THE CONVEXITY OF COMPRESSIBLE SUBSONIC IMPINGING JET FLOWS

XIAOHUI WANG

Department of Mathematics,
Chengdu University of Technology, Chengdu 610059, P. R. China.

Abstract. We proved the existence and uniqueness of comporessible subsonic impinging jet flow in the work [27]. As a continuation, in this paper, we investigate the shape of free boundary to the impinging jet flow established in [27]. More specifically, if the nozzle wall is concave to the fluid, then the free boundary of flow will be convex to the fluid. On another hand, the higher regularity of free boundary at separation point is obtained, provided that the nozzle satisfies corresponding hypotheses.

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1. Introduction and main Theorems

We first give a symmetric semi-infinitely long nozzle and a plate, which are assumed to be impermeable. Define the upper nozzle wall as

$$N : y = g(x) > 0, \quad g(x) \in C^{2,\alpha}(-\infty, -a] \text{ with } a > 0 \text{ for some } \alpha \in (0, 1). \tag{1.1}$$

Assume that

$$\lim_{x\to-\infty} g(x) = H, \quad \text{and} \quad g(-a) = 1. \tag{1.2}$$

Denote the plate $N_0 = \{(0, y) \mid y \geq 0\}$. The symmetric axis $T = \{(x, 0) \mid -\infty < x \leq 0\}$ and $A = (-a, 1)$ (see Figure 1).

In this paper, we investigate the convexity of free boundary to the steady isentropic inviscid subsonic impinging jet flow in two dimensions, which can be represented by

$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ \partial_x(\rho u^2) + \partial_y(\rho uv) + \partial_x p = 0, \\ \partial_x(\rho uv) + \partial_y(\rho v^2) + \partial_y p = 0, \end{cases} \tag{1.3}$$

E-Mail: xiaohuiwang1@126.com (X. Wang).

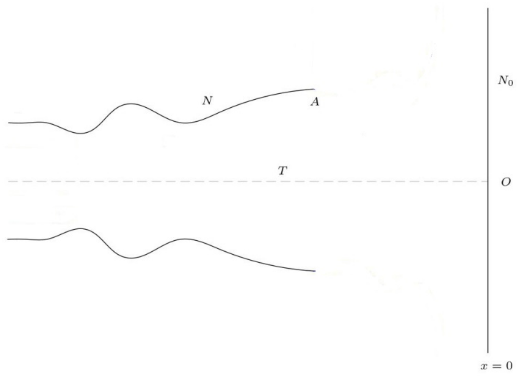

FIGURE 1. The symmetric nozzle and plate

with

$$v_x - u_y = 0, \tag{1.4}$$

that is, the flow is irrotational, where $(x, y) \in \mathbb{R}^2$, (u, v) and ρ denotes the velocity field and density of flow, respectively. Pressure satisfies the γ-law

$$p = p(\rho) = \mathcal{A}\rho^{\gamma}, \tag{1.5}$$

here $\mathcal{A} > 0$, and $\gamma > 1$ is the so-called adiabatic exponent. Defined $c(\rho) = \sqrt{p'(\rho)}$ as the sound speed, $\mathcal{M} = \dfrac{q}{c(\rho)}$ as the Mach number, here $q = \sqrt{u^2 + v^2}$ is the fluid speed. The compressible flow is subsonic as $\mathcal{M} < 1$, sonic as $\mathcal{M} = 1$ and supersonic as $\mathcal{M} > 1$.

Since N and N_0 are impermeable, and T is the axis of symmetry, then the flow satisfies the boundary conditions

$$(u, v) \cdot \vec{n} = 0 \ \text{ on } N \cup T, \quad \text{and} \quad u = 0 \ \text{ on } N_0, \tag{1.6}$$

where $\vec{n}$ is the outer normal of $N \cup T$. It follows from $(1.3)_1$ and (1.6) that for any $x = x_0 < -a$ the mass flux

$$m_0 = \int_0^{g(x_0)} (\rho u)(x_0, y) dy > 0. \tag{1.7}$$

It is not difficult to find that q and ρ satisfy the Bernoulli's law

$$\frac{q^2}{2} + \frac{\mathcal{A}\gamma}{\gamma - 1}\rho^{\gamma-1} = \mathcal{B}, \tag{1.8}$$

where $\mathcal{B}$ is the Bernoulli's constant.

