INTEGRAL REPRESENTATIONS AND INEQUALITIES OF EXTENDED CENTRAL BINOMIAL COEFFICIENTS

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Dedicated to people facing and battling COVID-19

Abstract. In the paper, the author presents three integral representations of extended central binomial coefficient, proves decreasing and increasing properties of two power-exponential functions involving extended (central) binomial coefficients, derives several double inequalities for bounding extended (central) binomial coefficient, and compares with known results.

Contents

1. Motivations

In this paper, we use the following natation:

$$\mathbb{Z}=\{0,\pm1,\pm2,\dots\},\qquad \mathbb{N}=\{1,2,\dots\},$$
$$\mathbb{N}_0=\{0,1,2,\dots\},\qquad \mathbb{N}_-=\{-1,-2,\dots\}.$$

The classical Euler's gamma function $\Gamma(z)$ can be defined [21, Chapter 3] by

$$\Gamma(z)=\lim_{n\to+\infty}\frac{n!n^z}{\prod_{k=0}^{n}(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}. \tag{1.1}$$

It is known [21, Chapter 3] that

(1) the gamma function $\Gamma(x)$ is positive on the intervals $(0,+\infty)$ and $(-2k,1-2k)$ for $k\in\mathbb{N}$, see [21, p. 44, Figure 3.1];
(2) the gamma function $\Gamma(x)$ is negative on the intervals $(1-2k,2-2k)$ for $k\in\mathbb{N}$, see [21, p. 44, Figure 3.1];

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(3) the gamma function $\Gamma(z)$ is single-valued and analytic over the punctured complex plane $\mathbb{C}\setminus\{1-k, k\in\mathbb{N}\}$, see [1, p. 255, 6.1.3];
(4) the gamma function $\Gamma(z)$ has simple poles in the left half-plane at the points $1-k$ and the residue at $1-k$ is $\frac{(-1)^{k-1}}{(k-1)!}$ for $k\in\mathbb{N}$, see [21, p. 44];
(5) the reciprocal $\frac{1}{\Gamma(z)}$ is an entire function possessing simple zeros at the points $1-k$ for $k\in\mathbb{N}$, see [1, p. 255, 6.1.3].

The extended binomial coefficient $\binom{z}{w}$ for $z,w\in\mathbb{C}$ is defined by

$$\binom{z}{w}=\begin{cases}\dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z\notin\mathbb{N}_-, \quad w, z-w\notin\mathbb{N}_-;\\ 0, & z\notin\mathbb{N}_-, \quad w\in\mathbb{N}_- \text{ or } z-w\in\mathbb{N}_-;\\ \dfrac{\langle z\rangle_w}{w!}, & z\in\mathbb{N}_-, \quad w\in\mathbb{N}_0;\\ \dfrac{\langle z\rangle_{z-w}}{(z-w)!}, & z,w\in\mathbb{N}_-, \quad z-w\in\mathbb{N}_0;\\ 0, & z,w\in\mathbb{N}_-, \quad z-w\in\mathbb{N}_-;\\ \infty, & z\in\mathbb{N}_-, \quad w\notin\mathbb{Z},\end{cases} \tag{1.2}$$

where

$$\langle\alpha\rangle_n=\prod_{k=0}^{n-1}(\alpha-k)=\begin{cases}\alpha(\alpha-1)\cdots(\alpha-n+1), & n\in\mathbb{N}\\ 1, & n=0\end{cases}$$

is called the falling factorial.

When defining extended binomial coefficient $\binom{z}{w}$ in (1.2), we considered [9, Theorem 1] and [12, Theorem 1.2], in which, among other things, the limit formula

$$\lim_{z\to-k}\frac{\Gamma(nz)}{\Gamma(qz)}=(-1)^{(n-q)k}\frac{q}{n}\frac{(qk)!}{(nk)!}$$

was obtained for $k\in\mathbb{N}_0$ and $n,q\in\mathbb{N}$.

