SOLVABILITY FOR TIME-FRACTIONAL SEMILINEAR PARABOLIC EQUATIONS WITH SINGULAR INITIAL DATA

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Abstract. We discuss the existence and nonexistence of a local and global-in-time solution to the fractional problem

$$\begin{cases} \partial_t^\alpha u = \Delta u + f(u) & x \in \Omega,\ 0 < t < T, \\ u = 0 & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with C^2 boundary, $0 < T \leq \infty$, $u_0 \in L^r(\Omega)$ $(1 \leq r < \infty)$ and ∂_t^α $(0 < \alpha < 1)$ is the Caputo fractional derivative. We assume that $f(u)$ is a continuous function such that for some $p > 1$ one has $|f(\xi) - f(\eta)| \leq C(1 + |\xi| + |\eta|)^{p-1}|\xi - \eta|$ for all $\xi, \eta \in \mathbb{R}$. Particular attention is paid to the doubly critical case $(p, r) = (1 + 2/N, 1)$.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with C^2 boundary and $0 < \alpha < 1$. We are interested in existence and nonexistence of a solution of the time fractional spatially homogeneous parabolic problem

$$\begin{cases} \partial_t^\alpha u = \Delta u + f(u) & x \in \Omega,\ 0 < t < T, \\ u = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.1}$$

where $0 < T \leq \infty$. The operator ∂_t^α denotes the Caputo fractional derivative defined by

$$\partial_t^\alpha u(t) = \frac{d}{dt} \int_0^t \frac{1}{\Gamma(1-\alpha)} \frac{u(s) - u(0)}{(t-s)^\alpha} ds,$$

where Γ is the usual Gamma function. If $u(t)$ is smooth enough, then the above derivative can be written as

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u(s)}{(t-s)^\alpha} ds.$$

In the last two decades various models involving fractional time derivatives have been devised to discuss real life phenomena ranging from hydrology [3] and earth sciences [29] to medical image enhancemen [18] and biosciences [20].

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From mathematical point of view, the Caputo fractional derivative has been recently discussed in the frame of diffusion equation [2,8], Hamilton-Jacobi equation [6], predator-prey models [11], transport equation [19], acoustic wave equations [14], porous medium equation [26] and Ginzburg-Landau equation [27]. A maximum principle for differential equations involving Caputo fractional derivative is studied in [15].

In the present work we investigate the influence of the Caputo fractional derivative in the parabolic problem (1.1). We assume that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function that fulfils:

(F1) there is $C>0$ such that

$$|f(\xi)-f(\eta)|\le C(1+|\xi|+|\eta|)^{p-1}|\xi-\eta| \text{ for } \xi,\eta\in\mathbb{R}.$$

When we discuss the existence of a global-in-time solution, we replace (F1) above with the slightly stronger condition, namely

(F1') $f(0)=0$ and there is $C>0$ such that

$$|f(\xi)-f(\eta)|\le C(|\xi|+|\eta|)^{p-1}|\xi-\eta| \text{ for } \xi,\eta\in\mathbb{R}.$$

In defining the notion of a solution to (1.1) we follow [10, Section 3]. We consider the following Wright type function

$$\Phi_\alpha(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{n!\Gamma(-\alpha n+1-\alpha)},\quad z\in\mathbb{C}.$$

It is known that $\Phi_\alpha(t)$ satsifies

$$\Phi_\alpha(t)\ge 0 \text{ for } t\ge 0 \text{ and } \int_0^\infty \Phi_\alpha(t)dt=1,$$

and hence $\Phi_\alpha(t)$ is a probability density function. Moreover,

$$\int_0^\infty t^p\Phi_\alpha(t)dt=\frac{\Gamma(p+1)}{\Gamma(\alpha p+1)},\quad p>-1,\quad 0<\alpha<1.$$

Let Δ be the Dirichlet Laplacian on Ω in the Lebesgue $L^r(\Omega)$, $1\le r<\infty$, equipped with the norm $\|\cdot\|_r$ whose domain is $\{u\in W^{2,r}(\Omega);\ u=0 \text{ in } \partial\Omega\}$. Then Δ generates a strongly continuous semigroup $\{S(t)\}_{t\ge0}$ on $L^r(\Omega)$. For $t>0$ we define

$$S_\alpha(t),P_\alpha(t):L^r(\Omega)\to L^r(\Omega)$$

by

$$\begin{cases} S_\alpha(t)v=\displaystyle\int_0^\infty \Phi_\alpha(\tau)S(\tau t^\alpha)v\,d\tau, \\ P_\alpha(t)v=\alpha t^{\alpha-1}\displaystyle\int_0^\infty \tau\Phi_\alpha(\tau)S(\tau t^\alpha)v\,d\tau, \end{cases}\quad \text{for all } v\in L^r(\Omega). \tag{1.2}$$

Let us consider the nonhomogeneous problem

$$\begin{cases} \partial_t^\alpha u = \Delta u + f(x,t,u), & x \in \Omega,\ 0 < t < T, \\ u = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $f : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a measurable function that satisfies, instead of (F1) and (F1') above, the nonhomogeneous conditions:

(i) there exists $c : \Omega \times [0,\infty) \to (0,\infty)$ and $p \geq 1$ such that for all $\xi \in \mathbb{R}$ and a.e $(x,t) \in \Omega \times (0,T)$ there holds

$$|f(x,t,\xi)| \leq c(x,t)(1 + |\xi|^p); \tag{1.4}$$

(ii) For all $\xi, \eta \in \mathbb{R}$ and a.e. $(x,t) \in \Omega \times (0,T)$ there holds

$$|f(x,t,\xi) - f(x,t,\eta)| \leq c(x,t)(1 + |\xi| + |\eta|)^{p-1}|\xi - \eta|. \tag{1.5}$$

Here,

$$\left(\int_0^T \|c(\,\cdot\,,t)\|_{L^{q_1}(\Omega)}^{q_2}\, dt\right)^{1/q_2} < \infty, \tag{1.6}$$

for $q_1, q_2 \in [1,\infty)$, with the obvious modifications when $q_1 = \infty$ and $q_2 = \infty$.

We adopt the following definition of a mild solution of (1.3) from [10, Definition 3.1.1].

Definition 1.1. We say that a measurable function u a solution of (1.3) if there exists $T \in (0,\infty]$ such that the following conditions (a)–(e) hold:

(a) $u(\,\cdot\,,t) \in L^1(\Omega)$ for all $0 < t < T$;

(b) $f(\,\cdot\,,t,u(\,\cdot\,,t)) \in L^1(\Omega)$ for a.e. $0 < t < T$;

(c) $\displaystyle\int_0^t \|f(\,\cdot\,,s,u(\,\cdot\,,s))\|_1\, ds < \infty$ for $0 < t < T$;

(d) $u(\,\cdot\,,t)$ satisfies

$$u(\,\cdot\,,t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f(\,\cdot\,,s,u(\,\cdot\,,s))ds \quad \text{for} \quad 0 < t < T,$$

where the integral in the above equality is an absolutely converging Bochner integral in $L^1(\Omega)$;

(e) The initial condition holds in the following sense:

$$\|u(\,\cdot\,,t) - u_0\|_r \to 0 \qquad \text{as } t \to 0$$

for $u_0 \in L^r(\Omega)$ if $1 \leq r < \infty$.

It follows from [10, Remark 3.1.2] that the property (e) in the Definition 1.1 holds if and only if

$$\|u(\,\cdot\,,t) - S_\alpha(t)u_0\|_r \to 0 \qquad \text{as } t \to 0, \tag{1.7}$$

which is equivalent to the convergence to zero in the norms of the integral term in Definition 1.1 (d).

