ON THE MINIMALITY OF QUASI-SUM PRODUCTION MODELS IN MICROECONOMICS

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Abstract. Historically, the minimality of surfaces is extremely important in mathematics and the study of minimal surfaces is a central problem, which has been widely concerned by mathematicians. Meanwhile, the study of the shape and the properties of the production models is a great interest subject in economic analysis. The aim of this paper is to study the minimality of quasi-sum production functions as graphs in a Euclidean space. We obtain minimal characterizations of quasi-sum production functions with two or three factors as hypersurfaces in Euclidean spaces. As a result, our results also give a classification of minimal quasi-sum hypersurfaces in dimensions two and three.

1. Introduction

The theory of minimal surfaces not only has important theoretical significance in mathematics, but also has been applied to other disciplines such as physics, architecture, engineering and so on. In fact, the study of minimal surfaces has always been a core problem, and has been widely concerned by scientists and engineers.

A production function is a positive non-constant function that specifies the output of a firm, an industry, or an entire economy for all combinations of inputs. In the field of economic analysis, some researchers often use a production function model to solve the engineering and managerial problems associated with a particular production process, see [10].

In 1928, the well-known Cobb-Douglas (CD) production function in economics was first introduced by C. W. Cobb and P. H. Douglas [10] as follows:

$$Y = bL^kC^{1-k}, \tag{1.1}$$

where L, C, b and Y represent the labor input, the capital input, the total factor productivity and the total production, respectively. By use of this two-factor CD production function, Cobb-Douglas described the distribution of the national income in the national income of the United States.

It seems natural to extend the concept of CD production function to the more general case with arbirary variables. To be more precise, the generalized CD production function depending on n inputs was defined by [5]

$$F(x_1, \dots, x_n) = Ax_1^{\alpha_1} \cdots x_n^{\alpha_n}, \tag{1.2}$$

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where $A,\ \alpha_1,\ \alpha_2,\ldots,\alpha_n>0$ are constants.

In 1961, K. J. Arrow, H. B. Chenery, B. S. Minhas and R. M. Solow [4] introduced another two-input production function, called ACMS production function, given by

$$Q=b(aK^r+(1-a)L^r)^{\frac{1}{r}}, \tag{1.3}$$

where $r=(s-1)/s$ with s being the elasticity of substitution, both K and L are the primary production factors, $Q,\ b$ and a stand for the output, the factor productivity and the share parameter, respectively. Furthermore, a generalized ACMS production function with n inputs was defined by H. Uzawa [16] and D. McFadden [14]

$$F(x_1,\ldots,x_n)=A\Big(\sum_{i=1}^{n}a_i^{\rho}x_i^{\rho}\Big)^{\frac{\gamma}{\rho}}, \tag{1.4}$$

where $A,\ a_i,\ \gamma>0$ are constants and ρ is a non-zero constant.

One of the most common economic indicators, called the Hicks elasticity of substitution (HES), was originally introduced by Hicks [13] and Robinson [15] independently. Let $f:\mathbb{R}_+^n\to\mathbb{R}_+$ denote a differentiable production function of class $\mathcal{C}^2$ with nowhere zero first derivatives. Set

$$H_{ij}(\mathbf{x})=\frac{\frac{1}{x_if_{x_i}}+\frac{1}{x_jf_{x_j}}}{-\frac{f_{x_ix_i}}{f_{x_i}^2}+\frac{2f_{x_ix_j}}{f_{x_i}f_{x_j}}-\frac{f_{x_jx_j}}{f_{x_j}^2}} \tag{1.5}$$

for $\mathbf{x}\in\mathbb{R}_+^n$ and $x_i\neq x_j\ (1\le i,j\le n)$, where f_{x_i} and $f_{x_jx_j}$ denote the first and the second partial derivatives of f with respect to the corresponding independent variables, respectively. A production function f is said to have the constant elasticity of substitution (CES) property if there exists a nonzero real number σ satisfying the following relation:

$$H_{ij}(\mathbf{x})\equiv\sigma \tag{1.6}$$

for $\mathbf{x}\in\mathbb{R}_+^n$ and $x_i\neq x_j\ (1\le i,j\le n)$.

It is easy to check that $H_{ij}(\mathbf{x})\equiv 1$ for the generalized CD production function, and $H_{ij}(\mathbf{x})\equiv\frac{1}{1-\rho}\neq 1$ for the generalized ACMS production function. Therefore, both the generalized CD production function and the generalized ACMS production function have the CES property (c.f. [3, 7]).

In [17], G. E. Vîlcu revealed a relationship between some basic concepts in the theory of production functions and the differential geometry of hypersurfaces by proving that the generalized CD production function has constant return to scale if and only if the corresponding hypersurface is developable. Furthermore, G. E. Vîlcu and A. D. Vîlcu [18] proved that the generalized ACMS production function has constant return to scale if and only if the corresponding hypersurface is developable.

We say that a production function f is quasi-sum [1, 8] if there exists continuous strict monotone functions $h_i:\mathbb{R}_+\to\mathbb{R},\ i=1,2,\cdots,n$, and there exists a continuous strict monotone increasing function $F:I\to\mathbb{R}_+$ such that $h_1(x_1)+h_2(x_2)+\cdots+h_n(x_n)\in I$ for any $\mathbf{x}=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$, and

$$f(\mathbf{x})=F(h_1(x_1)+h_2(x_2)+\cdots+h_n(x_n)).$$

Moreover, a quasi-sum production function is said to be quasi-linear if at most one of $F,\ h_1,\ h_2,\cdots,h_n$ in (1.7) is a nonlinear function. In [8], B. Y. Chen derived a geometric characterization for a quasi-sum production function to be quasilinear via its production hypersurface and gave a complete classification of the quasi-sum production hypersurface having vanishing Gauss-Kronecker curvature. In addition, B. Y. Chen proved in [6] that if a quasi-sum production function f satisfies the CES property, then the graph of f is a flat space if and only if f is either a linearly homogeneous generalized ACMS function or a linearly homogeneous generalized Cobb-Douglas function. Moreover, A. D. Vîlcu and G. E. Vîlcu [19] obtained a classification of quasi-sum production functions with constant elasticity of production with respect to any factor of production and with proportional marginal rate of substitution.

In recent years, there are also some interesting results on quasi-product production functions and homogeneous production functions, see for examples [2, 9, 11, 20] and references therein.

Motivated by the above results, this work is intended to investigate the minimality of quasi-sum production hypersurfaces. In fact, the study of minimal production hypersurfaces is not only widely used in economics, but also of great significance in the field of differential geometry. By definition quasi-sum production hypersurfaces include all the translation hypersurfaces as subclass, hence it is very interesting to investigate the geometry and classification of quasi sum production hypersurfaces. Meanwhile, it should be pointed out that, generally, it is difficult to classify completely minimal hypersurfaces for arbitrary dimension. In this paper, we are able to deal with minimal quasi-sum production hypersurfaces of $n = 2$ and $n = 3$, and obtain several classification results concerning a quasi-sum production function, see Theorem 3.1, 3.2 and 4.1.

2. The minimality in theory of production hypersurfaces

Geometrically speaking, each production function $f(x_1, \dots, x_n)$ can be identified with a graph of a non-parametric hypersurface in an Euclidean space $\mathbb{E}^{n+1}$ given by

$$L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n)). \tag{2.1}$$

For simplicity of notation, we write $f_{i_1 i_2 \cdots i_n}$ instead of the nth-order partial derivatives $\frac{\partial^n f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_n}}$.

We recall a well-known result concerning a graph of hypersurface (2.1) in $\mathbb{E}^{n+1}$. For a production hypersurface of $\mathbb{E}^{n+1}$ defined by (2.1), the mean curvature H is given by (see [21])

$$H = \frac{1}{nW}\Big(\sum_i f_{ii} - \frac{1}{W^2}\sum_{i,j} f_i f_j f_{ij}\Big), \tag{2.2}$$

where $W = \sqrt{1 + \sum_{i=1}^n f_i^2}$. It is well known that a hypersurface is called minimal if its mean curvature H vanishes identically.

Definition 2.1. ([8]) A production function F is called quasi-sum if it is given by

$$f(\mathbf{x}) = F(h_1(x_1) + h_2(x_2) + \cdots + h_n(x_n)), \tag{2.3}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+$, and $F, h_1, h_2, \dots, h_n$ are continuous strict monotone function with $F' > 0$.

We will be concerned with the minimality of quasi-sum production hypersurfaces in a Euclidean space. For $i, j = 1, 2, \cdots, n$ and $i \neq j$, we have

$$f_i = F' h_i', \tag{2.4}$$

$$f_{ii} = F''h_i'^2 + F'h_i'', \quad (2.5)$$
$$f_{ij} = F''h_i'h_j'. \quad (2.6)$$

In what follows, we always suppose that a quasi-sum production hypersurface is minimal, i.e. $H = 0$. It follows from (2.2) that

$$\Big(\sum_{i=1}^{n} f_{ii}\Big)\Big(1 + \sum_{i=1}^{n} f_i^2\Big) = \sum_{i,j} f_i f_j f_{ij}.$$

An equivalent form of the above equation is:

$$\sum_{i=1}^{n} f_{ii} + \sum_{i\neq j}\big({f_i}^2 f_{jj} - f_i f_j f_{ij}\big) = 0, \quad (2.7)$$

where we have made use of ${f_i}^2 f_{jj} - f_i f_j f_{ij} \equiv 0$ for $i = j$. Substituting (2.4)-(2.6) into (2.7), we obtain

$$F''\sum_{i=1}^{n} h_i'^2 + F'\sum_{i=1}^{n} h_i'' + F'^3\sum_{i\neq j} h_i'^2 h_j'' = 0. \quad (2.8)$$

In order to solve the above equation, let us introduce the following notations:

$$G = \frac{1}{F'^2}, \quad u_i = h_i(x_i), \quad X_i = h_i'^2, \quad 1 \le i \le n. \quad (2.9)$$

By (2.9) it is obvious that

$$G' = -\frac{2F''}{F'^3}, \quad X_i' = \frac{dX_i}{du_i} = \frac{dX_i}{dx_i}\cdot\frac{dx_i}{du_i} = 2h_i''. \quad (2.10)$$

Thus, equation (2.8) can be expressed as

$$G'\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} X_i'\left(G + \sum_{j\neq i} X_j\right) = 0. \quad (2.11)$$

We will restrict the discussions to the solution of the differential equation (2.11) for $n = 2$ and $n = 3$, which will be dealt with separately.