It is easy to see that

$$\binom{2z}{z}=\begin{cases}\dfrac{\Gamma(2z+1)}{\Gamma^2(z+1)}, & z\ne-\dfrac{1}{2},-1,-\dfrac{3}{2},-2,\dots;\\ 0, & z\in\mathbb{N}_-;\\ \infty, & z=-\dfrac{1}{2},-\dfrac{3}{2},\dots.\end{cases}$$

The above properties of the gamma function $\Gamma(z)$ and its reciprocal $\frac{1}{\Gamma(z)}$ mean that

(1) extended central binomial coefficient $\binom{2z}{z}$ is a single-valued and analytic function over the punctured complex plane $\mathbb{C}\setminus\left\{\frac{1}{2}-k, k\in\mathbb{N}\right\}$;
(2) extended central binomial coefficient $\binom{2z}{z}$ has simple poles at $\frac{1}{2}-k$ and has simple zeros at the points $-k$ over $\mathbb{C}$ for $k\in\mathbb{N}$;
(3) extended central binomial coefficient $\binom{2x}{x}$ is positive on the intervals $\left(-\frac{1}{2},+\infty\right)$ and $\left(\frac{1}{2}-2k,1-2k\right)$ for $k\in\mathbb{N}$;
(4) extended central binomial coefficient $\binom{2x}{x}$ is negative on intervals $\left(1-2k,\frac{3}{2}-2k\right)$ for $k\in\mathbb{N}$.

In the literature, there have been a number of estimates and inequalities of central binomial coefficient $\binom{2n}{n}$ for $n\in\mathbb{N}$. See [8, 20, 22], for example.

In [3, Corollary 3.2], [7, p. 2, Eq. (10)], [10, Theorem 3.1], and [18, Section 4.2], among other things, the integral representations

$$\begin{aligned}\binom{2n}{n} &= \frac{1}{\pi}\int_0^4 \sqrt{\frac{x}{4-x}}\, x^{n-1}\operatorname{d}x \\ &= \frac{2}{\pi}\int_0^{\pi/2} (2\sin x)^{2n}\operatorname{d}x \\ &= \frac{1}{\pi}\int_0^{+\infty} \frac{1}{(1/4+x^2)^{n+1}}\operatorname{d}x\end{aligned} \tag{1.3}$$

for $n \in \{0\} \cup \mathbb{N}$ were established and applied. See also [16, p. 57], [17, Section 2.4, Theorem 7], and [19, Lemma 2.5 and Theorem 5.5].

In [6, p. 116, (10)], Merkle obtained that the double inequality

$$6^x < \frac{\Gamma(2(1+x))}{\Gamma^2(1+x)} < (1+x)3^x$$

holds for $x \in (0,1)$ and its reversed inequality is valid for $x > 1$. In other words, the double inequality

$$\frac{6^x}{2x+1} > \binom{2x}{x} > \frac{(x+1)3^x}{2x+1} \tag{1.4}$$

is valid for $x > 1$ and its reversed version holds for $0 < x < 1$.

In this paper, we aim to extend integral representations in (1.3) and to extend and refine the double inequality (1.4).

2. Integral representations

In this section, we extend integral representations in (1.3).

Theorem 2.1. *For $z \in \mathbb{C}$ such that $\Re(z) > -\frac{1}{2}$, we have*

$$\begin{aligned}\binom{2z}{z} &= \frac{1}{\pi}\int_0^4 \sqrt{\frac{x}{4-x}}\, x^{z-1}\operatorname{d}x \\ &= \frac{2}{\pi}\int_0^{\pi/2} (2\sin x)^{2z}\operatorname{d}x \\ &= \frac{1}{\pi}\int_0^{+\infty} \frac{1}{(1/4+x^2)^{z+1}}\operatorname{d}x.\end{aligned} \tag{2.1}$$