The aim of the paper is to develop an L^r-theory for the Cauchy-Dirichlet problem (1.1), $1 \le r < \infty$ with a particular focus on the case $r = 1$. A solution u of (1.1) is defined similarly to Definition 1.1 above where $f(x,t,u) = f(t,u)$. The homogeneous version of [10, Theorem 3.1.4] that applies to (1.1) reads as follows:

Proposition 1.2 (Local existence in $L^r(\Omega)$). *Let $N \ge 1$ and $0 < \alpha < 1$. Assume that $f \in C(\mathbb{R})$ satisfies (F1) and that one of the following holds:*

(i) (Subcritical case) $1 \le r < \infty$ *and* $1 \le p < 1 + 2r/N$,

(ii) (Critical case) $1 < r < \infty$ *and* $1 < p = 1 + 2r/N$.

Then, for each initial function $u_0 \in L^r(\Omega)$, (1.1) has a local-in-time solution in the sense of Definition 1.1.

On the other hand, the nonexistence of a solution to (1.1) is not completely understood, apart from the case of pure-power nonlinearities $f(u)$ (see [23]). The first main result of this work discusses the nonexistence of a nonnegative solution of (1.1) which shows the sharpness of the two conditions (i), (ii) in Proposition 1.2.

Theorem 1.3 (Nonexistence in $L^r(\Omega)$, $1 \le r < \infty$). *Let $N \ge 1$ and $0 < \alpha < 1$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is nonnegative and nondecreasing. If $1 \le r < \infty$ and*

$$\limsup_{s\to\infty} \frac{f(s)}{s^{1+2r/N}} = \infty, \tag{1.8}$$

then there exists a nonnegative $u_0 \in L^r(\Omega)$ such that (1.1) has no nonnegative solution in the sense of Definition 1.1.

For

$$f(x,t,u) = c(x,t)|u|^{p-1}u \quad (x,t,u) \in \Omega \times [0,\infty) \times \mathbb{R}, \tag{1.9}$$

the nonexistence of a local-in-time solution to (1.3) was obtained in [23]. To the best of our knowledge, Theorem 1.3 is new for general nonlinearities $f(u)$.

For nonlinearities $f(x,t,u)$ given by (1.9), the nonexistence of a solution to (1.3) with a general partial differential operator A instead of Δ is conjectured in [10, Problem 2 in p.154]. In the Laplacian case the conjecture reads as follows: Let $q_1, q_2 \in [1,\infty]$ be given in (1.6). If

$$\frac{N}{2q_1} + \frac{1}{\alpha q_2} + \frac{N(p-1)}{2r} > 1, \tag{1.10}$$

then (1.3) does not have a solution for some initial data $u_0 \in L^r(\Omega)$. When $q_1 = q_2 = \infty$, (1.10) becomes $p > 1 + 2r/N$. By (1.8) we see that Theorem 1.3 gives an affirmative answer for the homogeneous nonlinearity $f = f(u)$.

Let us consider the case $f(u) = |u|^{p-1}u$. For each $r \in (1,\infty)$, Theorem 1.3 indicates that Proposition 1.2 is sharp in this case, i.e., if $1 < p \le 1 + 2r/N$, then (1.1) with $u_0 \in L^r(\Omega)$ has a solution. On the other hand, if $p > 1 + 2r/N$, there is a nonnegative initial function $u_0 \in L^r(\Omega)$ such that (1.1) has no nonnegative solution. The semilinear case $\alpha = 1$ is discussed in Weissler [25].

Let us next comment on the case $r = 1$ in the setting of pure power nonlinearities $f(u) = |u|^{p-1}u$. By Proposition 1.2 and Theorem 1.3 we see the following:

- If $1 \le p < 1 + 2/N$, then (1.1) with $u_0 \in L^1(\Omega)$ has a solution;
- If $p > 1 + 2/N$, then (1.1) with $u_0 \in L^1(\Omega)$ does not always have a nonnegative solution.

The doubly critical case $(p, r) = (1 + 2/N, 1)$ is not covered by neither the above Proposition 1.2 and Theorem 1.3. This requires a more delicate analysis as both exponents reach the critical threshold.

In the semilinear case $\alpha = 1$ and $(p, r) = (1 + 2/N, 1)$, it is known that there is a nonnegative initial function $u_0 \in L^1(\Omega)$ such that (1.1) has no nonnegative solution; see [5, 7] for a nonexistence result in the case $\Omega = \mathbb{R}^N$ and [7, 17] in the case where Ω is bounded.

In the recent paper [21] it is shown that the problem

$$\begin{cases} \partial_t u = \Delta u + |u|^{2/N} u & x \in \mathbb{R}^N,\ 0 < t < T, \\ u(\cdot, 0) = u_0 \in L^1(\mathbb{R}^N) & x \in \mathbb{R}^N, \end{cases}$$

has solutions if and only if

$$\int_{\mathbb{R}^N} |u_0| \left(\log(|u_0| + e)\right)^{N/2} dx < \infty.$$

Local existence of a solution to (1.1) is studied in an abstract setting in [1, 12, 24]. However, those methods are not applicable to our present setting, since a local Lipschitz (or Hölder) condition in a function space is assumed.

Our second main result concerns the existence of a local-in-time solution to (1.1) in the critical case $(p, r) = (1 + 2/N, 1)$.

Theorem 1.4 (Local solution in $L^1(\Omega)$)**.** *Let $N \ge 1$, $0 < \alpha < 1$ and $p = 1 + 2/N$. Suppose that $f \in C(\mathbb{R})$ satisfies (F1).*

Then, for each initial function $u_0 \in L^1(\Omega)$, (1.1) has a local-in-time solution in the sense of Definition 1.1.

Theorem 1.4 shows a sharp contrast in terms of existence of local-in-time solutions to (1.1) in comparison to the semilinear case $\alpha = 1$.

Our third main result discusses the existence of a global-in-time solution to (1.1) in the critical case $(p, r) = (1 + 2/N, 1)$. This time we require f satisfies (F1') instead (F1).

Theorem 1.5 (Global solution in $L^1(\Omega)$)**.** *Let $N \ge 1$, $0 < \alpha < 1$ and $p = 1 + 2/N$. Suppose that $f \in C(\mathbb{R})$ satisfies (F1').*

Then, there exists $\varepsilon > 0$ such that for all $u_0 \in L^1(\Omega)$ with $\|u_0\|_1 \le \varepsilon$ there exists a global-in-time solution u of (1.1) in the sense of Definition 1.1 with $T = \infty$.

Moreover, if u_0 is nontrivial and nonnegative and $\|u_0\|_1 \le \varepsilon$, then (1.1) has a nontrivial nonnegative global-in-time solution.

When $\alpha = 1$, $f(u) = u^p$ and $\Omega = \mathbb{R}^N$, Fujita [9] showed the following: If $1 < p < 1 + 2/N$, then (1.1) with a nontrivial nonnegative initial data blows up in finite time. If $p > 1 + 2/N$, then there is a small nontrivial nonnegative initial function u_0 such that (1.1) has a global-in-time solution. It is proved in [13,16] that a nontrivial nonnegative solution blows up in the threshold case $p = 1 + 2/N$.

This shows again a sharp contrast with the fractional case $0 < \alpha < 1$. For $f(u) = u^p$ and $\Omega = \mathbb{R}^N$ it is shown in [28] that (1.1) may have a nontrivial nonnegative global-in-time solution. Our Theorem 1.5 is in line with the result in [28] in the sense that the existence of a global-in-time solution still holds for small L^1 data in the case of bounded smooth domains Ω.