3. The case $n = 2$

Let us consider the case $n = 2$. Note that the production surface is given by

$$f(\mathbf{x}) = F(h_1(x_1) + h_2(x_2)), \quad (3.1)$$

and the equation (2.11) becomes

$$G'(X_1 + X_2) - X_1'(G + X_2) - X_2'(G + X_1) = 0. \quad (3.2)$$

Firstly, we will restrict ourselves to a special case: $X_1' = 0$, $X_2' = 0$ or $G' = 0$.

Theorem 3.1. *For a quasi-sum production function f with minimality stated as* (3.1)*, if at least one of the terms X_1', X_2' and G' vanishes identically, then production function f can be expressed in any of the following three forms:*
(1) *a linear production function;*
(2) *a Scherk type production function given by*

$$f(x_1, x_2) = \frac{1}{a}\ln\left|\frac{\cos ax_2}{\cos ax_1}\right| + b,$$

where a, b are constants;
(3) *a production function given by*

$$f(x_1, x_2) = \frac{1}{a} \arctan \sqrt{\frac{be^{2ax_1}}{\cos^2 ax_2} - 1},$$

where a, b are constants with $b > 0$.

Proof. When $G' = 0$, then we can deduce from (2.10) that

$$F(w) = c_1 w + c_2, \tag{3.3}$$

where c_1, c_2 are constants with $c_1 > 0$. Furthermore, (3.3) enables us to write (3.2) in the following form:

$$\frac{X_1'}{X_1 + \frac{1}{c_1^2}} = -\frac{X_2'}{X_2 + \frac{1}{c_1^2}}. \tag{3.4}$$

Hence, there exists a constant c_3 such that

$$\frac{X_1'}{X_1 + \frac{1}{c_1^2}} = -\frac{X_2'}{X_2 + \frac{1}{c_1^2}} = c_3. \tag{3.5}$$

If $c_3 = 0$, then $X_1' = X_2' = 0$, which means that the production function f is a linear function. Therefore, we obtain the case (1) in Theorem 3.1.

In the following we may assume $c_3 \neq 0$. Integrating (3.5) we can get

$$X_1 = c_4 e^{c_3 u_1} - \frac{1}{c_1^2}, \quad X_2 = c_4' e^{-c_3 u_2} - \frac{1}{c_1^2},$$

where $c_4, c_4' > 0$ are real constants. An equivalent formulation of the above equations are:

$$h_1'^2 = c_4 e^{c_3 h_1} - \frac{1}{c_1^2}, \quad h_2'^2 = c_4' e^{-c_3 h_2} - \frac{1}{c_1^2}. \tag{3.6}$$

By solving the two equations in (3.6) we can get

$$h_1(x_1) = \frac{1}{c_3} \ln \Big(\frac{1}{c_1^2 c_4} + \frac{1}{c_4} \big(\frac{1}{c_1} \tan \frac{c_3(x_1 + c_5)}{2c_1} \big)^2 \Big), \tag{3.7}$$

$$h_2(x_2) = -\frac{1}{c_3} \ln \Big(\frac{1}{c_1^2 c_4'} + \frac{1}{c_4'} \big(\frac{1}{c_1} \tan \frac{c_3(x_2 + c_5')}{2c_1} \big)^2 \Big), \tag{3.8}$$

where c_5, c_5' are constants.
According to (3.7) and (3.8) we have

$$w = h_1(x_1) + h_2(x_2) = \frac{2}{c_3} \ln \left| \frac{\cos \frac{c_3(x_2 + c_5')}{2c_1}}{\cos \frac{c_3(x_1 + c_5)}{2c_1}} \right| + \frac{1}{c_3} \ln \frac{c_4'}{c_4}. \tag{3.9}$$

Substituting (3.9) into (3.3), with some translation transformation, we obtain the case (2) in Theorem 3.1.

When $X_1' = 0$ or $X_2' = 0$. Without loss of generality we can assume that $X_1' = 0$. Then we can derive from the second expression in (2.10) that

$$h_1(x_1) = c_1 x_1 + c_2, \tag{3.10}$$

where c_1, c_2 are constants. Since $X_1' = 0$, the equation (3.2) can be rewritten as

$$\frac{G'}{G + c_1^2} = \frac{X_2'}{X_2 + c_1^2}. \tag{3.11}$$

Taking the partial derivative of (3.11) with respect to u_1 one gets

$$\Big(\frac{G'}{G+c_1^2}\Big)'=0,$$

which indicates that there exists a constant c_3 such that

$$\frac{G'}{G+c_1^2}=\frac{X_2'}{X_2+c_1^2}=c_3. \tag{3.12}$$

A direct computation show that the solutions of (3.12) are given by

$$G=c_4e^{c_3w}-c_1^2,\quad X_2=c_4'e^{c_3u_2}-c_1^2,$$

that is to say,

$$F'^2=\frac{1}{c_4e^{c_3w}-c_1^2},\quad h_2'^2=c_4'e^{c_3h_2}-c_1^2, \tag{3.13}$$

where $c_4, c_4'>0$ are constants.
By solving the two equations in (3.13) it follows that

$$F(w)=\frac{2}{c_1c_3}\arctan\frac{\sqrt{c_4e^{c_3w}-c_1^2}}{c_1}+c_5, \tag{3.14}$$

$$h_2(x_2)=\frac{1}{c_3}\ln\frac{c_1^2}{c_4'}-\frac{2}{c_3}\ln\Big|\cos\frac{c_1c_3(x_2+c_6)}{2}\Big|, \tag{3.15}$$

where $c_5, c_6>0$ are constants.
When (3.10) and (3.15) are substituted in (3.14) we can get

$$f(x_1,x_2)=\frac{2}{c_1c_3}\arctan\sqrt{\frac{c_4e^{c_2c_3}}{c_4'}\times\frac{e^{c_1c_3x_1}}{\cos^2\frac{c_1c_3(x_2+c_6)}{2}}-1}+c_5. \tag{3.16}$$

After making some translation, we can rewrite the equation (3.16) as the case (3) in Theorem 3.1. □

From now on we make the assumption: $G'X_1'X_2'\neq 0$. Differentiating (3.2) with respect to u_1 and u_2 respectively, one has

$$G''(X_1+X_2)-X_1''(G+X_2)=X_2'(G'+X_1'), \tag{3.17}$$

$$G''(X_1+X_2)-X_2''(G+X_1)=X_1'(G'+X_2'). \tag{3.18}$$

From (3.2), (3.17) and (3.18) we find that

$$M(X_1+X_2)G'=-X_1'X_2'G'[X_2''(G'+X_1')+X_1''(G'+X_2')], \tag{3.19}$$

$$M(G+X_2)X_1'=-X_1'X_2'G'[X_2''(G'+X_1')+G''(X_1'-X_2')], \tag{3.20}$$

$$M(G+X_1)X_2'=-X_1'X_2'G'[X_1''(G'+X_2')-G''(X_1'-X_2')], \tag{3.21}$$

where $M=X_1''X_2''G'-X_1'X_2''G''-X_1''X_2'G''$.
Differentiating (3.17) and (3.18) with respect to u_1 and u_2 respectively, one gets three equations

$$G'''(X_1+X_2)-X_1'''(G+X_2)-X_1''(G'+X_2')+G''(X_1'-X_2')=0, \tag{3.22}$$

$$G'''(X_1+X_2)-X_1''(G'+X_2')-X_2''(G'+X_1')=0, \tag{3.23}$$

$$G'''(X_1+X_2)-X_2'''(G+X_1)-X_2''(G'+X_1')-G''(X_1'-X_2')=0. \tag{3.24}$$

Subtracting equation (3.23) from equation (3.22) and (3.24) respectively, we get

$$X_1'''(G+X_2)=X_2''(G'+X_1')+G''(X_1'-X_2'), \tag{3.25}$$

$$X_2'''(G+X_1) = X_1''(G'+X_2') - G''(X_1'-X_2'). \tag{3.26}$$

Substituting (3.23), (3.25) and (3.26) into (3.19), (3.20) and (3.21) respectively, we see that

$$\begin{aligned}
M(X_1+X_2)G' &= -X_1'X_2'G'(X_1+X_2)G''', \\
M(G+X_2)X_1' &= -X_1'X_2'G'(G+X_2)X_1''', \\
M(G+X_1)X_2' &= -X_1'X_2'G'(G+X_1)X_2'''.
\end{aligned}$$