Proof. In [4, p. 19], Kazarinoff proved that

$$\int_0^{\pi/2} \sin^\alpha x\operatorname{d}x = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{\alpha+1}{2}\right)}{\Gamma\left(\frac{\alpha+2}{2}\right)}, \quad -1 < \alpha < +\infty. \tag{2.2}$$

See also [10, p. 112, Remark 5], [11, p. 16, Section 2.3, (2.18)], [15, p. 34, Remark 11.1], and [17, p. 5, (16)]. Replacing $\alpha > -1$ by $2t > -1$ in (2.2) gives

$$\int_0^{\pi/2} \sin^{2t} x\operatorname{d}x = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{2t+1}{2}\right)}{\Gamma(t+1)} = \frac{\pi}{2^{2t+1}}\frac{\Gamma(2t+1)}{\Gamma^2(t+1)} = \frac{\pi}{2^{2t+1}}\binom{2t}{t} \tag{2.3}$$

for $-\frac{1}{2} < t < +\infty$. Further making use of the uniqueness theorem of analytic functions in the theory of complex functions [5, p. 62, Theorem 3.2], we can extend the integral representation in (2.3) from $-\frac{1}{2} < t < +\infty$ to $\Re(z) > -\frac{1}{2}$. The second integral representation (2.1) is thus proved.

The first and third integrals in (2.1) can be derived via variable substitutions of definite integrals from the second integral in (2.1), as done in the proof of [14, Theorem 1.3]. The proof of Theorem 2.1 is complete. $\square$

Remark 2.1. The proof of Theorem 2.1 provides an alternative proof of the integral representation of the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. Several integral representations of the Catalan numbers C_n have been reviewed and surveyed in the paper [17].

3. Monotonicity and inequalities

In this section, we present decreasing and increasing properties of two power-exponential functions involving extended central binomial coefficient $\binom{2x}{x}$, derive several inequalities for bounding extended central binomial coefficient $\binom{2x}{x}$, and compare some of these inequalities with the double inequality (1.4).

Theorem 3.1. *Let $k \in \mathbb{N}$. The function*

$$F_1(x) = \begin{cases} \left[\dbinom{2x}{x}(2x+1)\right]^{1/x}, & x \neq 0 \\ \mathrm{e}^2, & x = 0 \end{cases} \tag{3.1}$$

is decreasing on the interval $\left(-\frac{1}{2}, +\infty\right)$, with the limits

$$\lim_{x \to (-1/2)^+} F_1(x) = \pi^2 \tag{3.2}$$

and

$$\lim_{x \to +\infty} F_1(x) = 4. \tag{3.3}$$

Proof. By straightforward computation, we obtain

$$\ln F_1(x) = \frac{1}{x}\left[\ln \frac{\Gamma(2x+1)}{\Gamma^2(x+1)} + \ln(2x+1)\right],$$

$$[x \ln F_1(x)]' = 2\left[\psi(2x+1) - \psi(x+1) + \frac{1}{2x+1}\right],$$

and

$$[x \ln F_1(x)]'' = 2\left[2\psi'(2x+1) - \psi'(x+1) - \frac{2}{(2x+1)^2}\right].$$

Utilizing the duplication formula

$$\psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi\left(z + \frac{1}{2}\right) + \ln 2, \quad \Re(z) > 0 \tag{3.4}$$

in [1, p. 259, 6.3.8] gives

$$\psi'(2x+1) = \frac{1}{4}\left[\psi'\left(x + \frac{1}{2}\right) + \psi'(x+1)\right], \quad x > -\frac{1}{2}. \tag{3.5}$$

Hence, we have

$$[x \ln F_1(x)]'' = \psi'\left(x + \frac{1}{2}\right) - \psi'(x+1) - \frac{1}{\left(x + \frac{1}{2}\right)^2}, \quad x > -\frac{1}{2}.$$