There is a long history of research on the free streamline problems in fluid mechanics, which are very interesting and have attracted the attention of many mathematicians and hydrodynamicist in recent years. We first recall some classical results on the incompressible free streamline problems as follows. In 1952, P. Garabedian, H. Lewy and M. Schiffer [21] proved the existence of axisymmetric incompressible cavity flows in three dimensions. In the mid-20th century, G. Birkhoff and E. Zarantonello [8], D. Gilbarg [24], and M. Gurevich [25] reviewed the research progress of free boundary problems, such as the incompressible jets, cavities and wakes. In 1980's, combined variational approach and geometric measure theory, H. Alt, L. Caffarelli and A. Friedman studied the free boundary problems of linear elliptic equation [1], and established the regularity theory of free boundary. On the basis of this, the authors obtained many meaningful well-posedness results about the incompressible jet flows, such as asymmetrical jets in two dimensions [2], three-dimensional axisymmetric jets [4] and jets with gravity [3]. The impinging jet problem was first introduced by A. Friedman in the outstanding work [20]. J. Cheng, L. Du and Y. Wang investigated the plane asymmetry [11] and three-dimensional axisymmetric impinging jet flows [12] for incompressible case.

Later on, the regularity theory of free boundary for quasilinear elliptic equation was established by H. Alt, L. Caffarelli and A. Friedman in [5]. With the aid of the regularity theory of free boundary, the existence and uniqueness of compressible subsonic jets and cavities were proved in [6]. Recently, J. Cheng, L. Du and W. Xiang etc. also studied the subsonic impinging jet flow for the two-dimensional asymmetry case (the nozzle is concave to the fluid) in [13] and the three-dimensional axisymmetric case in [14]. We also obtained the existence and uniqueness of subsonic impinging jet flows in an arbitrary semi-infinitely long nozzle [27]. In addition, please refer to literatures [9, 10], [16]-[19] and [28]-[30] for the compressible flows in the infinitely long nozzle, and [26] in the finitely long nozzle. However, so far most of the results are focus on the well-posedness of the jet flows. There are relatively few results concerning the geometric properties of free boundary. We will investigate the shape of free boundary in this paper.

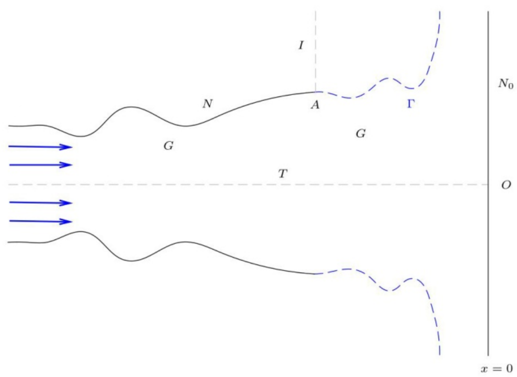

FIGURE 2. The impinging jet flow

For the plane symmetric subsonic impinging jet flow with the free boundary Γ, which initiates smoothly from A to infinity in y-direction (see Figure 2), we only need to study the fluid in the upper half plane. Since the free boundary Γ is a streamline, the boundary condition (1.6) holds on Γ. The pressure satisfies

$$p = p_{atm} \quad \text{on } \Gamma, \tag{1.9}$$

here $p_{atm} > 0$ is atmospheric pressure. The γ-law (1.5) indicates that $\rho = \rho_{con} = \left(\frac{p_{atm}}{\mathcal{A}}\right)^{\frac{1}{\gamma}}$ is a constant on Γ.

In addition, we assume

$$y = g(x) \text{ is star-shaped with respect to } O = (0,0). \tag{1.10}$$

We would like to mention that the assumption (1.10) is to establish the uniqueness of flow.

According to the arguments in Theorem 1.1 in [27], we can obtain the well-posedness of the compressible impinging jet flow in the following theorem.

