Matrix methods for perfect signal recovery underlying range space of operators

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Abstract. The most important purpose of this article is to investigate perfect reconstruction underlying range space of operators in finite dimensional Hilbert spaces by a new matrix method. To this end, first we obtain more structures of the canonical K-dual. Then, we survey the problem of recovering and robustness of signals when the erasure set satisfies the minimal redundancy condition or the K-frame is maximal robust. Furthermore, we show that the error rate is reduced under erasures if the K-frame is of uniform excess. Toward the protection of encoding frame (K-dual) against erasures, we introduce a new concept so called (r, k)-matrix to recover lost data and solve the perfect recovery problem via matrix equations. Moreover, we discuss the existence of such matrices by using minimal redundancy condition on decoding frames for operators. We exhibit several examples that illustrate the advantage of using the new matrix method with respect to the previous approaches in existence construction. And finally, we provide the numerical results to confirm the main results in the case noise-free and test sensitivity of the method with respect to noise.

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1. Introduction and preliminaries

The theory of frames has established efficient algorithms for a wide range of applications in the last twenty years [6, 8, 9, 11, 23]. In most of those applications, they deal with dual frames to reconstruct the modified data and compare it with the original data. In frame theory setting, an original signal f is encoded by the measurements $\theta_F^* f$ (encoded coefficients), where θ_F^* is the analysis operator of a frame F. Then, from these measurements f can be recovered applying a reconstruction formula by a dual frame G (decoding frame) as $\theta_G \theta_F^* f$. In real applications, in these transmissions, usually a part of the data vectors are corrupted or lost, and we may have to perform the reconstruction by using the partial information at hand. So, searching for the best dual frames that minimize the reconstruction errors when erasures occur, optimal dual problem is one of the most important problems in frame theory that was introduced by Han et al. in [24, 26]. To state the optimal dual problem, we first recall some basic notations of finite classical frames. Let $\mathcal{H}_n$ be an n-dimensional Hilbert space and $I_m = \{1, 2, ..., m\}$. A sequence $F := \{f_i\}_{i\in I_m} \subseteq \mathcal{H}_n$ is called a *frame* for $\mathcal{H}_n$ whenever $span\{f_i\}_{i\in I_m} = \mathcal{H}_n$. The *synthesis operator* $\theta_F : l^2(I_m) \to \mathcal{H}_n$ is defined by $\theta_F\{c_i\} = \sum_{i\in I_m} c_i f_i$. If $\{f_i\}_{i\in I_m}$ is a frame, then $S_F = \theta_F\theta_F^*$ is called frame operator where $\theta_F^* : \mathcal{H}_n \to l^2(I_m)$, the adjoint of θ_F, is given by $\theta_F^* f = \{\langle f, f_i\rangle\}_{i\in I_m}$ and is known as the *analysis operator.* A sequence $G := \{g_i\}_{i\in I_m} \subseteq \mathcal{H}_n$ is called a *dual* for $\{f_i\}_{i\in I_m}$ if $\theta_G\theta_F^* = I_{\mathcal{H}_n}$. A special dual frame as $\{S_F^{-1}f_i\}_{i\in I_m}$ is called the canonical dual of F. It is well known that $\{g_i\}_{i\in I_m}$ is a dual frame of $\{f_i\}_{i\in I_m}$ if and only if $g_i = S_F^{-1}f_i + u_i$, for all $i \in I_m$ where $U = \{u_i\}_{i\in I_m}$ satisfies $\theta_F\theta_U^* = 0$. We refer the reader to [10] for more information on frame theory. The optimal dual problem propounds the following problem: let $F = \{f_i\}_{i\in I_m}$ be a frame for $\mathcal{H}_n$ (encoding frame), find a dual frame of F that minimize the reconstruction errors when erasures occur. If $G = \{g_i\}_{i\in I_m}$ is a dual of F (decoding frame) and $\Lambda \subset I_m$, then the error operator E_Λ is defined by

$$E_\Lambda = \sum_{i\in\Lambda} f_i \otimes g_i = \theta_G^* D\theta_F,$$

where D is an $m \times m$ diagonal matrix with $d_{ii} = 1$ for $i \in \Lambda$ and 0 otherwise. Let

$$d_r(F, G) = \max\{\|\theta_G^* D\theta_F\| : D \in \mathcal{D}_r\} = \max\{\|E_\Lambda\| : |\Lambda| = r\}, \tag{1.1}$$

in which $|\Lambda|$ is the cardinality of Λ, the norm used in (1.1) is the operator norm, $1 \le r < m$ is a natural number and $\mathcal{D}_r$ is the set of all $m\times m$ diagonal matrices with r $1's$ and $m-r$ $0's$. Then, $d_r(F, G)$ is the largest possible error when r-erasures fall out. Indeed, G is called an optimal dual frame of F for 1-erasure or 1-loss optimal dual if

$$d_1(F, G) = \min\left\{d_1(F, Y) : Y \ \textit{is a dual of } F\right\}. \tag{1.2}$$

Inductively, for $r > 1$, a dual frame G is called an *optimal dual* of F for r-erasures (r-loss optimal dual) if it is optimal for $(r-1)$-erasures and

$$d_r(F, G) = \min\left\{d_r(F, Y) : Y \ \textit{is a dual of } F\right\}.$$

See [3, 4, 21, 24, 26, 29] and references therein for more details and information on optimal reconstruction problem and identification of optimal dual frames.