By the integral representations

$$\frac{\Gamma(z)}{s^z}=\int_0^\infty t^{z-1}\,\mathrm{e}^{-st}\,\mathrm{d}\,t,\quad s>0,\quad \Re(z)>0$$

and

$$\psi^{(n)}(z)=(-1)^{n+1}\int_0^\infty \frac{t^n}{1-\mathrm{e}^{-t}}\,\mathrm{e}^{-zt}\,\mathrm{d}\,t,\quad \Re(z)>0,\quad n\in\mathbb{N} \tag{3.6}$$

in [21, p. 49] and [1, p. 260, 6.4.1], respectively, we acquire

$$[x\ln F_1(x)]''=\int_0^\infty\biggl(\frac{1}{1+\mathrm{e}^{-t/2}}-1\biggr)t\,\mathrm{e}^{-(x+1/2)t}\,\mathrm{d}\,t<0,\quad x>-\frac12.$$

Therefore, the first derivative $[x\ln F_1(x)]'$ is decreasing on $\bigl(-\frac12,+\infty\bigr)$.

For $a,b\in\mathbb{R}$ with $a<b$, let $f(x)$ and $g(x)$ be continuous on $[a,b]$, differentiable on (a,b), and $g'(x)\ne0$ on (a,b). Theorem 1.25 in [2, pp. 10–11] reads that, if the ratio $\frac{f'(x)}{g'(x)}$ is increasing (decreasing) on (a,b), so are $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$. Taking $[a,b]=[a,0]\subset\bigl(-\frac12,0\bigr]$ and letting

$$f(x)=x\ln F_1(x)=\ln\frac{\Gamma(2x+1)}{\Gamma^2(x+1)}+\ln(2x+1)$$

and $g(x)=x$, we acquire

$$\frac{f'(x)}{g'(x)}=[x\ln F_1(x)]',$$

which is decreasing on $\bigl(-\frac12,+\infty\bigr)$. Accordingly, the ratio

$$\frac{f(x)}{g(x)}=\ln F_1(x)$$

is decreasing on $(a,0)$. Due to $a\in\bigl(-\frac12,0\bigr)$ is arbitrary, the function $\ln F_1(x)$ is decreasing on $\bigl(-\frac12,0\bigr)$. Similarly, if taking $[a,b]=[0,b]\subset[0,+\infty)$, due to $b\in(0,+\infty)$ is arbitrary, we can find that the function $\ln F_1(x)$ is decreasing on $(0,+\infty)$. In conclusion, the function $F_1(x)$ is decreasing on the interval $\bigl(-\frac12,+\infty\bigr)$.

It is easy to see that

$$\begin{aligned}\lim_{x\to(-1/2)^+}\ln F_1(x)&=-2\lim_{x\to(-1/2)^+}\bigl[\ln\Gamma(2x+2)-\ln\Gamma^2(x+1)\bigr]\\&=-2\biggl[\ln\Gamma(1)-2\ln\Gamma\biggl(\frac12\biggr)\biggr]\\&=2\ln\pi.\end{aligned}$$

The limit (3.2) is thus proved.

By virtue of the L'Hôpital rule, we find

$$\begin{aligned}\lim_{x\to\infty}\ln F_1(x)&=\lim_{x\to\infty}[x\ln F_1(x)]'\\&=2\lim_{x\to\infty}[\psi(2x+1)-\psi(x+1)]\\&=2\lim_{x\to\infty}[\psi(2x+1)-\psi(x+1)]\\&=2\lim_{x\to\infty}\biggl[\frac12\psi\biggl(x+\frac12\biggr)-\frac12\psi(x+1)+\ln2\biggr]\\&=2\ln2,\end{aligned}$$

where we used the duplication formula (3.4) and the limit formula

$$\lim_{x\to\infty}\left(x^{k+1}\left[\psi^{(k)}(x+a)-\psi^{(k)}(x)\right]\right)=(-1)^k k!a,\quad k\in\mathbb{N}_0,\quad a\ge 0 \tag{3.7}$$

in [13, Lemma 2.5]. The limit (3.3) is thus proved. □

Corollary 3.1. *For $x\in(0,+\infty)$, the double inequality*

$$\frac{2^{2x}}{2x+1}<\binom{2x}{x}<\frac{\mathrm{e}^{2x}}{2x+1}$$

is sharp in the sense that the bases 2 *and* e *cannot be replaced by any larger and smaller constants, respectively.*

