The right expression of the equivalent integral equation and non-uniqueness of solution of impulsive fractional order system

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Abstract: The fractional derivatives are not equal for different expressions of the same piecewise function, which caused that the equivalent integral equations of impulsive fractional order system (IFrOS) proposed in existing papers are incorrect. Thus we reconsider two generalized IFrOSs that both have the corresponding impulsive Caputo fractional order system and the corresponding impulsive Riemann-Liouville fractional order system as their special cases, and discover that their equivalent integral equations are two integral equations with some arbitrary constants, which reveal the non-uniqueness of solution of the two generalized IFrOSs. Finally, two numerical examples are offered for explaining the non-uniqueness of solution to the two generalized IFrOSs.

Key Words: impulsive fractional differential equations; non-uniqueness of solution; fractional differential equations; initial value problems

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1 Introduction

Since the idea of impulses was brought in the Caputo fractional order system, there appeared a lot of studies to focus on the subject of two kinds of impulsive fractional order systems (IFrOSs) (for details see [7–30] and references therein)

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\alpha} z(t) = h(t, z(t)), & t \in (t_0, S] \text{ and } t \neq t_k \ (k = 1, 2, ..., K), \\ z(t_k^+) - z(t_k^-) = \phi_k(z(t_k^-)), & k = 1, 2, ..., K, \\ z(t_0) = z_0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} {}_{t_0}^{C}\mathcal{D}_t^{\alpha} z(t) = h(t, z(t)), & t \in (t_0, S] \text{ and } t \neq t_k \ (k = 1, 2, ..., K), \\ z(t_k^+) = \psi_k(z(t_k^-)), & k = 1, 2, ..., K, \\ z(t_0) = z_0, \end{cases} \tag{1.2}$$

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where ${}_{t_0}^{C}\mathcal{D}_t^{\alpha}$ ($\alpha \in (0,1)$) represents the left-sided Caputo fractional derivative, $h : [t_0, S] \times \mathbb{R} \to \mathbb{R}$, $0 \le t_0 < t_1 < ... < t_K < t_{K+1} = S$, $\phi_k : \mathbb{R} \to \mathbb{R}$ and $\psi_k : \mathbb{R} \to \mathbb{R}$ ($k = 1, 2, ..., K$).

For (1.1) and (1.2), the piecewise function

$$z(t) = \begin{cases} z_0 + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau, z(\tau)) d\tau & \text{for } t \in [t_0, t_1], \\ z(t_k^+) + \int_{t_k}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau, z(\tau)) d\tau & \text{for } t \in (t_k, t_{k+1}], k = 1, ..., K, \end{cases} \tag{1.3}$$

was often used to discuss their properties (such as existence of solution, numerical solution, stability and controllability etc) in many existing papers. But, (1.3) does not satisfy the condition of fractional derivative in (1.1) and (1.2), and its fractional derivative is

$$\begin{aligned} {}_{t_0}^{C}\mathcal{D}_t^{\alpha} z(t)\big|_{t \in (t_k, t_{k+1}]} &= \int_{t_0}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} z'(s) ds \quad \text{for } t \in (t_k, t_{k+1}] \ (k = 1, 2, ..., K) \\ &= \int_{t_0}^{t_1} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} z'(s) ds + \int_{t_1}^{t_2} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} z'(s) ds + ... + \int_{t_k}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} z'(s) ds \\ &= \int_{t_0}^{t_1} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left[z_0 + \int_{t_0}^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right]' ds \\ &\quad + \int_{t_1}^{t_2} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left[z(t_1^+) + \int_{t_1}^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right]' ds \\ &\quad + ... + \int_{t_k}^{t} \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \left[z(t_k^+) + \int_{t_k}^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right]' ds \\ &\neq h(t, z(t)) \quad \text{for } t \in (t_k, t_{k+1}] \ (k = 1, 2, ..., K). \end{aligned} \tag{1.4}$$

In fact, it has always been the incorrect understanding for the piecewise function (1.3) in existing studies. Let us reconsider another piecewise expression of (1.3):

$$\begin{aligned} z(t) = &\begin{cases} z_0 + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau, z(\tau)) d\tau, & t \in [t_0, t_1], \\ 0, & t \in (t_1, S], \end{cases} \\ &+ \sum_{k=1}^{K} \begin{cases} 0, \quad t \in (t_0, t_k], \\ z(t_k^+) + \int_{t_k}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h(\tau, z(\tau)) d\tau, \quad t \in (t_k, t_{k+1}], \\ 0, \quad t \in (t_{k+1}, S], \end{cases} \end{aligned} \tag{1.5}$$

which its fractional derivative is

$$\begin{aligned} {}_{t_0}^{C}\mathcal{D}_t^{\alpha} z(t) = &\begin{cases} h(t, z(t)), & t \in [t_0, t_1], \\ 0, & t \in (t_1, S], \end{cases} + \sum_{k=1}^{K} \begin{cases} 0, & t \in (t_0, t_k], \\ h(t, z(t)), & t \in (t_k, t_{k+1}], \\ 0, & t \in (t_{k+1}, S], \end{cases} \\ &= h(t, z(t)), \quad t \in \left([t_0, t_1] \cup \cup_{k=1}^{K} (t_k, t_{k+1}] \right). \end{aligned} \tag{1.6}$$

Thus (1.5) meets the condition of fractional derivative in (1.1) and (1.2).

Remark 1.1. Although (1.5) and (1.3) in value are equal, (1.5) meets the fractional derivative in (1.1) and (1.2) but (1.3) does not satisfy fractional derivative in (1.1) and (1.2).

Moreover, the above incorrect understanding also appeared in studies of two impulsive Riemann-Liouville fractional order systems [31, 32]:

$$\begin{cases} {}_{t_0}^{RL}\mathcal{D}_t^{\alpha} z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_k \ (k = 1, 2, ..., K), \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha} z(t)\big|_{t=t_k^+} - {}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha} z(t)\big|_{t=t_k^-} = \phi_k(z(t_k^-)), \quad k = 1, ..., K, \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha} z(t)\big|_{t\to t_0+} = z_0, \end{cases} \tag{1.7}$$

and

$$\begin{cases} {}_{t_0}^{RL}\mathcal{D}_t^{\alpha} z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_k \ (k = 1, 2, ..., K), \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha} z(t)\big|_{t=t_k^+} = \psi_k(z(t_k^-)), \quad k = 1, 2, ..., K, \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha} z(t)\big|_{t\to t_0+} = z_0, \end{cases} \tag{1.8}$$

where ${}_{t_0}^{RL}\mathcal{D}_t^{\alpha}$ and ${}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha}$ represents the left-sided Riemann-Liouville fractional derivative and the Riemann-Liouville fractional integral respectively.

The above incorrect understanding for (1.3) caused that the equivalent integral equations of the above four IFrOSs proposed in existing papers are incorrect. Motivated by the above discussion, we integrate the above impulsive Caputo fractional order systems and impulsive Riemann-Liouville fractional order systems by using the Hilfer generalized fractional derivative to reconsider the equivalent integral equation of two IFrOSs:

$$\begin{cases} {}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta} z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_k \ (k = 1, 2, ..., K), \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^+} - {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^-} = \phi_k(z(t_k^-)), \quad k = 1, ..., K, \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t\to t_0+} = z_0, \end{cases} \tag{1.9}$$

and

$$\begin{cases} {}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta} z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_k \ (k = 1, 2, ..., K), \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^+} = \psi_k(z(t_k^-)), \quad k = 1, 2, ..., K, \\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t\to t_0+} = z_0, \end{cases} \tag{1.10}$$

where ${}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}$ ($0 < \alpha < 1$ and $0 \leq \beta \leq 1$) denotes the Hilfer generalized fractional derivative and ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}$ ($\gamma = \alpha + \beta - \alpha\beta$) represents the Riemann-Liouville fractional integral, ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^+} = \lim_{\varepsilon\to 0^+} {}_{t_0}^{RL}\mathcal{I}_{t_k+\varepsilon}^{1-\gamma} z(t_k+\varepsilon)$ and ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^-} = \lim_{\varepsilon\to 0^-} {}_{t_0}^{RL}\mathcal{I}_{t_k+\varepsilon}^{1-\gamma} z(t_k+\varepsilon)$.

Remark 1.2. (1.1) and (1.7) are two special cases of (1.9) when $\beta = 1$ and $\beta = 0$ respectively, and (1.2) and (1.8) are two special cases of (1.10) when $\beta = 1$ and $\beta = 0$ respectively.

2 Preliminaries

Let $-\infty < t_0 < S < \infty$, and give some notations: the spaces of Lebesgue integrable functions $L^p(t_0, S)$ ($p \geq 1$), the spaces of continuous $C[t_0, S]$, the spaces of absolute continuous $AC[t_0, S]$ and the spaces of n-times continuously differentiable functions $C^n[t_0, S]$. And let

$$C_\gamma[t_0, S] = \{z : (t_0, S] \to \mathbb{R} : (t - t_0)^\gamma z(t) \in C[t_0, S]\} \quad (0 \leq \gamma < 1)$$

and

$$C_\gamma^n[t_0,S]=\left\{z\in C^{n-1}[t_0,S]: z^{(n)}\in C_\gamma[t_0,S]\right\}\ (n\in\mathbb{N}) \text{ and } C_\gamma^0[t_0,S]=C_\gamma[t_0,S].$$

Definition 2.1([1–3]). Let $z\in L^1(t_0,S)$. The Riemann-Liouville fractional integral ${}_{t_0}^{RL}\mathcal{I}_t^{\alpha}z(t)$ is defined by

$${}_{t_0}^{RL}\mathcal{I}_t^{\alpha}z(t)=\int_{t_0}^{t}\frac{(t-s)^{\alpha-1}x(s)}{\Gamma(\alpha)}ds\ \ (t>t_0,\ \alpha>0).$$

Definition 2.2 ([1–3]). The expression

$${}_{t_0}^{RL}\mathcal{D}_t^{\alpha}z(t)=D{}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha}z(t)\ \ (t>t_0,\ 0<\alpha<1,\ D=\frac{d}{dt}),$$

provided $D{}_{t_0}^{RL}\mathcal{I}_t^{1-\alpha}z(t)$ exists, is called the Riemann-Liouville fractional derivative.