Theorem 1.6 (Uniqueness of solution)**.** *Let $N \geq 1$, $0 < \alpha < 1$, $p = 1 + 2/N$ and $\beta = \alpha/p$. Suppose that $f \in C(\mathbb{R})$ satisfies (F1).*

Then, (1.1) *has a unique solution in the class*

$$K := \left\{ u(t) \in C([0,T), L^1(\Omega)) \text{ for some } 0 < T \leq \infty \text{ and } \sup_{0<t<T_1} t^{\beta} \|u(t)\|_p < \infty \right\},$$

in the sense that if $u(t) \in C([0,T_1), L^1(\Omega))$ and $v(t) \in C([0,T_2), L^1(\Omega))$ are two solutions in the above class K then

$$u(t) = v(t) \text{ in } L^1(\Omega) \text{ for } 0 \leq t < \min\{T_1, T_2\}.$$

The remaining of the paper is organised as follows. In Section 2 we recall the L^r-theory ($1 < r < \infty$) for (1.3) and a monotone iterative method. A supersolution to (1.1) is constructed by using a solution of (1.3). Then, a monotone iterative method enables us to construct a nonnegative solution of (1.1) under a mild condition on f. In Section 3 we describe the L^p-L^q estimates for $S_\alpha(t)$, $P_\alpha(t)$ and construct a local-in-time solution of (1.1) in $L^1(\Omega)$. We also construct a nonnegative solution of (1.1) in $L^1(\Omega)$ under a mild condition on f using a monotone iterative method. In Section 4 an existence of a global-in-time solution of (1.1) for a small initial data in $L^1(\Omega)$ is proved. In Section 5 we establish a nonexistence theorem in $L^r(\Omega)$, $1 \leq r < \infty$. The method used in Section 5 is based on the approach developped in [17,23]. Combining existence and nonexistence results, we obtain a necessarily and sufficient condition on the growth of f for the existence of a nonnegative solution to (1.1) in $L^r(\Omega)$. Finally, in Section 6 we prove the uniqueness of a solution of (1.1) in the set K given in Theorem 1.6.

2. Exsitence of a solution in $L^r(\Omega)$, $1 < r < \infty$

Let us start with a quick overview on the monotone iteration techniques in the fractional setting. Consider the problem

$$\begin{cases} \partial_t^\alpha u = \Delta u + f_0(x,t,u), & x \in \Omega,\ 0 < t < T, \\ u = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with smooth boundary and $f_0 : \Omega \times [0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that:

(F2) $\qquad f_0(x,t,\xi)$ is nonnegative and it is nondecreasing in ξ.

Definition 2.1. We sat that a measurable function u a solution of (2.1) if there exists $T \in (0,\infty]$ such that the following conditions (a)–(e) hold:

(a) $u(\,\cdot\,,t) \in L^1(\Omega)$ for all $0 < t < T$;

(b) $f_0(\,\cdot\,,t,u(\,\cdot\,,t)) \in L^1(\Omega)$ for a.e. $0 < t < T$;

(c) $\displaystyle\int_0^t \|f_0(\,\cdot\,,s,u(\,\cdot\,,s))\|_1\, ds < \infty$ for $0 < t < T$;

(d) $u(\,\cdot\,,t)$ satisfies

$$u(\,\cdot\,,t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f_0(\,\cdot\,,s,u(\,\cdot\,,s))ds \quad \text{for} \quad 0 < t < T,$$

where the integral in the above equality is an absolutely converging Bochner integral in $L^1(\Omega)$;

(e) The initial condition holds in the following sense:

$$\|u(\,\cdot\,,t) - u_0\|_r \to 0 \qquad \text{as } t \to 0$$

for $u_0 \in L^r(\Omega)$ if $1 \le r < \infty$.

We call $\bar{u}$ a supersolution of (2.1) if $\bar{u}$ satisfies Definition 2.1 (a)–(c) and the following (d') and (e') hold:

(d') $\bar{u}$ satisfies the inequality:

$$\bar{u}(\,\cdot\,,t) \ge S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f_0(\,\cdot\,,s,\bar{u}(\,\cdot\,,s))ds \quad \text{for} \quad 0 < t < T,$$

(e') $\bar{u}$ satisfies

$$\left\| \int_0^t P_\alpha(t-s)f_0(\,\cdot\,,s,\bar{u}(\,\cdot\,,s))ds \right\|_r \to 0 \qquad \text{as } t \to 0,$$

where $u_0 \in L^r(\Omega)$ for some $1 \le r < \infty$.

Proposition 2.2. *Let $0 < T \le \infty$. Suppose that the function $f_0(x,t,\xi)$ satisfies (F2).*

If (2.1) has a nonnegative supersolution $\bar{u}(t)$ on $0 < t < T$, then (2.1) has a solution $u(t)$ in the sense of Definition 1.1 on $0 < t < T$ such that $0 \le u(t) \le \bar{u}(t)$.

For the reader's convenience we provide a quick proof (see [22, Theorem 2.1] for details).

Proof. Let $\bar{u}$ be a nonnegative supersolution of (2.1) for $0 < t < T$. Set $u_1 = S(t)u_0$ and for $n \ge 2$ define u_n by

$$u_n = \mathcal{F}(u_{n-1}),$$

where

$$\mathcal{F}(u) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s)f_0(\,\cdot\,,s,u(\,\cdot\,,s))ds.$$

Then, an induction argument yields

$$0 \le u_1 \le u_2 \le \cdots \le u_n \le \cdots \le \bar{u} < \infty \quad \text{for a.e. } x \in \mathbb{R}^N,\ 0 < t < T.$$

This indicates that the limit $u(x,t) := \lim_{n\to\infty} u_n(x,t)$ exists for almost all $x \in \mathbb{R}^N$ and $0 < t < T$. By the monotone convergence theorem we see that

$$\lim_{n\to\infty} \mathcal{F}(u_{n-1}) = \mathcal{F}(u),$$

and hence $u = \mathcal{F}(u)$. It is clear that

$$0 \le u(t) \le \bar{u}(t) \ \text{ for a.e } x \in \mathbb{R}^N,\ 0 < t < T. \tag{2.2}$$

We check that $u(t)$ satisfies the conditions in Definition 1.1. Since $\bar{u}$ satisfies Definition 1.1 (a)–(c), by (F2) and (2.2) we see that u also satisfies (a)–(c). Since $u = \mathcal{F}(u)$, u satisfies (d). By (F2) and (2.2) we obtain

$$\begin{aligned}\|u(t) - S_\alpha(t)u_0\|_r &= \left\| \int_0^t P_\alpha(t-s) f_0(\,\cdot\,, s, u(\,\cdot\,,s))ds \right\|_r \\ &\le \left\| \int_0^t P_\alpha(t-s) f_0(\,\cdot\,, s, \bar{u}(\,\cdot\,,s))ds \right\|_r .\end{aligned}$$

Since $\bar{u}$ satisfies (e'), we have

$$\|u(t) - S_\alpha(t)u_0\|_r \to 0 \quad \text{ as } t \to 0,$$

and hence (e) holds. Thus, u satisfies Definition 1.1 (a)–(e), and hence u is a solution of (2.1). □

Theorem 2.3. *Suppose that $f_0(x,t,\xi)$ satisfies (F2). Suppose that there exists a function $f_1(x,t,\xi)$ such that f_1 satisfies (1.4) and (1.5) with*

$$p \in [1,\infty),\ q_1 \in [1,\infty] \cap (\frac{N}{2},\infty],\ q_2 \in (\frac{1}{\alpha},\infty]$$

and

$$f_0(x,t,\xi) \le f_1(x,t,\xi) \ \textit{for } x \in \Omega,\ t \ge 0 \ \textit{and } \xi \ge 0. \tag{2.3}$$