Since $X_1'X_2'G' \neq 0$, it follows from the above three equations that

$$\frac{G'''}{G'} = \frac{X_1'''}{X_1'} = \frac{X_2'''}{X_2'} = K, \tag{3.27}$$

for a constant K. Next, the proof will be divided into three parts.
Case 1. $K > 0$. Let $K = k^2, k > 0$. The solutions of (3.27) are of the form

$$\begin{aligned}
G(w) &= a_0 + b_0\cosh(kw) + c_0\sinh(kw), \\
X_1(u_1) &= a_1 + b_1\cosh(ku_1) + c_1\sinh(ku_1), \\
X_2(u_2) &= a_2 + b_2\cosh(ku_2) + c_2\sinh(ku_2),
\end{aligned} \tag{3.28}$$

where $a_i, b_i, c_i \in \mathbb{R}, 0 \leq i \leq 2$. Observe that (3.28) must satisfy (3.2), these parameters a_i, b_i and c_i are restricted. Substituting (3.28) into (3.2) leads to

$$\begin{aligned}
&P_1\sinh(ku_1) + P_2\sinh(ku_2) + P_3\cosh(ku_1) + P_4\cosh(ku_2) \\
&+P_5\sinh(ku_1+ku_2) + P_6\cosh(ku_1+ku_2) = 0.
\end{aligned}$$

Since the above hyperbolic functions $\sinh(ku_1)$, $\sinh(ku_2)$, $\cosh(ku_1)$, $\cosh(ku_2)$, $\sinh(ku_1+ku_2)$ and $\cosh(ku_1+ku_2)$ are linearly independent, all of P_i $(1 \leq i \leq 6)$ must be vanishing identically. We can deduce that

$$\begin{cases}
(a_0+a_1)b_2 - b_0b_1 + c_0c_1 = 0, \\
(a_0+a_2)b_1 - b_0b_2 + c_0c_2 = 0, \\
(a_0+a_1)c_2 - b_1c_0 + b_0c_1 = 0, \\
(a_0+a_2)c_1 - b_2c_0 + b_0c_2 = 0, \\
(a_1+a_2)b_0 - b_1b_2 - c_1c_2 = 0, \\
(a_1+a_2)c_0 - b_1c_2 - b_2c_1 = 0.
\end{cases} \tag{3.29}$$

Case 2. $K < 0$. Let $K = -k^2, k > 0$. The solutions of (3.27) are given by

$$\begin{aligned}
G(w) &= a_0 + b_0\cos(kw) + c_0\sin(kw), \\
X_1(u_1) &= a_1 + b_1\cos(ku_1) + c_1\sin(ku_1), \\
X_2(u_2) &= a_2 + b_2\cos(ku_2) + c_2\sin(ku_2)
\end{aligned} \tag{3.30}$$

for $a_i, b_i, c_i \in \mathbb{R},\ 0 \leq i \leq 2$. Substituting (3.30) into (3.2) yields

$$\begin{aligned}
&P_1\sin(ku_1) + P_2\sin(ku_2) + P_3\cos(ku_1) + P_4\cos(ku_2) \\
&+P_5\sin(ku_1+ku_2) + P_6\cos(ku_1+ku_2) = 0.
\end{aligned}$$

Since the above functions $\sin(ku_1), \sin(ku_2), \cos(ku_1)$, $\cos(ku_2)$, $\sin(ku_1+ku_2)$ and $\cos(ku_1+ku_2)$ are linearly independent, all of P_i $(1 \leq i \leq 6)$ must be zero. So the corresponding parameters a_i, b_i and c_i should satisfy the following relation:

$$\begin{cases} (a_0+a_1)b_2 - b_0b_1 - c_0c_1 = 0, \\ (a_0+a_2)b_1 - b_0b_2 - c_0c_2 = 0, \\ (a_0+a_1)c_2 - b_1c_0 + b_0c_1 = 0, \\ (a_0+a_2)c_1 - b_2c_0 + b_0c_2 = 0, \\ (a_1+a_2)b_0 - b_1b_2 + c_1c_2 = 0, \\ (a_1+a_2)c_0 - b_1c_2 - b_2c_1 = 0. \end{cases} \tag{3.31}$$

Case 3. $K = 0$. Clearly, the solutions of (3.27) are

$$\begin{aligned} G(w) &= a_0 + b_0w + c_0w^2, \\ X_1(u_1) &= a_1 + b_1u_1 + c_1u_1^2, \\ X_2(u_2) &= a_2 + b_2u_2 + c_2u_2^2 \end{aligned} \tag{3.32}$$

for $a_i, b_i, c_i \in \mathbb{R}, 0 \le i \le 2$. By inserting (3.32) into (3.2) one gets

$$P_0 + P_1u_1 + P_2u_2 + P_3u_1u_2 + P_4u_1^2 + P_5u_2^2 + P_6u_1^2u_2 + P_7u_1u_2^2 = 0.$$

It is easy to see that

$$\begin{cases} (a_1+a_2)b_0 - (a_0+a_2)b_1 - (a_0+a_1)b_2 = 0, \\ 2(a_1+a_2)c_0 - 2(a_0+a_2)c_1 - (b_0+b_1)b_2 = 0, \\ 2(a_1+a_2)c_0 - 2(a_0+a_1)c_2 - (b_0+b_2)b_1 = 0, \\ (b_1-b_2)c_0 - (b_0+b_2)c_1 = 0, \\ (b_2-b_1)c_0 - (b_0+b_1)c_2 = 0, \\ (b_0+b_2)c_1 + (b_0+b_1)c_2 = 0, \\ c_0c_1 + c_1c_2 + c_0c_2 = 0. \end{cases} \tag{3.33}$$

Consequently, we obtain the following characterization result:

Theorem 3.2. *Let f be a smooth quasi-sum production function given by* (3.1) *with $G'X_1'X_2' \neq 0$. Then the production hypersurface of f is minimal if and only if one of the following three cases holds:*
(1) $K > 0$. *The functions X_1, X_2 and G are given by* (3.28) *and a_i, b_i, c_i must satisfy* (3.29)*;*
(2) $K < 0$. *The functions X_1, X_2 and G are given by* (3.30) *and a_i, b_i, c_i must satisfy* (3.31)*;*
(3) $K = 0$. *The functions X_1, X_2 and G are given by* (3.32) *and a_i, b_i, c_i must satisfy* (3.33).

Remark 3.3. Here we use the similar technique developed in [12], where the authors investigated zero mean curvature surfaces in 3-dimensional Lorentz-Minkowski space. With this method, one could construct many examples of minimal surfaces, which are of great interest to geometers. For more examples on minimal production surfaces, see Section 5.

4. The case $n = 3$

In this section, we focus on the case with three factors, namely $n = 3$. In this case, the quasi-sum production takes the form

$$f(\mathbf{x}) = F(h_1(x_1) + h_2(x_2) + h_3(x_3)). \tag{4.1}$$

Meanwhile, the equation (2.11) can be rewritten as

$$\begin{aligned}&G'(X_1+X_2+X_3)-X_1'(G+X_2+X_3)\\&-X_2'(G+X_1+X_3)-X_3'(G+X_1+X_2)=0.\end{aligned}\tag{4.2}$$

Differentiating (4.2) with respect to u_1 , u_2 and u_3 respectively, we get

$$G''(X_1+X_2+X_3)-X_1''(G+X_2+X_3)\tag{4.3}$$
$$-(X_2'+X_3')(G'+X_1')=0,$$
$$G''(X_1+X_2+X_3)-X_2''(G+X_1+X_3)\tag{4.4}$$
$$-(X_1'+X_3')(G'+X_2')=0,$$
$$G''(X_1+X_2+X_3)-X_3''(G+X_1+X_2)\tag{4.5}$$
$$-(X_1'+X_2')(G'+X_3')=0.$$

By differentiating (4.3) with respect to u_2 and u_3 respectively, we obtain

$$G'''(X_1+X_2+X_3)+G''X_2'-X_1''(G'+X_2')\tag{4.6}$$
$$-X_2''(G'+X_1')-(X_2'+X_3')G''=0,$$
$$G'''(X_1+X_2+X_3)+G''X_3'-X_1''(G'+X_3')\tag{4.7}$$
$$-X_3''(G'+X_1')-(X_2'+X_3')G''=0.$$

Subtracting equation (4.6) from equation (4.7) gives

$$(G''-X_1'')(X_2'-X_3')=(G'+X_1')(X_2''-X_3'').\tag{4.8}$$

The proof will be divided into the following situations:
Case A. At least two of the terms X_1', X_2' and X_3' are equal to each other. We can certainly assume $X_2'=X_3'=c_1$. Thus,

$$X_i=c_1u_i+d_i,\quad i=2,3,\tag{4.9}$$

where c_1, d_2 and d_3 are real constants. For $c_1\neq 0$, taking into account (2.9) and integrating (4.9) we have

$$h_i(x_i)=\frac{c_1}{4}(x_i+c_i')^2-\frac{d_i}{c_1},\quad i=2,3,\tag{4.10}$$

where c_2', c_3' are real constants.

Subtracting equation (4.3) from equation (4.4) yields

$$(X_1''+c_1)(G'+c_1)=X_1''(G+X_2+X_3)+2c_1(G'+X_1').\tag{4.11}$$

Differentiating (4.11) with respect to u_2 gives

$$G''(X_1'-c_1)=X_1''(G'+c_1).\tag{4.12}$$

Subcase A.1. $X_1'=c_1$. We thus have

$$X_1=c_1u_1+d_1\tag{4.13}$$

for a constant d_1. At this time, (4.2) turns into

$$G'(c_1w+\sum_{i=1}^{3}d_i)-3c_1G=2c_1(c_1w+\sum_{i=1}^{3}d_i).\tag{4.14}$$

If $c_1=0$, then $X_1'=X_2'=X_3'=0$. From (4.14) we have $G'=0$. Hence, the production function f is a linear function.