For $x\in\left(-\frac{1}{2},0\right)$, the double inequality

$$\frac{\pi^{2x}}{2x+1}<\binom{2x}{x}<\frac{\mathrm{e}^{2x}}{2x+1}$$

is sharp in the sense that the bases π and e *cannot be replaced by any larger and smaller constants, respectively.*

Proof. This follows from the limits (3.2) and (3.3), the definition $F_1(0)=\mathrm{e}^2$, and decreasing property of $F_1(x)$ on the interval $\left(-\frac{1}{2},+\infty\right)$. □

Remark 3.1. From the limit (3.3), $F_1(0)=\mathrm{e}^2$, and $F_1(1)=6$, by the decreasing property of $F_1(x)$, we acquire

$$6=F_1(1)\le F_1(x)\le F_1(0)=\mathrm{e}^2,\quad x\in[0,1]$$

and

$$4=\lim_{x\to+\infty}F_1(x)<F_1(x)\le F_1(1)=6,\quad x\in[1,+\infty).$$

Equivalently, we obtain

$$\frac{6^x}{2x+1}\le\binom{2x}{x}\le\frac{\mathrm{e}^{2x}}{2x+1},\quad x\in[0,1] \tag{3.8}$$

and

$$\frac{4^x}{2x+1}<\binom{2x}{x}\le\frac{6^x}{2x+1},\quad x\in[1,+\infty). \tag{3.9}$$

When $0<x<x_1<1$, the upper bound in (3.8) is better than the corresponding one in (1.4), where x_1 is the unique positive root of the equation $\frac{\ln(x+1)}{x}=2-\ln 3$.

When $x>x_2>1$, the lower bound in (3.9) is better than the corresponding one in (1.4), where x_2 is the unique positive root of the equation $\frac{\ln(x+1)}{x}=2\ln 2-\ln 3$.

Theorem 3.2. *Let $k\in\mathbb{N}$. The function*

$$F_2(x)=\begin{cases}\dbinom{2x+1}{x}^{1/x}, & x\ne 0\\ \mathrm{e}, & x=0\end{cases} \tag{3.10}$$

is increasing on the interval $\left(-\frac{3}{2},+\infty\right)$, with the limits

$$\lim_{x\to(-3/2)^+}F_2(x)=0 \tag{3.11}$$

and

$$\lim_{x\to+\infty}F_2(x)=4. \tag{3.12}$$

Proof. Taking the logarithm of $F_2(x)$ and differentiating yield

$$\begin{aligned}
\ln F_2(x) &= \frac{1}{x}\ln\binom{2x+1}{x} \\
&= \frac{1}{x}\ln\frac{\Gamma(2x+2)}{\Gamma(x+2)\Gamma(x+1)}, \\
[x\ln F_2(x)]' &= 2\psi(2x+2)-\psi(x+2)-\psi(x+1), \\
[x\ln F_2(x)]'' &= 4\psi'(2x+2)-\psi'(x+2)-\psi'(x+1).
\end{aligned}$$

Making use of the formulas (3.5) and (3.6), we arrive at

$$\begin{aligned}
[x\ln F_2(x)]'' &= \psi'\left(x+\frac{3}{2}\right)-\psi'(x+2) \\
&= \int_0^\infty \frac{t}{1-\mathrm{e}^{-t}}\left(1-\mathrm{e}^{-t/2}\right)\mathrm{e}^{-(x+3/2)t}\,\mathrm{d}\,t \\
&> 0.
\end{aligned}$$

Thus, the first derivative $[x\ln F_2(x)]'$ is increasing on $\left(-\frac{3}{2},+\infty\right)$.