Definition 2.3 ([4, 5]). Let $0<\alpha<1$ and $0\le\beta\le1$. The Hilfer generalized fractional derivative is defined by

$${}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}={}_{t_0}^{RL}\mathcal{I}_t^{\beta(1-\alpha)}D\left({}_{t_0}^{RL}\mathcal{I}_t^{(1-\beta)(1-\alpha)}\right),$$

which is the **Riemann-Liouville fractional derivative** and the **Caputo fractional derivative** when $\beta=0$ and $\beta=1$, respectively.

Define

$$C_{1-\gamma}^{\alpha,\beta}[t_0,S]=\left\{z\in C_{1-\gamma}[t_0,S],\ {}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}z\in C_{1-\gamma}[t_0,S]\right\}$$

and

$$C_{1-\gamma}^{\gamma}[t_0,S]=\left\{z\in C_{1-\gamma}[t_0,S],\ {}_{t_0}^{RL}\mathcal{D}_t^{\gamma}z\in C_{1-\gamma}[t_0,S]\right\}.$$

Theorem 2.4 ([6]). Let $\alpha\in(0,1)$, $\beta\in[0,1]$ and $\gamma=\alpha+\beta-\alpha\beta$. Let $f:(t_0,S]\times\mathbb{R}\to\mathbb{R}$ satisfy $f(\cdot,z(\cdot))\in C_{1-\gamma}[t_0,S]$ for any $z(\cdot)\in C_{1-\gamma}[t_0,S]$. If $z(t)\in C_{1-\gamma}^{\gamma}[t_0,S]$ satisfies the fractional differential equations

$$\begin{cases}{}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}z(t)=f(t,z(t)), & t\in(t_0,S],\\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t\to t_0+}=z_0,\end{cases}$$

iff $z(t)$ satisfies

$$z(t)=\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau,z(\tau))d\tau,\quad t\in(t_0,S].$$

Remark 2.5. In order to the existence of ${}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}\left[\int_{t_0}^{t}(t-\tau)^{\alpha-1}f(\tau,z(\tau))d\tau\right]$ in **Theorem 2.4**, we need add an assumption that for function $f:[t_0,S]\times\mathbb{R}\to\mathbb{R}$ there exist two positive constants L and M such that

$$|f(t,y)-f(s,z)|\le L|t-s|+M|y-z| \text{ for } \forall s,t\in[t_0,S] \text{ and } \forall y,z\in\mathbb{R}.$$

3 The equivalent integral equations of two IFrOSs

For simplicity, let $J_k=(t_k,t_{k+1}]$ $(k=0,1,...,K)$, $h=h(\tau,z(\tau))$,

$$\Lambda(t)=\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau \tag{3.1}$$

and

$$\Upsilon_k(t) = \frac{z_0 + \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_k)^{\gamma-1} + \int_{t_k}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau, \ k = 1, ..., K. \tag{3.2}$$

And for $0 \le \gamma < 1$ further define some function spaces:

$$\begin{aligned}
&\widehat{C}_{1-\gamma}[t_0, S] := \left\{ z : (t_0, S] \to \mathbb{R} : \ [t - t_i]^{1-\gamma} z(t) \in C[t_i, t_{i+1}], \ i = 0, 1, ..., K \right\}, \\
&\widehat{C}^{\gamma}_{1-\gamma}[t_0, S] := \left\{ z \in \widehat{C}_{1-\gamma}[t_0, S], \ {}^{RL}_{t_0}\mathcal{D}^{\gamma}_t z(t) \in \widehat{C}_{1-\gamma}[t_0, S] \right\}, \\
&IC([t_0, S], \mathbb{R}) := \Big\{ z \in \widehat{C}^{\gamma}_{1-\gamma}[t_0, S], \ {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \in C^1([t_0, t_1] \cup \cup_{k=1}^{K} (t_k, t_{k+1}]), \\
&\qquad \lim_{t \to t_k^+} \left[\frac{d}{dt} {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \right] < \infty, \lim_{t \to t_k^-} \left[\frac{d}{dt} {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \right] = \frac{d}{dt} {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \Big|_{t=t_k} < \infty \Big\}.
\end{aligned}$$

For (1.9) and (1.10), there are some hidden properties:

$$\begin{aligned}
(a) \quad & \lim_{\phi_k(z(t_k^-)) \to 0 \text{ for all } k \in \{1,2,...,K\}} \{\text{system (1.9)}\} \\
&= \begin{cases} {}^{HR}_{t_0}\mathcal{D}^{\alpha,\beta}_t z(t) = h(t, z(t)), \quad t \in (t_0, S], \\ {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \Big|_{t \to t_0+} = z_0. \end{cases} \\
&\Leftrightarrow z(t) = \frac{z_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} h d\tau, \quad t \in (t_0, S].
\end{aligned}$$

$$\begin{aligned}
(b) \quad & \lim_{t_k \to t_r \text{ for all } k \in \{1,2,...,K\} \text{ and } \forall r \in \{1,2,...,K\}} \{\text{system (1.9)}\} \\
&= \begin{cases} {}^{HR}_{t_0}\mathcal{D}^{\alpha,\beta}_t z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_r, \\ {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \Big|_{t = t_r^+} - {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \Big|_{t = t_r^-} = \sum_{k=1}^{K} \phi_k(z(t_r^-)), \\ {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \Big|_{t \to t_0+} = z_0. \end{cases}
\end{aligned}$$

$$\begin{aligned}
(c) \quad & \lim_{\substack{\left[\psi_k(z(t_k^-)) - z_0 - \frac{1}{\Gamma(\alpha-\gamma+1)} \int_{t_0}^{t_k} (t_k-\tau)^{\alpha-\gamma} h d\tau\right] \to 0 \\ \text{for all } k \in \{1,2,...,K\}}} \{\text{system (1.10)}\} \\
&= \begin{cases} {}^{HR}_{t_0}\mathcal{D}^{\alpha,\beta}_t z(t) = h(t, z(t)), \quad t \in (t_0, S], \\ {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z(t) \Big|_{t \to t_0+} = z_0. \end{cases} \\
&\Leftrightarrow z(t) = \frac{z_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} h d\tau, \quad t \in (t_0, S].
\end{aligned}$$

Remark 3.1. The property (c) of (1.10) is corresponding to the property (a) of (1.9), and no property of (1.10) is corresponding to the property (b) of (1.9). In particular, (1.9) is equivalence with (1.10) under $K = 1$ and $\phi_1(z) = \psi_1(z) - {}^{RL}_{t_0}\mathcal{I}^{1-\gamma}_t z$.

To seek the integral solution of (1.9), we consider fractional derivative of (1.9) in each subinterval to find a piecewise function

$$\widetilde{z}(t)=\begin{cases}\dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-\tau)^{\alpha-1}hd\tau, & t\in J_0,\\ 0, \quad t\in(t_1,S] & \end{cases}$$
$$+\sum_{k=1}^{K}\begin{cases}0, & t\in(t_0,t_k],\\ \dfrac{{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^+}}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^{t}(t-\tau)^{\alpha-1}hd\tau, & t\in J_k,\\ 0, & t\in(t_{k+1},S],\end{cases}\tag{3.3}$$

with ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^+}={}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^-}+\phi_k(z(t_k^-))$ and $1\leq k\leq K$.

Although $\widetilde{z}(t)$ meets these conditions of initial value, impulses and fractional derivative in (1.9), but $\widetilde{z}(t)$ reject the property (a) to be only regarded as **an approximate solution** of (1.9).

And then we discover the equivalence between (1.9) and the Volterra integral equation of the second kind by calculating the error between $\widetilde{z}(t)$ and the exact solution of (1.9).

Theorem 3.2. Let $0<\alpha<1$, $0\leq\beta\leq 1$ and $\gamma=\alpha+\beta-\alpha\beta$. And let function $h(\cdot,z(\cdot))$ satisfy

$$|h(t,x)-h(s,y)|\leq L|t-s|+M|x-y| \text{ for } \forall s,t\in[t_0,S] \text{ and } \forall x,y\in\mathbb{R},$$

where L and M are two positive constants.
If $z(t)\in IC([t_0,S],\mathbb{R})$ meets system (1.9) iff $z(t)$ meets

$$z(t)=\begin{cases}\Lambda(t), & t\in J_0,\\ \vdots & \\ \Lambda(t), & t\in J_K,\end{cases}+\sum_{k=1}^{K}\begin{cases}0, & t\in(t_0,t_k],\\ \dfrac{\phi_k(z(t_k^-))}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}, & t\in(t_k,S],\end{cases}$$
$$+\,\xi\phi_1(z(t_1^-))\left[\begin{cases}\Lambda(t), & t\in J_0,\\ 0, & t\in(t_1,S],\end{cases}+\begin{cases}0, & t\in J_0,\\ \Upsilon_1(t), & t\in J_1,\\ \vdots & \\ \Upsilon_1(t), & t\in J_K,\end{cases}-\begin{cases}\Lambda(t), & t\in J_0,\\ \Lambda(t), & t\in J_1,\\ \vdots & \\ \Lambda(t), & t\in J_K,\end{cases}\right]+\ldots\tag{3.4}$$
$$+\,\xi\phi_K(z(t_K^-))\left[\begin{cases}\Lambda(t), & t\in J_0,\\ \vdots & \\ \Lambda(t), & t\in J_{K-1},\\ 0, & t\in J_K,\end{cases}+\begin{cases}0, & t\in(t_0,t_K],\\ \Upsilon_K(t), & t\in J_K,\end{cases}-\begin{cases}\Lambda(t), & t\in J_0,\\ \Lambda(t), & t\in J_1,\\ \vdots & \\ \Lambda(t), & t\in J_K,\end{cases}\right]$$

where ξ is an arbitrary constant.

