# Phragmén-Lindelöf alternative results in time-dependent double-diffusive Darcy plane flow

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**Abstract**

This paper investigates the spatial behavior of the solutions of the double-diffusive Darcy plane flow in a semi-infinite channel. Using the energy estimate method and the differential inequality technology, a differential inequality about the solutions is derived. By solving this differential inequality, it is proved that the solutions grow polynomially or decay exponentially with spatial variable. In the case of decay, we obtain the upper bound for the total energy. We also give some remarks to generalize the results of this paper.

KEY WORDS: Phragmén-Lindelöf alternative results; Darcy plane flow; Energy estimation.

# 1 Introduction

The Darcy equations are often used to describe flow in a porous medium which have been discussed in the books of Nield and Bejan [1], and Straughan [2]. Many scholars in the literature have paid attention to the spatial attenuation of the Darcy equations on a semi-infinite cylinder. Song [3] and Payne and Song [4] considered the fluid in porous media controlled by Darcy equations in a semi-infinite cylinder. In another paper, Song [5] considered the time-dependent double-diffusive convective Darcy flow in a semi-infinite channel and the Saint-Venant type decay of the solutions on a is obtained. For more works, one can see [6–12]. However,these papers need to assume that the solutions satisfy certain a priori assumptions at the infinity of the cylinder or the channel.

The classical Phragmén-Lindelöf alternative theorem does not need such a priori assumption, but proves that the solution either decays exponentially or increases exponentially with the distance from

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the finite end of the cylinder. In the past decades, the phragmén-lindelöf alternative research has received a lot of attention (see [13–19]). These papers above always assumed that the generators of the semi-infinite cylinder or the channel parallel to the coordinate axis.

In this paper, we define a new channel

$$R = \Big\{(x_1, x_2)|x_1 > a, 0 < x_2 < h(x_1)\Big\},$$

where $a$ is a positive constant and $h(x_1)$ is a smooth curve in the plane. Obviously, the generatrix of $R$ is no longer parallel to the coordinate axis. We investigate the time-dependent double-diffusive Darcy flow of a fluid through a porous medium in $R$. The Darcy plane flow can be written as

$$u_\alpha = -p_{,\alpha} + g_\alpha T + h_\alpha C, \ in \ R \times (0, \tau), \tag{1.1}$$

$$u_{\alpha,\alpha} = 0, \ in \ R \times (0, \tau), \tag{1.2}$$

$$\partial_t T + u_\alpha T_{,\alpha} = \Delta T, \ in \ R \times (0, \tau), \tag{1.3}$$

$$\partial_t C + u_\alpha C_{,\alpha} = \Delta C + \sigma \Delta T, \ in \ R \times (0, \tau), \tag{1.4}$$

where $\alpha = 1, 2$, $u_\alpha, p, T, C$ represent velocity, pressure, temperature, and concentration of the flow. $g_\alpha, h_\alpha$ are bounded functions. $\sigma$ is a material positive constant. For simplicity, we assume $\boldsymbol{g}$ satisfies

$$|g_\alpha g_\alpha| \le 1, \quad |h_\alpha h_\alpha| \le 1.$$

In this paper, we also use the summation convention summed from 1 to 2, and a comma is used to indicate differentiation. e.g., $u_{\alpha,\beta}u_{\alpha,\beta} = \sum_{\alpha,\beta=1}^{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2$.

The equations (1.1)-(1.4) also satisfy the following initial-boundary conditions

$$u_i(x_1, 0, t) = u_i(x_1, h(x_1), t) = 0, \ x_1 \ge a, 0 < t < \tau, \tag{1.5}$$

$$T(x_1, 0, t) = T(x_1, h(x_1), t) = 0, \ x_1 \ge a, 0 < t < \tau, \tag{1.6}$$

$$C(x_1, 0, t) = C(x_1, h(x_1), t) = 0, \ x_1 \ge a, 0 < t < \tau, \tag{1.7}$$

$$u_i(a, x_2, t) = F_i(x_2, t), \ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.8}$$

$$T(a, x_2, t) = H(x_2, t), C(a, x_2, t) = \widetilde{H}(x_2, t), \ 0 < x_2 < h(a), 0 < t < \tau, \tag{1.9}$$

$$u_i(x_1, x_2, 0) = 0, \ T(x_1, x_2, 0) = C(x_1, x_2, 0) = 0, \ (x_1, x_2) \in R, \tag{1.10}$$

where $F_i$ and $H$ are differentiable functions which are assumed to satisfy appropriate compatibility conditions

$$F_i(h(a), t) = H(h(a), t) = \widetilde{H}(h(a), t) = 0.$$

We also introduce the notation:

$$R_z = \Big\{(x_1, x_2)|x_1 \ge z > a, 0 < x_2 < h(x_1)\Big\},$$

$$L_z = \Big\{(x_1, x_2)|x_1 = z \geq a, 0 < x_2 < h(z)\Big\},$$

where $z$ is a running variable along the $x_1$ axis.

Different from paper [5], this paper studies the Phragmén-Lindelöf type alternative theorem of equations (1.1)-(1.10) on $R$. In the remarks, we consider four types of channel and on each channel we proof that the solutions either grow exponentially (polynomially, logarithmical) or decay exponentially (polynomially, logarithmical) as $z \to \infty$. Because our model contains nonlinear term and pressure term that are difficult to deal with, how to set the energy function is the key. As far as we know, there are few relevant results on the Phragmén-Lindelöf type alternative results of such nonlinear equations. Therefore, the research of this paper is very meaningful and can provide reference for the alternative research of other types of nonlinear equations.

# 2 Preliminary

In order to eliminate the pressure term, we introduce the stream function $\boldsymbol{v}(x_1, x_2, t) = (v_1, v_2)$ satisfied

$$(u_1, u_2) = \nabla^{\perp}\boldsymbol{v},$$

where $\nabla^{\perp} = (\partial_{x_1}, -\partial_{x_2})$. The equations (1.1)-(1.10) can be rewritten as

$$\Delta v = -\nabla^{\perp} \cdot \boldsymbol{g}T - \boldsymbol{g} \cdot \nabla^{\perp}T - \nabla^{\perp} \cdot \boldsymbol{h}C - \boldsymbol{h} \cdot \nabla^{\perp}C, \ in \ R \times (0, \tau), \tag{2.1}$$

$$\partial_t T + \nabla^{\perp}\boldsymbol{v} \cdot \nabla T = \Delta T, \ in \ R \times (0, \tau), \tag{2.2}$$

$$\partial_t C + \nabla^{\perp}\boldsymbol{v} \cdot \nabla C = \Delta C + \sigma \Delta T, \ in \ R \times (0, \tau), \tag{2.3}$$

$$v(x_1, 0, t) = v(x_1, h(x_1), t) = 0, \ x_1 \geq a, 0 < t < \tau, \tag{2.4}$$

$$v_n(x_1, h(x_1), t) = v_n(x_1, h(x_1), t) = 0, \ x_1 \geq a, 0 < t < \tau, \tag{2.5}$$

$$T(x_1, 0, t) = T(x_1, h(x_1), t) = 0, \ x_1 \geq a, 0 < t < \tau, \tag{2.6}$$

$$C(x_1, 0, t) = C(x_1, h(x_1), t) = 0, \ x_1 \geq a, 0 < t < \tau, \tag{2.7}$$

$$v(a, x_2, t) = \widetilde{F}_1(x_2, t) = \int_0^{x_2} F_1(s, t)ds, \ 0 < x_2 < h(a), 0 < t < \tau, \tag{2.8}$$

$$v_{,1}(a, x_2, t) = \widetilde{F}_2(x_2, t) = -F_2(x_2, t), \ 0 < x_2 < h(a), 0 < t < \tau, \tag{2.9}$$

$$T(a, x_2, t) = H(x_2, t), \ C(a, x_2, t) = \widetilde{H}(x_2, t), \ 0 < x_2 < h(a), 0 < t < \tau, \tag{2.10}$$

$$v_{,\alpha}(x_1, x_2, 0) = 0, \ T(x_1, x_2, 0) = 0, \ (x_1, x_2) \in R, \tag{2.11}$$

where $v_n$ is the outward normal derivative of $v$, $\boldsymbol{g} = (g_1, g_2)$, $\boldsymbol{h} = (h_1, h_2)$.