Theorem A. If the nozzle wall N satisfies the conditions (1.1), (1.2) and (1.10), then for some appropriate incoming mass flux $m_0 > 0$, there exist a unique two-dimensional symmetric subsonic impinging jet flow (u, v, ρ, Γ) and a unique p_0, such that the free boundary Γ: $y = f(x) \in C^1([-a, -b))$ (with $f(x) \to +\infty$ as $x \to (-b)^-$) initiates smoothly from A, goes to infinity in y-direction, and the pressure balances to p_{atm} on Γ. Moreover, the flow satisfies

(1) $(u, v, \rho) \in \left(C^{1,\alpha}(G) \cap C(\overline{G})\right)^3$ solves the Euler system (1.3) in G, bounded by N, N_0, T and Γ.
(2) the horizontal velocity $u > 0$ in $\overline{G} \setminus N_0$.
(3) The asymptotic behavior at upstream satisfies

$$(u(x,y), v(x,y), \rho(x,y)) \to \left(\frac{m_0}{\rho_0 H}, 0, \rho_0\right), \text{ and } \nabla(u,v,\rho) \to 0 \text{ as } x \to -\infty$$

uniformly in any compact subset of $(0, H)$, where $\rho_0 = \left(\frac{p_0}{\mathcal{A}}\right)^{\frac{1}{\gamma}}$. We have at downstream,

$$(u(x,y), v(x,y), \rho(x,y)) \to \left(0, \frac{m_0}{\rho_{con} b}, \rho_{con}\right), \text{ and } \nabla(u,v,\rho) \to 0 \text{ as } y \to +\infty$$

uniformly in any compact subset of $(-b, 0)$, where b is the asymptotic width of flow at downstream.

The main purpose of this paper is to show that the free boundary of the flow obtained in Theorem A is convex to the fluid, provided that the nozzle wall $N : y = g(x)$ satisfies the following condition,

$$g''(x) \le 0 \text{ in } (-\infty, -a], \quad \text{and } H > a, \tag{1.11}$$

that is, the nozzle wall is concave to the fluid.

Now, we state the main results of this paper as follows.

Theorem 1.1. *Under the assumptions of Theorem A hold, if the nozzle wall N satisfies the additional condition* (1.11)*, then the free boundary of the subsonic impinging jet flow established in Theorem A is convex to the fluid (see Figure 3), that is,*

$$f''(x) > 0 \textit{ for any } x \in (-a, -b).$$

On the other hand, we will show the optimal regularity of free boundary at the separation point A in the following theorem.

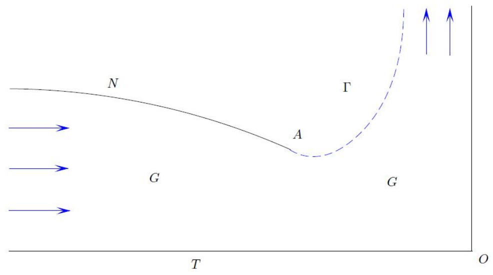

FIGURE 3. The convexity of free boundary

Theorem 1.2. *Under the assumptions of Theorem 1.1 hold, if the function $g(x)$ satisfies*

$$g(x) \in C^{3,\alpha}[-a-\epsilon, -a] \quad \textit{and} \quad g''(-a-0) < 0,$$

for some small enough $\epsilon > 0$, then $N \cup \Gamma$ is $C^{1,\frac{1}{2}}$ at A and $f''(-a+0) = +\infty$.

2. THE CONVEXITY OF FREE BOUNDARY

In this section, based on the existence and uniqueness of the solution to subsonic impinging jet flow in Theorem A, the convexity of free boundary will be obtained by using the maximum principle in [22, 23].

2.1. **Free boundary value problem.**

The equation $(1.3)_1$ yields that there exists a stream function $\psi(x,y)$, such that

$$\rho u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \rho v = -\frac{\partial \psi}{\partial x}. \tag{2.1}$$

It follows from the condition (1.6) that ψ is an invariant along the boundaries. Let us just assume that

$$\psi = 0 \quad \text{on } T \cup N_0, \quad \text{and} \quad \psi = m_0 \quad \text{on } N \cup \Gamma. \tag{2.2}$$

Then the free boundary of impinging jet flow can be defined as

$$\Gamma = \Omega \cap \{x > -a\} \cap \partial\{\psi < m_0\}, \tag{2.3}$$

where the domain Ω is bounded by N, N_0, T and $I = \{(-a, y) \mid y \geq 1\}$.