Now we assume $c_1 \neq 0$. Solving (4.14), we conclude that the solution is of the type

$$G = c\big(c_1 w + \sum_{i=1}^{3} d_i\big)^3 - \big(c_1 w + \sum_{i=1}^{3} d_i\big), \tag{4.15}$$

where c is a real constant. Since $X_i = h_i'^2$ and $u_i = h_i$, by integrating (4.13) we find

$$h_1(x_1) = \frac{c_1}{4}(x_1 + c_1')^2 - \frac{d_1}{c_1}, \tag{4.16}$$

where c_1' is a real constant. Taking into account (4.10), (4.15) and (4.16), we have

$$f(x_1, x_2, x_3) = F(\frac{c_1}{4}\sum_{i=1}^{3}(x_i + c_i')^2 - \frac{1}{c_1}\sum_{i=1}^{n} d_i),$$

where $F = \int \frac{1}{\sqrt{c(c_1 w + \sum_{i=1}^{3} d_i)^3 - (c_1 w + \sum_{i=1}^{3} d_i)}} dw + c'$ with a real constant c'.

Subcase A.2. $G' = -c_1$. It follows that

$$G = -c_1 w + d_0, \tag{4.17}$$

where d_0 is a real constant. Therefore, equation (4.2) can be written as

$$(c_1 u_1 - d_0 - d_2 - d_3)X_1' - 3c_1 X_1 = -2c_1(c_1 u_1 - d_0 - d_2 - d_3). \tag{4.18}$$

If $c_1 = 0$, then $X_1' = X_2' = X_3' = G' = 0$, which implies that the production function f is a linear function.

Now we assume $c_1 \neq 0$. By the method of variation of parameters, we see that the solution of (4.18) takes the form

$$X_1 = c_1'(c_1 u_1 - d_0 - d_2 - d_3)^3 + (c_1 u_1 - d_0 - d_2 - d_3), \tag{4.19}$$

where c_1' is a real constant. Since $X_1 = h_1'^2(x_1)$, $u_1 = h_1(x_1)$, (4.19) becomes

$$h_1'^2 = c_1'(c_1 h_1 - d_0 - d_2 - d_3)^3 + (c_1 h_1 - d_0 - d_2 - d_3), \tag{4.20}$$

By integrating (4.17) one has

$$F(w) = -\frac{2}{c_1}\sqrt{-c_1 w + d_0} + c,$$

where c is a real constant. Consequently,

$$f(x_1, x_2, x_3) = \Big(-\frac{4}{c_1}h_1(x_1) - (x_2 + c_2')^2 - (x_3 + c_3')^2 + \frac{4}{c_1^2}(d_0 + d_2 + d_3)\Big)^{\frac{1}{2}} + c,$$

where $h_1(x_1)$ is a solution of equation (4.20).

Subcase A.3. $X_1' \neq c_1$ and $G' \neq -c_1$. Equation (4.12) can be written as

$$\frac{G''}{G' + c_1} = \frac{X_1''}{X_1' - c_1}. \tag{4.21}$$

Differentiating (4.21) with respect to u_2, we get

$$(\frac{G''}{G' + c_1})' = 0. \tag{4.22}$$

There exists a constant c_2 such that

$$\frac{G''}{G' + c_1} = \frac{X_1''}{X_1' - c_1} = c_2. \tag{4.23}$$

Equation (4.4) can be written as

$$\frac{X_1' + c_1}{X_1 + X_2 + X_3} = \frac{G''}{G' + c_1} = c_2. \tag{4.24}$$

Hence,

$$X_1' + c_1 = c_2(X_1 + X_2 + X_3). \tag{4.25}$$

Differentiating (4.25) with respect to u_1 yields

$$X_1'' = c_2 X_1'. \tag{4.26}$$

By inserting (4.26) into (4.23), we thus get $c_1 c_2 = 0$.

(1). When $c_2 = 0$, we can obtain from (4.23) and (4.25) that $G'' = 0$ and $X_1' = -c_1$. The equation (4.3) can be written as $c_1(G' - c_1) = 0$.

If $c_1 = 0$, then $X_1' = X_2' = X_3' = 0$. From (4.2) we get that $G' = 0$, which implies that the production function f is a linear function.

If $c_1 \neq 0$, then $G' = c_1$. Hence,

$$\begin{aligned} G &= c_1 w + d_0, \\ X_1 &= -c_1 u_1 + d_1. \end{aligned}$$

By integrating the above two expressions, one obtains

$$F(w) = \frac{2}{c_1}\sqrt{c_1 w + d_0} + c, \tag{4.27}$$

$$h_1(u_1) = h_1(x_1) = -\frac{c_1}{4}(x_1 + c_1')^2 + \frac{d_1}{c_1}, \tag{4.28}$$

where c, c_1' are real constants.

At this time, the equation (4.2) reduces to $d_0 + d_1 - d_2 - d_3 = 0$. By (4.10), (4.27) and (4.28) we have $f(x_1, x_2, x_3) = \sqrt{(x_2 + c_2')^2 + (x_3 + c_3')^2 - (x_1 + c_1')^2} + c$.

(2). If $c_1 = 0$, then (4.25) reduces to

$$X_1' = c_2(X_1 + d_2 + d_3). \tag{4.29}$$

If $c_2 = 0$, we have actually proved that the production function f must be a linear function.

If $c_2 \neq 0$, by solving (4.29), we conclude that $X_1 = ce^{c_2 u_1} - d_2 - d_3$ with $c > 0$ being a constant. From (4.24) we see that $G = c_0 e^{c_2 w} + c_3$, where c_0, c_3 are real constants. At this time, the equation (4.2) reduces to $c_3 + d_2 + d_3 = 0$. Then we obtain the following equations:

$$\begin{aligned} G &= c_0 e^{c_2 w} - d_2 - d_3, \\ X_1 &= ce^{c_2 u_1} - d_2 - d_3, \\ X_2 &= d_2, \\ X_3 &= d_3. \end{aligned}$$

Taking into account (2.9), solving the above equations give

$$\begin{aligned} F(w) &= \frac{2}{c_2\sqrt{d_2 + d_3}} \arctan \sqrt{\frac{c_0 e^{c_2 w}}{d_2 + d_3} - 1} + c_0', \\ h_1(x_1) &= \frac{1}{c_2} \ln \Big[\frac{1}{c}(d_2 + d_3) \sec^2 \Big(\frac{c_2\sqrt{d_2 + d_3}}{2}(x_1 + c_1')\Big)\Big], \end{aligned}$$

$$h_2(x_2) = \sqrt{d_2}(x_2 + c'_2),$$
$$h_3(x_3) = \sqrt{d_3}(x_3 + c'_3),$$

where c'_0, c'_1, c'_2, c'_3 are constants. Hence, it follows that

$$f(x_1, x_2, x_3) = \frac{2}{c_2\sqrt{d_2 + d_3}} \arctan \sqrt{\frac{c_0}{c} \sec^2 \left(\frac{c_2\sqrt{d_2 + d_3}}{2} x_1\right) e^{\sqrt{d_2}x_2 + \sqrt{d_3}x_3} - 1},$$

where c, c_0, d_2, d_3 are positive constants.

Case B. $G' = -X'_1$. Differentiating this equation with respect to u_2, we get $G'' = 0$. Suppose that $G' = -X'_1 = c_1$ for a real constant c_1. Then,

$$\begin{aligned} G(w) &= c_1 w + d_0, \\ X_1(u_1) &= -c_1 u_1 + d_1, \end{aligned} \tag{4.30}$$

where d_0, d_1 are real constants. For $c_1 \neq 0$, it follows from (2.9) and (4.30) that

$$F(w) = \frac{2}{c_1}\sqrt{c_1 w + d_0} + c, \tag{4.31}$$

$$h_1(x_1) = -\frac{c_1}{4}(x_1 + c'_1)^2 + \frac{d_1}{c_1}, \tag{4.32}$$

where c, c'_1 are real constants. It is easy to see that the equation (4.4) reduces to

$$X''_2(G + X_1 + X_3) + (X'_2 + c_1)(X'_3 - c_1) = 0. \tag{4.33}$$

Differentiating the equation (4.33) with respect to u_3 leads to

$$X''_2(X'_3 + c_1) + X''_3(X'_2 + c_1) = 0. \tag{4.34}$$

Subcase B.1. $X'_2 = -c_1$ or $X'_3 = -c_1$. Without loss of generality, we can certainly suppose $X'_2 = -c_1$. So $X_2(u_2) = -c_1 u_2 + d_2$ for some real constant d_2, and the equation (4.2) reduces to

$$X'_3 - \frac{3c_1}{c_1 u_3 + d_0 + d_1 + d_2} X_3 = 2c_1. \tag{4.35}$$

If $c_1 = 0$, then $X'_1 = X'_2 = X'_3 = G' = 0$, hence the production function f is a linear function. If $c_1 \neq 0$, solving (4.35) gives

$$X_3 = c(c_1 u_3 + d_0 + d_1 + d_2)^3 - (c_1 u_3 + d_0 + d_1 + d_2), \tag{4.36}$$

where c is a real constant. Now we have

$$h'^2_2 = -c_1 h_2 + d_2, \tag{4.37}$$

$$h'^2_3 = c(c_1 h_3 + d_0 + d_1 + d_2)^3 - (c_1 h_3 + d_0 + d_1 + d_2). \tag{4.38}$$