Here are some lemmas that will be often used in this paper.

**Lemma 1** [20, 21] If $w(x_1, 0) = w(x_1, h) = 0, w_n(x_1, 0) = w_n(x_1, h) = 0$, then the following

Wirtinger type inequality holds

$$\int_{L_z} (w_{,1})^2 dx_2 \le \frac{h^2}{\pi^2} \int_{L_z} (w_{,12})^2 dx_2.$$

Based on lemma 1, the following lemma can be obtained

**Lemma 2** If $w(x_1, 0) = w(x_1, h) = 0$, then

$$\Big[\int_{L_z} w^4 dx_2\Big]^{\frac{1}{2}} \le \frac{2h\sqrt{h}}{\pi} \int_{L_z} (w_{,2})^2 dx_2.$$

**Proof** Since $w(x_1, 0) = w(x_1, h) = 0$, we have

$$w^2(x_1, x_2) = 2\int_0^{x_2} w\frac{\partial}{\partial\zeta} w(x_1, \zeta) d\zeta = -2\int_{x_2}^{h} w\frac{\partial}{\partial\zeta} w(x_1, \zeta) d\zeta. \tag{2.12}$$

From (2.12) we have

$$w^2(x_1, x_2) \le 2\int_0^h \Big| w\frac{\partial}{\partial x_2} w(x_1, x_2)\Big| dx_2.$$

Therefore, we have

$$\Big[\int_0^h w^4(x_1, x_2) dx_2\Big]^{\frac{1}{2}} \le 2\sqrt{h}\int_0^h \Big| w\frac{\partial}{\partial x_2} w(x_1, x_2)\Big| dx_2.$$

Using the Hölder inequality, we obtain

$$\Big[\int_0^h w^4(x_1, x_2) dx_2\Big]^{\frac{1}{2}} \le 2\sqrt{h}\Big[\int_0^h w^2 dx_2 \int_0^h \Big(\frac{\partial w}{\partial x_2}\Big)^2 dx_2\Big]^{\frac{1}{2}}. \tag{2.13}$$

Using lemma 1 in (2.13), we can obtain lemma 2.

To get the Phragmén-Lindelöf type alternative result of the solutions to the equations (1.1)-(1.10), we first establish three energy functions

$$\begin{aligned} F_1(z,t) &= \beta_1 \int_0^t \int_{L_z} e^{-\delta\eta} v v_{,1} dx_2 d\eta + \beta_1 \int_0^t \int_{L_z} e^{-\delta\eta} g_2 T v dx_2 d\eta \\ &+ \beta_1 \int_0^t \int_{L_z} e^{-\delta\eta} h_2 C v dx_2 d\eta \\ &\doteq A_1 + A_2 + A_3, \end{aligned} \tag{2.14}$$

$$\begin{aligned} F_2(z,t) &= \beta_2 \int_0^t \int_{L_z} e^{-\delta\eta} T T_{,1} dx_2 d\eta - \frac{1}{2}\beta_2 \int_0^t \int_{L_z} e^{-\delta\eta} T^2 v_{,2} dx_2 d\eta \\ &\doteq B_1 + B_2, \end{aligned} \tag{2.15}$$

$$\begin{aligned} F_3(z,t) &= \int_0^t \int_{L_z} e^{-\delta\eta} C C_{,1} dx_2 d\eta - \frac{1}{2} \int_0^t \int_{L_z} e^{-\delta\eta} C^2 v_{,2} dx_2 d\eta \\ &+ \sigma \int_0^t \int_{L_z} e^{-\delta\eta} C T_{,1} dx_2 d\eta \\ &\doteq C_1 + C_2 + C_3, \end{aligned} \tag{2.16}$$

where $\beta_1, \beta_2, \delta$ are arbitrary positive constants.

Let $z_0$ be a positive constant which satisfies $z > z_0 \geq 0$. Using the divergence theorem, the equations (2.1)-(2.11), we have

$$
\begin{aligned}
F_1(z,t) - F_1(z_0,t) &= -\beta_1 \int_0^t \int_{z_0}^z \int_{L_\xi} e^{-\omega\eta}(vv_{,\alpha})_{,\alpha} dx_2 d\xi d\eta + \beta_1 \int_0^t \int_{L_z} e^{-\delta\eta} g_2 T v dx_2 d\eta \\
&+ \beta_1 \int_0^t \int_{L_z} e^{-\delta\eta} h_2 C v dx_2 d\eta \\
&= \beta_1 \int_0^t \int_{z_0}^z \int_{L_\xi} e^{-\omega\eta} v_{,\alpha} v_{,\alpha} dx_2 d\xi d\eta + \beta_1 \int_0^t \int_{z_0}^z \int_{L_\xi} e^{-\omega\eta} \boldsymbol{g} \cdot \nabla^\perp v T dx_2 d\xi d\eta \\
&+ \beta_1 \int_0^t \int_{z_0}^z \int_{L_\xi} e^{-\omega\eta} \boldsymbol{h} \cdot \nabla^\perp v C dx_2 d\xi d\eta. \qquad (2.17)
\end{aligned}
$$

Similar, we have

$$
\begin{aligned}
F_2(z,t) - F_2(z_0,t) = \beta_2 \int_0^t \int_{z_0}^z \int_{L_\xi} \Big[\frac{1}{2}\delta T^2 + T_{,\alpha} T_{,\alpha}\Big] dx_2 d\xi d\eta \\
+ \frac{1}{2}\beta_2 e^{-\omega t} \int_{z_0}^z \int_{D_\xi} T^2 dx_2 d\xi, \qquad (2.18)
\end{aligned}
$$

and

$$
\begin{aligned}
F_3(z,t) - F_3(z_0,t) = \int_0^t \int_{z_0}^z \int_{L_\xi} \Big[\frac{1}{2}\delta C^2 + C_{,\alpha} C_{,\alpha}\Big] dx_2 d\xi d\eta \\
+ \frac{1}{2} e^{-\omega t} \int_{z_0}^z \int_{D_\xi} C^2 dx_2 d\xi + \sigma \int_0^t \int_{z_0}^z \int_{L_\xi} T_{,\alpha} C_{,\alpha} dx_2 d\xi d\eta, \qquad (2.19)
\end{aligned}
$$