Taking $[a,b]=[a,0]\subset\left(-\frac{3}{2},0\right]$ and letting

$$f(x)=x\ln F_2(x)=\ln\binom{2x+1}{x}$$

and $g(x)=x$, we acquire

$$\frac{f'(x)}{g'(x)}=[x\ln F_2(x)]',$$

which is increasing on $\left(-\frac{3}{2},+\infty\right)$. Accordingly, in the light of Theorem 1.25 in [2, pp. 10–11] mentioned in the proof of Theorem 3.1 above, the ratio

$$\frac{f(x)}{g(x)}=\ln F_2(x)$$

is increasing on $(a,0)$. Due to $a\in\left(-\frac{3}{2},0\right)$ is arbitrary, the function $\ln F_2(x)$ is increasing on $\left(-\frac{3}{2},0\right)$. Similarly, if taking $[a,b]=[0,b]\subset[0,+\infty)$, due to $b\in(0,+\infty)$ is arbitrary, we can find that the function $\ln F_2(x)$ is increasing on $(0,+\infty)$. In conclusion, the function $F_2(x)$ is increasing on the interval $\left(-\frac{3}{2},+\infty\right)$.

It is straightforward that

$$\begin{aligned}
\lim_{x\to(-3/2)^+}\ln F_2(x) &= -\frac{2}{3}\left(\ln\left[\lim_{x\to(-3/2)^+}\frac{\Gamma(2x+2)}{\Gamma(x+1)}\right]-\ln\Gamma\left(\frac{1}{2}\right)\right) \\
&= -\frac{2}{3}\left[\ln\frac{\lim_{x\to(-1)^+}\Gamma(x)}{\Gamma(-1/2)}-\ln\Gamma\left(\frac{1}{2}\right)\right] \\
&= -\infty,
\end{aligned}$$

where we used the facts that

$$\Gamma\left(-\frac{1}{2}\right)=-2\sqrt{\pi}$$

and that

$$\lim_{x\to(-1)^+}\Gamma(x)=-\infty.$$

This limit can be seen from the definition (1.1) or can be deduced from

$$\lim_{z\to -n}[(z+n)\Gamma(z)] = \frac{(-1)^n}{n!}, \quad n\in\mathbb{N}_0$$

in [21, p. 44]. The limit (3.11) is thus proved.

By virtue of the L'Hôpital rule, we obtain

$$\begin{aligned}\lim_{x\to\infty}\ln F_2(x) &= \lim_{x\to\infty}[x\ln F_2(x)]' \\ &= \lim_{x\to\infty}[2\psi(2x+2)-\psi(x+2)-\psi(x+1)] \\ &= \lim_{x\to\infty}\left[\psi\left(x+\frac{3}{2}\right)-\psi(x+2)+2\ln 2\right] \\ &= 2\ln 2,\end{aligned}$$

where we used the duplication formula (3.4) and the limit formula (3.7). The limit (3.12) is thus proved. □

Corollary 3.2. *For $x\in(0,+\infty)$, the double inequality*

$$\mathrm{e}^x < \binom{2x+1}{x} = \binom{2x}{x}\frac{2x+1}{x+1} < 4^x$$

is sharp in the sense that the bases e *and* 4 *cannot be replaced by any larger and smaller constants, respectively.*

For $x\in\left(-\frac{3}{2},0\right)$, the double inequality

$$\mathrm{e}^x < \binom{2x+1}{x} = \binom{2x}{x}\frac{2x+1}{x+1} < +\infty$$

is sharp in the sense that the base e *and the symbol $+\infty$ cannot be replaced by any larger and smaller constants.*