By solving (4.37), we have

$$h_2(x_2) = -\frac{c_1}{4}(x_2 + c'_2)^2 + \frac{d_2}{c_1}. \tag{4.39}$$

According to (4.31), (4.32), (4.38) and (4.39), it may be concluded that

$$f(x_1, x_2, x_3) = \sqrt{-(x_1 + c'_1)^2 - (x_2 + c'_2)^2 + \frac{4}{c_1} h_3(x_3) + \frac{4}{c_1^2}(d_0 + d_1 + d_2) + c},$$

where $h_3(x_3)$ is a solution of equation (4.38).
Subcase B.2. $X_2' \neq -c_1$ and $X_3' \neq -c_1$. By (4.34) we get

$$\frac{X_2''}{X_2'+c_1} = -\frac{X_3''}{X_3'+c_1}. \tag{4.40}$$

Differentiating equation (4.40) with respect to u_2, we get

$$\left(\frac{X_2''}{X_2'+c_1}\right)' = 0,$$

and hence

$$\frac{X_2''}{X_2'+c_1} = -\frac{X_3''}{X_3'+c_1} = c_2. \tag{4.41}$$

(1). Consider $c_2 = 0$. Then $X_2'' = X_3'' = 0$. Hence,

$$\begin{aligned} X_2(u_2) &= a_2u_2 + d_2, \\ X_3(u_3) &= a_3u_3 + d_3, \end{aligned} \tag{4.42}$$

for some constants a_2, a_3, d_2, d_3, and $a_2 \neq -c_1, a_3 \neq -c_1$. By substituting (4.30) and (4.42) into (4.2), we get

$$Au_2 + Bu_3 + C = 0,$$

where $A = (a_2+c_1)(c_1-a_3)$, $B = (a_3+c_1)(c_1-a_2)$, $C = c_1(d_0+d_1+2d_2+2d_3) - a_2(d_0+d_1+d_3) - a_3(d_0+d_1+d_2)$. Since u_2, u_3 are linearly independent, A, B and C vanish identically. So we see that $a_2 = a_3 = c_1 \neq 0$, and $d_0 + d_1 - d_2 - d_3 = 0$. Now we have

$$\begin{aligned} h_2'^2 &= c_1h_2 + d_2, \\ h_3'^2 &= c_1h_3 + d_3. \end{aligned}$$

By solving the above two equations, we obtain

$$h_2(x_2) = \frac{c_1}{4}(x_2 + c_2')^2 - \frac{d_2}{c_1}, \tag{4.43}$$

$$h_3(x_3) = \frac{c_1}{4}(x_3 + c_3')^2 - \frac{d_3}{c_1}, \tag{4.44}$$

where c_2', c_3' are real constants. Combining (4.31), (4.32), (4.43) and (4.44) we can assert that

$$f(x_1, x_2, x_3) = \sqrt{-(x_1+c_1')^2 + (x_2+c_2')^2 + (x_3+c_3')^2 + c}.$$

(2). In the case of $c_2 \neq 0$, a straightforward computation shows that the solutions of (4.41) are of the forms

$$\begin{aligned} X_2(u_2) &= a_2e^{c_2u_2} - c_1u_2 + d_2, \\ X_3(u_3) &= a_3e^{-c_2u_3} - c_1u_3 + d_3, \end{aligned}$$

where a_2, a_3 are real constants. Since $X_2', X_3' \neq -c_1$, we obtain that $a_2, a_3 \neq 0$. Then the expressions (4.4) and (4.5) may be rewritten as

$$\begin{aligned} Au_2 + B &= 0, \\ Cu_3 + D &= 0, \end{aligned}$$

where $A = a_2c_1c_2^2$, $B = -2a_2c_1c_2 + a_2c_2^2(d_0+d_1+d_3)$, $C = a_3c_1c_2^2$, $D = 2a_3c_1c_2 + a_3c_2^2(d_0+d_1+d_2)$. We must ensure that A, B, C and D are vanishing identically.

Note that c_2, a_2 and a_3 are not equal to zero. It is easy to see that $d_0+d_1+d_2=0$ and $d_0+d_1+d_3=0$. An trivial verification shows that the equation (4.2) holds identically. Hence, we have

$$\begin{aligned}
&\frac{1}{F'^2} = d_0,\\
&h_1'^2 = d_1,\\
&h_2'^2 = a_2 e^{c_2 h_2} - d_0 - d_1,\\
&h_3'^2 = a_3 e^{-c_2 h_3} - d_0 - d_1,
\end{aligned}$$

where a_2, a_3, d_0, d_1 are positive real constants. Integrating the above differential equations gives rise to

$$\begin{aligned}
F(w) &= \frac{1}{\sqrt{d_0}} w + c_0',\\
h_1(x_1) &= \sqrt{d_1}(x_1 + c_1'),\\
h_2(x_2) &= \frac{1}{c_2} \ln \Big[\frac{d_0+d_1}{a_2} \sec^2\big(\frac{c_2\sqrt{d_0+d_1}}{2}(x_2+c_2')\big)\Big],\\
h_3(x_3) &= -\frac{1}{c_2} \ln \Big[\frac{d_0+d_1}{a_3} \sec^2\big(\frac{c_2\sqrt{d_0+d_1}}{2}(x_3+c_3')\big)\Big],
\end{aligned}$$

where c_0', c_1', c_2' and c_3' are real constants. We thus get

$$f(x_1,x_2,x_3) = \frac{1}{\sqrt{d_0}}\left[\sqrt{d_1}(x_1+c_1') + \frac{2}{c_2}\ln\frac{\cos\big(\frac{c_2\sqrt{d_0+d_1}}{2}(x_3+c_3')\big)}{\cos\big(\frac{c_2\sqrt{d_0+d_1}}{2}(x_2+c_2')\big)} + \frac{1}{c_2}\ln\frac{a_3}{a_2}\right] + c_0'.$$

Case C. X_1', X_2' and X_3' are not equal to each other and $G' \neq -X_1', -X_2', -X_3'$. We can write (4.8) in the form

$$\frac{G''-X_1''}{G'+X_1'} = \frac{X_2''-X_3''}{X_2'-X_3'}. \tag{4.45}$$

By differentiating (4.45) with respect to u_1 one has

$$G'G''' - G''^2 + X_1'G''' - X_1'''G' - X_1'X_1''' + X_1''^2 = 0. \tag{4.46}$$

Analysis similar to that in the proof of (4.46) shows that

$$G'G''' - G''^2 + X_2'G''' - X_2'''G' - X_2'X_2''' + X_2''^2 = 0. \tag{4.47}$$

Subtracting equation (4.46) from equation (4.47), we get

$$\begin{aligned}
&G'''(X_1'-X_2') - G'(X_1'''-X_2''')\\
&-(X_1'X_1'''-X_2'X_2''') + (X_1''^2 - X_2''^2) = 0.
\end{aligned} \tag{4.48}$$

Subcase C.1. $G'' \neq 0$. By taking the partial derivative of (4.48) with respect to u_3, we obtain

$$\frac{G^{(4)}}{G''} = \frac{X_1'''-X_2'''}{X_1'-X_2'}. \tag{4.49}$$

In the same way, one can see that

$$\frac{G^{(4)}}{G''} = \frac{X_1'''-X_3'''}{X_1'-X_3'} = \frac{X_2'''-X_3'''}{X_2'-X_3'} = \frac{X_1'''-X_2'''}{X_1'-X_2'} = K, \tag{4.50}$$

where K is a real constant. It follows from (4.50) that

$$\begin{aligned} G''' &= KG' + d, && (4.51)\\ X_1''' &= KX_1' + m, && (4.52)\\ X_2''' &= KX_2' + m, && (4.53)\\ X_3''' &= KX_3' + m, && (4.54) \end{aligned}$$

where $m, d \in \mathbb{R}$ are constants.
Substituting (4.52) into (4.46), we obtain

$$G'G''' - G''^2 - mG' = X_1'X_1''' - X_1''^2 - dX_1'. \tag{4.55}$$

Similarly,

$$G'G''' - G''^2 - mG' = X_2'X_2''' - X_2''^2 - dX_2', \tag{4.56}$$

$$G'G''' - G''^2 - mG' = X_3'X_3''' - X_3''^2 - dX_3'. \tag{4.57}$$

It follows from (4.55)-(4.57) that

$$G'G''' - G''^2 - mG' = k_1, \tag{4.58}$$

where $k_1 \in \mathbb{R}$ is a constant. Substituting (4.51) into (4.58), we obtain

$$KG'^2 - G''^2 + (d - m)G' = k_1. \tag{4.59}$$

Differentiating (4.59) with respect to u_1 implies that

$$2KG' - 2G''' + (d - m) = 0. \tag{4.60}$$

Substituting (4.51) into (4.60), we obtain

$$m + d = 0.$$

Hence, we can rewrite (4.51) as

$$G''' = KG' - m. \tag{4.61}$$

By solving (4.52)-(4.54) and (4.61), we get the following three cases:

When $K = k^2 > 0$ with $k > 0$, the solutions to equations (4.51)-(4.54) are given by

$$\begin{aligned} G &= a_0 \cosh(kw) + b_0 \sinh(kw) + \alpha w + c_0,\\ X_i &= a_i \cosh(ku_i) + b_i \sinh(ku_i) - \alpha u_i + c_i, \end{aligned} \tag{4.62}$$

where a_i, b_i, c_i $(0 \le i \le 3)$ and $\alpha = \frac{m}{k^2} = \frac{m}{K}$ are real constants. Since (4.62) must satisfy (4.2), these parameters are restricted. Substituting (4.62) into (4.2), we have

$$\begin{aligned} &\sum_{i=1}^{3} A_i \sinh(ku_i) + \sum_{i=1}^{3} B_i \cosh(ku_i) - \alpha k \sum_{i=1}^{3} a_i u_i \sinh(ku_i) - \alpha k \sum_{i=1}^{3} b_i u_i \cosh(ku_i)\\ &+ \sum_{i\neq j}^{3} C_i \sinh(ku_i + ku_j) + \sum_{i\neq j}^{3} D_i \cosh(ku_i + ku_j) + A_4 \sinh(ku_1 + ku_2 + ku_3)\\ &+ B_4 \cosh(ku_1 + ku_2 + ku_3) - a_0 \alpha k(u_1 + u_2 + u_3) \sinh(ku_1 + ku_2 + ku_3)\\ &- b_0 \alpha k(u_1 + u_2 + u_3) \cosh(ku_1 + ku_2 + ku_3) + E = 0. \end{aligned}$$

Since these hyperbolic functions are linearly independent, all of the coefficients must be zero.