We also define

$$
F(z,t) = F_1(z,t) + F_2(z,t) + F_3(z,t). \qquad (2.20)
$$

Combining (2.17)-(2.20), we have

$$
\begin{aligned}
\frac{\partial}{\partial z} F(z,t) = \int_0^t \int_{L_z} e^{-\omega\eta} \Big[\frac{1}{2}\beta_2 \delta T^2 + \frac{1}{2}\delta C^2 + \beta_1 v_{,\alpha} v_{,\alpha} + \beta_2 T_{,\alpha} T_{,\alpha} + C_{,\alpha} C_{,\alpha}\Big] dx_2 d\eta \\
+ \frac{1}{2} e^{-\omega t} \int_{L_z} \Big[\beta_2 T^2 + C^2\Big] dx_2 + \sigma \int_0^t \int_{L_z} T_{,\alpha} C_{,\alpha} dx_2 d\eta \\
+ \beta_1 \int_0^t \int_{L_z} e^{-\omega\eta} \boldsymbol{g} \cdot \nabla^\perp v T dx_2 d\eta + \beta_1 \int_0^t \int_{L_z} e^{-\omega\eta} \boldsymbol{h} \cdot \nabla^\perp v C dx_2 d\eta. \qquad (2.21)
\end{aligned}
$$

Based on (2.20) and (2.21), we have the following lemma.

**Lemma 3** For the function $F(z,t)$ defined in (2.20), the following differential inequality is satisfied

$$
|F(z,t)| \leq b_1 \Big[h \frac{\partial F}{\partial z}(z,t)\Big] + b_2 \Big[h \frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}}, \qquad (2.22)
$$

where $b_1 = \frac{1}{\pi} \max\Big\{1 + \frac{\sqrt{2\beta_1}}{\beta_2 \delta} + \frac{\sqrt{2\beta_1}}{\delta}, 1 + \frac{\sigma}{\sqrt{\beta_2}}\Big\}, b_2 = \frac{2\sqrt{2}}{\delta\pi\sqrt{\beta_1}}$.

**Proof** Using the Hölder inequality and the Young inequality, we have

$$
\Big|\sigma \int_0^t \int_{L_z} T_{,\alpha} C_{,\alpha} dx_2 d\eta\Big| \leq \frac{1}{2}\sigma^2 \int_0^t \int_{L_z} e^{-\omega\eta} T_{,\alpha} T_{,\alpha} dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_z} e^{-\omega\eta} C_{,\alpha} C_{,\alpha} dx_2 d\eta. \qquad (2.23)
$$

Similarly, we have

$$\Big|\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}\boldsymbol{g}\cdot\nabla^{\perp}vTdx_2d\eta\Big| \le \beta_1\int_0^t\int_{L_z} e^{-\omega\eta}T^2dx_2d\eta + \frac{1}{4}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta, \quad (2.24)$$

$$\Big|\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}\boldsymbol{g}\cdot\nabla^{\perp}vTdx_2d\eta\Big| \le \beta_1\int_0^t\int_{L_z} e^{-\omega\eta}C^2dx_2d\eta + \frac{1}{4}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta. \quad (2.25)$$

Inserting (2.23)-(2.25) into (2.21) and choosing $\delta > \max\{2\beta_1, \frac{2\beta_1}{\beta_2}\}, \beta_2 > \sigma^2$, we have

$$\begin{aligned}\frac{\partial}{\partial z}F(z,t) \le& \int_0^t\int_{L_z} e^{-\omega\eta}\Big[\frac{3}{4}\beta_2\delta T^2 + \frac{3}{4}\delta C^2 + \frac{3}{2}\beta_1 v_{,\alpha}v_{,\alpha} + \frac{3}{2}\beta_2 T_{,\alpha}T_{,\alpha} + \frac{3}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2d\eta\\ &+\frac{1}{2}e^{-\omega t}\int_{L_z}\Big[\beta_2T^2 + C^2\Big]dx_2, \end{aligned}\quad (2.26)$$

and

$$\begin{aligned}\frac{\partial}{\partial z}F(z,t) \ge& \int_0^t\int_{L_z} e^{-\omega\eta}\Big[\frac{1}{4}\beta_2\delta T^2 + \frac{1}{4}\delta C^2 + \frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha} + \frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha} + \frac{1}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2d\eta\\ &+\frac{1}{2}e^{-\omega t}\int_{L_z}\Big[\beta_2T^2 + C^2\Big]dx_2. \end{aligned}\quad (2.27)$$

Using the Hölder inequality, the Young inequality and lemma 1, we have

$$\begin{aligned}|A_1| \le& \beta_1\Big[\int_0^t\int_{L_z} e^{-\omega\eta}v^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}(v_{,1})^2dx_2d\eta\Big]^{\frac{1}{2}}\\ \le& \frac{h}{\pi}\beta_1\Big[\int_0^t\int_{L_z} e^{-\omega\eta}(v_{,2})^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}(v_{,1})^2dx_2d\eta\Big]^{\frac{1}{2}}\\ \le& \frac{h}{2\pi}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta, \end{aligned}\quad (2.28)$$

$$\begin{aligned}|A_2| \le& \beta_1\Big[\int_0^t\int_{L_z} e^{-\omega\eta}v^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}T^2dx_2d\eta\Big]^{\frac{1}{2}}\\ \le& \frac{h\sqrt{2\beta_1}}{\beta_2\delta\pi}\Big[\frac{1}{2}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta + \int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{4}\beta_2\delta T^2dx_2d\eta\Big], \end{aligned}\quad (2.29)$$

$$|A_3| \le \frac{h\sqrt{2\beta_1}}{\delta\pi}\Big[\frac{1}{2}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta + \int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{4}\delta C^2dx_2d\eta\Big], \quad (2.30)$$

Inserting (2.28)-(2.30) into (2.14), we have

$$\begin{aligned}|F_1(z,t)| \le& \frac{h}{\pi}\Big[1 + \frac{\sqrt{2\beta_1}}{\beta_2\delta} + \frac{\sqrt{2\beta_1}}{\delta}\Big]\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha}dx_2d\eta\\ &+\frac{h\sqrt{2\beta_1}}{\beta_2\delta\pi}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{4}\beta_2\delta T^2dx_2d\eta + \frac{h\sqrt{2\beta_1}}{\delta\pi}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{4}\delta C^2dx_2d\eta. \end{aligned}\quad (2.31)$$

Using the Hölder inequality, the Young inequality, lemma 1 and lemma 3, we have

$$\begin{aligned}|B_1| \le& \frac{h}{\pi}\beta_2\Big[\int_0^t\int_{L_z} e^{-\omega\eta}(T_{,2})^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}(T_{,1})^2dx_2d\eta\Big]^{\frac{1}{2}}\\ \le& \frac{h}{2\pi}\beta_2\int_0^t\int_{L_z} e^{-\omega\eta}T_{,\alpha}T_{,\alpha}dx_2d\eta, \end{aligned}\quad (2.32)$$

$$\begin{aligned}|B_2| \le& \beta_2\Big[\int_0^t\int_{L_z} e^{-\omega\eta}(v_{,2})^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}T^4dx_2d\eta\Big]^{\frac{1}{2}}\\ \le& \frac{2h\sqrt{2h}}{\delta\pi\sqrt{\beta_1}}\Big[\frac{1}{2}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta\Big]^{\frac{1}{2}}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}\beta_2T_{,\alpha}T_{,\alpha}dx_2d\eta. \end{aligned}\quad (2.33)$$

