and (20) into the derivative of (19) as

$$\frac{d^2<\hat{x}(t)>_{rel}}{dt^2}+i\beta_{rel}^{(2)}\frac{d<x(t)>_{rel}}{dt}+\left(\omega_{rel}^{(2)2}-\frac{\beta_{rel}^{(2)2}}{4}\right)<\hat{x}(t)>_{rel}=0, \tag{21}$$

in which the second-order relativistic vibrational frequency $\omega_{rel}^{(2)}$ is equal to $[(4-3n)-0.5(z\alpha)^2(6-5n)]\omega_0$ and includes the nonrelativistic term $\omega_{vib}^{(2)}=(4-3n)\omega_0$ of Ref [13], as well. The second-order term $\beta_{rel}^{(2)}=(3+2.5z^2\alpha^2)\omega_0$ must be ignored because it is only a phase term associated with the infinite number of vibrational axis directions (here the typical x-axis has been chosen).

As a result, the common features of the first and second-order RVHs $\hat{H}_{rel}^{(1)}$ and $\hat{H}_{rel}^{(2)}$ defined by (12) and (18) and their corresponding motion equations (15) and (21) [$\beta_{rel}^{(2)} = 0$] imply that the general RVH (10) must imitate the motion equation of a quantum harmonic oscillator (QHO) in the usual form [13, 23]

$$\frac{d^2 <\hat{x}(t)>_{rel}}{dt^2} + (\omega_{vib}^{rel})^2 <\hat{x}(t)>_{rel} = 0, \tag{22}$$

in which the general relativistic vibrational frequency ω_{vib}^{rel} is the same as the corresponding relativistic rotational frequency ω_{n}^{rel} given by (6). The solution of equation (22) simultaneously yields the temporal relativistic variations of position and linear momentum of the oscillating electron as

$$<\hat{x}(t)>_{rel} = r_n^{rel} \left[\cos\left(\omega_{vib}^{rel} t\right) + \sin\left(\omega_{vib}^{rel} t\right) \right] \tag{23}$$

and

$$<\hat{p}(t)>_{rel} = \gamma m_e \frac{d <\hat{x}(t)>_{rel}}{dt} = \gamma m_e r_n^{rel} \omega_{vib}^{rel} \left[\cos\left(\omega_{vib}^{rel} t\right) - \sin\left(\omega_{vib}^{rel} t\right) \right], \tag{24}$$

where the relativistic amplitude of electron vibration r_n^{rel} has already been derived by using the wave nature of rotating electron (standing wave condition) in the relation (5). It should be noticed that the Lorentz factor γ is independent of time according to the relation (17).

Finally, It is necessary to justify the accuracy of rotational and vibrational relations (1)-(24) by demonstrating the energy conservation relation in the form

$$E_{mechanical}^{rel} = E_{kinetic}^{rel}(t) + E_{potential}^{rel}(t), \tag{25}$$

in which the relativistic mechanical energy $E_{mechanical}^{rel}$ must remain constant in all vibrational time t due to the conservative nature of the Coulomb force. The relativistic kinetic energy $E_{kinetic}^{rel}(t)$ is determined by substituting the relativistic linear momentum $<\hat{p}(t)>_{rel}$ from (24) into the general relation of relativistic kinetic energy (1). The relativistic potential energy $E_{potential}^{rel}(t)$ is defined by the usual relativistic relation $0.5 \, m_e (\omega_{vib}^{rel})^2 <x(t)>_{rel}^2$ so that its numerical value is obtained by applying $<\hat{x}(t)>_{rel}$ from the solution (23) and implementing the relativistic frequency relation (6) in which $\omega_{vib}^{rel} = \omega_{n}^{rel} = \gamma \omega_n$. The temporal variations of relativistic vibrational kinetic and potential energies $E_{kinetic}^{rel}(t)$ and $E_{potential}^{rel}(t)$ are plotted in Fig. 3 for the Hydrogen atom with the physical characteristic $z = 1$, $n=2$, $r_2^{rel} = 1.32 \times 10^{-7} \, cm$,