According to the Bernoulli's law (1.8), we have that ψ satisfies

$$\frac{|\nabla \psi|^2}{2\rho^2} + \frac{\mathcal{A}\gamma}{\gamma - 1}\rho^{\gamma-1} = \mathcal{B} \quad \text{in } G.$$

It follows that the momentum is a constant on Γ, denoted this constant as λ, namely,

$$|\nabla \psi| = \frac{\partial \psi}{\partial \nu} = \lambda \quad \text{on } \Gamma, \tag{2.4}$$

where ν is the outer normal to Γ. It is easy to verify that there exist the critical quantities with respect to λ,

$$\varrho_{cr}(\lambda) = \left(\frac{2\mathcal{B}(\lambda^2)}{\mathcal{A}\gamma}\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}, \quad \varrho_{max}(\lambda) = \left(\frac{\mathcal{B}(\lambda^2)(\gamma-1)}{\mathcal{A}\gamma}\right)^{\frac{1}{\gamma-1}}, \quad q_{cr}(\lambda) = \left(2\mathcal{B}(\lambda^2)\frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{2}},$$

such that the fluid is subsonic if and only if $q < q_{cr}(\lambda)$ or $\varrho_{cr}(\lambda) < \rho \leq \varrho_{max}(\lambda)$.

Denote $\digamma_\lambda = \varrho_{cr}(\lambda)q_{cr}(\lambda)$ as the critical value of momentum. Along the arguments similar to those in [15], for any $\lambda < \lambda_{cr} = \left(\mathcal{A}\gamma\rho_{con}^{\gamma+1}\right)^{\frac{1}{2}}$, we have

$$\lambda < \digamma_\lambda, \quad \lambda_{cr} = \digamma_{\lambda_{cr}},$$

and

$$\rho \text{ is a decreasing function of } |\nabla\psi|^2 \in [0, \digamma_\lambda^2). \tag{2.5}$$

Then ρ can be denoted as $\rho = \rho_\lambda(|\nabla\psi|^2)$ for any $\lambda < \lambda_{cr}$. Denote

$$\mathcal{H}_\lambda(s) = \rho_\lambda(s) \text{ and } \mathcal{H}'_\lambda(s) = \frac{\partial\rho_\lambda(s)}{\partial s} \quad \text{for any } s \in \left(0, \digamma_\lambda^2\right).$$

In view of (1.4), the system (1.3) is reduced to a single quasilinear equation

$$Q_\lambda\psi = A_{ij}^\lambda(\nabla\psi)\partial_{ij}\psi = 0 \quad \text{in } G, \tag{2.6}$$

where $A_{ij}^\lambda(\nabla\psi) = \mathcal{H}_\lambda(|\nabla\psi|^2)\delta_{ij} - 2\mathcal{H}'_\lambda(|\nabla\psi|^2)\partial_i\psi\partial_j\psi$. Therefore, we can restate the subsonic impinging jet flow problem into the following free boundary problem.

The free boundary problem: For any given $m_0 > 0$ and $\lambda < \digamma_\lambda$, find a pair (ψ, Γ), such that

$$\begin{cases} Q_\lambda\psi = 0 \quad \text{in } \Omega \cap \{\psi < m_0\}, \\ \psi = 0 \quad \text{on } T \cup N_0, \quad \psi = m_0 \quad \text{on } N \cup \Gamma, \\ \dfrac{\partial\psi}{\partial\nu} = \lambda \quad \text{on } \Gamma, \end{cases} \tag{2.7}$$

with $\sup\limits_{X \in \Omega \cap \{\psi < m_0\}} |\nabla\psi(X)| < \digamma_\lambda$.

It follows from Theorem 3.1 in [27] that the problem (2.7) has a unique subsonic solution (ψ, Γ), and the free boundary $\Gamma = \Omega \cap \partial\{\psi < m_0\}$.