Then, for each nonnegative initial function $u_0 \in L^r(\Omega)$, (2.1) has a nonnegative local-in-time solution if one of the following holds:

(i) *$p \in (1,\infty)$, $r \in (1,\infty)$ and*

$$\frac{N}{2q_1} + \frac{1}{\alpha q_2} + \frac{(p-1)N}{2r} \le 1. \tag{2.4}$$

(ii) *$p \in [1,\infty)$, $r \in [1,\infty)$ and*

$$\frac{N}{2q_1} + \frac{1}{\alpha q_2} + \frac{(p-1)N}{2r} < 1.$$

Proof. Since all the assumptions in [10, Theorem 3.1.4] are satisfied in both of the above cases (i) and (ii), it follows from [10, Theorem 3.1.4] that the problem

$$\begin{cases} \partial_t^\alpha u = \Delta u + f_1(x,t,u), & x \in \Omega,\ 0 < t < T, \\ u = 0, & x \in \partial\Omega,\ 0 < t < T, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{2.5}$$

has a local-in-time solution, which is denoted by $\bar{u}(t)$, in the sense of Definition 1.1. By (F2) and (2.3) we see that $f_1(x,t,\xi) \ge 0$ for $\xi \ge 0$. Since $u_0 \ge 0$, it is clear from Definition 1.1 (d) that the solution $\bar{u}(t)$ is nonnegative. We show that $\bar{u}$ is a supersolution of (2.1). Since $\bar{u}$ is a solution of (2.5), it is clear that $\bar{u}$ satisfies Definition 1.1 (a)–(c) with f_0. Because of (2.3), (d') clearly holds. Using the equivalence of (e) and (1.7), we have

$$\left\|\int_0^t P_\alpha(t-s)f_0(\,\cdot\,,s,\bar{u}(\,\cdot\,,s))ds\right\|_r \le \left\|\int_0^t P_\alpha(t-s)f_1(\,\cdot\,,s,\bar{u}(\,\cdot\,,s))ds\right\|_r$$
$$= \|\bar{u}(t)-S_\alpha(t)u_0\|_r \to 0.$$

Therefore, (e') holds, and hence $\bar{u}$ is a supersolution. Since $\bar{u}(t)$ is a supersolution of (2.1), by Proposition 2.2 we see that (2.1) has a nonnegative solution. □

Theorem 2.4. *Suppose that $f \in C(\mathbb{R})$ is nonnegative and nondecreasing. Then, for each nonnegative initial function $u_0 \in L^r(\Omega)$, (1.1) has a nonnegative local-in-time solution if one of the following holds:*

(i) $1 < r < \infty$ *and*

$$\limsup_{s\to\infty} \frac{f(s)}{s^{1+2r/N}} < \infty. \tag{2.6}$$

(ii) $r = 1$ *and there exists* $p \in (0, 1+2r/N)$ *such that*

$$\limsup_{s\to\infty} \frac{f(s)}{s^p} < \infty.$$

Proof. First, we prove the case (i). Let $p = 1 + 2r/N$. We define $f_0(x,t,\xi) = f(u)$. Because of (2.6), there exists $A > 0$ such that $f(\xi) \le A(1+|\xi|^p)$ for $\xi \ge 0$. We define $f_1(x,t,\xi) = A(1+|\xi|^p)$. It is clear that $f_1(x,t,\xi)$, which is defined on $\Omega\times[0,\infty)\times\mathbb{R}$, satisfies (1.4) and (1.5). Because f_1 does not depend on (x,t), $c(x,t)$'s in (1.4) and (1.5) are constants, and hence $q_1 = \infty$ and $q_2 = \infty$. Then, p, q_1, q_2, r satisfy all assumptions in Theorem 2.3 (i), including (2.4). Then, (2.1) has a nonnegative local-in-time solution for every nonnegative initial $u_0 \in L^r(\Omega)$. The proof of the case (i) is complete.

Using Theorem 2.3 (ii), one can prove the case (ii) in a similar way. We omit the details. The proof of the theorem is complete. □

In Section 3 we show that Theorem 2.4 holds for $r = 1$ and $p = 1 + 2/N$.

3. Existence of a solution in $L^1(\Omega)$

We recall the following estimates on $S_\alpha(t)$ and $P_\alpha(t)$.

Proposition 3.1. *(i) Let $N \ge 1$ and $1 \le p \le q \le \infty$. Then, for each $\phi \in L^p(\Omega)$,*

$$\|S(t)\phi\|_q \le t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi\|_p \quad \textit{for} \;\; t>0.$$

(ii) Let $N \ge 1$ and $1 \le p < q \le \infty$. Then, for each $C_* > 0$ and $\phi \in L^p(\Omega)$, there is $t_0 = t_0(C_*, \phi_0)$ such that

$$\|S(t)\phi\|_q \le C_* t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \textit{for} \;\; 0<t<t_0.$$

The multiplicative constant in (i) can be taken to be 1 (see [5, Lemma 7]). The constant C_* in (ii) can be taken arbitrarily small. The proof of Proposition 3.1 (ii) can be found in [5, Lemma 8].

Proposition 3.2. (see [10, Lemma 2.2.2])

Let $N \geq 1$ and $1 \leq p \leq q \leq \infty$. Then the following hold:
(i) If $\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right) < 1$, then there exists $C > 0$ such that, for each $\phi \in L^p(\Omega)$,

$$\|S_\alpha(t)\phi\|_q \leq Ct^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\phi\|_p \quad \text{for } \; t > 0.$$

(ii) If $\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right) < 2$, then there exists $C > 0$ such that, for each $\phi \in L^p(\Omega)$,

$$\left\|t^{1-\alpha}P_\alpha(t)\phi\right\|_q \leq Ct^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\phi\|_p \quad \text{for } \; t > 0.$$

A similar result to Proposition 3.2 (ii) holds for $S_\alpha(t)$.

Lemma 3.3. *Let $N \geq 1$ and $1 \leq p < q \leq \infty$ such that $\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right) < 1$. Then, for each $C_* > 0$ and $\phi \in L^p(\Omega)$, there is $t_0 = t_0(C_*, \phi)$ such that*

$$\|S_\alpha(t)\phi\|_q \leq C_* t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for } \; 0 < t < t_0.$$

We can take an arbitrarily small $C_* > 0$.

Proof. Let $\varepsilon > 0$ be fixed. By Proposition 3.1 (ii) we see that there is $t_1 > 0$ such that

$$\|S(t)\phi\|_q \leq \varepsilon t^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for } \; 0 < t < t_1.$$

Then, if $0 < \tau t^\alpha < t_1$, then

$$\|S(\tau t^\alpha)\phi\|_q \leq \varepsilon \tau^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for } \; 0 < \tau t^\alpha < t_1,$$

and hence

$$\begin{aligned} \int_0^{t_1 t^{-\alpha}} \Phi_\alpha(\tau) \|S(\tau t^\alpha)\phi\|_q \, d\tau &\leq \varepsilon t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \int_0^{t_1 t^{-\alpha}} \Phi_\alpha(\tau) \tau^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} d\tau \\ &\leq \varepsilon t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \frac{\Gamma\left(1-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)}{\Gamma\left(1-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)}. \end{aligned} \tag{3.1}$$

Note that (3.1) holds for all $t > 0$. On the other hand, there is $t_2 > 0$ such that if $0 < t < t_2$, then

$$\begin{aligned} \int_{t_1 t^{-\alpha}}^\infty \Phi_\alpha(\tau) \|S(\tau t^\alpha)\phi\|_q \, d\tau &\leq \|\phi\|_p t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \int_{t_1 t^{-\alpha}}^\infty \Phi_\alpha(\tau) \tau^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} d\tau \\ &\leq \varepsilon t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \|\phi\|_p, \end{aligned} \tag{3.2}$$

since

$$\int_0^\infty \Phi_\alpha(\tau) \tau^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} d\tau < \infty.$$