Inserting (2.32) and (2.33) into (2.15), we obtain

$$|F_2(z,t)| \leq \frac{h}{\pi}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha}dx_2d\eta$$
$$+\frac{2h\sqrt{2h}}{\delta\pi\sqrt{\beta_1}}\Big[\frac{1}{2}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta\Big]^{\frac{1}{2}}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha}dx_2d\eta. \tag{2.34}$$

Similarly, we have

$$|C_1| \leq \frac{h}{\pi}\Big[\int_0^t\int_{L_z} e^{-\omega\eta}(C_{,2})^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}(C_{,1})^2dx_2d\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{h}{2\pi}\int_0^t\int_{L_z} e^{-\omega\eta}C_{,\alpha}C_{,\alpha}dx_2d\eta, \tag{2.35}$$
$$|C_2| \leq \Big[\int_0^t\int_{L_z} e^{-\omega\eta}(v_{,2})^2dx_2d\eta\int_0^t\int_{L_z} e^{-\omega\eta}C^4dx_2d\eta\Big]^{\frac{1}{2}}$$
$$\leq \frac{2h\sqrt{2h}}{\delta\pi\sqrt{\beta_1}}\Big[\frac{1}{2}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta\Big]^{\frac{1}{2}}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}C_{,\alpha}C_{,\alpha}dx_2d\eta, \tag{2.36}$$
$$|C_3| \leq \frac{h}{\pi}\frac{\sigma}{\sqrt{\beta_2}}\int_0^t\int_{L_z} e^{-\omega\eta}C_{,\alpha}C_{,\alpha}dx_2d\eta+\frac{h}{\pi}\frac{\sigma}{\sqrt{\beta_2}}\int_0^t\int_{L_z} e^{-\omega\eta}(T_{,1})^2dx_2d\eta. \tag{2.37}$$

Inserting (2.35)-(2.37) into (2.16), we obtain

$$|F_3(z,t)| \leq \frac{h}{\pi}\Big[1+\frac{\sigma}{\sqrt{\beta_2}}\Big]\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}C_{,\alpha}C_{,\alpha}dx_2d\eta$$
$$+\frac{2h\sqrt{2h}}{\delta\pi\sqrt{\beta_1}}\Big[\frac{1}{2}\beta_1\int_0^t\int_{L_z} e^{-\omega\eta}v_{,\alpha}v_{,\alpha}dx_2d\eta\Big]^{\frac{1}{2}}\int_0^t\int_{L_z} e^{-\omega\eta}\frac{1}{2}C_{,\alpha}C_{,\alpha}dx_2d\eta$$
$$+\frac{h}{\pi\sqrt{\beta_2}}\sigma\int_0^t\int_{L_z} e^{-\delta\eta}(T_{,1})^2dx_2d\eta. \tag{2.38}$$

Combining (2.20), (2.27), (2.31), (2.34) and (2.38), we obtain lemma 3.

# 3 Main result

Based on lemma 4, we can get the following theorem.

**Theorem 3.1.** Let $(v,T,C)$ be a solution of the equations (2.1)-(2.11) in $R$, where $h'(z) \leq 0,\ h \geq h_0 > 0$, then for fixed $t$ either

$$\lim_{r\to\infty}\Big\{\Big(\int_{z_0}^z\frac{1}{h(\zeta)}d\zeta\Big)^{-3}$$
$$\cdot\Big[\int_0^t\int_{z_0}^z\int_{L_\xi} e^{-\omega\eta}\Big(\frac{3}{4}\beta_2\delta T^2+\frac{3}{4}\delta C^2+\frac{3}{2}\beta_1 v_{,\alpha}v_{,\alpha}+\frac{3}{2}\beta_2 T_{,\alpha}T_{,\alpha}+\frac{3}{2}C_{,\alpha}C_{,\alpha}\Big)dx_2d\xi d\eta$$
$$+\frac{1}{2}e^{-\omega t}\int_{z_0}^z\int_{L_\xi}\Big(\beta_2T^2+C^2\Big)dx_2d\xi\Big]\Big\} \geq \gamma_1 \tag{3.1}$$

holds or

$$\int_0^t\int_z^\infty\int_{L_\xi} e^{-\omega\eta}\Big[\frac{1}{4}\beta_2\delta T^2+\frac{1}{4}\delta C^2+\frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha}+\frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha}+\frac{1}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+\frac{1}{2}e^{-\omega t}\int_z^\infty\int_{L_\xi}\Big[\beta_2 T^2+C^2\Big]dx_2d\xi$$
$$\le b_6Q^2(a,t)e^{-\frac{2}{b_5}\int_a^z\frac{1}{h(\zeta)}d\zeta}+b_5Q(a,t)e^{-\frac{1}{b_5}\int_a^z\frac{1}{h(\zeta)}d\zeta} \tag{3.2}$$

holds, where $\gamma_1, b_5, b_6$ are positive constants and $Q(a,t)$ will be defined in (3.16).

**Proof** We consider (2.22) for two cases.

Case I. $\exists\ z_0\ge a$ such that $F(z_0,t)\ge 0$.

From (2.27) we know that $\frac{\partial}{\partial z}F(z,t)\ge 0$. So, we have $F(z,t)\ge F(z_0,t)\ge 0,\ z\ge z_0$. Therefore, (2.22) can be written as

$$F(z,t)\le b_1\Big[h\frac{\partial F}{\partial z}(z,t)\Big]+b_2\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}},\ z\ge z_0. \tag{3.3}$$

Using the Young inequality, we have

$$\Big[h\frac{\partial F}{\partial z}(z,t)\Big]=\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{4}\cdot\frac{2}{3}}\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}\cdot\frac{1}{3}}$$
$$\le\frac{2}{3}\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{4}}+\frac{1}{3}\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}}. \tag{3.4}$$

Inserting (3.4) into (3.3), we have

$$F(z,t)\le b_3\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{4}}+b_4\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}},\ z\ge z_0,$$

where $b_3=\frac{2}{3}b_1,\ b_4=\frac{1}{3}b_1+b_2$. Therefore we have

$$F(z,t)+\frac{b_3^2}{4b_4}\le b_4\Big[\Big(h\frac{\partial F}{\partial z}(z,t)\Big)^{\frac{3}{4}}+\frac{b_3}{2b_4}\Big]^2,\ z\ge z_0. \tag{3.5}$$

From (3.5) it follows that

$$h\frac{\partial F}{\partial z}(z,t)\ge\Big[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{\frac{4}{3}},\ z\ge z_0.$$

So, we have

$$\Big\{2b_4\frac{1}{\Big[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{\frac{1}{3}}}+b_3\frac{1}{\Big[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{\frac{4}{3}}}\Big\}$$
$$\cdot d\Big[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]\ge\frac{1}{h},\ z\ge z_0. \tag{3.6}$$