2.2. **The convexity of free boundary.** In order to prove the conclusion of Theorem 1.1, we first introduce the following lemma.

Lemma 2.1. *On the boundaries $T \cup N_0$ and $N \cup \Gamma$, we have*

$$\frac{\partial q}{\partial\nu} + \kappa q = 0, \tag{2.8}$$

where ν is the outer normal to $T \cup N_0$ and $N \cup \Gamma$, and κ is the curvature of boundaries. Moreover, if the streamline is concave to the fluid, then $\kappa \geq 0$.

Remark 2.1. The proof of Lemma 2.1 is similar to Lemma 7.2 in [6], and we omit it here. It follows from Lemma 2.1 that $\kappa = -\dfrac{g''(x)}{(1 + g'(x)^2)^{\frac{3}{2}}} \geq 0$ on N. We will obtain that the free boundary Γ is strictly convex to the fluid, provided that we can show

$$\kappa = -\frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}} < 0 \text{ on } \Gamma.$$

Next, by using the conclusion in Lemma 2.1 and the geometric properties of the nozzle wall N, we will establish the convexity of free boundary in the following proposition.

Proposition 2.2. *If the nozzle wall N satisfies the assumptions* (1.11)*, then the free boundary Γ is convex to the fluid, namely,*

$$f''(x) > 0 \text{ for any } x \in (-a, -b).$$

Proof. According to (2.5) and the relationship $q = |\nabla\phi|^2 = \dfrac{|\nabla\psi|^2}{\rho^2}$ (ϕ is the velocity potential of flow), it is easy to check that

$$\begin{gathered} |\nabla\psi| \text{ attains maximum at a point } X_0 \\ \text{if and only if } q \text{ attains maximum at } X_0. \end{gathered} \tag{2.9}$$

Based on the arguments in [6], one has

$$\mathcal{Q}q^2 = D_i\left(e^{\beta q^2} a_{ij}^{\lambda}(X) D_j q^2\right) \geq 0 \quad \text{in } G,$$

for some $\beta > 0$, where

$$a_{ij}^{\lambda}(X) = \mathcal{H}_{\lambda}(|\nabla\phi|^2)\delta_{ij} + 2\mathcal{H}_{\lambda}'(|\nabla\phi|^2) D_i\phi(X) D_j\phi(X).$$

Therefore, q^2 is a subsolution of the quasilinear elliptic equation $\mathcal{Q}q^2 = 0$, and the maximum principle in [23] gives that q cannot take its maximum in G. Then q takes the maximum on $N \cup N_0 \cup T \cup \Gamma$ or at the far fields.

Next, we claim that

$$q \text{ takes its maximum on } \Gamma. \tag{2.10}$$

We will show the claim (2.10) by excluding the following four cases.

Case 1. q takes the maximum at a point X_1 of the nozzle wall N. By the maximum principle, one has

$$\frac{\partial q}{\partial \nu} > 0 \quad \text{at } X_1,$$

where ν is the outer normal vector. Therefore, it follows from Lemma 2.1 that

$$\kappa = -\frac{1}{q}\frac{\partial q}{\partial \nu} < 0 \quad \text{at } X_1,$$

which contracts the fact that $\kappa \geq 0$ on N.

Case 2. q takes the maximum at a point X_2 of $N_0 \cup T \setminus O$. The maximum principle implies that

$$\frac{\partial q}{\partial \nu} > 0 \quad \text{at } X_2,$$

where ν is the outer normal vector. Therefore, it follows from Lemma 2.1 that

$$\kappa = -\frac{1}{q}\frac{\partial q}{\partial \nu} < 0 \quad \text{at } X_2,$$

which is impossible, since both T and N_0 are straight lines, $\kappa = 0$ on $N_0 \cup T \setminus O$.