By (3.1) and (3.2) we have

$$\|S_\alpha(t)\phi\|_q \leq \varepsilon C_1 t^{-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for all } \; 0 < t < t_1,$$

where

$$C_1 = \frac{\Gamma\left(1-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)}{\Gamma\left(1-\frac{N\alpha}{2}\left(\frac{1}{p}-\frac{1}{q}\right)\right)} + \|\phi\|_p.$$

We can take an arbitrarily small $\varepsilon > 0$ if $t_1 > 0$ is small enough. Therefore, we can take $\varepsilon > 0$ such that $C_* = C_1\varepsilon$, and hence the conclusion holds. $\square$

Proof of Theorem 1.4. Let

$$E = L^\infty((0,T), L^1(\Omega)) \cap \left\{u \in L^\infty_{\mathrm{loc}}((0,T), L^p(\Omega));\ t^\beta u \in L^\infty((0,T), L^p(\Omega))\right\}, \tag{3.3}$$

where $\beta = \frac{N\alpha}{2}\left(1 - \frac{1}{p}\right) = \frac{\alpha}{p} < 1$ and T will determined later.

We define the metric $d : E \times E \to [0,\infty)$ by

$$d(u,v) = \sup_{0<t<T} t^\beta \|u(t) - v(t)\|_p \quad \text{for all } u, v \in E. \tag{3.4}$$

Let $M \geq \|u_0\|_1$ and

$$K = \left\{u \in E:\ \|u(t)\|_1 \leq C_1 M + 1 \text{ and } t^\beta \|u(t)\|_p \leq \delta \text{ for } 0 < t < T\right\},$$

where $C_1 > 0$ and $\delta > 0$ are chosen later. Then, (K, d) is a nonempty complete metric space. We define $\mathcal{F} : K \to E$ by

$$\mathcal{F}(u)(t) = S_\alpha(t)u_0 + \int_0^t P_\alpha(t-s) f(u(s))ds \quad \text{for all } t \geq 0. \tag{3.5}$$

Because of (F1), the following holds:

$$\text{There exists } C > 0 \text{ such that } |f(\xi)| \leq C(1 + |\xi|^p) \ \text{ for } \ \xi \in \mathbb{R}. \tag{3.6}$$

By (3.6) it follows that

$$\|f(u(s))\|_1 \leq C(1 + \|u(s)\|_p^p). \tag{3.7}$$

Using the estimate (3.7) together with Proposition 3.2(ii) (for $p = q = 1$) we have

$$\begin{aligned}
\|\mathcal{F}(u)(t)\|_1 &\leq C_1 \|u_0\|_1 + C_2 \int_0^t (t-s)^{-1+\alpha} \left(1 + \|u(s)\|_p^p\right) ds \\
&\leq C_1 \|u_0\|_1 + \frac{C_2}{\alpha} t^\alpha + C_2 \int_0^t (t-s)^{-1+\alpha} s^{-\alpha} \left(s^\beta \|u(s)\|_p\right)^p ds \\
&\leq C_1 \|u_0\|_1 + \frac{C_2}{\alpha} t^\alpha + C_2 \left(\sup_{0<t<T} t^\beta \|u(t)\|_p\right)^p \int_0^1 (1-\tau)^{-1+\alpha} \tau^{-\alpha} d\tau \\
&\leq C_1 \|u_0\|_1 + \frac{C_2}{\alpha} t^\alpha + C_2' \delta^p. \qquad (3.8)
\end{aligned}$$

Here, we used $p\beta = \alpha$ and the fact that with the change of variable $s = \tau t$ one has

$$\int_0^t (t-s)^{-1+\alpha} s^{-\alpha} ds = \int_0^1 (1-\tau)^{-1+\alpha} \tau^{-\alpha} d\tau < \infty.$$

Therefore,

$$\|\mathcal{F}(u)(t)\|_1 < C_1 M + 1 \tag{3.9}$$

if

$$C_2' \delta^p < \frac{1}{2} \tag{3.10}$$

and

$$C_1 \alpha^{-1} T^\alpha < \frac{1}{2}. \tag{3.11}$$

Next, by Proposition3.2(ii) (with $q=1$) and (3.7) we have

$$\begin{aligned}
t^{\beta}\|\mathcal{F}(u)(t)\|_p &\le \sup_{0<t<T} t^{\beta}\|S_{\alpha}(t)u_0\|_p + C_3 t^{\beta}\int_0^t (t-s)^{-1+\alpha-\beta}\left(1+\|u(s)\|_p^p\right)ds\\
&\le \sup_{0<t<T} t^{\beta}\|S_{\alpha}(t)u_0\|_p + \frac{C_3}{\alpha-\beta}t^{\alpha}\\
&\quad + C_3 t^{\beta}\left(\sup_{0<t<T} t^{\beta}\|u(t)\|_p\right)^p \int_0^t (t-s)^{-1+\alpha-\beta}s^{-p\beta}ds\\
&\le \sup_{0<t<T} t^{\beta}\|S_{\alpha}(t)u_0\|_p + \frac{C_3}{\alpha-\beta}t^{\alpha} + C_3\delta^p\int_0^1 (1-\tau)^{-1+\alpha-\beta}\tau^{-\alpha}d\tau\\
&\le \sup_{0<t<T} t^{\beta}\|S_{\alpha}(t)u_0\|_p + \frac{C_3}{\alpha-\beta}t^{\alpha} + C_3'\delta^p. \qquad (3.12)
\end{aligned}$$

In the above sequence of estimates we used the change of variable $s=\tau t$ so that

$$\int_0^t (t-s)^{-1+\alpha-\beta}s^{-p\beta}ds = t^{-\beta}\int_0^1 (1-\tau)^{-1+\alpha-\beta}\tau^{-\alpha}d\tau \le Ct^{-\beta}.$$

Therefore,

$$\sup_{0<t<T} t^{\beta}\|\mathcal{F}(u)(t)\|_p \le \sup_{0<t<T} t^{\beta}\|S_{\alpha}(t)u_0\|_p + \frac{\delta}{2} \qquad (3.13)$$

if

$$C_3'\delta^{p-1} < \frac{1}{4} \qquad (3.14)$$

and

$$\frac{C_3}{\alpha-\beta}T^{\alpha} < \frac{\delta}{4}. \qquad (3.15)$$

Since

$$\begin{aligned}
\int_{\Omega}|f(u(s))-f(v(s))|dx &\le C_4\int_{\Omega}(1+|u(s)|+|v(s)|)^{p-1}|u(s)-v(s)|dx\\
&\le C_4'\int_{\Omega}\left(1+|u(s)|^{p-1}+|v(s)|^{p-1}\right)|u(s)-v(s)|dx\\
&\le C_4''\|u(s)-v(s)\|_p\\
&\quad + C_4'\left(\|u(s)\|_p^{p-1}+\|v(s)\|_p^{p-1}\right)\|u(s)-v(s)\|_p,
\end{aligned}$$

we have

$$\begin{aligned}
t^{\beta}\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\|_p &\le C_3 t^{\beta}\int_0^t (t-s)^{-1+\alpha-\beta}\|f(u(s))-f(v(s))\|_1 ds\\
&\le C_3 t^{\beta}\left(C_4''\int_0^t (t-s)^{-1+\alpha-\beta}s^{-\beta}ds\right.\\
&\quad \left.+2\delta^{p-1}C_4'\int_0^t (t-s)^{-1+\alpha-\beta}s^{-p\beta}ds\right)\sup_{0<t<T} t^{\beta}\|u(t)-v(t)\|_p\\
&\le C_3\left(C_4''C_5 t^{\alpha-\beta}+2C_4'C_6\delta^{p-1}\right)\sup_{0<t<T} t^{\beta}\|u(t)-v(t)\|_p, \qquad (3.16)
\end{aligned}$$

where

$$C_5 = \int_0^1 (1-\tau)^{-1+\alpha-\beta}\tau^{-\beta}d\tau \quad \text{and} \quad C_6 = \int_0^1 (1-\tau)^{-1+\alpha-\beta}\tau^{-\alpha}d\tau.$$