Integrating (3.6) from $z_0$ to $z$, we have

$$3b_4\Big\{\Big[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{\frac{2}{3}}-\Big[\sqrt{\frac{1}{b_4}F(z_0,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{\frac{2}{3}}\Big\}$$
$$-3b_3\Big\{\Big[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{-\frac{1}{3}}-\Big[\sqrt{\frac{1}{b_4}F(z_0,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\Big]^{-\frac{1}{3}}\Big\}$$
$$\ge\int_{z_0}^z\frac{1}{h(\zeta)}d\zeta,\ z\ge z_0. \tag{3.7}$$

We drop the second and third terms at the left end of (3.7). In the first term of (3.7) we use the following inequality

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b},\ a, b \geq 0,$$

to have

$$3b_4\Big[\sqrt{\frac{1}{b_4}F(z,t)}\Big]^{\frac{2}{3}} \geq \int_{z_0}^{z} \frac{1}{h(\zeta)}d\zeta - 3b_3\Big[\sqrt{\frac{1}{b_4}F(z_0,t) + \frac{b_3^2}{4b_4^2}} - \frac{b_3}{2b_4}\Big]^{-\frac{1}{3}}.$$

Therefore we have

$$F(z,t) \geq \Big\{\frac{1}{3\sqrt[3]{b_4^2}}\int_{z_0}^{z} \frac{1}{h(\zeta)}d\zeta - \frac{b_3}{\sqrt[3]{b_4^2}}\Big[\sqrt{\frac{1}{b_4}F(z_0,t) + \frac{b_3^2}{4b_4^2}} - \frac{b_3}{2b_4}\Big]^{-\frac{1}{3}}\Big\}^3. \tag{3.8}$$

On the other hand, we integrate (2.13) from $z_0$ to $z$ to obtain

$$\begin{aligned} F(z,t) - F(z_0,t) \leq & \int_0^t \int_{z_0}^{z} \int_{L_\xi} e^{-\omega\eta}\Big[\frac{3}{4}\beta_2\delta T^2 + \frac{3}{4}\delta C^2 + \frac{3}{2}\beta_1 v_{,\alpha}v_{,\alpha} + \frac{3}{2}\beta_2 T_{,\alpha}T_{,\alpha} + \frac{3}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2 d\xi d\eta \\ & + \frac{1}{2}e^{-\omega t}\int_{z_0}^{z}\int_{L_\xi}\Big[\beta_2 T^2 + C^2\Big]dx_2 d\xi. \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9), we can obtain (3.1).

Case II. $\forall\ z \geq a$ such that $F(z,t) < 0$, then we have from (2.22)

$$-F(z,t) \leq b_1\Big[h\frac{\partial F}{\partial z}(z,t)\Big] + b_2\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}},\ z \geq a. \tag{3.10}$$

Using the Young inequality again, we have

$$\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{3}{2}} = \Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{\frac{1}{2}}\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^{2\cdot\frac{1}{2}} \leq \frac{1}{2}\Big[h\frac{\partial F}{\partial z}(z,t)\Big] + \frac{1}{2}\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^2. \tag{3.11}$$

Inserting (3.11) into (3.10), we have

$$-F(z,t) \leq b_5\Big[h\frac{\partial F}{\partial z}(z,t)\Big] + b_6\Big[h\frac{\partial F}{\partial z}(z,t)\Big]^2,\ z \geq a, \tag{3.12}$$

where $b_5 = b_1 + \frac{1}{2}b_2,\ b_6 = \frac{1}{2}b_2$. It follows from (3.12) that

$$h\frac{\partial F}{\partial z}(z,t) \geq \sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}.$$

So, we have

$$\Big\{-2b_6 - b_5\frac{1}{\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}}\Big\}d\Big\{\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\Big\} \geq \int_a^z \frac{1}{h(\zeta)}d\zeta,\ z \geq a. \tag{3.13}$$

Integrating (3.13) from 0 to $z$, we have

$$\begin{aligned} & 2b_6\Big[\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}}\Big] \\ & + b_5 ln\Big[\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\Big] - b_5 ln\Big[\sqrt{-\frac{F(a,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\Big] \\ & \leq -\int_a^z \frac{1}{h(\zeta)}d\zeta. \end{aligned} \tag{3.14}$$

Dropping the first term on the left of (3.14), we have

$$b_5 ln\Big[\sqrt{-\frac{F(z,t)}{b_6}+\frac{b_5^2}{4b_6^2}}-\frac{b_5}{2b_6}\Big]\leq -\int_a^z\frac{1}{h(\zeta)}d\zeta+2b_6\sqrt{-\frac{F(a,t)}{b_6}+\frac{b_5^2}{4b_6^2}}$$
$$+b_5 ln\Big[\sqrt{-\frac{F(a,t)}{b_6}+\frac{b_5^2}{4b_6^2}}-\frac{b_5}{2b_6}\Big].$$

Therefore, we obtain

$$\sqrt{-\frac{F(z,t)}{b_6}+\frac{b_5^2}{4b_6^2}}\leq Q(a,t)e^{-\frac{1}{b_5}\int_a^z\frac{1}{h(\zeta)}d\zeta}+\frac{b_5}{2b_6}, \tag{3.15}$$

where

$$Q(a,t)=\Big[\sqrt{-\frac{F(a,t)}{b_6}+\frac{b_5^2}{4b_6^2}}-\frac{b_5}{2b_6}\Big]e^{\frac{2b_6}{b_5}\sqrt{-\frac{F(a,t)}{b_6}+\frac{b_5^2}{4b_6^2}}} \tag{3.16}$$

Squaring (3.15), we have

$$-F(z,t)\leq b_6Q^2(a,t)e^{-\frac{2}{b_5}\int_a^z\frac{1}{h(\zeta)}d\zeta}+b_5Q(a,t)e^{-\frac{1}{b_5}\int_a^z\frac{1}{h(\zeta)}d\zeta}, \tag{3.17}$$

So, we have $\lim_{z\to\infty}\Big[-F(z,t)\Big]=0$. Now, we integrating (2.27) from $z$ to $\infty$ to obtain

$$-F(z,t)\geq\int_0^t\int_z^\infty\int_{L_\xi}e^{-\omega\eta}\Big[\frac{1}{4}\beta_2\delta T^2+\frac{1}{4}\delta C^2+\frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha}+\frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha}+\frac{1}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2d\xi d\eta$$
$$+\frac{1}{2}e^{-\omega t}\int_z^\infty\int_{L_\xi}\Big[\beta_2T^2+C^2\Big]dx_2d\xi. \tag{3.18}$$

Combining (3.17) and (3.18), we can obtain (3.2).

**Remark 3.2.** If $h(z)=h$ is a positive constant, then

$$\int_{z_0}^z\frac{1}{h(\zeta)}d\zeta=\frac{1}{h}(z-z_0),\ \int_a^z\frac{1}{h(\zeta)}d\zeta=\frac{1}{h}(z-a).$$

In this case, theorem 3.1 shows that the solutions of equations (2.1)-(2.11) grow polynomially or decay exponentially as $z\to\infty$, where the growth rate is at least as fast as $z^3$.