$v_2 = 1.09 \times 10^8 \ cm/s$, $\omega_0^{rel} = 6.59 \times 10^{15} \ Hz$, $\omega_2^{rel} = 8.24 \times 10^{14} \ Hz$, and $\gamma = 1.0000067$. The sum of relativistic kinetic (red color) and potential (blue color) energies of the oscillating electron is always constant, as illustrated with black color.
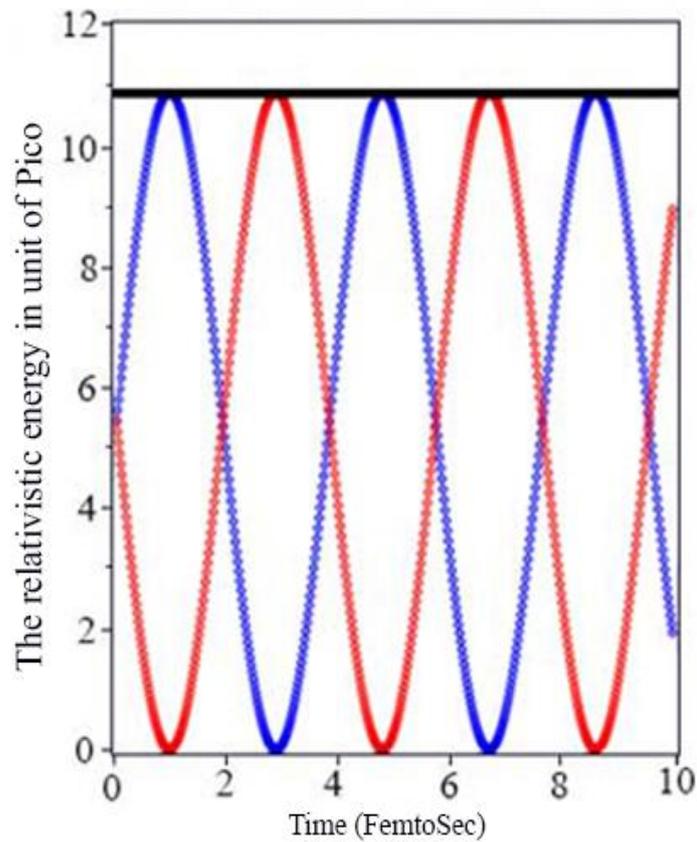

Fig. 3. The temporal variations of relativistic kinetic (red colour) and potential (blue colour) energies of oscillating electron are demonstrated for Hydrogen atom in the typical excited state *n=2*. The sum of these two energies gives the mechanical energy (black colour) with a constant value due to the conservative property of Coulomb force.

6 Conclusions

The relativistic rotational and vibrational features of the oscillating electron over Bohr's circular orbits have simultaneously been studied by using its wave-particle nature, relativistic vibrational Hamiltonian, and Heisenberg equation of motion. The wave-particle property has first been implemented to calculate the relativistic de Broglie wavelength λ_n^{rel}, reduced radius r_n^{rel}, and increased angular frequency ω_n^{rel} of rotating electron in the relations (4)-(6), respectively. The relativistic potential energy V_{rel} is then gained by applying the reduced radius r_n^{rel} from (5) and the expansion of the Lorentz factor γ from (17) into the Coulomb potential (7). The relativistic kinetic energy T_{rel} has similarly been determined by substituting the relativistic linear momentum $p_{rel} = \gamma m_e v_n$ into the general relativistic kinetic energy (1). The relativistic vibrational energy E_{vib}^{rel} will finally be specified as the sum of relativistic kinetic T_{rel} and potential V_{rel} energies in the relation (8).

The Heisenberg picture now begins by constructing the relativistic vibrational Hamiltonian (RVH) $\hat{H}_{vib}^{rel}$ similar to the nonrelativistic vibrational Hamiltonian $\hat{H}_{vib}$.[13] By applying the first-order RVH $\hat{H}_{rel}^{(1)}$ (12) to the Heisenberg equation, the relation (16) was turned out for the normalized relativistic natural frequency $\omega_0^{ref} / \omega_0$ associated with the lowest vibrational state $n = 1$. Although the ratio $\omega_0^{ref} / \omega_0$ is negligibly raised by increasing the atomic number z, but this increment becomes sensible for the elements with $z \geq 20$, as illustrated in Fig. 1. Similar incremental behavior has exactly been reported for the linear entropy of HLAS in Fig. 4 of Refs [9] and [22], and for the relativistic to the nonrelativistic ratio of position and linear momentum of the oscillating electron in Fig. 1 of Ref [22]. It is here discussed how the relativistic behavior of these quantities can be interpreted according to the variations of the Lorentz factor γ plotted in Fig. 2.

The second-order RVH $\hat{H}_{rel}^{(2)}$ plays a different key role in determining the temporal variations of two important variables $< \hat{x}(t) >_{rel}$ and $< \hat{p}(t) >_{rel}$ as presented in the relations (23) and (24), respectively. These two variables have been implemented to investigate the accuracy of results by confirming the energy conservation relation (25). The sum of relativistic vibrational kinetic and potential energies $E_{kinetic}^{rel}(t)$ and $E_{potential}^{rel}(t)$ is conserved in all oscillation time t, as illustrated in Fig. 3.

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