The proof of **Theorem 3.2** will be given in the section of appendix.

Remark 3.3. (3.4) in value is equal to

$$z(t)=\begin{cases}\dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau, \quad t\in J_0,\\ \dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau+\sum_{i=1}^{k}\frac{\phi_i(z(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}\\ \quad +\xi\displaystyle\sum_{i=1}^{k}\phi_i(z(t_i^-))\left[\frac{z_0+\int_{t_0}^{t_i}\frac{(t_i-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}+\int_{t_i}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right.\\ \quad \left.-\dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}-\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right], \quad t\in J_k, k=1,...,K.\end{cases} \tag{3.5}$$

But (3.5) does not satisfy the condition of fractional derivative in (1.9).

Next we consider the integral solution of (1.10). Similarly, (3.1) with ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^+}=\psi_k(z(t_k^-))$ (here $1\le k\le K$)

$$\begin{aligned}\widehat{z}(t)=&\begin{cases}\dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-\tau)^{\alpha-1}hd\tau, \quad t\in J_0,\\ 0, \quad t\in(t_1,S],\end{cases}\\ &+\sum_{k=1}^{K}\begin{cases}0, \quad t\in(t_0,t_k],\\ \dfrac{\psi_k(z(t_k^-))}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}+\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_k}^{t}(t-\tau)^{\alpha-1}hd\tau, \quad t\in J_k,\\ 0, \quad t\in(t_{k+1},S],\end{cases}\end{aligned} \tag{3.6}$$

is **an approximate solution** of (1.10), which $\widehat{z}(t)$ satisfies these conditions of initial value, impulses and fractional derivative in (1.10), but it dissatisfies the property (c).

On the other hand, we consider fractional derivative of (1.10) on whole interval $(t_0,S]$ to discover **a particular solution** of (1.10):

$$z(t)=\begin{cases}\Lambda(t), \quad t\in J_0,\\ \Lambda(t), \quad t\in J_1,\\ \quad\vdots\\ \Lambda(t), \quad t\in J_K\end{cases}+\sum_{k=1}^{K}\begin{cases}0, \quad t\in(t_0,t_k],\\ \dfrac{\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}, \quad t\in J_k,\\ 0, \quad t\in(t_{k+1},S].\end{cases} \tag{3.7}$$

The next theorem yields the equivalence between the Cauchy problem (1.10) and the Volterra integral equation of the second kind.

Theorem 3.4. Let $0<\alpha<1$, $0\le\beta\le1$ and $\gamma=\alpha+\beta-\alpha\beta$. And let function $h(\cdot,z(\cdot))$ satisfy

$$|h(t,x)-h(s,y)|\le L|t-s|+M|x-y| \text{ for } \forall s,t\in[t_0,S] \text{ and } \forall x,y\in\mathbb{R},$$

where L and M are two positive constants.

If $z(t) \in IC([t_0, S], \mathbb{R})$ satisfies (1.10) iff $z(t)$ meets

$$
\begin{aligned}
z(t) = & \begin{cases} \Lambda(t), & t \in J_0, \\ \Lambda(t), & t \in J_1, \\ \quad \vdots & \\ \Lambda(t), & t \in J_K, \end{cases} + \sum_{k=1}^{K} \begin{cases} 0, & t \in (t_0, t_k], \\ \dfrac{\psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_k)^{\gamma-1}, & t \in J_k, \\ 0, & t \in (t_{k+1}, S], \end{cases} \\
& + \eta_1 \left[\psi_1(z(t_1^-)) - z_0 - \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau \right] \\
& \times \left[\begin{cases} \Lambda(t), & t \in J_0, \\ 0, & t \in (t_1, S], \end{cases} + \begin{cases} 0, & t \in J_0, \\ \Upsilon_1(t), & t \in J_1, \\ 0, & t \in (t_2, S], \end{cases} - \begin{cases} \Lambda(t), & t \in J_0, \\ \Lambda(t), & t \in J_1, \\ 0, & t \in (t_2, S], \end{cases} \right] \\
& + \ldots + \eta_K \left[\psi_K(z(t_K^-)) - z_0 - \int_{t_0}^{t_K} \frac{(t_K-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau \right] \\
& \times \left[\begin{cases} \Lambda(t), & t \in J_0, \\ \quad \vdots & \\ \Lambda(t), & t \in J_{K-1}, \\ 0, & t \in J_K, \end{cases} + \begin{cases} 0, & t \in (t_0, t_K], \\ \Upsilon_K(t), & t \in J_K, \end{cases} - \begin{cases} \Lambda(t), & t \in J_0, \\ \Lambda(t), & t \in J_1, \\ \quad \vdots & \\ \Lambda(t), & t \in J_K, \end{cases} \right]
\end{aligned} \tag{3.8}
$$

where η_k $(1 \le k \le K)$ are some arbitrary constants.

The proof of **Theorem 3.4** will be given in the section of appendix.

Remark 3.5. (3.8) in value is equal with

$$
z(t) = \begin{cases} \dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} + \displaystyle\int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau, \quad t \in J_0, \\ \dfrac{\psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_k)^{\gamma-1} + \dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} \\ \quad + \displaystyle\int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \eta_k \left[\psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau \right] \\ \quad \times \left\{ \dfrac{z_0 + \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_k)^{\gamma-1} + \displaystyle\int_{t_k}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right. \\ \quad \left. - \dfrac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} - \displaystyle\int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right\}, \quad t \in J_k, k = 1, \ldots, K, \end{cases} \tag{3.9}
$$

but (3.9) does not satisfy the condition of fractional derivative in (1.10).

4 Applications

Let $\alpha = \frac{1}{2}$, $\beta \in [0, 1]$ and $\gamma = \frac{1}{2}(\beta + 1)$ in this section, and we provide two IFrOSs to expound **Theorems 3.2** and **3.4**.

Example 4.1. Consider the following IFrOS

$$\begin{cases} {}_0^{HR}\mathcal{D}_t^{\frac{1}{2},\beta}x(t) = t, \quad t \in (0,2] \text{ and } t \neq 1, \\ {}_0^{RL}\mathcal{I}_t^{\frac{1}{2}(1-\beta)}x(t)\Big|_{t=1^+} - {}_0^{RL}\mathcal{I}_t^{\frac{1}{2}(1-\beta)}x(t)\Big|_{t=1^-} = 1, \\ {}_0^{RL}\mathcal{I}_t^{\frac{1}{2}(1-\beta)}x(t)\Big|_{t\to 0+} = 1. \end{cases} \tag{4.1}$$

By **Theorem 3.2**, we compute the solution of (4.1):

$$\begin{aligned} x(t) &= \begin{cases} \Lambda(t), & t \in (0,1], \\ \Lambda(t), & t \in (1,2], \end{cases} + \begin{cases} 0, & t \in (0,1], \\ \dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}(t-1)^{\frac{1}{2}(\beta-1)}, & t \in (1,2], \end{cases} \\ &+ \xi\left[\begin{cases} \Lambda(t), & t \in (0,1], \\ 0, & t \in (1,2], \end{cases} + \begin{cases} 0, & t \in (0,1], \\ \Upsilon_1(t), & t \in (1,2], \end{cases} - \begin{cases} \Lambda(t), & t \in (0,1], \\ \Lambda(t), & t \in (1,2], \end{cases}\right] \end{aligned} \tag{4.2}$$

where ξ is an arbitrary constant,

$$\Lambda(t) = \frac{t^{\gamma-1}}{\Gamma(\gamma)} + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\tau d\tau = \frac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)} + \frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} \tag{4.3}$$

and

$$\begin{aligned} \Upsilon_1(t) &= \frac{1 + \int_0^1 \frac{(1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}\tau d\tau}{\Gamma(\gamma)}(t-1)^{\gamma-1} + \int_1^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\tau d\tau \\ &= \frac{1 + \frac{1}{\Gamma(3-\frac{1}{2}\beta)}}{\Gamma(\frac{1}{2}(\beta+1))}(t-1)^{\frac{1}{2}(\beta-1)} + \frac{1}{\Gamma(\frac{5}{2})}(t-1)^{\frac{3}{2}} + \frac{1}{\Gamma(\frac{3}{2})}(t-1)^{\frac{1}{2}}. \end{aligned} \tag{4.4}$$

Remark 4.2. (4.2) in value is equal to

$$x(t) = \begin{cases} \dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)} + \dfrac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}, \quad t \in (0,1], \\ \dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)} + \dfrac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} + \dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}(t-1)^{\frac{1}{2}(\beta-1)} \\ \quad + \xi\left\{ \dfrac{1 + \frac{1}{\Gamma(3-\frac{1}{2}\beta)}}{\Gamma(\frac{1}{2}(\beta+1))}(t-1)^{\frac{1}{2}(\beta-1)} + \dfrac{1}{\Gamma(\frac{5}{2})}(t-1)^{\frac{3}{2}} + \dfrac{1}{\Gamma(\frac{3}{2})}(t-1)^{\frac{1}{2}} \right. \\ \quad \left. - \dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)} - \dfrac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}} \right\}, \quad t \in (1,2], \end{cases} \tag{4.5}$$

Moreover, (4.1) without impulses is presented by

$$\begin{cases} {}_0^{HR}\mathcal{D}_t^{\frac{1}{2},\beta}x(t) = t, \quad t \in (0,2], \\ {}_0^{RL}\mathcal{I}_t^{\frac{1}{2}(1-\beta)}x(t)\Big|_{t\to 0+} = 1, \end{cases} \tag{4.6}$$

and by **Theorem 2.4** the solution of (4.6) is

$$x(t)=\frac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)}+\frac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}},\quad t\in(0,2]. \tag{4.7}$$

We find that (4.1) has many solutions because its solution (4.2) is with an arbitrary constant ξ, and we use the numerical simulation to show the non-uniqueness of solution of (4.1) and compare some solution trajectories of (4.1) with the solution trajectory of (4.6). Fig.1-4 show the solution trajectories of (4.1) and (4.6) with $\beta=0,\ 0.5,\ 0.8,\ 1$ respectively. In each figure, 'No impulse' denotes the solution trajectory of (4.6) with the corresponding β, and these curves '$\xi=0$', '$\xi=1$' and '$\xi=-1$' which are drawn by numerical simulation of (4.2) with $\xi=0,\ 1,\ -1$ respectively, represent three solution trajectories of (4.1) with the corresponding β.