**Remark 3.3.** If $h(z)$ satisfies

$$h(z)\leq k_1z^{\tau_1}, 0<\tau_1<1,\ k_1>0, \tag{3.19}$$

then

$$\int_{z_0}^z\frac{1}{h}d\zeta=\frac{1}{k_1}\int_{z_0}^z\zeta^{-\tau_1}d\zeta=\frac{1}{k_1(1-\tau_1)}[z^{1-\tau_1}-z_0^{1-\tau_1}],$$
$$\int_{z_0}^z\frac{1}{h}d\zeta=\frac{1}{k_1(1-\tau_1)}[z^{1-\tau_1}-a^{1-\tau_1}].$$

In this case, theorem 3.1 shows that the solutions of equations (2.1)-(2.11) grow polynomially or decay exponentially as $z\to\infty$, where the growth rate is at least as fast as $z^{3(1-\tau_1)}$. Obviously, the growth rate and decay rate are slightly slower than those in Remark 3.2.

**Remark 3.4.** If $\tau_1 = 1$ in (3.19), then

$$\int_{z_0}^{z} \frac{1}{h} d\zeta = \frac{1}{k_1} \int_{z_0}^{z} \frac{1}{\zeta} d\zeta = \frac{1}{k_1} \ln\left(\frac{z}{z_0}\right),$$
$$\int_{z_0}^{z} \frac{1}{h} d\zeta = \frac{1}{k_1} \ln\left(\frac{z}{a}\right)$$

In this case, theorem 3.1 shows that the solutions of equations (2.1)-(2.11) grow logarithmic or decay polynomially as $z \to \infty$, where the growth rate is at least as fast as $(\ln z)^3$ and the decay rate is at least as fast as $z^{-\frac{1}{k_1 b_5}}$.

**Remark 3.5.** If $h(z)$ satisfies

$$h(z) \leq k_2 z (\ln z)^{\tau_2}, 0 < \tau_2 < 1,\ k_2 > 0, \tag{3.20}$$

then

$$\int_{z_0}^{z} \frac{1}{h} d\zeta = \frac{1}{k_2} \int_{z_0}^{z} \frac{1}{\zeta (\ln \zeta)^{\tau_2}} ds = \frac{1}{(1-\tau_2) k_2} \Big[ (\ln z)^{1-\tau_2} - (\ln z_0)^{1-\tau_2} \Big],$$
$$\int_{a}^{z} \frac{1}{h} d\zeta = \frac{1}{(1-\tau_2) k_2} \Big[ (\ln z)^{1-\tau_2} - (\ln a)^{1-\tau_2} \Big].$$

In this case, theorem 3.1 shows that the solutions of equations (2.1)-(2.11) grow logarithmic or decay exponentially as $z \to \infty$, where the growth rate is at least as fast as $(\ln z)^{3(1-\tau_2)}$ and the decay rate is at least as fast as $e^{-(\ln z)^{1-\tau_2}}$.

**Remark 3.6.** If $\tau_2 = 1$ in (3.20), then

$$\int_{z_0}^{z} \frac{1}{h(\zeta)} d\zeta = \frac{1}{k_2} \int_{z_0}^{z} \frac{1}{\zeta \ln \zeta} ds = \frac{1}{k_2} \ln\left(\frac{\ln z}{\ln z_0}\right), z_0 > \max\{a, e\}.,$$
$$e^{-\int_a^z \frac{1}{h(\zeta)} d\zeta} = \left(\frac{\ln z}{\ln a}\right)^{-\frac{1}{k_2}}.$$

In this case, theorem 3.1 shows that the solutions of equations (2.1)-(2.11) grow logarithmic or decay logarithmic as $z \to \infty$. Obviously, the growth rate and decay rates are slower than those in Remark 3.2 to 3.5.

**Remark 3.7.** To make decay estimate explicit, we have to derive the upper bounds for $-F(a,t)$. We will derive the upper bound in the next section.

# 4 The upper bounds for the total energy

In the case of decay, we choose $z = a$ in (3.18) to obtain

$$\begin{aligned} -F(a,t) \geq & \int_0^t \int_R e^{-\omega\eta} \Big[ \frac{1}{4}\beta_2 \delta T^2 + \frac{1}{4}\delta C^2 + \frac{1}{2}\beta_1 v_{,\alpha} v_{,\alpha} + \frac{1}{2}\beta_2 T_{,\alpha} T_{,\alpha} + \frac{1}{2} C_{,\alpha} C_{,\alpha} \Big] dx_2 d\xi d\eta \\ & + \frac{1}{2} e^{-\omega t} \int_R \Big[ \beta_2 T^2 + C^2 \Big] dx_2 d\xi. \end{aligned} \tag{4.1}$$

Also, in (2.14)-(2.16) and (2.21) we choose $z = a$ to have

$$-F_1(a,t) = -\beta_1 \int_0^t \int_{L_a} e^{-\delta\eta} v v_{,1} dx_2 d\eta - \beta_1 \int_0^t \int_{L_a} e^{-\delta\eta} g_2 T v dx_2 d\eta$$
$$- \beta_1 \int_0^t \int_{L_a} e^{-\delta\eta} h_2 C v dx_2 d\eta, \tag{4.2}$$
$$-F_2(a,t) = -\beta_2 \int_0^t \int_{L_a} e^{-\delta\eta} T T_{,1} dx_2 d\eta + \frac{1}{2}\beta_2 \int_0^t \int_{L_a} e^{-\delta\eta} T^2 v_{,2} dx_2 d\eta \tag{4.3}$$
$$-F_3(a,t) = - \int_0^t \int_{L_z} e^{-\delta\eta} C C_{,1} dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_z} e^{-\delta\eta} C^2 v_{,2} dx_2 d\eta$$
$$- \sigma \int_0^t \int_{L_z} e^{-\delta\eta} C T_{,1} dx_2 d\eta, \tag{4.4}$$

and

$$-F(a,t) = -F_1(a,t) - F_2(a,t) - F_3(a,t). \tag{4.5}$$

Now we introduce three auxiliary functions

$$S(x_1,x_2,t) = \Big\{ \widetilde{F}_1(\widetilde{\tau},t) + (x_1 - a)\Big[\widetilde{F}_2(\widetilde{\tau},t) + \sigma_1 \widetilde{F}_1(\widetilde{\tau},t) - \frac{\partial \widetilde{F}_1}{\partial \widetilde{\tau}}(\widetilde{\tau},t)\frac{\partial \widetilde{\tau}}{\partial x_1}\Big]\Big\} e^{-\sigma_1(x_1-a)},$$
$$\Gamma(x_1,x_2,t) = H(x_1,x_2,t) e^{-\sigma_2(x_1-a)},$$
$$\mathcal{C}(x_1,x_2,t) = \widetilde{H}(x_1,x_2,t) e^{-\sigma_3(x_1-a)},$$

where $\sigma_1, \sigma_2, \sigma_3$ are positive arbitrary constants and

$$\widetilde{\tau} = \frac{x_2 h(a)}{h(x_1)}.$$

Obviously, $S, \Gamma, \mathcal{C}$ have the same boundary conditions to $v, T, C$, respectively. Therefore, (4.2)-(4.4) can be written as

$$-F_1(a,t) = -\beta_1 \int_0^t \int_{L_a} e^{-\delta\eta} S v_{,1} dx_2 d\eta - \beta_1 \int_0^t \int_{L_a} e^{-\delta\eta} g_2 \Gamma S dx_2 d\eta$$
$$- \beta_1 \int_0^t \int_{L_a} e^{-\delta\eta} h_2 \mathcal{C} S dx_2 d\eta, \tag{4.6}$$
$$-F_2(a,t) = -\beta_2 \int_0^t \int_{L_a} e^{-\delta\eta} \Gamma T_{,1} dx_2 d\eta + \frac{1}{2}\beta_2 \int_0^t \int_{L_a} e^{-\delta\eta} \Gamma^2 v_{,2} dx_2 d\eta \tag{4.7}$$
$$-F_3(a,t) = - \int_0^t \int_{L_z} e^{-\delta\eta} \mathcal{C} C_{,1} dx_2 d\eta + \frac{1}{2} \int_0^t \int_{L_z} e^{-\delta\eta} \mathcal{C}^2 v_{,2} dx_2 d\eta$$
$$- \sigma \int_0^t \int_{L_z} e^{-\delta\eta} \mathcal{C} T_{,1} dx_2 d\eta. \tag{4.8}$$