Proof. This follows from the limits (3.11) and (3.12), the definition $F_2(0)=\mathrm{e}$, and increasing property on the interval $\left(-\frac{3}{2},+\infty\right)$. □

Remark 3.2. From the limit (3.12), $F_2(0)=\mathrm{e}$, $F_2(1)=3$, and the increasing property of $F_2(z)$ on the interval $\left(-\frac{3}{2},+\infty\right)$, we arrive at

$$\mathrm{e}=F_2(0)\le F_2(x)\le F_2(1)=3, \quad x\in[0,1]$$

and

$$3=F_2(1)\le F_2(x)<\lim_{x\to\infty}F_2(x)=4, \quad x\in[1,+\infty).$$

Equivalently, we acquire

$$\frac{(x+1)\,\mathrm{e}^x}{2x+1}\le\binom{2x}{x}\le\frac{(x+1)3^x}{2x+1}, \quad x\in[0,1] \tag{3.13}$$

and

$$\frac{(x+1)3^x}{2x+1}\le\binom{2x}{x}<\frac{(x+1)4^x}{2x+1}, \quad x\in[1,+\infty). \tag{3.14}$$

When $0<x<x_3<1$, the lower bound in (3.13) is better than the corresponding one in (1.4), where x_3 is the unique positive root of the equation $\frac{\ln(x+1)}{x}=\ln 6-1$.

When $x>x_4>1$, the upper bound in (3.14) is better than the corresponding one in (1.4), where x_4 is the unique positive root of the equation $\frac{\ln(x+1)}{x}=\ln 6-2\ln 2$.

4. Two problems

In this section, we pose two open problems to interested readers.

Problem 4.1. *From poles and zeros of the gamma function* $\Gamma(z)$*, the reciprocal* $\frac{1}{\Gamma(z)}$*, and the extended binomial coefficient* $\binom{2z}{z}$ *mentioned on pages 1 and 2 in Section 1, we can immediately deduce the limits*

$$\lim_{x\to -k^+} F_1(x) = +\infty$$

and

$$\lim_{x\to(-1/2-k)^-} F_1(x) = 0$$

for $k \in \mathbb{N}$*. The function* $F_1(x)$ *defined by* (3.1) *is also decreasing on the intervals* $\left(-1-k, -\frac{1}{2}-k\right)$ *for* $k \in \mathbb{N}$*.*

Problem 4.2. *From poles and zeros of the gamma function* $\Gamma(z)$*, the reciprocal* $\frac{1}{\Gamma(z)}$*, and the extended binomial coefficient* $\binom{2z}{z}$ *mentioned on pages 1 and 2 in Section 1, we can straightforwardly deduce the limits*

$$\lim_{x\to(-1/2-k)^+} F_2(x) = 0$$

and

$$\lim_{x\to(-1-k)^-} F_2(x) = +\infty$$

for $k \in \mathbb{N}$*. The function* $F_2(x)$ *defined by* (3.10) *is also increasing on the intervals* $\left(-\frac{3}{2}-k, -1-k\right)$ *for* $k \in \mathbb{N}$*.*

5. Declarations

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References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, Reprint of the 1972 edition, Dover Publications, Inc., New York, 1992.

[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[3] T. Dana-Picard and D. G. Zeitoun, *Parametric improper integrals, Wallis formula and Catalan numbers*, Internat. J. Math. Ed. Sci. Tech. **43** (2012), no. 4, 515–520; available online at `https://doi.org/10.1080/0020739X.2011.599877`.

[4] D. K. Kazarinoff, *On Wallis' formula*, Edinburgh Math. Notes **1956** (1956), no. 40, 19–21.

[5] S. Lang, *Complex Analysis*, Fourth edition, Graduate Texts in Mathematics, **103**, Springer-Verlag, New York, 1999; available online at `https://doi.org/10.1007/978-1-4757-3083-8`.

[6] M. Merkle, *On log-convexity of a ratio of gamma functions*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **8** (1997), 114–119; available online at `https://www.jstor.org/stable/43666390`.