Example 4.3. Consider another IFrOS

$$\begin{cases} {}_0^{HR}\mathcal{D}_t^{\frac{1}{2},\beta}x(t)=t, & t\in(0,2] \text{ and } t\neq 1,\\ {}_0^{RL}\mathcal{I}_t^{\frac{1}{2}(1-\beta)}x(t)\Big|_{t=1+}=1, & \\ {}_0^{RL}\mathcal{I}_t^{\frac{1}{2}(1-\beta)}x(t)\Big|_{t\to 0+}=1. & \end{cases} \tag{4.8}$$

By **Theorem 3.4**, we calculate the solution of (4.8):

$$\begin{aligned} x(t)=&\begin{cases}\Lambda(t), & t\in(0,1],\\ \Lambda(t), & t\in(1,2],\end{cases}+\begin{cases}0, & t\in(0,1],\\ \dfrac{-1}{\Gamma(\frac{1}{2}(\beta+1))\Gamma(3-\frac{1}{2}\beta)}(t-1)^{\frac{1}{2}(\beta-1)}, & t\in(1,2],\end{cases}\\ &-\frac{\eta}{\Gamma(3-\frac{1}{2}\beta)}\left[\begin{cases}\Lambda(t), & t\in(0,1],\\ 0, & t\in(1,2],\end{cases}+\begin{cases}0, & t\in(0,1],\\ \Upsilon_1(t), & t\in(1,2],\end{cases}-\begin{cases}\Lambda(t), & t\in(0,1],\\ \Lambda(t), & t\in(1,2],\end{cases}\right]\end{aligned} \tag{4.9}$$

where η is an arbitrary constant, and $\Lambda(t)$ and $\Upsilon_1(t)$ are defined by (4.3) and (4.4).

Remark 4.4. (4.9) in value is equal to

$$x(t)=\begin{cases}\dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)}+\dfrac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}, \quad t\in(0,1],\\ \dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)}+\dfrac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}-\dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))\Gamma(3-\frac{1}{2}\beta)}(t-1)^{\frac{1}{2}(\beta-1)}\\ \quad -\dfrac{\eta}{\Gamma(3-\frac{1}{2}\beta)}\left\{\dfrac{1+\frac{1}{\Gamma(3-\frac{1}{2}\beta)}}{\Gamma(\frac{1}{2}(\beta+1))}(t-1)^{\frac{1}{2}(\beta-1)}+\dfrac{1}{\Gamma(\frac{5}{2})}(t-1)^{\frac{3}{2}}\right.\\ \quad \left.+\dfrac{1}{\Gamma(\frac{3}{2})}(t-1)^{\frac{1}{2}}-\dfrac{1}{\Gamma(\frac{1}{2}(\beta+1))}t^{\frac{1}{2}(\beta-1)}-\dfrac{1}{\Gamma(\frac{5}{2})}t^{\frac{3}{2}}\right\}, \quad t\in(1,2],\end{cases} \tag{4.10}$$

Similarly, (4.8) has many solutions owing to arbitrariness of η in (4.9). We also apply the numerical simulation to show the non-uniqueness of solution of (4.8) and compare some solution trajectories of (4.8) with the solution trajectory of (4.6). Fig. 5-8 represent the solution trajectories of (4.8) and

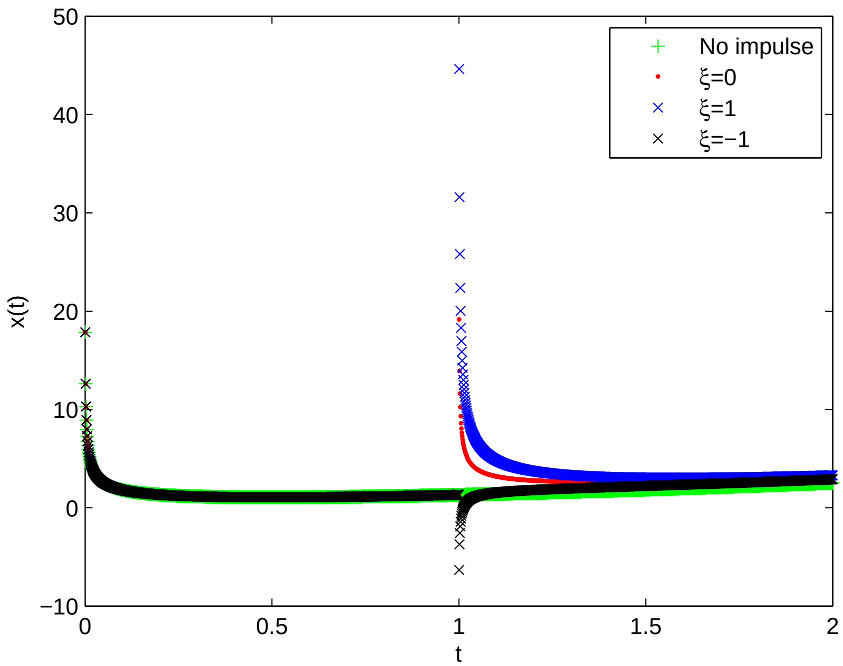

Fig. 1: The solution trajectories of (4.1) with $\beta = 0$

(4.6) with $\beta = 0,\ 0.5,\ 0.8,\ 1$ respectively. In each figure, 'No impulse' denotes the solution trajectory of (4.6) with the corresponding β, and these curves '$\eta = 0$', '$\eta = 1$' and '$\eta = -1$' which are drawn by the numerical simulation of (4.9) with $\eta = 0,\ 1,\ -1$ respectively, represent three solution trajectories of (4.8) with the corresponding β.

5 Appendix

We first prove **Theorem 4.2**.

Proof. 'Sufficiency' (it prove that the solution of (1.9) satisfies (3.4) by using mathematical induction). By **Theorem 2.4**, the solution of (1.9) as $t \in (t_0, t_1]$ satisfies

$$z(t) = \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}h d\tau, \quad t \in (t_0, t_1]. \tag{A.1}$$

Therefore the solution of (1.9) satisfies (3.4) as $t \in (t_0, t_1]$.

By (A.1) we have

$$\begin{aligned} {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_1^+} &= {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_1^-} + \phi_1(z(t_1^-)) \\ &= z_0 + \phi_1(z(t_1^-)) + \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau. \end{aligned} \tag{A.2}$$