Using the divergence theorem, equation (2.1)-(2.11), the Hölder inequality and the Young inequality, we can get

$$
\begin{aligned}
-F_1(a,t) = & -\beta_1\int_0^t\int_R e^{-\delta\eta}(Sv_{,\alpha})_{,\alpha}dx_2d\xi d\eta-\beta_1\int_0^t\int_{L_a}e^{-\delta\eta}g_2\Gamma Sdx_2d\eta\\
& -\beta_1\int_0^t\int_{L_a}e^{-\delta\eta}h_2\mathcal{C}Sdx_2d\eta\\
= & \beta_1\int_0^t\int_R e^{-\delta\eta}S_{,\alpha}v_{,\alpha}dx_2d\xi d\eta+\beta_1\int_0^t\int_R e^{-\delta\eta}\boldsymbol{g}\cdot\nabla^{\perp}STdx_2d\xi d\eta\\
& +\beta_1\int_0^t\int_R e^{-\delta\eta}\boldsymbol{h}\cdot\nabla^{\perp}SCdx_2d\xi d\eta\\
\le & \frac{1}{2}\varepsilon_1\beta_1\int_0^t\int_R e^{-\delta\eta}v_{,\alpha}v_{,\alpha}dx_2d\xi d\eta+\frac{1}{2\varepsilon_1}\beta_1\int_0^t\int_R e^{-\delta\eta}S_{,\alpha}S_{,\alpha}dx_2d\xi d\eta\\
& +\frac{1}{2}\varepsilon_2\beta_1\int_0^t\int_R e^{-\delta\eta}T^2dx_2d\xi d\eta+\frac{1}{2\varepsilon_2}\beta_1\int_0^t\int_R e^{-\delta\eta}S_{,\alpha}S_{,\alpha}dx_2d\xi d\eta\\
& +\frac{1}{2}\varepsilon_3\beta_1\int_0^t\int_R e^{-\delta\eta}C^2dx_2d\xi d\eta+\frac{1}{2\varepsilon_3}\beta_1\int_0^t\int_R e^{-\delta\eta}S_{,\alpha}S_{,\alpha}dx_2d\xi d\eta,
\end{aligned}
\tag{4.9}
$$

where $\varepsilon_1,\varepsilon_2,\varepsilon_3$ are positive constants to be determined later. For $F_2(z,t)$, we have

$$
\begin{aligned}
-F_2(a,t) = & -\beta_2\int_0^t\int_{L_a}e^{-\delta\eta}\Gamma T_{,1}dx_2d\eta+\frac{1}{2}\beta_2\int_0^t\int_{L_a}e^{-\delta\eta}\Gamma^2v_{,2}dx_2d\eta\\
= & \beta_2\int_0^t\int_R e^{-\delta\eta}(\Gamma T_{,\alpha})_{,\alpha}dx_2d\xi d\eta-\beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma\nabla\Gamma\cdot\nabla^{\perp}v_{,2}dx_2d\xi d\eta\\
= & \beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma_{,\alpha}T_{,\alpha}dx_2d\xi d\eta+\frac{1}{2}\beta_2e^{-\delta t}\int_R\Gamma Tdx_2d\xi\\
& -\beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma_\eta Tdx_2d\xi d\eta+\beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma\nabla^{\perp}v\cdot\nabla Tdx_2d\xi d\eta\\
& -\beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma\nabla\Gamma\cdot\nabla^{\perp}vdx_2d\xi d\eta\\
\le & \frac{1}{2}\beta_2\varepsilon_4\int_0^t\int_R e^{-\delta\eta}T_{,\alpha}T_{,\alpha}dx_2d\xi d\eta+\frac{1}{2\varepsilon_4}\beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma_{,\alpha}\Gamma_{,\alpha}dx_2d\xi d\eta\\
& +\frac{1}{4}\varepsilon_5\beta_2e^{-\delta t}\int_R T^2dx_2d\xi+\frac{1}{4\varepsilon_5}\beta_2e^{-\delta t}\int_R\Gamma^2dx_2d\xi\\
& +\frac{1}{2}\varepsilon_6\beta_2\int_0^t\int_R e^{-\delta\eta}T^2dx_2d\xi d\eta+\frac{1}{2\varepsilon_6}\beta_2\int_0^t\int_R e^{-\delta\eta}\Gamma_\eta^2dx_2d\xi d\eta\\
& +\frac{1}{2}\varepsilon_7\beta_2\Gamma_M\int_0^t\int_R e^{-\delta\eta}T_{,\alpha}T_{,\alpha}dx_2d\xi d\eta+\frac{1}{2\varepsilon_7}\beta_2\Gamma_M\int_0^t\int_R e^{-\delta\eta}v_{,\alpha}v_{,\alpha}dx_2d\xi d\eta\\
& +\frac{1}{2}\varepsilon_8\beta_2\int_0^t\int_R e^{-\delta\eta}v_{,\alpha}v_{,\alpha}dx_2d\xi d\eta+\beta_2\frac{1}{2\varepsilon_8}\int_0^t\int_R e^{-\delta\eta}|\Gamma\nabla\Gamma|^2dx_2d\xi d\eta,
\end{aligned}
\tag{4.10}
$$

where $\Gamma_M=\max_{R\times[0,\tau]}\Gamma$, $\varepsilon_i(i=4,5,6,7,8)$ are positive constants to be determined later. Similar to (4.10) we have