Therefore,

$$\sup_{0<t<T} t^\beta \left\| \mathcal{F}(u)(t) - \mathcal{F}(v)(t) \right\|_p \le \frac{1}{2} d(u,v) \tag{3.17}$$

if

$$2C_3C_4'C_6\delta^{p-1} < \frac{1}{4} \tag{3.18}$$

and

$$C_3C_4''C_5T^{\alpha-\beta} < \frac{1}{4}. \tag{3.19}$$

We fix any small $\delta > 0$ such that (3.18), (3.14) and (3.10) hold. Because of Lemma 3.3, we can choose $T > 0$ such that (3.19), (3.15) and (3.11) hold and

$$\sup_{0<t<T} t^\beta \left\| S_\alpha(t)u_0 \right\|_p \le \frac{\delta}{2}. \tag{3.20}$$

Then, by (3.20), (3.13) and (3.9) we see that $\mathcal{F} : K \to K$. Moreover, it follows from (3.17) that $\mathcal{F}$ is a contraction mapping on K. Thus, $\mathcal{F} : K \to K$ has a unique fixed point in K which is denoted by $u(t)$.

We check that $u(t)$ satisfies Definition 1.1 (a)–(e). Since $u(t) \in K$, (a) holds. Since $t^\beta \left\| u(t) \right\|_p \le \delta$ and $p\beta = \alpha$, by (3.6) we have

$$\left\| f(u(t)) \right\|_1 \le C(1 + \left\| u(t) \right\|_p^p) \le C(1 + \delta t^{-\alpha}) < \infty \quad \text{for} \quad 0 < t < T. \tag{3.21}$$

Thus, (b) holds. Let $t \in (0,T)$. Integrating (3.21) over $(0,t)$, we see that (c) holds. We can easily check that $\int_0^t \left\| P_\alpha(t-s)f(u(s)) \right\|_1 ds < \infty$ for $0 < t < T$ which indicates that $u(t) \in C((0,T), L^1(\Omega))$. Therefore, $u(t) = \mathcal{F}(u)(t)$ for a.e. $x \in \Omega$ and $0 < t < T$. Hence, (d) holds. As in (3.8) we have

$$\left\| u(t) - S_\alpha(t)u_0 \right\|_1 \le C_1\alpha^{-1}t^\alpha + C_2'\delta^p.$$

Since $\delta > 0$ and $t > 0$ can be chosen arbitrarily small, by the local uniqueness of the fixed point we have

$$\left\| u(t) - S_\alpha(t)u_0 \right\|_1 \to 0,$$

which indicates that (e) holds. Thus, $u(t)$ is a solution of (1.1) in the sense of Definition 1.1. □

Corollary 3.4. *Suppose that $f \in C(\mathbb{R})$ is nonnegative and nondecreasing. Then, for each nonnegative initial function $u_0 \in L^1(\Omega)$, (1.1) has a nonnegative solution if*

$$\limsup_{s\to\infty} \frac{f(s)}{s^{1+2/N}} < \infty. \tag{3.22}$$

Proof. The proof is similar to that of Theorem 2.4. Let $p = 1+2/N$. Because of (3.22), there is $C > 0$ such that $f(\xi) \le C(1+\xi^p)$ for $\xi \ge 0$. Let $f_1(\xi) = C(1+|\xi|^p)$. Then it follows from Theorem 1.4 that (1.1) with f_1 has a nonnegative solution $\bar{u}$ for each nonnegative initial data $u_0 \in L^1(\Omega)$. By Proposition 2.2 we can construct a nonnegative solution for (1.1) with f, since $\bar{u}$ is a supersolution for (1.1). The details are omitted. □

4. Global-in-time solution in $L^1(\Omega)$

Proof of Theorem 1.5. We follow a similar strategy to that in the proof of Theorem 1.4. Let E (resp. d) be defined by (3.3) with $T=\infty$ (resp. by (3.4) with $T=\infty$). Here, $\beta=\frac{N\alpha}{2}\left(1-\frac{1}{p}\right)=\frac{\alpha}{p}<1$. We define K by

$$K=\left\{u\in E;\ \sup_{t>0}\|u(t)\|_1+\sup_{t>0}t^\beta\|u(t)\|_p\le\delta\right\},$$

where $\delta>0$ is determined later. Then, (K,d) is a nonempty complete metric space. We define $\mathcal{F}$ by (3.5). Hereafter we assume that $\|u_0\|_1\le\varepsilon$. By (F1') we see that there is $C>0$ such that $|f(u)|\le C|u|^p$. By a calculation similar to (3.8) we have

$$\|\mathcal{F}(u)(t)\|_1\le C_1\|u_0\|+C_2'\delta^p.$$

Therefore,

$$\|\mathcal{F}(u)(t)\|_1\le\delta \tag{4.1}$$

if

$$C_1\varepsilon\le\frac{\delta}{2} \tag{4.2}$$

and

$$C_2'\delta^{p-1}\le\frac{1}{2}. \tag{4.3}$$

By a calculation similar to (3.12) we have

$$t^\beta\|\mathcal{F}(u)(t)\|_p\le\sup_{t>0}t^\beta\|S_\alpha(t)u_0\|_p+C_3'\delta^p.$$

Therefore,

$$\sup_{t>0}t^\beta\|\mathcal{F}(u)(t)\|_p\le\sup_{t>0}t^\beta\|S_\alpha(t)u_0\|_p+\frac{\delta}{2} \tag{4.4}$$

if

$$C_3'\delta^{p-1}\le\frac{1}{2}. \tag{4.5}$$

By a calculation similar to (3.16) we have

$$t^\beta\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\|_p\le 2C_3C_4'C_6\delta^{p-1}\sup_{t>0}t^\beta\|u(t)-v(t)\|_p.$$

Here we used (F1'), and hence the term $C_3C_4''C_5t^{\alpha-\beta}$ in (3.16) does not appear. Therefore,

$$\sup_{t>0}t^\beta\|\mathcal{F}(u)(t)-\mathcal{F}(v)(t)\|_p\le\frac{1}{2}d(u,v) \tag{4.6}$$

if

$$2C_3C_4'C_6\delta^{p-1}\le\frac{1}{2}. \tag{4.7}$$

We fix $\delta > 0$ such that (4.7), (4.5) and (4.3) hold. By Proposition 3.2 (i) we see that

$$\sup_{t>0} t^{\beta} \left\| S_\alpha(t)u_0 \right\|_p \le C\varepsilon. \tag{4.8}$$

We can choose $\varepsilon > 0$ such that $C\varepsilon \le \frac{\delta}{2}$ and (4.2) holds. Then, it follows from (4.8) and (4.4) that

$$\sup_{t>0} t^{\beta} \left\| \mathcal{F}(u)(t) \right\|_p \le \delta. \tag{4.9}$$

By (4.9) and (4.1) we see that $\mathcal{F} : K \to K$. Moreover, by (4.6) we see that $\mathcal{F}$ is a contraction mapping on K. Thus, $\mathcal{F} : K \to K$ has a unique fixed point in K which is denoted by $u(t)$.