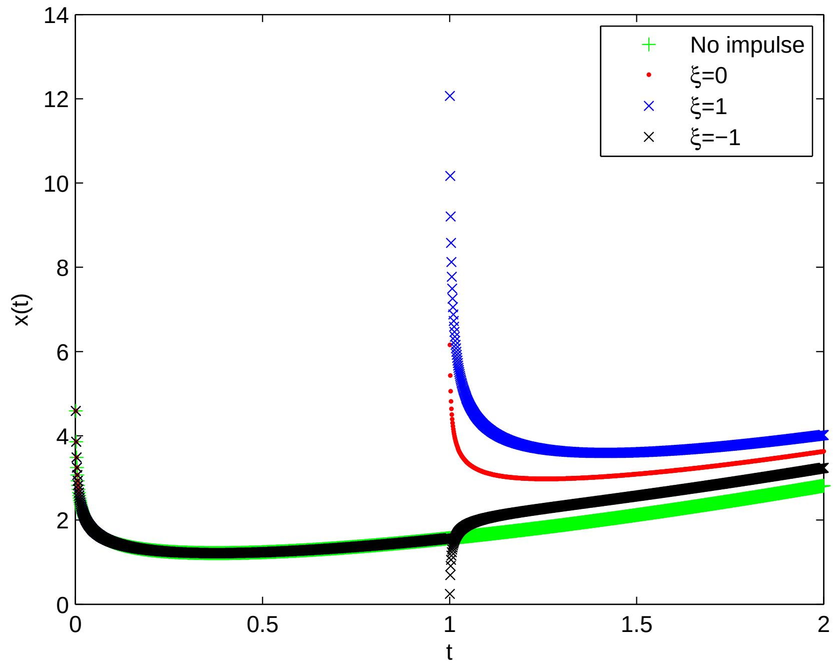

Fig. 2: The solution trajectories of (4.1) with $\beta = 0.5$

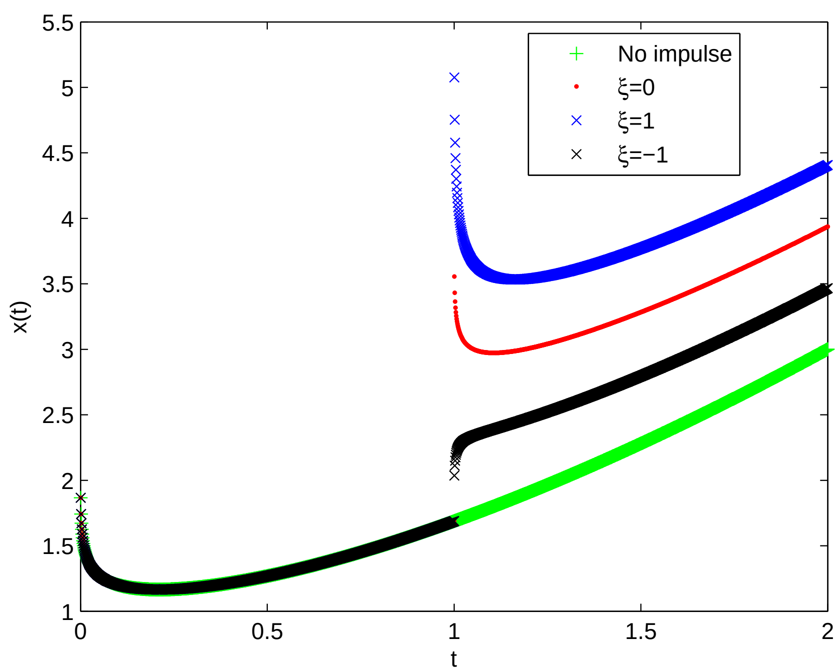

Fig. 3: The solution trajectories of (4.1) with $\beta = 0.8$

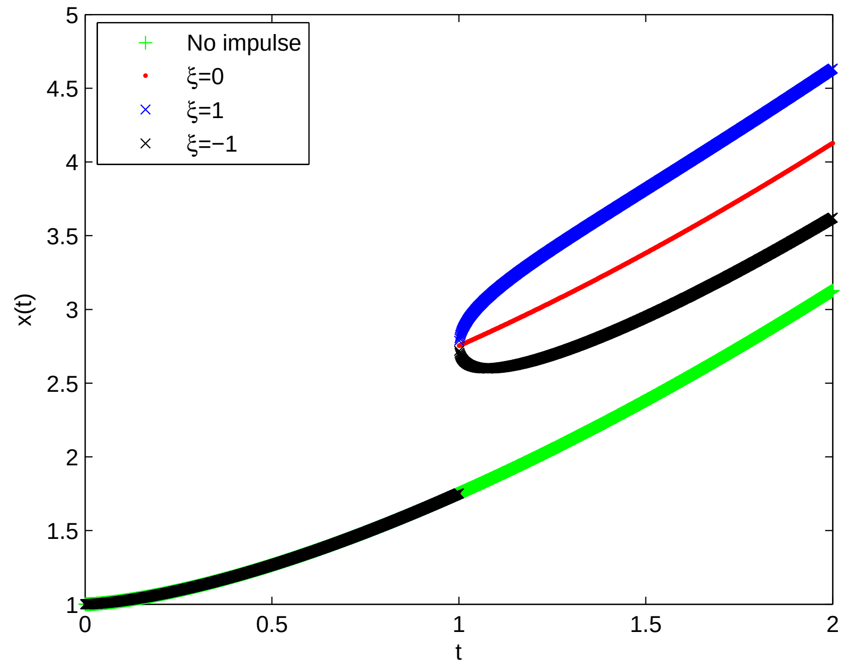

Fig. 4: The solution trajectories of (4.1) with $\beta = 1$

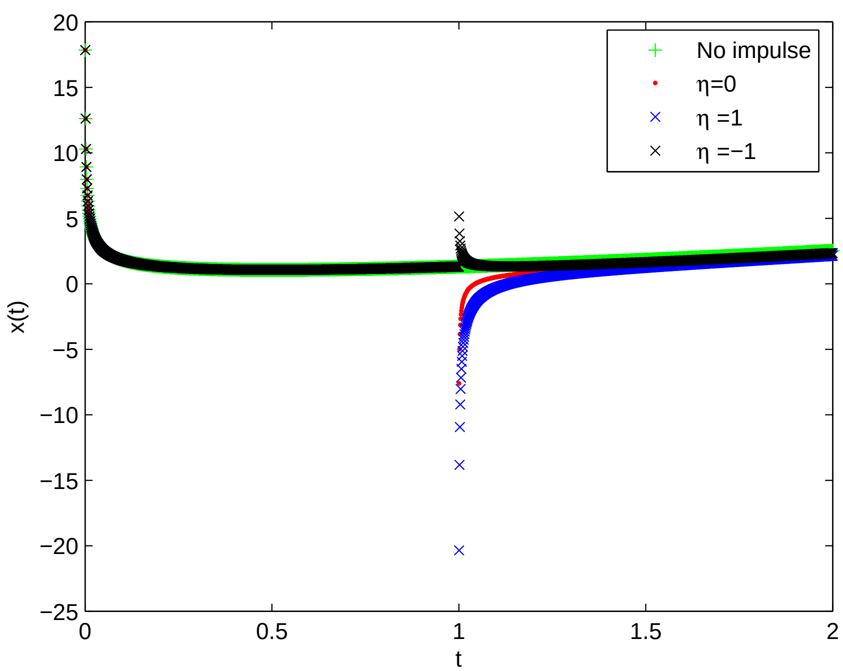

Fig. 5: The solution trajectories of (4.8) with $\beta = 0$

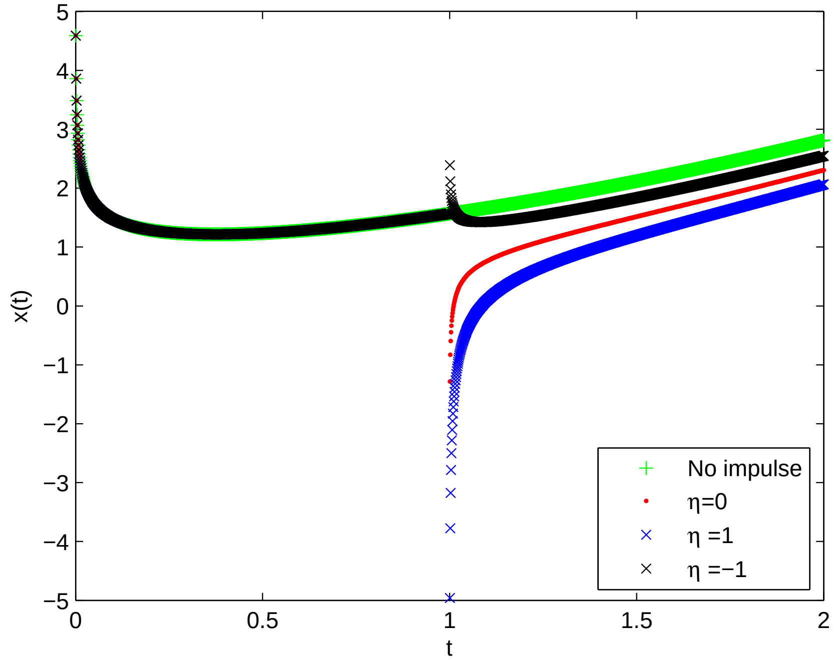

Fig. 6: The solution trajectories of (4.8) with $\beta = 0.5$

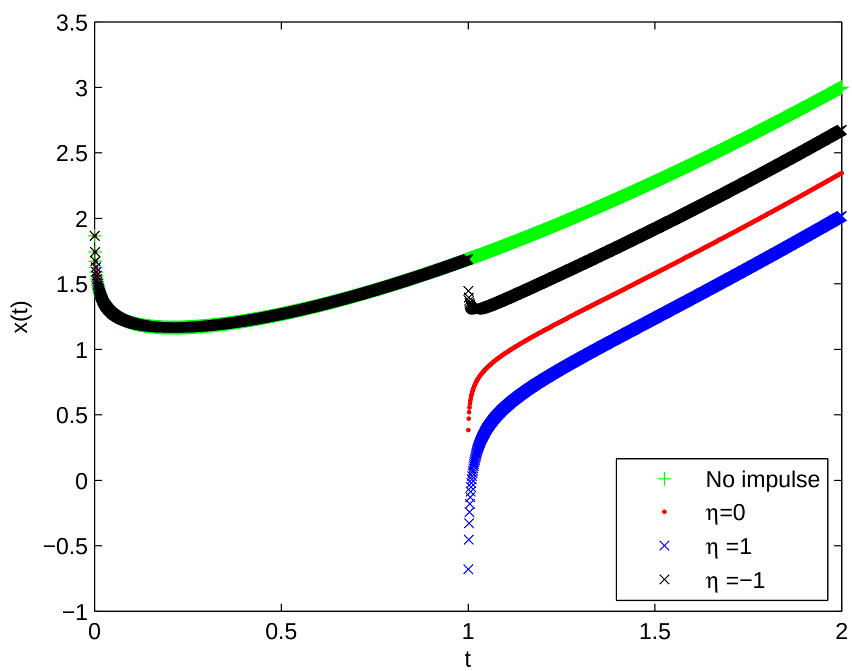

Fig. 7: The solution trajectories of (4.8) with $\beta = 0.8$

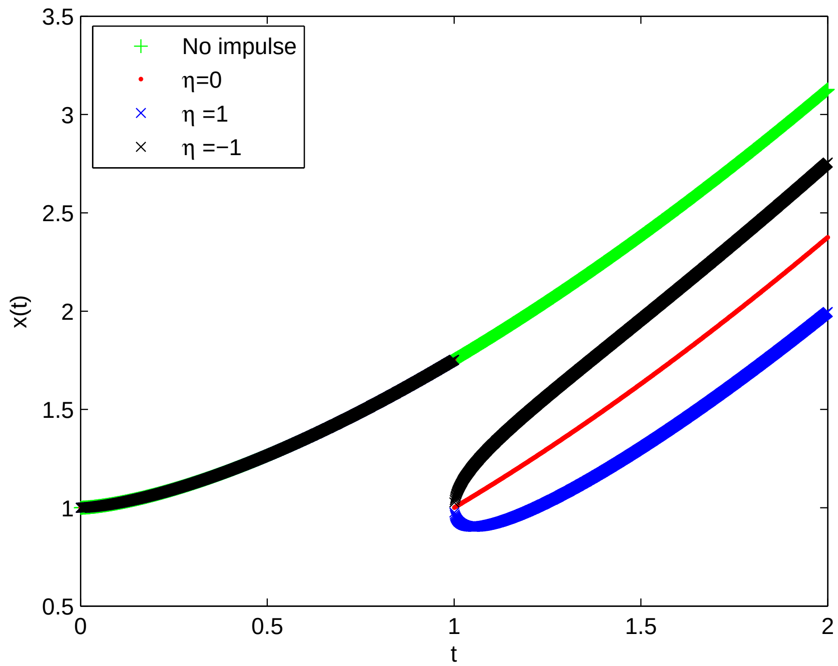

Fig. 8: The solution trajectories of (4.8) with $\beta = 1$

We substitute (A.2) into (3.3) to obtain

$$\widetilde{z}(t) = \frac{z_0 + \phi_1(z(t_1^-)) + \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_1)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t} (t-\tau)^{\alpha-1} h d\tau \quad \text{for } t \in (t_1, t_2]. \tag{A.3}$$

Let $e_k(t) = z(t) - \widetilde{z}(t)$ for $t \in (t_k, t_{k+1}]$ (here $1 \le k \le K$) denote the error between the approximate solution $\widetilde{z}(t)$ and the exact solution of (1.9) as $t \in (t_k, t_{k+1}]$.

By (A.1), the exact solution $z(t)$ of (1.9) as $t \in (t_1, t_2]$ satisfies

$$\lim_{\phi_1(z(t_1^-)) \to 0} z(t) = \frac{z_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} h d\tau, \quad t \in (t_1, t_2]. \tag{A.4}$$

Applying (A.3) and (A.4), we obtain

$$\begin{aligned} \lim_{\phi_1(z(t_1^-)) \to 0} e_1(t) &= \lim_{\phi_1(z(t_1^-)) \to 0} \{z(t) - \widetilde{z}(t)\} \\ &= -\Bigg\{ \frac{z_0 + \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_1)^{\gamma-1} + \int_{t_1}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \\ &\quad - \frac{z_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \Bigg\}, \quad t \in (t_1, t_2]. \end{aligned} \tag{A.5}$$

By (A.5), we assume

$$
\begin{aligned}
e_1(t) &= \varpi\left(\phi_1(z(t_1^-))\right) \lim_{\phi_1(z(t_1^-))\to 0} e_1(t) \\
&= -\varpi\left(\phi_1(z(t_1^-))\right)\left\{ \frac{z_0 + \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_1)^{\gamma-1} + \int_{t_1}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right. \\
&\quad \left. - \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right\}, \quad t \in (t_1, t_2],
\end{aligned} \tag{A.6}
$$

where $\varpi(\cdot)$ is an undetermined function.

By (A.3) and (A.6), we have

$$
\begin{aligned}
z(t) &= \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \frac{\phi_1(z(t_1^-))}{\Gamma(\gamma)}(t-t_1)^{\gamma-1} \\
&\quad + \left[1-\varpi\left(\phi_1(z(t_1^-))\right)\right]\left\{ \frac{z_0 + \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_1)^{\gamma-1} + \int_{t_1}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right. \\
&\quad \left. - \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right\}, \quad t \in (t_1, t_2].
\end{aligned} \tag{A.7}
$$

Letting $\beta = 1$ in (1.9) (${}_{t_0}^{HR}\mathcal{D}_t^{\alpha,1}$ is the Caputo fractional derivative), we get $1 - \varpi(\phi_1(z(t_1^-))) = \xi\phi_1(z(t_1^-))$ (here ξ is an arbitrary constant) by using the method in **Theorem 2.1** in [13]. Thus we rewrite (A.7) into

$$
\begin{aligned}
z(t) &= \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \frac{\phi_1(z(t_1^-))}{\Gamma(\gamma)}(t-t_1)^{\gamma-1} \\
&\quad + \xi\phi_1(z(t_1^-))\left\{ \frac{z_0 + \int_{t_0}^{t_1} \frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_1)^{\gamma-1} + \int_{t_1}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right. \\
&\quad \left. - \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right\}, \quad t \in (t_1, t_2].
\end{aligned} \tag{A.8}