If $\alpha \neq 0$, then $a_i, b_i = 0, 0 \le i \le 3$, which contradicts to $G'' \neq 0$.

If $\alpha=0$, we have

$$\begin{cases} a_0a_1-b_0b_1-a_2a_3-b_2b_3=0,\\ a_0a_2-b_0b_2-a_1a_3-b_1b_3=0,\\ a_0a_3-b_0b_3-a_1a_2-b_1b_2=0,\\ a_1b_0-a_0b_1-a_2b_3-a_3b_2=0,\\ a_2b_0-a_0b_2-a_1b_3-a_3b_1=0,\\ a_3b_0-a_0b_3-a_1b_2-a_2b_1=0,\\ b_0(c_1+c_2+c_3)=0,\\ b_1(c_0+c_2+c_3)=0,\\ b_2(c_0+c_1+c_3)=0,\\ b_3(c_0+c_1+c_2)=0,\\ a_0(c_1+c_2+c_3)=0,\\ a_1(c_0+c_2+c_3)=0,\\ a_2(c_0+c_1+c_3)=0,\\ a_3(c_0+c_1+c_2)=0, \end{cases} \tag{4.63}$$

and the equations (4.62) reduce to

$$G=a_0\cosh(kw)+b_0\sinh(kw)+c_0, \tag{4.64}$$
$$X_i=a_i\cosh(ku_i)+b_i\sinh(ku_i)+c_i,\quad 1\le i\le 3. \tag{4.65}$$

Since $G''\neq 0$, at most one of a_0 and b_0 is zero. According to the seventh and the eleventh equations of (4.63), we can obtain $c_1+c_2+c_3=0$. The proof falls naturally into four subcases.
(1). $X_1''=X_2''=X_3''=0$. Then $a_i=b_i=0, 1\le i\le 3$, and $X_1'=X_2'=X_3'=0$, which contradicts to the assumption of Case C.
(2). One of the terms X_1'', X_2'', X_3'' is not equal to 0. Without loss of generality we can assume that $X_1''=X_2''=0$ and $X_3''\neq 0$. Then $a_1=b_1=a_2=b_2=0$, and $X_1'=X_2'=0$, which contradicts the assumption of Case C.
(3). Two of the terms X_1'', X_2'', X_3'' are not equal to 0. Without loss of generality, we may suppose that $X_1''=0$, $X_2''\neq 0$, $X_3''\neq 0$. Then $a_1=b_1=0$, at most one of the terms a_2, b_2 is 0, at most one of a_3, b_3 is 0 and $c_0+c_1+c_2=0$, $c_0+c_1+c_3=0$.

If $a_0, b_0, a_2, b_2, a_3, b_3\neq 0$, the first, second and sixth equations of (4.63) reduce to

$$\begin{cases} a_2a_3+b_2b_3=0,\\ a_0a_2-b_0b_2=0,\\ a_3b_0-a_0b_3=0. \end{cases} \tag{4.66}$$

Obviously, the above three equations are contradictory.

If $a_0=0, b_0\neq 0$ (or $a_0\neq 0, b_0=0$), then $a_2=b_2=a_3=b_3=0$, which contradicts to $X_2''\neq 0$ and $X_3''\neq 0$.

If $a_2=0, b_2\neq 0$ (or $a_2\neq 0, b_2=0$), then $a_0=b_0=a_3=b_3=0$, which contradicts to $G''\neq 0$ and $X_3''\neq 0$.

If $a_3=0, b_3\neq 0$ (or $a_3\neq 0, b_3=0$), then $a_0=b_0=a_2=b_2=0$, which contradicts to $G''\neq 0$ and $X_2''\neq 0$.
(4). All of the terms X_1'', X_2'' and X_3'' are not equal to 0. Then (4.63) can be

rewritten as $c_0 = c_1 = c_2 = c_3 = 0$ and

$$\begin{cases} a_0a_1 - b_0b_1 - a_2a_3 - b_2b_3 = 0, \\ a_0a_2 - b_0b_2 - a_1a_3 - b_1b_3 = 0, \\ a_0a_3 - b_0b_3 - a_1a_2 - b_1b_2 = 0, \\ a_1b_0 - a_0b_1 - a_2b_3 - a_3b_2 = 0, \\ a_2b_0 - a_0b_2 - a_1b_3 - a_3b_1 = 0, \\ a_3b_0 - a_0b_3 - a_1b_2 - a_2b_1 = 0, \end{cases} \tag{4.67}$$

where at least one of a_i, b_i is nonzero for any $0 \le i \le 3$.

When $K = -k^2 < 0$ with $k > 0$, the solutions of (4.51)-(4.54) are given by

$$\begin{aligned} G &= a_0 \cos(kw) + b_0 \sin(kw) + \alpha w + c_0, \\ X_i &= a_i \cos(ku_i) + b_i \sin(ku_i) - \alpha u_i + c_i, \end{aligned} \tag{4.68}$$

where a_i, b_i, c_i $(0 \le i \le 3)$ and $\alpha = -\frac{m}{k^2} = \frac{m}{K}$ are real constants. Since (4.68) must satisfy (4.2), these parameters are restricted. Substituting (4.68) into (4.2) gives rise to

$$\begin{aligned} &\sum_{i=1}^{3} A_i \sin(ku_i) + \sum_{i=1}^{3} B_i \cos(ku_i) + \alpha k \sum_{i=1}^{3} a_i u_i \sin(ku_i) - \alpha k \sum_{i=1}^{3} b_i u_i \cosh(ku_i) \\ &+ \sum_{i \neq j}^{3} C_i \sin(ku_i + ku_j) + \sum_{i \neq j}^{3} D_i \cos(ku_i + ku_j) + A_4 \sin(ku_1 + ku_2 + ku_3) \\ &+ B_4 \cos(ku_1 + ku_2 + ku_3) + a_0 \alpha k (u_1 + u_2 + u_3) \sin(ku_1 + ku_2 + ku_3) \\ &- b_0 \alpha k (u_1 + u_2 + u_3) \cos(ku_1 + ku_2 + ku_3) + E = 0. \end{aligned}$$

Since these trigonometric functions are linearly independent, all of the coefficients must be zero.

If $\alpha \neq 0$, then $a_i, b_i = 0, 0 \le i \le 3$, which contradicts to $G'' \neq 0$.

If $\alpha = 0$, we have

$$\begin{cases} a_0a_1 + b_0b_1 - a_2a_3 + b_2b_3 = 0, \\ a_0a_2 + b_0b_2 - a_1a_3 + b_1b_3 = 0, \\ a_0a_3 + b_0b_3 - a_1a_2 + b_1b_2 = 0, \\ a_1b_0 - a_0b_1 - a_2b_3 - a_3b_2 = 0, \\ a_2b_0 - a_0b_2 - a_1b_3 - a_3b_1 = 0, \\ a_3b_0 - a_0b_3 - a_1b_2 - a_2b_1 = 0, \\ b_0(c_1 + c_2 + c_3) = 0, \\ b_1(c_0 + c_2 + c_3) = 0, \\ b_2(c_0 + c_1 + c_3) = 0, \\ b_3(c_0 + c_1 + c_2) = 0, \\ a_0(c_1 + c_2 + c_3) = 0, \\ a_1(c_0 + c_2 + c_3) = 0, \\ a_2(c_0 + c_1 + c_3) = 0, \\ a_3(c_0 + c_1 + c_2) = 0, \end{cases} \tag{4.69}$$

and the equations (4.68) reduce to

$$\begin{aligned} G &= a_0\cos(kw) + b_0\sin(kw) + c_0, \\ X_i &= a_i\cos(ku_i) + b_i\sin(ku_i) + c_i, \quad 1 \le i \le 3. \end{aligned} \tag{4.70}$$

Similarly, there is one possibility: $X_1'', X_2'', X_3'' \neq 0$. Then (4.69) can be rewritten as $c_0 = c_1 = c_2 = c_3 = 0$ and

$$\begin{cases} a_0a_1 + b_0b_1 - a_2a_3 + b_2b_3 = 0, \\ a_0a_2 + b_0b_2 - a_1a_3 + b_1b_3 = 0, \\ a_0a_3 + b_0b_3 - a_1a_2 + b_1b_2 = 0, \\ a_1b_0 - a_0b_1 - a_2b_3 - a_3b_2 = 0, \\ a_2b_0 - a_0b_2 - a_1b_3 - a_3b_1 = 0, \\ a_3b_0 - a_0b_3 - a_1b_2 - a_2b_1 = 0, \end{cases} \tag{4.71}$$

where at least one of a_i, b_i is nonzero for any $0 \le i \le 3$.