We can check that $u(t)$ satisfies Definition 1.1 (a)–(e) with $T = \infty$ in the same way as in the proof of Theorem 1.4. We omit details.

The existence of a nontrivial nonnegative solution with small nonnegative initial function easily follows from a nonnegativity of a solution with nontrivial nonnegative initial function.

The proof is complete. $\square$

Corollary 4.1. *Suppose that $f \in C(\mathbb{R})$ is nonnegative and nondecreasing. If*

$$\sup_{t>0} \frac{f(s)}{s^{1+2/N}} < \infty, \tag{4.10}$$

then there is $\varepsilon > 0$ such that for each nonnegative initial function $u_0 \in L^1(\Omega)$ with $\|u_0\|_1 \le \varepsilon$, there exists a global-in-time solution $u(t)$ of (1.1) in the sense of Definition 1.1 with $T = \infty$.

Proof. Let $p = 1 + 2/N$. Because of (4.10), there is $C > 0$ such that $f(\xi) \le C\xi^p$ for $\xi \ge 0$. Let $f_1(\xi) = C\xi^p$. Since f_1 satisfies (F1'), it follows from Theorem 1.5 that there is $\varepsilon > 0$ such that (1.1) with f_1 has a nonnegative global-in-time solution $\bar{u}$ for each nonnegative initial data $u_0 \in L^1(\Omega)$ with $\|u_0\|_1 \le \varepsilon$. By Proposition 2.2 we can construct a nonnegative global-in-time solution for (1.1) with f, since $\bar{u}$ is a supersolution of (1.1) with $T = \infty$. The proof is complete. $\square$

5. Nonexsitence of a solution in $L^r(\Omega)$, $1 \le r < \infty$

Let $G(x, y, t)$ denote the Dirichlet heat kernel on Ω. When $u_0 \in L^r(\Omega)$, we see that $S(t)u_0(x) = \int_\Omega G(x, y, t)u_0(y)dy$ and that $S(t)u_0$ gives a solution of the heat equation $\partial_t u = \Delta u$ on Ω with the Dirichlet boundary condition.

Proposition 5.1 (see e.g., [4, Corollary 2.2])**.** *Let $\delta > 0$ be such that $B_{2\delta} \subset \Omega$. Then,*

$$G(x, y, t) \ge c_* t^{-N/2} \text{ for all } x, y \in B_\delta \text{ and } 0 \le t \le \delta^2$$

such that $|x - y| \le \sqrt{t}$. Here, c_* depends on N and δ.

Proof of Theorem 1.3. Let $p = 1 + 2r/N$ and denote by ω_N the volume of the unit ball in $\mathbb{R}^N$. If c_* is the constant given in Proposition 5.1, we define

$$c_0 = \frac{c_* \omega_N}{2^N} \int_{1/4}^{1} \Phi_\alpha(\tau)d\tau,$$

which depends only on c_*, N and α. By (1.8) we can choose $\{a_k\}_{k=1}^\infty$ such that

$$a_k \geq k \quad \text{and} \quad f(c_0 a_k) \geq a_k^p e^{k/r} \qquad \text{for all } k \geq 1. \tag{5.1}$$

Let $\rho_k = \varepsilon a_k^{-r/N} k^{-2r/N}$. By taking $\varepsilon > 0$ small enough, we may assume $B_{3\rho_k} \subset \Omega$ for all $k \geq 1$. Let

$$u_0(x) = \sum_{k=1}^\infty u_k \quad \text{where} \quad u_k = a_k \chi_{\rho_k},$$

where for $\rho > 0$, χ_ρ denotes the indicator function of the ball $B_\rho \subset \mathbb{R}^N$.

Since

$$\|u_k\|_r^r = \omega_N \rho_k^N a_k^r = \omega_N \frac{\varepsilon^N}{k^{2r}},$$

we have

$$\|u_0\|_r \leq \sum_{k=1}^\infty \|u_k\|_r = \omega_N^{1/N} \varepsilon^{N/r} \sum_{k=1}^\infty \frac{1}{k^2} < \infty.$$

Suppose that (1.1) with the above defined initial data $u_0 \in L^r(\mathbb{R}^N)$ has a nonnegative solution $u(t)$ for small $t > 0$. Then,

$$u(t) = S_\alpha(t) u_0 + \int_0^t P_\alpha(t-s) f(u(s)) ds.$$

Since $u \geq 0$ and $f \geq 0$, we have

$$u(t) \geq S_\alpha(t) u_0 \geq S_\alpha(t) u_k.$$

Hence, for any $k \geq 1$,

$$u(t) \geq \int_0^t P_\alpha(t-s) f(S_\alpha(s) u_k) ds, \tag{5.2}$$

since f is nondecreasing.

For $k \geq 1$, let $s > 0$ be small such that $\sqrt{s^\alpha} \leq \rho_k/2$ and $0 < \tau \leq 1$. By Proposition 5.1 applied for $t = \tau s^\alpha$ and $\delta = t^2$ one has

$$\begin{aligned} S(\tau s^\alpha)\chi_{\rho_k} &= \int_{|y| \leq \rho_k} G(x, y, \tau s^\alpha) dy \\ &\geq \chi_{\sqrt{\tau s^\alpha}/2} \int_{|x-y| \leq \sqrt{\tau s^\alpha}/2} c_* (\tau s^\alpha)^{-N/2} dy \\ &= \frac{c_* \omega_N}{2^N} \chi_{\sqrt{\tau s^\alpha}/2}. \end{aligned}$$

Hence, from (1.2) we find

$$\begin{aligned} S_\alpha(s) u_k &\geq \int_{1/4}^1 \Phi_\alpha(\tau) S(\tau s^\alpha)[a_k \chi_{\rho_k}] d\tau \\ &\geq \frac{c_* \omega_N a_k}{2^N} \int_{1/4}^1 \Phi_\alpha(\tau) \chi_{\sqrt{\tau s^\alpha}/2} d\tau \\ &\geq \frac{c_* \omega_N a_k}{2^N} \chi_{\sqrt{s^\alpha}/4} \int_{1/4}^1 \Phi_\alpha(\tau) d\tau = c_0 a_k \chi_{\sqrt{s^\alpha}/4}. \end{aligned} \tag{5.3}$$

On the other hand, if $t>0$ is small and

$$\frac{\sqrt{\tau(t-s)^\alpha}}{2} \le \frac{\sqrt{s^\alpha}}{8}, \quad \frac{t}{3} \le s \le \frac{t}{2} \tag{5.4}$$

then

(5.5)

$$\begin{aligned} S(\tau(t-s)^\alpha)[\chi_{\sqrt{s^\alpha}/4}] &= \int_{|y|\le\sqrt{s^\alpha}/4} G(x,y,\tau(t-s)^\alpha)dy \\ &\ge \chi_{\sqrt{\tau(t-s)^\alpha}/2} \int_{|x-y|\le\sqrt{\tau(t-s)^\alpha}/2} c_*\{\tau(t-s)^\alpha\}^{-N/2}dy \\ &= \frac{c_*\omega_N}{2^N}\chi_{\sqrt{\tau(t-s)^\alpha}/2} > \ c_0\chi_{\sqrt{\tau(t-s)^\alpha}/2}. \end{aligned}$$

Let $\tau_0 = \frac{1}{16\cdot 2^\alpha}$. If $\tau_0/4 \le \tau \le \tau_0$, then

$$\sqrt{\tau(t-s)^\alpha}/2 \le \sqrt{s^\alpha}/8 \quad \text{ for all } s,t>0 \text{ such that } t/3 \le s \le t/2.$$