$$

Thus the solution of (1.9) satisfies (3.4) as $t \in (t_1, t_2]$.

Next suppose that the solution of (1.9) as $t \in (t_k, t_{k+1}]$ (here $1 \le k \le m$) satisfies

$$
\begin{aligned}
z(t) = &\begin{cases} \Lambda(t), & t \in J_0, \\ \vdots & \\ \Lambda(t), & t \in J_K, \end{cases} + \sum_{i=1}^{k} \begin{cases} 0, & t \in (t_0, t_i], \\ \dfrac{\phi_i(z(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}, & t \in (t_i, S], \end{cases} \\
&+ \xi\phi_1(z(t_1^-)) \left[\begin{cases} \Lambda(t), & t \in J_0, \\ 0, & t \in (t_1, S], \end{cases} + \begin{cases} 0, & t \in J_0, \\ \Upsilon_1(t), & t \in J_1, \\ \vdots & \\ \Upsilon_1(t), & t \in J_K, \end{cases} - \begin{cases} \Lambda(t), & t \in J_0, \\ \Lambda(t), & t \in J_1, \\ \vdots & \\ \Lambda(t), & t \in J_K, \end{cases} \right] + ... \\
&+ \xi\phi_k(z(t_k^-)) \left[\begin{cases} \Lambda(t), & t \in J_0, \\ \vdots & \\ \Lambda(t), & t \in J_{k-1}, \\ 0, & t \in (t_k, S], \end{cases} + \begin{cases} 0, & t \in (t_0, t_k], \\ \Upsilon_k(t), & t \in J_k, \\ \vdots & \\ \Upsilon_k(t), & t \in J_K, \end{cases} - \begin{cases} \Lambda(t), & t \in J_0, \\ \Lambda(t), & t \in J_1, \\ \vdots & \\ \Lambda(t), & t \in J_K, \end{cases} \right]
\end{aligned} \tag{A.9}
$$

By applying (A.9), we have

$$
\begin{aligned}
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_{k+1}^+} &= {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_{k+1}^-} + \phi_{k+1}(z(t_{k+1}^-)) \\
&= z_0 + \int_{t_0}^{t_{k+1}} \frac{(t_{k+1}-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau + \sum_{i=1}^{k+1} \phi_i(z(t_i^-)) + \sum_{i=1}^{k} \frac{\xi\phi_i(z(t_i^-))}{\Gamma(\alpha-\gamma+1)} \\
&\quad \times \left[\int_{t_0}^{t_i} (t_i-\tau)^{\alpha-\gamma} h d\tau + \int_{t_i}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-\gamma} h d\tau - \int_{t_0}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-\gamma} h d\tau \right].
\end{aligned} \tag{A.10}
$$

Plugging (A.10) into (3.3), we obtain

$$
\begin{aligned}
\widetilde{z}(t) = &\frac{(t-t_{k+1})^{\gamma-1}}{\Gamma(\gamma)} \Bigg\{ z_0 + \int_{t_0}^{t_{k+1}} \frac{(t_{k+1}-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau + \sum_{i=1}^{k+1} \phi_i(x(t_i^-)) + \sum_{i=1}^{k} \frac{\xi\phi_i(z(t_i^-))}{\Gamma(\alpha-\gamma+1)} \\
&\times \left[\int_{t_0}^{t_i} (t_i-\tau)^{\alpha-\gamma} h d\tau + \int_{t_i}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-\gamma} h d\tau - \int_{t_0}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-\gamma} h d\tau \right] \Bigg\} \\
&+ \int_{t_{k+1}}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau, \quad t \in (t_{k+1}, t_{k+2}].
\end{aligned} \tag{A.11}
$$

On the other hand, by (A.9), the exact solution $z(t)$ of (1.9) as $t \in (t_{k+1}, t_{k+2}]$ satisfies

$$
\lim_{\phi_i(z(t_i^-)) \to 0 \text{ for all } i \in \{1,\ldots,k+1\}} x(t) = \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau, \ t \in (t_{k+1}, t_{k+2}], \tag{A.12}
$$

and

$$\begin{aligned}
&\lim_{\phi_j(z(t_j^-))\to 0 \text{ for } j\in\{1,\ldots,k+1\}} z(t)\\
&\quad= \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} + \int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau + \sum_{\substack{1\le i\le k+1\\ \text{and } i\neq j}}\frac{\phi_i(z(t_i^-))}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}\\
&\qquad+\xi\sum_{\substack{1\le i\le k+1\\ \text{and } i\neq j}}\phi_i(z(t_i^-))\left\{\frac{z_0+\int_{t_0}^{t_i}\frac{(t_i-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}+\int_{t_i}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right.\\
&\qquad\left.-\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right\},\quad t\in(t_{k+1},t_{k+2}].
\end{aligned}\tag{A.13}$$