$$
\begin{aligned}
-F_3(a,t) \leq & \frac{1}{2}\varepsilon_9 \int_0^t \int_R e^{-\delta\eta} C_{,\alpha} C_{,\alpha} dx_2 d\xi d\eta + \frac{1}{2\varepsilon_9} \int_0^t \int_R e^{-\delta\eta} \mathcal{C}_{,\alpha} \mathcal{C}_{,\alpha} dx_2 d\xi d\eta \\
& + \frac{1}{4}\varepsilon_{10} e^{-\delta t} \int_R C^2 dx_2 d\xi + \frac{1}{4\varepsilon_{10}} \beta_2 e^{-\delta t} \int_R \mathcal{C}^2 dx_2 d\xi \\
& + \frac{1}{2}\varepsilon_{11} \int_0^t \int_R e^{-\delta\eta} C^2 dx_2 d\xi d\eta + \frac{1}{2\varepsilon_{11}} \int_0^t \int_R e^{-\delta\eta} \Gamma_\eta^2 dx_2 d\xi d\eta \\
& + \frac{1}{2}\varepsilon_{12} \mathcal{C}_M \int_0^t \int_R e^{-\delta\eta} C_{,\alpha} C_{,\alpha} dx_2 d\xi d\eta + \frac{1}{2\varepsilon_{12}} \mathcal{C}_M \int_0^t \int_R e^{-\delta\eta} v_{,\alpha} v_{,\alpha} dx_2 d\xi d\eta \\
& + \frac{1}{2}\varepsilon_{13} \int_0^t \int_R e^{-\delta\eta} v_{,\alpha} v_{,\alpha} dx_2 d\xi d\eta + \frac{1}{2\varepsilon_{13}} \int_0^t \int_R e^{-\delta\eta} |\mathcal{C} \nabla \mathcal{C}|^2 dx_2 d\xi d\eta \\
& + \frac{1}{2}\varepsilon_{14} \sigma \int_0^t \int_R e^{-\delta\eta} T_{,\alpha} T_{,\alpha} dx_2 d\xi d\eta + \frac{1}{2\varepsilon_{14}} \sigma \int_0^t \int_R e^{-\delta\eta} \mathcal{C}_{,\alpha} \mathcal{C}_{,\alpha} dx_2 d\xi d\eta,
\end{aligned} \tag{4.11}
$$

where $\mathcal{C}_M = \max_{R\times[0,\tau]} \mathcal{C}$, $\varepsilon_i (i = 9, 10, 11, 12, 13, 14)$ are positive constants to be determined later.

Next, we choose $\varepsilon_i (i = 1, 2, 3, ..., 14)$ small enough and $\delta$ large enough such that

$$
\begin{aligned}
& \frac{1}{2}\varepsilon_1 + \frac{1}{2\varepsilon_7}\beta_2 \Gamma_M + \frac{1}{2}\varepsilon_8 \beta_2 + \frac{1}{2\varepsilon_{12}} + \frac{1}{2}\varepsilon_{13} \leq \frac{1}{2}\beta_1, \\
& \frac{1}{2}\varepsilon_2 \beta_1 + \frac{1}{2}\varepsilon_6 \beta_2 \leq \frac{1}{8}\beta_2 \delta, \ \frac{1}{2}\varepsilon_4 \beta_4 + \frac{1}{2}\varepsilon_{11} \mathcal{C}_M \leq \frac{1}{8}\delta, \\
& \frac{1}{2}\varepsilon_2 \beta_1 + \frac{1}{2}\varepsilon_7 \beta_2 \Gamma_M + \frac{1}{2}\varepsilon_{14} \sigma \leq \frac{1}{4}\beta_2, \\
& \varepsilon_5, \varepsilon_{10} \leq 1, \frac{1}{2}\varepsilon_9 + \frac{1}{2}\varepsilon_{12} \leq \frac{1}{4}.
\end{aligned}
$$

Inserting (4.9)-(4.11) into (4.5) and recalling (4.1), we have

$$
-F(a,t) \leq \frac{1}{2}\Big[ -F(a,t) \Big] + \frac{1}{2} r(t), \tag{4.12}
$$

where

$$
\begin{aligned}
r(t) = & \Big[ \frac{1}{4\varepsilon_1}\beta_1 + \frac{1}{4\varepsilon_2}\beta_1 + \frac{1}{4\varepsilon_3}\beta_1 \Big] \int_0^t \int_R e^{-\delta\eta} S_{,\alpha} S_{,\alpha} dx_2 d\xi d\eta \\
& + \frac{1}{8\varepsilon_4}\beta_2 \int_0^t \int_R e^{-\delta\eta} \Gamma_{,\alpha} \Gamma_{,\alpha} dx_2 d\xi d\eta + \frac{1}{8\varepsilon_5}\beta_2 e^{-\delta t} \int_R \Gamma^2 dx_2 d\xi \\
& + \frac{1}{4\varepsilon_6}\beta_2 \int_0^t \int_R e^{-\delta\eta} \Gamma_\eta^2 dx_2 d\xi d\eta + \beta_2 \frac{1}{4\varepsilon_8} \int_0^t \int_R e^{-\delta\eta} |\Gamma \nabla \Gamma|^2 dx_2 d\xi d\eta \\
& + \frac{1}{4\varepsilon_9} \int_0^t \int_R e^{-\delta\eta} \mathcal{C}_{,\alpha} \mathcal{C}_{,\alpha} dx_2 d\xi d\eta + \frac{1}{8\varepsilon_{10}} \beta_2 e^{-\delta t} \int_R \mathcal{C}^2 dx_2 d\xi \\
& + \frac{1}{4\varepsilon_{11}} \int_0^t \int_R e^{-\delta\eta} \Gamma_\eta^2 dx_2 d\xi d\eta + \frac{1}{4\varepsilon_{13}} \int_0^t \int_R e^{-\delta\eta} |\mathcal{C} \nabla \mathcal{C}|^2 dx_2 d\xi d\eta \\
& + \frac{1}{4\varepsilon_{14}} \sigma \int_0^t \int_R e^{-\delta\eta} \mathcal{C}_{,\alpha} \mathcal{C}_{,\alpha} dx_2 d\xi d\eta.
\end{aligned}
$$

From (4.12) we obtain

$$
-F(a,t) \leq r(t). \tag{4.13}
$$

Combining (4.1) and (4.13), we obtain the following theorem.

**Theorem 4.1** Assume that $(v,T)$ are solutions of (2.1)-(2.11). If $F(z,t)<0$ for any $z\geq a$, then

$$\begin{aligned}\int_0^t\int_R e^{-\omega\eta}\Big[\frac{1}{4}\beta_2\delta T^2+\frac{1}{4}\delta C^2+\frac{1}{2}\beta_1 v_{,\alpha}v_{,\alpha}+\frac{1}{2}\beta_2 T_{,\alpha}T_{,\alpha}+\frac{1}{2}C_{,\alpha}C_{,\alpha}\Big]dx_2 d\xi d\eta\\ +\frac{1}{2}e^{-\omega t}\int_R\Big[\beta_2 T^2+C^2\Big]dx_2 d\xi\\ \leq r(t).\end{aligned}\tag{4.14}$$

# 5 Conclusion

In this paper, the Darcy equations (1.1)-(1.3) are reconsidered in a semi-infinite cylinder and the Phragmén-Lindelöf alternative result is obtained. However, there are still some deeper problems to be studied in this paper. We note that Quintanilla [14] considered the spatial selectivity of solutions of several kinds of partial differential equations with radius defined in the outer region of the sphere, in which the so-called outer region of the sphere is

$$\Omega=\{(x_1,x_2,x_3)|x_1^2+x_2^2+x_3^2\geq R_0^2, R_0>0\}.$$

Li et al. [18, 22] studied the selectivity of the wave equation in the region $\Omega$, and obtained the rapid attenuation rate and growth rate. However, this type of research has not received sufficient attention. Therefore, it will be a meaningful topic to study the spatial properties of solutions of Dacry equations on $\Omega$.

# Competing interests

The authors declare that they have no competing interests.

# Availability of data and materials

This paper focuses on theoretical analysis, not involving experiments and data.

# Author's contributions

The authors have equal contributions to each part of this paper. All authors read and approved the final manuscript.

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