Consider $K = 0$. For $1 \le i \le 3$, the solutions of (4.51)-(4.54) are given by

$$\begin{aligned} G &= -\frac{m}{6}w^3 + a_0w^2 + b_0w + c_0, \\ X_i &= \frac{m}{6}u_i^3 + a_iu_i^2 + b_iu_i + c_i, \quad 1 \le i \le 3. \end{aligned} \tag{4.72}$$

Substituting (4.72) into (4.2), we have

$$\begin{aligned} &3\beta^2\sum_{i\neq j}^{3} u_i^4u_j + 6\beta^2\sum_{i\neq j}^{3} u_i^2u_j^3 + 18\beta^2\sum_{i\neq j\neq k}^{3} u_iu_j^2u_k^2 + 12\beta^2\sum_{i\neq j\neq k}^{3} u_iu_ju_k^3 \\ &-\beta\sum_{i=1}^{3}(a_0+a_i)u_i^4 - 4a_0\beta\sum_{i\neq j}^{3} u_iu_j^3 - 6a_0\beta\sum_{i\neq j}^{3} u_i^2u_j^2 \\ &+6\beta(a_1+a_2+a_3-a_0)\sum_{i\neq j\neq k}^{3} u_iu_ju_k^2 - 2\beta\sum_{i=1}^{3}(b_i+b_0)u_i^3 + \sum_{i\neq j}^{3} A_iu_iu_j^2 \\ &-4a_0(a_1+a_2+a_3)u_1u_2u_3 + \sum_{i=1}^{3} B_iu_i^2 + \sum_{i\neq j}^{3} C_iu_iu_j + \sum_{i=1}^{3} D_iu_i + E = 0, \end{aligned}$$

where $\beta = \frac{m}{6}$ is a real constant. Since these functions are linearly independent, all of the coefficients must be zero. It is clear that $\beta = \frac{m}{6} = 0$. Then we have

$$\begin{aligned} &b_0(c_1+c_2+c_3) - b_1(c_0+c_2+c_3) - b_2(c_0+c_1+c_3) - b_3(c_0+c_1+c_2) = 0, \\ &2a_0(c_1+c_2+c_3) - 2a_1(c_0+c_2+c_3) - (b_0+b_1)(b_2+b_3) = 0, \\ &2a_0(c_1+c_2+c_3) - 2a_2(c_0+c_1+c_3) - (b_1+b_3)(b_0+b_2) = 0, \\ &2a_0(c_1+c_2+c_3) - 2a_3(c_0+c_1+c_2) - (b_1+b_2)(b_0+b_3) = 0, \\ &a_0(b_1-b_2-b_3) - a_1(b_0+b_2+b_3) = 0, \\ &a_0(b_2-b_1-b_3) - a_2(b_0+b_1+b_3) = 0, \\ &a_0(b_3-b_1-b_2) - a_3(b_0+b_1+b_2) = 0, \\ &a_1(b_0+b_2) + a_2(b_0+b_1) + a_0b_3 = 0, \\ &a_1(b_0+b_3) + a_3(b_0+b_1) + a_0b_2 = 0, \\ &a_2(b_0+b_3) + a_3(b_0+b_2) + a_0b_1 = 0, \end{aligned}$$

$$a_1(a_0+a_2)+a_0a_2=0,$$
$$a_1(a_0+a_3)+a_0a_3=0,$$
$$a_2(a_0+a_3)+a_0a_3=0,$$
$$a_0(a_1+a_2+a_3)=0,$$

and the equations (4.72) reduce to

$$\begin{aligned} &G=a_0w^2+b_0w+c_0,\\ &X_i=a_iu_i^2+b_iu_i+c_i, \quad 1\le i\le 3. \end{aligned} \tag{4.73}$$

Notice that $G''\neq 0$, we have $a_0\neq 0$, and then $a_1+a_2+a_3=0$. We next analyze the four subcases.
(1). $X_1''=X_2''=X_3''=0$. Then $a_i=0, 1\le i\le 3$. This gives $b_1=b_2=b_3=0, c_1+c_2+c_3=0$. This contradicts to the fact that $X_i=h_i'^2=c_i>0, 1\le i\le 3$.
(2). Only one of the terms X_1'', X_2'' and X_3'' is not equal to 0. Without loss of generality, suppose $X_1''=X_2''=0$, $X_3''\neq 0$. Then $a_1=a_2=0, a_3\neq 0$. This contradicts to the fact that $a_1+a_2+a_3=0$.
(3). Two of the terms X_1'', X_2'' and X_3'' are not equal to 0. Without loss of generality, we suppose $X_1''=0$, $X_2''\neq 0$, $X_3''\neq 0$. Then $a_1=0, a_2\neq 0, a_3\neq 0$. This clearly forces $a_0=0$, which contradicts to the fact that $G''\neq 0$.
(4). $X_1'', X_2'', X_3''\neq 0$. Then $a_i\neq 0, 1\le i\le 3$. It follow that $a_1=a_2=a_3$, and then $a_1=a_2=a_3=0$, a contradiction.
Subcase C.2. $G''=0$. It follows that

$$G=cw+d, \tag{4.74}$$

where c, d are real constants. Therefore, equation (4.46) reduces to

$$-(c+X_1')X_1'''+X_1''^2=0. \tag{4.75}$$

By solving (4.75), the result is

$$X_1=-cu_1+a_1e^{b_1u_1}+c_1. \tag{4.76}$$

Similarly,

$$X_i=-cu_i+a_ie^{b_iu_i}+c_i, \quad 1\le i\le 3. \tag{4.77}$$

Then, substituting (4.77) into (4.2), we get

$$\begin{aligned} &-c\sum_{i=1}^{3}a_ib_iu_ie^{b_iu_i}+\sum_{i\neq j\neq k}^{3}[3a_ic-a_ib_i(d+c_j+c_k)]e^{b_iu_i}\\ &-\sum_{i\neq j}^{3}a_ia_j(b_i+b_j)e^{b_iu_i+b_ju_j}+3c(d+c_1+c_2+c_3)=0. \end{aligned}$$

Since these exponential functions are linearly independent, all of the coefficients must vanish.

If $c \neq 0$, and hence

$$\begin{cases} d + c_1 + c_2 + c_3 = 0, \\ a_1 b_1 = a_2 b_2 = a_3 b_3 = 0, \\ 3a_1 c - a_1 b_1 (d + c_2 + c_3) = 0, \\ 3a_2 c - a_2 b_2 (d + c_1 + c_3) = 0, \\ 3a_3 c - a_3 b_3 (d + c_1 + c_2) = 0, \\ a_1 a_2 (b_1 + b_2) = 0, \\ a_1 a_3 (b_1 + b_3) = 0, \\ a_2 a_3 (b_2 + b_3) = 0. \end{cases} \tag{4.78}$$

If $a_1 = a_2 = a_3 = 0$, (4.78) reduce to $d + c_1 + c_2 + c_3 = 0$; if $a_1 = a_2 = b_3 = 0$, according to (4.78), we obtain $a_3 = 0$. Similarly, if $a_1 = b_2 = b_3 = 0$ or $b_1 = b_2 = b_3 = 0$, we can deduce that $a_1 = a_2 = a_3 = 0$. Hence, we obtain $X_1' = X_2' = X_3' = -c$, which contradicts to the assumption that the terms X_1', X_2' and X_3' are not equal to each other.

If $c = 0$, and hence

$$\begin{cases} a_1 b_1 (d + c_2 + c_3) = 0, \\ a_2 b_2 (d + c_1 + c_3) = 0, \\ a_3 b_3 (d + c_1 + c_2) = 0, \\ a_1 a_2 (b_1 + b_2) = 0, \\ a_1 a_3 (b_1 + b_3) = 0, \\ a_2 a_3 (b_2 + b_3) = 0. \end{cases} \tag{4.79}$$

Moreover, the equations (4.77) and (4.74) reduce to

$$X_i = a_i e^{b_i u_i} + c_i, \quad 1 \leq i \leq 3, \tag{4.80}$$

$$G = d. \tag{4.81}$$

As $G = \frac{1}{F'^2}$ we have $G = d > 0$. According to the assumption that X_1', X_2' and X_3' are not equal to each other and $G' \neq -X_1', -X_2', -X_3'$, we can obtain that $a_i, b_i \neq 0$ for $1 \leq i \leq 3$, which contradicts to (4.79).

In summary, we obtain the following theorem.

Theorem 4.1. *Let f be a differentiable quasi-sum production function given by (4.1) and denote $X_i = h_i'^2$ and $G = \frac{1}{F'^2}$. Then the production hypersurface of f is minimal if and only if, up to a suitable translation, one of the following cases holds:*
(1) *The production function f is a linear function;*
(2) *The function is given by*

$$f(x_1, x_2, x_3) = F(\frac{c_1}{4} \sum_{i=1}^{3} x_i^2 - \frac{1}{c_1} \sum_{i=1}^{n} d_i),$$

where $c_1 \neq 0$ and $F = \int \frac{1}{\sqrt{c(c_1 w + \sum_{i=1}^{3} d_i)^3 - (c_1 w + \sum_{i=1}^{3} d_i)}} dw$;
(3) *The function is given by*

$$f(x_1, x_2, x_3) = \sqrt{-\frac{4}{c_1} h_1(x_1) - x_2^2 - x_3^2 + \frac{4}{c_1^2}(d_0 + d_2 + d_3)},$$

where $c_1 < 0$ *and* $h_1(x_1)$ *is a solution of equation* (4.20)*;*
(4) *The function is given by*

$$f(x_1, x_2, x_3) = \sqrt{x_2^2 + x_3^2 - x_1^2};$$

(5) *The function is given by*

$$f(x_1, x_2, x_3) = \frac{2}{c_2\sqrt{d_2 + d_3}} \arctan \sqrt{\frac{c_0}{c} \sec^2 \left(\frac{c_2\sqrt{d_2 + d_3}}{2} x_1\right) e^{\sqrt{d_2}x_2 + \sqrt{d_3}x_3} - 1},$$

where c, c_0, d_2, d_3 *are positive constants;*
(6) *The function is given by*

$$f(x_1, x_2, x_3) = \sqrt{-x_1^2 - x_2^2 + \frac{4}{c_1} h_3(x_3) + \frac{4}{c_1^2}(d_0 + d_1 + d_2)},$$

where $c_1 \neq 0$ *and* $h_3(x_3)$ *is a solution of equation* (4.38)*;*
(7) *The function is given by*

$$f(x_1, x_2, x_3) = \frac{1}{\sqrt{d_0}} \left[\sqrt{d_1} x_1 + \frac{2}{c_2} \ln \frac{\cos\left(\frac{c_2\sqrt{d_0+d_1}}{2} x_3\right)}{\cos\left(\frac{c_2\sqrt{d_0+d_1}}{2} x_2\right)} + \frac{1}{c_2} \ln \frac{a_3}{a_2} \right],$$

where d_0, d_1 *are positive constants, and* a_2, a_3, c_2 *are nonzero constants;*
(8) *Case* $K = k^2, k > 0$.

$$\begin{aligned} G &= a_0 \cosh(kw) + b_0 \sinh(kw), \\ X_i &= a_i \cosh(ku_i) + b_i \sinh(ku_i), \quad 1 \leq i \leq 3, \end{aligned}$$

where $a_i, b_i (0 \leq i \leq 3)$ *satisfy the relations* (4.67)*;*
(9) *Case* $K = -k^2, k > 0$.

$$\begin{aligned} G &= a_0 \cos(kw) + b_0 \sin(kw), \\ X_i &= a_i \cos(ku_i) + b_i \sin(ku_i), \quad 1 \leq i \leq 3, \end{aligned}$$

where $a_i, b_i (0 \leq i \leq 3)$ *satisfy the relations* (4.71).