Thus by (A.11)-(A.13), we have

$$\begin{aligned}
&\lim_{\phi_j(z(t_j^-))\to 0 \text{ for all } j\in\{1,\ldots,k+1\}} e_{k+1}(t) = \lim_{\phi_j(z(t_j^-))\to 0 \text{ for all } j\in\{1,\ldots,k+1\}}\{z(t)-\widetilde{z}(t)\}\\
&\quad= -\frac{z_0+\int_{t_0}^{t_{k+1}}\frac{(t_{k+1}-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_{k+1})^{\gamma-1}-\int_{t_{k+1}}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\\
&\qquad+\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau,\quad t\in(t_{k+1},t_{k+2}],
\end{aligned}\tag{A.14}$$

and

$$\begin{aligned}
&\lim_{\phi_j(z(t_j^-))\to 0 \text{ for } j\in\{1,\ldots,k+1\}} e_{k+1}(t) = \lim_{\phi_j(z(t_j^-))\to 0 \text{ for } j\in\{1,\ldots,k+1\}}\{z(t)-\widetilde{z}(t)\}\\
&= -\frac{z_0+\int_{t_0}^{t_{k+1}}\frac{(t_{k+1}-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_{k+1})^{\gamma-1}-\int_{t_{k+1}}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau+\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}\\
&\quad+\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau+\sum_{\substack{1\le i\le k+1\\ \text{and } i\neq j}}\xi\phi_i(z(t_i^-))\left[\frac{z_0+\int_{t_0}^{t_i}\frac{(t_i-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_i)^{\gamma-1}\right.\\
&\quad\left.+\int_{t_i}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau-\frac{x_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right]-\frac{(t-t_{k+1})^{\gamma-1}}{\Gamma(\gamma)}\\
&\quad\times\sum_{\substack{1\le i\le k+1\\ \text{and } i\neq j}}\frac{\xi\phi_i(z(t_i^-))}{\Gamma(\alpha-\gamma+1)}\left[\int_{t_0}^{t_i}(t_i-\tau)^{\alpha-\gamma}hd\tau+\int_{t_i}^{t_{k+1}}(t_{k+1}-\tau)^{\alpha-\gamma}hd\tau\right.\\
&\quad\left.-\int_{t_0}^{t_{k+1}}(t_{k+1}-\tau)^{\alpha-\gamma}hd\tau\right]+\sum_{\substack{1\le i\le k+1\\ \text{and } i\neq j}}\frac{\phi_i(z(t_i^-))}{\Gamma(\gamma)}\left[(t-t_i)^{\gamma-1}-(t-t_{k+1})^{\gamma-1}\right],\\
&\quad t\in(t_{k+1},t_{k+2}].
\end{aligned}\tag{A.15}$$

By (A.14) and (A.15), we get

$$
\begin{aligned}
e_{k+1}(t) = & -\frac{z_0 + \int_{t_0}^{t_{k+1}} \frac{(t_{k+1}-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_{k+1})^{\gamma-1} - \int_{t_{k+1}}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \frac{z_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} \\
& + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \sum_{i=1}^{k+1} \xi \phi_i(z(t_i^-)) \Bigg[\frac{z_0 + \int_{t_0}^{t_i} \frac{(t_i-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_i)^{\gamma-1} \\
& + \int_{t_i}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau - \frac{x_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \Bigg] - \frac{(t-t_{k+1})^{\gamma-1}}{\Gamma(\gamma)} \\
& \times \sum_{i=1}^{k+1} \frac{\xi \phi_i(z(t_i^-))}{\Gamma(\alpha-\gamma+1)} \Bigg[\int_{t_0}^{t_i} (t_i-\tau)^{\alpha-\gamma} h d\tau + \int_{t_i}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-\gamma} h d\tau \\
& - \int_{t_0}^{t_{k+1}} (t_{k+1}-\tau)^{\alpha-\gamma} h d\tau \Bigg] + \sum_{i=1}^{k+1} \frac{\phi_i(z(t_i^-))}{\Gamma(\gamma)} \left[(t-t_i)^{\gamma-1} - (t-t_{k+1})^{\gamma-1} \right], \\
& t \in (t_{k+1}, t_{k+2}].
\end{aligned} \tag{A.16}
$$

Using (A.11) and (A.16), we obtain

$$
\begin{aligned}
z(t) = & \frac{z_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \sum_{i=1}^{k+1} \frac{\phi_i(z(t_i^-))}{\Gamma(\gamma)} (t-t_i)^{\gamma-1} \\
& + \xi \sum_{i=1}^{k+1} \phi_i(z(t_i^-)) \Bigg[\frac{z_0 + \int_{t_0}^{t_i} \frac{(t_i-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)} (t-t_i)^{\gamma-1} + \int_{t_i}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \\
& - \frac{x_0}{\Gamma(\gamma)} (t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \Bigg], \quad t \in (t_{k+1}, t_{k+2}].
\end{aligned} \tag{A.17}
$$

Therefore, the solution of (1.9) satisfies (3.4).

'Necessity'. We compute the fractional derivative and the fractional integral of (3.4):

$$
\begin{aligned}
{}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta} z(t) = & \begin{cases} h(t,z(t)), & t \in J_0, \\ \quad \vdots \\ h(t,z(t)), & t \in J_K, \end{cases} + \sum_{k=1}^{K} \begin{cases} 0, & t \in (t_0, t_k], \\ 0, & t \in J_k, \\ 0, & t \in (t_{k+1}, S], \end{cases} + \xi \phi_1(z(t_1^-)) \\
& \times \left[\begin{cases} h(t,z(t)), & t \in J_0, \\ 0, & t \in (t_1, S], \end{cases} + \begin{cases} 0, & t \in J_0, \\ h(t,z(t)), & t \in J_1, \\ \vdots \\ h(t,z(t)), & t \in J_K, \end{cases} - \begin{cases} h(t,z(t)), & t \in J_0, \\ h(t,z(t)), & t \in J_1, \\ \vdots \\ h(t,z(t)), & t \in J_K, \end{cases} \right] \\
& + \ldots + \xi \phi_K(z(t_K^-)) \\
& \times \left[\begin{cases} h(t,z(t)), & t \in J_0, \\ \quad \vdots \\ h(t,z(t)), & t \in J_{K-1}, \\ 0, & t \in J_K, \end{cases} + \begin{cases} 0, & t \in (t_0, t_K], \\ h(t,z(t)), & t \in J_K, \end{cases} - \begin{cases} h(t,z(t)), & t \in J_0, \\ h(t,z(t)), & t \in J_1, \\ \quad \vdots \\ h(t,z(t)), & t \in J_K, \end{cases} \right] \\
= & \ h(t,z(t)), \quad t \in J_k \ (k = 0, 1, \ldots K),
\end{aligned} \tag{A.18}
$$

and

$$
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)=\begin{cases} z_0+\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau, \ t\in J_0,\\ \qquad\vdots \\ z_0+\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau, \ t\in J_K, \end{cases} +\sum_{k=1}^{K}\begin{cases} 0, \quad t\in(t_0,t_k],\\ \phi_k(z(t_k^-)), \quad t\in J_k,\\ 0, \quad t\in(t_{k+1},S], \end{cases}
$$
$$
\begin{aligned}
&+\xi\phi_1(z(t_1^-))\\
&\times\begin{cases} 0, \quad t\in J_0,\\ \displaystyle\int_{t_0}^{t_1}\frac{(t_1-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau+\int_{t_1}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau,\\ \quad t\in J_1,\\ 0, \quad t\in(t_2,S], \end{cases}\\
&+\ldots+\xi\phi_K(z(t_K^-))\\
&\times\begin{cases} 0, \quad t\in(t_0,t_K],\\ \displaystyle\int_{t_0}^{t_K}\frac{(t_K-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau+\int_{t_K}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau,\\ \quad t\in J_K. \end{cases}
\end{aligned}
\tag{A.19}
$$

Thus ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^+}-{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^-}=\phi_k(z(t_k^-))$ (here $1\le k\le K$) and ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t\to t_0+}=z_0$.

Finally, we consider two limit cases of (3.4):

$$
\begin{aligned}
&\lim_{\phi_k(z(t_k^-))\to0 \text{ for all } k\in\{1,2,\ldots,m\}}\{\text{equation (3.4)}\}\\
&\qquad\Leftrightarrow z(t)=\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}hd\tau, \quad t\in(t_0,S].\\
&\qquad\Leftrightarrow\begin{cases} {}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}z(t)=h(t,z(t)), \quad t\in(t_0,S],\\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t\to t_0+}=z_0. \end{cases}\\
&\qquad=\lim_{\phi_k(z(t_k^-))\to0 \text{ for all } k\in\{1,2,\ldots,K\}}\{\text{system (1.9)}\}
\end{aligned}
\tag{A.20}
$$

and

$$
\begin{aligned}
&\lim_{t_k\to t_p \text{ for all } k\in\{1,2,\ldots,K\} \text{ and } \forall p\in\{1,2,\ldots,K\}}\{\text{equation (3.4)}\}\\
&\Leftrightarrow z(t)=\begin{cases} \Lambda(t), \quad t\in(t_0,t_p],\\ \Lambda(t), \quad t\in(t_p,S], \end{cases}+\begin{cases} 0, \quad t\in(t_0,t_p],\\ \dfrac{\sum_{k=1}^{K}\phi_k(z(t_p^-))}{\Gamma(\gamma)}(t-t_p)^{\gamma-1}, \quad t\in(t_p,S], \end{cases}\\
&\quad+\xi\sum_{k=1}^{K}\phi_k(z(t_p^-))\\
&\quad\times\left[\begin{cases} \Lambda(t), \quad t\in(t_0,t_p],\\ 0, \quad t\in(t_p,S], \end{cases}+\begin{cases} 0, \quad t\in(t_0,t_p],\\ \Upsilon_p(t), \quad t\in(t_p,S], \end{cases}-\begin{cases} \Lambda(t), \quad t\in(t_0,t_p],\\ \Lambda(t), \quad t\in(t_p,S], \end{cases}\right].
\end{aligned}
\tag{A.21}
$$

Therefore, (3.4) satisfies all conditions of (1.9). The proof is completed. $\square$

Now we prove **Theorem 3.4**.