Case 3. q takes the maximum at point O. Denote θ_0 as the angle formed by T and N_0, it is easy to see that $\theta_0 = \frac{\pi}{2} < \pi - 3\sigma$ for some sufficiently small $\sigma > 0$. Now let's set up a polar coordinate system, and think of O as the pole, the polar axis is the ray issuing from O along the negative direction of x-axis, and the clockwise is defined as positive direction. Let

$$G_{O,\eta} = \{(r,\theta) \in G : r = \sqrt{x^2+y^2} < \eta\},$$

for small $\eta > 0$. It is easy to see that $G_{O,\eta} \subset \{x : |x| < \eta, \ -\sigma < \theta < \theta_0 + \sigma\}$. Similar to the arguments in [18], for sufficiently small $\beta > 0$, we have

$$0 \le \psi(x,y) \le Cr^{1+\beta} \quad \text{in } G_{O,\eta},$$

and therefore, ψ is $C^{1,\beta}$-smooth at O, which implies that

$$|\nabla\psi| = 0 \quad \text{at } O.$$

Case 4. q takes the maximum at upstream. As $x \to -\infty$, $0 < y < H$. It follows from the asymptotic behavior of flow at upstream (see Theorem A) that

$$\nabla\psi \to \nabla\psi_0 = \left(0, \frac{m_0}{H}\right) \ \text{uniformly in any compact subset of } (0,H) \text{ as } x \to -\infty$$

It follows that

$$\lambda_0 = |\nabla\psi| = \frac{m_0}{H} < \frac{m_0}{a} < \frac{m_0}{b} = \lambda.$$

As $y \to +\infty$, $-b < x < 0$. By using the asymptotic behavior of flow at downstream (see Theorem A), we have

$$\psi(x,y) \to -\lambda x \text{ for any } x \in (-b,0). \tag{2.11}$$

Then

$$|\nabla\psi(x,y)| \to \lambda.$$

Recalling that $|\nabla\psi| = \lambda$ on Γ. It follows from the above arguments that

$$\max_{\overline{G}} |\nabla\psi| = \lambda. \tag{2.12}$$

Combine the above, we obtain the claim (2.10), i.e., q takes its maximum on the free boundary Γ. According to Hopf's lemma, one has

$$\frac{\partial q}{\partial \nu} > 0 \quad \text{on } \Gamma,$$

where ν is the outer normal. Then it follows from Lemma 2.1 that

$$\kappa = -\frac{1}{q}\frac{\partial q}{\partial \nu} < 0 \quad \text{on } \Gamma,$$

which implies that the free boundary Γ is convex to the fluid. Therefore, we complete the proof of Theorem 1.1. □

3. The optimal regularity of free boundary at A

Based on the outstanding work [7] by H. Alt, L. Caffarelli and A. Friedman, the higher regularity of free boundary near the separation point A will be obtained, provided that the nozzle wall N satisfies corresponding hypothesis. We first introduce Theorem 1.1 in [7] in the following lemma, and omit the proof for simplicity.

Lemma 3.1. *If N is $C^{3,\alpha}$ near the end point A, then either*

(1) *$N \cup \Gamma$ is C^2 at A, or*
(2) *The optimal regularity of $N \cup \Gamma$ at A is only $C^{1,\frac{1}{2}}$ and the curvature of Γ goes to $\pm\infty$ as $x \to (-a)^+$.*

Since $g''(x) \leq 0$ in $(-\infty, -a)$ and $g''(-a-0) < 0$, we claim that

$$\text{the statement (1) in Lemma 3.1 is not true.} \tag{3.1}$$

We assume that the claim (3.1) is not true, then $N \cup \Gamma$ is C^2 at A, it follows that

$$f''(-a+0) = g''(-a-0) < 0. \tag{3.2}$$

In view of (3.2) and the analyticity of the free boundary Γ, one has

$$f''(x) < 0 \quad \text{in } (-a, -a+\epsilon),$$

for some small enough $\epsilon > 0$, which leads to a contradiction with $f''(x) > 0$ in $(-a, -b)$. Therefore, we proved the claim (3.1), the statement (2) in Lemma 3.1 gives that the optimal regularity of $N_0 \cup \Gamma$ at A is $C^{1,\frac{1}{2}}$ and $f''(0+0) = +\infty$. Consequently, Theorem 1.2 is established.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

ORCID

xiaohuiwang1@126.com. https://orcid.org/0000-0002-3534-8221.

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