Since (5.4) holds for $\tau_0/4 \le \tau \le \tau_0$, by (5.3) and (5.5) we see that if

$$\sqrt{s^\alpha} \le \rho_k/2,\ t/3 \le s \le t/2 \quad \text{ and } \quad t \ge 0 \ \text{ is small,}$$

then

$$\begin{aligned} P_\alpha(t-s)[f(S_\alpha(s)u_k)] &\ge P_\alpha(t-s)[f(c_0a_k)\chi_{\sqrt{s^\alpha}/4}] \\ &\ge f(c_0a_k)\alpha(t-s)^{\alpha-1}\int_{\tau_0/4}^{\tau_0} \tau\Phi_\alpha(\tau)S(\tau(t-s)^\alpha)[\chi_{\sqrt{s^\alpha}/4}]d\tau \\ &\ge f(c_0a_k)\alpha(t-s)^{\alpha-1}\int_{\tau_0/4}^{\tau_0} \tau\Phi_\alpha(\tau)c_0\chi_{\sqrt{\tau(t-s)^\alpha}/2}d\tau \\ &\ge f(c_0a_k)\alpha(t-s)^{\alpha-1}c_0\chi_{\sqrt{\tau(t-s)^\alpha}/4}\int_{\tau_0/4}^{\tau_0} \tau\Phi_\alpha(\tau)d\tau \\ &= c_1f(c_0a_k)\alpha(t-s)^{\alpha-1}\chi_{\sqrt{\tau_0(t-s)^\alpha}/4}. \end{aligned}$$

Therefore, if $t>0$ is small and

$$\sqrt{\left(\frac{t}{2}\right)^\alpha} \le \frac{\rho_k}{2}, \tag{5.6}$$

then

$$\begin{aligned} \int_0^t P_\alpha(t-s)[f(S_\alpha(s)u_k)]ds &\ge c_1f(c_0a_k)\alpha\int_{t/3}^{t/2}(t-s)^{\alpha-1}\chi_{\sqrt{\tau_0(t-s)^\alpha}/4}ds \\ &\ge c_1f(c_0a_k)\left\{\left(\frac{2}{3}\right)^\alpha - \left(\frac{1}{2}\right)^\alpha\right\}t^\alpha\chi_{\sqrt{\tau_0(t/2)^\alpha}/4}. \end{aligned}$$

Thus, from (5.1) and (5.2) we deduce

$$\|u(t)\|_r^r \ge c_2f(c_0a_k)^rt^{\alpha r}t^{N\alpha/2} \ge c_2a_k^{rp}t^{\alpha\left(r+\frac{N}{2}\right)}e^k.$$

It follows from (5.6) that $t \le 2^{1-2/\alpha}\rho_k^{2/\alpha}$, and hence we can choose $t = 2^{1-2/\alpha}\rho_k^{2/\alpha}$. Then,

$$\|u(t)\|_r^r \ge c_3a_k^{rp}\rho_k^{N+2r}e^k = c_4k^{-2rp}e^k \to \infty \text{ as } k\to\infty.$$

This means that $\|u(t)\|_r^r \to \infty$ as $t \to 0$, since $t = 2^{1-2/\alpha}\rho_k^{2/\alpha} \to 0$ as $k \to \infty$. Therefore, $\|u(t) - u_0\|_r \geq \|u(t)\|_r - \|u_0\|_r \to \infty$, and hence Definition 1.1 (e) does not hold. Thus, (1.1) with the initial function u_0 has no nonnegative solution. $\square$

Corollary 5.2. *Let $1 \leq r < \infty$. Suppose that $f \in C[0,\infty)$ is nonnegative and nondecreasing. The problem (1.1) has a nonnegative solution for every nonnegative initial data $u_0 \in L^r(\Omega)$ if and only if*

$$\limsup_{s\to\infty} \frac{f(s)}{s^{1+2r/N}} < \infty.$$

Proof. When $1 < r < \infty$, the sufficient part follows from Theorem 2.4 (i) and the necessary part follows from Theorem 1.3 with $1 < r < \infty$. When $r = 1$, the sufficient part follows from Corollary 3.4 and the necessary part follows from Theorem 1.3 with $r = 1$. $\square$

6. Uniqueness

In this section we provide the proof of Theorem 1.6. Let $u(t)$ and $v(t)$ be two solutions of (1.1) in the class K defined for $0 < t < T_1$ and $0 < t < T_2$, respectively. Without loss of generality we assume $T_1 \leq T_2$.

$$A = \left\{t \in [0, T_1];\ u(t) = v(t) \text{ in } L^1(\Omega)\right\}.$$

Since $u(0) = v(0) = u_0$, it follows that $0 \in A$, so A is nonempty. Let $U \subset [0, T_1]$ be the connected component of A that contains 0. Because of the definition, U is a closed set of $[0, T_1]$. Hereafter, we show that U is an open set. Since $[0, T_1]$ is a connected set, this yields $U = [0, T_1]$.

Suppose that $u(t) = v(t)$ in $L^1(\Omega)$ for $0 \leq t \leq T$ where $0 \leq T < T_1$. By the fact that $u(t), v(t) \in K$ there is $M > 0$ such that

$$\sup_{0<t<T_1} t^\beta \|u(t)\|_p < M \quad\text{and}\quad \sup_{0<t<T_2} t^\beta \|v(t)\|_p < M.$$

Let $0 < \varepsilon < T_1 - T$. With a similar calculation to (3.16), for $T \leq t \leq T + \varepsilon$ we have

$$\begin{aligned}
t^\beta \|u(t) - v(t)\|_p &\leq C_3 t^\beta \int_0^t (t-s)^{-1+\alpha-\beta} \|f(u(s)) - f(v(s))\|_1 \, ds \\
&\leq C_3 t^\beta \int_T^{T+\varepsilon} (t-s)^{-1+\alpha-\beta} \|f(u(s)) - f(v(s))\|_1 \, ds \\
&\leq C_3 t^\beta \Bigg(C_4'' \int_T^{T+\varepsilon} (t-s)^{-1+\alpha-\beta} s^{-\beta} ds \\
&\quad + 2M^{p-1} C_4' \int_T^{T+\varepsilon} (t-s)^{-1+\alpha-\beta} s^{-p\beta} ds \Bigg) \sup_{T\leq t<T+\varepsilon} t^\beta \|u(t) - v(t)\|_p .
\end{aligned}$$

Observe that as $\varepsilon \to 0$ we have

$$t^\beta \int_T^{T+\varepsilon} (t-s)^{-1+\alpha-\beta} s^{-\beta} ds \to 0 \quad\text{and}\quad t^\beta \int_T^{T+\varepsilon} (t-s)^{-1+\alpha-\beta} s^{-p\beta} ds \to 0.$$

Hence, from the above estimate it follows that

$$t^\beta \|u(t) - v(t)\|_p \leq o(1) \sup_{T\leq t<T+\varepsilon} t^\beta \|u(t) - v(t)\|_p \text{ as } \varepsilon \to 0.$$

By taking $\varepsilon > 0$ small one derives

$$\sup_{T\le t<T+\varepsilon} t^\beta \|u(t)-v(t)\|_p \le \frac{1}{2} \sup_{T\le t<T+\varepsilon} t^\beta \|u(t)-v(t)\|_p .$$

so $u(t) = v(t)$ in $L^p(\Omega)$ on $[0, T+\varepsilon]$. Because of the continuous embedding $L^p(\Omega) \hookrightarrow L^1(\Omega)$, we see that $u(t) = v(t)$ in $L^1(\Omega)$ for $0 \le t < T+\varepsilon$. Hence, U is an open set which yields $U = [0, T_1]$. This completes the proof. □

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