Proof. 'Sufficiency'. By **Theorem 2.4** the solution of (1.10) meets

$$z(t)=\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}hd\tau,\quad t\in(t_0,t_1]. \tag{A.22}$$

Next we seek the solution of (1.10) for $t\in(t_k,t_{k+1}]$ $(1\leq k\leq K)$. By (3.6), the approximate solution $\widehat{z}(t)$ when $t\in(t_k,t_{k+1}]$ is

$$\widehat{z}(t)=\frac{\psi_k(z(t_k^-))}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-\tau)^{\alpha-1}hd\tau,\quad t\in(t_k,t_{k+1}], \tag{A.23}$$

and $\widehat{e}_k(t)=z(t)-\widehat{z}(t)$ as $t\in(t_k,t_{k+1}]$ (here $1\leq k\leq K$) represents the piecewise error between $\widehat{z}(t)$ and the exact solution $z(t)$ of (1.10).

On the other hand, by the particular solution (3.7), the exact solution of (1.10) when $t\in(t_k,t_{k+1}]$ meets

$$\lim_{\left[\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau\right]\to0}z(t)=\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}h}{\Gamma(\alpha)}d\tau,\ t\in(t_k,t_{k+1}]. \tag{A.24}$$

By (A.23) and (A.24), we have

$$\begin{aligned}&\lim_{\left[\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau\right]\to0}\widehat{e}_k(t)=\lim_{\left[\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau\right]\to0}\{z(t)-\widehat{z}(t)\}\\&=-\left[\frac{z_0+\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}+\int_{t_k}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right.\\&\left.\quad-\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right].\end{aligned} \tag{A.25}$$

According to (A.25), we assume

$$\begin{aligned}\widehat{e}_k(t)&=\kappa\left(\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau\right)\lim_{\left[\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau\right]\to0}\widehat{e}_k(t)\\&=\kappa\left(\psi_k(z(t_k^-))-z_0-\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau\right)\left[\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}\right.\\&\left.\quad+\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau-\frac{z_0+\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1}-\int_{t_k}^{t}\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}hd\tau\right],\end{aligned} \tag{A.26}$$

where $\kappa(\cdot)$ is an undetermined function.

Using (A.23) and (A.26), we get

$$
\begin{aligned}
z(t) = {} & \frac{\psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1} + \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} \\
& + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \left[1 - \kappa\left(\psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau\right)\right] \\
& \times \left[\frac{z_0 + \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1} + \int_{t_k}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right. \\
& \left. - \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau\right], \quad t \in (t_k, t_{k+1}].
\end{aligned} \tag{A.27}
$$

To get $\kappa(\cdot)$ in (A.27), we discuss a special case of (1.10) (only an impulse in (1.10)) as:

$$
\begin{cases}
{}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta} z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_k, \\
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^+} = \psi_k(z(t_k^-)), \\
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t\to t_0+} = z_0.
\end{cases}
$$

$$
= \begin{cases}
{}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta} z(t) = h(t, z(t)), \quad t \in (t_0, S] \text{ and } t \neq t_k, \\
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^+} - {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^-} \\
\qquad = \psi_k(z(t_k^-)) - {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t=t_k^-} \\
\qquad = \psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau, \\
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma} z(t)\Big|_{t\to t_0+} = z_0.
\end{cases} \tag{A.28}
$$

By applying **Theorem 3.2** and (A.27) to (A.28) respectively, we get $1 - \kappa(y) = \eta_k y$ for $\forall y \in \mathbb{R}$ where η_k is an arbitrary constant. Therefore, (A.27) is rewritten as

$$
\begin{aligned}
z(t) = {} & \frac{\psi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1} + \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} \\
& + \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau + \eta_k \left[\Phi_k(z(t_k^-)) - z_0 - \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau\right] \\
& \times \left[\frac{z_0 + \int_{t_0}^{t_k} \frac{(t_k-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)} h d\tau}{\Gamma(\gamma)}(t-t_k)^{\gamma-1} + \int_{t_k}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau \right. \\
& \left. - \frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1} - \int_{t_0}^{t} \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} h d\tau\right], \quad t \in (t_k, t_{k+1}],
\end{aligned} \tag{A.29}
$$

which means that the solution of (1.10) satisfies (3.8) as $t \in (t_k, t_{k+1}]$ $(1 \leq k \leq K)$. Thus the solution of (1.10) satisfies (3.8).

Now we prove the necessity. We compute the fractional derivative and fractional integral of (3.8):

$$
\begin{aligned}
{}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}z(t) &= \begin{cases} h(t,z(t)), & t\in J_0,\\ \quad\vdots & \\ h(t,z(t)), & t\in J_K,\end{cases} + \sum_{k=1}^{K}\begin{cases} 0, & t\in(t_0,t_k],\\ 0, & t\in J_k,\\ 0, & t\in(t_{k+1},S],\end{cases}\\
&+\eta_1\left[\psi_1(z(t_1^-))-z_0-\int_{t_0}^{t_1}\frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau\right]\\
&\times\left[\begin{cases} h(t,z(t)), & t\in J_0,\\ 0, & t\in(t_1,S],\end{cases}+\begin{cases} 0, & t\in J_0,\\ h(t,z(t)), & t\in J_1,\\ 0, & t\in(t_2,S],\end{cases}-\begin{cases} h(t,z(t)), & t\in J_0,\\ h(t,z(t)), & t\in J_1,\\ 0, & t\in(t_2,S],\end{cases}\right]\\
&+\ldots+\eta_K\left[\psi_K(z(t_K^-))-z_0-\int_{t_0}^{t_K}\frac{(t_K-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau\right]\\
&\times\left[\begin{cases} h(t,z(t)), & t\in J_0,\\ \quad\vdots & \\ h(t,z(t)), & t\in J_{K-1},\\ 0, & t\in J_K,\end{cases}+\begin{cases} 0, & t\in(t_0,t_K],\\ h(t,z(t)), & t\in J_K,\end{cases}-\begin{cases} h(t,z(t)), & t\in J_0,\\ h(t,z(t)), & t\in J_1,\\ \quad\vdots & \\ h(t,z(t)), & t\in J_K,\end{cases}\right]\\
&= h(t,z(t)),\quad t\in J_k\ (k=0,1,\ldots K),
\end{aligned}
\tag{A.30}
$$

and

$$
\begin{aligned}
{}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t) &= \begin{cases} z_0+\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau, & t\in J_0,\\ \quad\vdots & \\ z_0+\displaystyle\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau, & t\in J_K,\end{cases} + \sum_{k=1}^{K}\begin{cases} 0, & t\in(t_0,t_k],\\ -z_0-\displaystyle\int_{t_0}^{t_k}\frac{(t_k-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau & \\ \quad+\psi_k(z(t_k^-)), & t\in J_k,\\ 0, & t\in(t_{k+1},S],\end{cases}\\
&+\eta_1\left[\psi_1(z(t_1^-))-z_0-\int_{t_0}^{t_1}\frac{(t_1-\tau)^{\alpha-\gamma}}{\Gamma(\alpha-\gamma+1)}hd\tau\right]\\
&\times\begin{cases} 0, & t\in J_0,\\ \displaystyle\int_{t_0}^{t_1}\frac{(t_1-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau+\int_{t_1}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau, & t\in J_1,\\ 0, & t\in(t_2,S],\end{cases}\\
&+\ldots+\eta_K\left[\psi_K(z(t_K^-))-z_0-\int_{t_0}^{t_K}\frac{(t_K-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau\right]\\
&\times\begin{cases} 0, & t\in(t_0,t_K],\\ \displaystyle\int_{t_0}^{t_K}\frac{(t_K-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau+\int_{t_K}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau-\int_{t_0}^{t}\frac{(t-\tau)^{\alpha-\gamma}h}{\Gamma(\alpha-\gamma+1)}d\tau, & t\in J_K.\end{cases}
\end{aligned}
\tag{A.31}
$$

Thus by (A.31) we have ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t=t_k^+}=\psi_k(z(t_k^-))$ (here $1\le k\le K$) and ${}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t\to t_0+}=z_0$.

Finally, consider a limiting case of (3.8)

$$\lim_{\substack{\left[\psi_k(z(t_k^-))-z_0-\frac{1}{\Gamma(\alpha-\gamma+1)}\int_{t_0}^{t_k}(t_k-\tau)^{\alpha-\gamma}hd\tau\right]\to 0\\ \text{for all } k\in\{1,2,\ldots,K\}}} \{\text{equation (3.8)}\}$$

$$\begin{aligned}
&\Leftrightarrow z(t)=\frac{z_0}{\Gamma(\gamma)}(t-t_0)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}hd\tau,\quad t\in(t_0,S].\\
&\Leftrightarrow \begin{cases} {}_{t_0}^{HR}\mathcal{D}_t^{\alpha,\beta}z(t)=h(t,z(t)), & t\in(t_0,S],\\ {}_{t_0}^{RL}\mathcal{I}_t^{1-\gamma}z(t)\Big|_{t\to t_0+}=z_0. \end{cases}\\
&=\lim_{\substack{\left[\psi_k(z(t_k^-))-z_0-\frac{1}{\Gamma(\alpha-\gamma+1)}\int_{t_0}^{t_k}(t_k-\tau)^{\alpha-\gamma}hd\tau\right]\to 0\\ \text{for all } k\in\{1,2,\ldots,K\}}} \{\text{system (1.10)}\}
\end{aligned} \tag{A.32}$$

Therefore (3.8) meets all conditions of (1.10). The proof is completed. □

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