5. Final remark and examples

Note that, in fact, Theorem 3.2 and Theorem 4.1 provide a new method to construct new minimal hypersurfaces in $\mathbb{E}^3$ and $\mathbb{E}^4$. Generally, it is hard to give explicit parametric equations of minimal hypersurfaces, but this is quite important in differential geometry. For $n > 3$, the situation becomes more complicated and we leave this problem for further study.

In the next, we state three examples of Theorem 3.2 and Theorem 4.1.

Example 5.1. Consider $K > 0$ in Theorem 3.2. Without loss of generality we can assume that $k = 1$. Consider the following constants:

$$\begin{aligned} &a_0 = 0, \quad b_0 = 1, \quad c_0 = -1, \\ &a_1 = 0, \quad b_1 = 1, \quad c_1 = -1, \\ &a_2 = 2, \quad b_2 = 2, \quad c_2 = 0. \end{aligned}$$

Then,

$$\begin{aligned} G(w) &= \cosh w - \sinh w = e^{-w}, \\ X_1(u_1) &= \cosh u_1 - \sinh u_1 = e^{-u_1}, \\ X_2(u_2) &= -2 + 2\cosh u_2. \end{aligned} \tag{5.1}$$

By solving (5.1), we get

$$\begin{aligned} F(w) &= 2e^{\frac{w}{2}} + c', \\ h_1(x_1) &= 2\ln\frac{(x_1 + c_1')}{2}, \\ h_2(x_2) &= 2\ln\tan\frac{(x_2 + c_2')}{2}, \end{aligned}$$

where c', c_1', c_2' are real constants. So we obtain a family of two-factor quasi-sum production functions with minimality as follows:

$$F(x_1, x_2) = (x_1 + c_1')\tan\frac{(x_2 + c_2')}{2} + c'.$$

Example 5.2. Consider $K = 0$ in Theorem 3.2. We choose the following constants:

$$\begin{aligned} &a_0 = 0, \quad b_0 = -4, \quad c_0 = 4, \\ &a_1 = 0, \quad b_1 = 4, \quad c_1 = 0, \\ &a_2 = 0, \quad b_2 = 4, \quad c_2 = 0. \end{aligned}$$

Then,

$$\begin{aligned} G(w) &= 4w^2 - 4w, \\ X_1(u_1) &= 4u_1, \\ X_2(u_2) &= 4u_2. \end{aligned} \tag{5.2}$$

By solving (5.2) we get

$$\begin{aligned} F(w) &= \frac{1}{2}\ln\left(2w - 1 + \sqrt{4w^2 - 4w}\right) + c', \\ h_1(x_1) &= (x_1 + c_1')^2, \\ h_2(x_2) &= (x_2 + c_2')^2, \end{aligned}$$

where c', c_1', c_2' are real constants. Hence one gets another a family of minimal quasi-sum production functions with two inputs as follows:

$$\begin{aligned} F(x_1, x_2) =& \frac{1}{2}\ln\Big(2\big((x_1 + c_1')^2 + (x_2 + c_2')^2\big) - 1 \\ &+ \sqrt{4\big((x_1 + c_1')^2 + (x_2 + c_2')^2\big)^2 - 4\big((x_1 + c_1')^2 + (x_2 + c_2')^2\big)}\Big) + c'. \end{aligned}$$

Example 5.3. Let $K > 0$ in Theorem 4.1. Without loss of generality we can assume that $k = 1$. Consider the following constants:

$$\begin{aligned} &a_0 = 1, \quad b_0 = 1, \\ &a_1 = 1, \quad b_1 = 1, \\ &a_2 = 1, \quad b_2 = 1, \\ &a_3 = 1, \quad b_3 = -1. \end{aligned}$$

Then,

$$\begin{aligned} G(w) &= \cosh w + \sinh w = e^{w}, \\ X_1(u_1) &= \cosh u_1 + \sinh u_1 = e^{u_1}, \\ X_2(u_2) &= \cosh u_2 + \sinh u_2 = e^{u_2}, \\ X_3(u_3) &= \cosh u_3 - \sinh u_3 = e^{-u_3}. \end{aligned} \tag{5.3}$$

Solving (5.3) gives

$$\begin{aligned} F(w) &= -2e^{-\frac{w}{2}} + c', \\ h_1(x_1) &= -2\ln\big(-\frac{x_1 + c_1'}{2}\big), \\ h_2(x_2) &= -2\ln\big(-\frac{x_2 + c_2'}{2}\big), \\ h_3(x_3) &= 2\ln\big(\frac{x_3 + c_3'}{2}\big), \end{aligned}$$

where c', c_1', c_2' and c_3' are constants. Hence we obtain a family of minimal quasi-sum production models with three inputs as follows:

$$F(x_1, x_2, x_3) = -\frac{(x_1 + c_1')(x_2 + c_2')}{x_3 + c_3'} + c'.$$

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References

[1] J. Aczél and G. Maksa, Solution of the rectangular $m \times n$ generalized bisymmetry equation and of the problem of consistent aggregation, J. Math. Anal. Appl. 203 (1996), 104–126.

[2] H. Alodan, B.-Y. Chen, S. Deshmukh and G. E. Vîcu, On some geometric properties of quasi-product production models, J. Math. Anal. Appl. 474(1) (2019), 693–711.

[3] H. Alodan, B. Y. Chen, S. Deshmukh and G. E. Vîcu, Solution of the system of nonlinear PDEs characterizing CES property under quasi-homogeneity conditions, Adv. Diff. Eq. (2021): 257 https://doi.org/10.1186/s13662-021-03417-6

[4] K. J. Arrow, H. B. Chenery, B. S. Minhas, R. M. Solow, Capital-labor substitution and economic efficiency, Rev. Econ. Stat. 43 (3) (1961), 225–250.

[5] K. V. Bhanu Murthy, Arguing a case for Cobb-Douglas production function, Rev. Commer. Stud. 20-21 (1) (2002), 75–94.

[6] B. Y. Chen, Geometry of quasi-sum production functions with constant elasticity of substitution property, J. Adv. Math. Stud. 5(2) (2012), 90–97.

[7] B. Y. Chen, Solutions to homogeneous Monge-Ampère equations of homothetic functions and their applications to production models in economics, J. Math. Anal. Appl. 411 (2014), 223–229.

[8] B. Y. Chen, On some geometric properties of quasi-sum production models, J. Math. Anal. Appl. 392 (2012), 192–199.

[9] B. Y. Chen and G. E. Vilcu, Geometric classifications of homogeneous production functions, Appl. Math. Comput. 225 (2013), 67–80.

[10] C. W. Cobb, P. H. Douglas, A theory of production, Am. Econ. Rev. 18 (1928), 139–165.

[11] Y. Fu, W. G. Wang, Geometric characterizations of quasi-product production models in economics, Filomat 31 (2017), 1601–1609.

[12] S. Kayar, R. López, Classification of zero mean curvature surfaces of separable type in Lorentz-Minkowski space, to appear in Tohoku Mathematical Journal, arXiv: 2005.07663v1

[13] J. R. Hicks, The theory of wages, Macmillan, London(1932).
[14] D. McFadden, Constant elasticity of substitution production functions, Rev. Econ. Stud. 30 (2) (1963), 73–83.
[15] J. V. Robinson, The economics of imperfect competition, Macmillan, London (1933).
[16] H. Uzawa, Production functions with constant elasticities of substitution, Rev. Econ. Stud. 29 (4) (1962), 291–299.
[17] G. E. Vîlcu, A geometric perspective on the generalized Cobb-Douglas production functions, Appl. Math. Lett. 24(5) (2011), 777–783.
[18] A. D. Vîlcu and G. E. Vîlcu, On some geometric properties of the generalized CES production functions, Appl. Math. Comput. 218(1) (2011), 124–129.
[19] A. D. Vîlcu and G. E. Vîlcu, Some characterizations of the quasi-sum production models with proportional marginal rate of substitution, Comptes Rendus Math. 353 (2015), pp. 1129–1133.
[20] X. Wang, A geometric characterization of homogeneous production models in economics, Filomat 30(13) (2016), 3465–3471.
[21] X. Wang and Y. Fu, Some characterizations of the Cobb-Douglas and CES production functions in microeconomics, Abstr. Appl. Anal. (2013), Article ID 761832, 6 pages.