[7] K. A. Penson and J.-M. Sixdeniers, *Integral representations of Catalan and related numbers*, J. Integer Seq. **4** (2001), no. 2, Article 01.2.5, 6 pages.

[8] A. Yu. Popov, *Two-sided estimates of the central binomial coefficient*, Chelyab. Fiz.-Mat. Zh. **5** (2020), no. 1, 56–69; available online at `https://doi.org/10.24411/2500-0101-2020-15105`. (Russian)

[9] A. Prabhu and H. M. Srivastava, *Some limit formulas for the Gamma and Psi (or Digamma) functions at their singularities*, Integral Transforms Spec. Funct. **22** (2011), no. 8, 587–592; available online at `https://doi.org/10.1080/10652469.2010.535970`.

[10] F. Qi, *An improper integral, the beta function, the Wallis ratio, and the Catalan numbers*, Probl. Anal. Issues Anal. **7 (25)** (2018), no. 1, 104–115; available online at `https://doi.org/10.15393/j3.art.2018.4370`.

[11] F. Qi, *Bounds for the ratio of two gamma functions*, J. Inequal. Appl. **2010** (2010), Article ID 493058, 84 pages; available online at `https://doi.org/10.1155/2010/493058`.

[12] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604; available online at `http://dx.doi.org/10.2298/FIL1304601Q`.

[13] F. Qi, *Lower bound of sectional curvature of Fisher–Rao manifold of beta distributions and complete monotonicity of functions involving polygamma functions*, Results Math. **77** (2022), in press; `https://doi.org/10.20944/preprints202011.0315.v1`.

[14] F. Qi, *Some properties of the Catalan numbers*, Ars Combin. (2022), in press; available online at `https://www.researchgate.net/publication/328891537`.

[15] F. Qi and R. P. Agarwal, *On complete monotonicity for several classes of functions related to ratios of gamma functions*, J. Inequal. Appl. **2019**, Paper No. 36, 42 pages; available online at `https://doi.org/10.1186/s13660-019-1976-z`.

[16] F. Qi, C.-P. Chen, and D. Lim, *Several identities containing central binomial coefficients and derived from series expansions of powers of the arcsine function*, Results Nonlinear Anal. **4** (2021), no. 1, 57–64; available online at `https://doi.org/10.53006/rna.867047`.

[17] F. Qi and B.-N. Guo, *Integral representations of the Catalan numbers and their applications*, Mathematics **5** (2017), no. 3, Article 40, 31 pages; available online at `https://doi.org/10.3390/math5030040`.

[18] F. Qi, W.-H. Li, J. Cao, D.-W. Niu, and J.-L. Zhao, *An analytic generalization of the Catalan numbers and its integral representation*, arXiv preprint (2020), available online at `https://arxiv.org/abs/2005.13515v1`.

[19] F. Qi, Q. Zou, and B.-N. Guo, *The inverse of a triangular matrix and several identities of the Catalan numbers*, Appl. Anal. Discrete Math. **13** (2019), no. 2, 518–541; available online at `https://doi.org/10.2298/AADM190118018Q`.

[20] Z. Sasvári, *Inequalities for binomial coefficients*, J. Math. Anal. Appl. **236** (1999), no. 1, 223–226; available online at `https://doi.org/10.1006/jmaa.1999.6420`.

[21] N. M. Temme, *Special Functions: An Introduction to Classical Functions of Mathematical Physics*, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1996; available online at `http://dx.doi.org/10.1002/9781118032572`.

[22] I. V. Tikhonov, V. B. Sherstyukov, and D. G. Tsvetkovich, *Comparative analysis of two-sided estimates of the central binomial coefficient*, Chelyab. Fiz.-Mat. Zh. **5** (2020), no. 1, 70–95; available online at `https://doi.org/10.24411/2500-0101-2020-15106`. (Russian)

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