This work was motivated by some recent methods of perfect recovery of signals from erasures corrupted frame coefficients at known or unknown locations [18, 19]. In all previous approaches presented on classical frames, erasures considered in frame coefficients (encoding frame coefficients). We are going to extend perfect recovery problem on K-frame theory, however we show that the methods previously used does not meet the requirements of reconstruction in K-frame setting. Hence, we consider the erasure coefficients on K-dual coefficients (as encoding frame instead of K-frame). Then we introduce a new concept, called (r, k)-matrix, to recover lost data and get perfect reconstruction. This also leads to some new method for recovery problem in ordinary frames, that sometimes work for frames better than the previous methods. Among other things, we demonstrate the advantages of using the new method in existence and construction with respect to the previous approaches.

The present paper is organized as follows. In Section 2, we recall some definitions and notations of finite K-frames. In Section 3, we provide more characterizations and structures of K-duals and particularly, the canonical K-duals. We present some concepts such as minimal redundancy condition and maximal robustness for K-frames and provide some necessary conditions for a finite set of indices which satisfies minimal redundancy condition. Moreover, we discuss the robustness of K-frame under operator perturbation, particularly when the erasure set satisfies the minimal redundancy condition or the K-frame is maximal robust, in Section 4. Then in Section 5, we introduce a new matrix called (r, k)-matrix and give the necessary condition for the existence of (r, k)-matrices. This notion leads to a new matrix equation which allows the signal vectors underlying the range space of a bounded operator to be exactly recovered. This approach not only assures that K-frames with uniform excess under some erasures of K-dual coefficients make complete reconstruction, but also provides a new method for erasure recovery by using ordinary frames, by changing encode and decode frames, which sometimes work better than the previous methods. In Section 6, we exhibit several examples to illustrate our results and the advantage of using (r, k)-matrices. Finally, in Section 7, we present the numerical results for the recovery of signals from erased and noisy K-dual frame coefficients to survey sensitivity of the method with respect to noise.

2. Finite K-frames

Atomic decomposition for a closed subspace $\mathcal{H}_0$ of a Hilbert space $\mathcal{H}$, as a new approach for reconstruction, was introduced by Feichtinger et al. with frame-like properties [13]. The sequences in atomic decompositions do not necessarily belong to $\mathcal{H}_0$, and this striking property is valuable, especially in sampling theory [28, 30]. K-frames were introduced to study atomic systems with respect to a bounded operator $K \in B(\mathcal{H})$ [15]. In fact, K-frames are equivalent with atomic systems for the operator K and help us to reconstruct elements from the range of a bounded linear operator K in a separable Hilbert space. In the sequel, we recall some definitions and notations of finite K-frames. A sequence $F := \{f_i\}_{i\in I_m} \subseteq \mathcal{H}_n$ is called a K-frame for $\mathcal{H}_n$, if $R(K) \subset R(\theta_F)$ or equivalently there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{i\in I_m} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}_n). \tag{2.1}$$

If K is an onto operator, then F is an ordinary frame and therefore K-frames arise as a generalization of ordinary frames. The constants A and B in (2.1) are called the lower and the upper bounds of F, respectively. Similar to ordinary frames, the synthesis operator can be defined by $\theta_F : l^2(I_m) \to \mathcal{H}$; $\theta_F(\{c_i\}_{i\in I_m}) = \sum_{i\in I_m} c_i f_i$. A matrix representation for this bounded operator is the matrix $F_{n\times m}$ whose ith column is the ith K-frame vector, i.e.,

$$F = [f_1, ..., f_m].$$

Notice that we sometimes denote a K-frame $F = \{f_i\}_{i\in I_m}$ by its synthesis matrix F. Also, the analysis operator is given by $\theta_F^*(f) = \{\langle f, f_i\rangle\}_{i\in I_m}$ and has the matrix representation as F^*. The frame operator is given by $S_F = \theta_F\theta_F^*$ with the matrix representation as FF^* and $\mathcal{G}_F = F^*F$ denotes the Gramian matrix with respect to the K-frame F. Unlike ordinary frames, the frame operator of a K-frame is not invertible in general. Although, in finite dimensional Hilbert spaces, K is a closed range operator so S_F from $R(K)$ onto $S_F(R(K))$ is an invertible operator [32]. When we need this restriction of the K-frame operator we use the notation $S_F|R(K)$. Suppose M_K denotes matrix representation of the operator $K \in B(\mathcal{H}_n)$ with respect to the standard orthonormal basis of $\mathcal{H}_n$. Then, a K-frame is said to be α-tight whenever $FF^* = \alpha M_K M_K^*$, Parseval if $\alpha = 1$ and equal norm (EN) if the columns of F have the equal norm.

The authors in [5] considered the notion of duality for K-frames and presented several methods for construction and characterization of K-frames and their duals. Indeed, a sequence $\{g_i\}_{i\in I_m} \subseteq \mathcal{H}_n$ is called a K-*dual* of $\{f_i\}_{i\in I_m}$ if

$$Kf = \sum_{i\in I_m} \langle f, g_i\rangle f_i, \quad (f \in \mathcal{H}_n), \tag{2.2}$$

or equivalently G is K-dual of F if $FG^* = M_K$. The following result is useful for the proof of our main results.

Theorem 2.1 (Douglas [12]**).** *Let $L_1 \in B(\mathcal{H}_1, \mathcal{H})$ and $L_2 \in B(\mathcal{H}_2, \mathcal{H})$ be bounded linear mappings on given Hilbert spaces. Then the following assertions are equivalent:*

(i) $R(L_1) \subseteq R(L_2)$;
(ii) $L_1L_1^* \leq \lambda^2 L_2L_2^*$, *for some* $\lambda > 0$;
(iii) *there exists a bounded linear mapping* $X \in L(\mathcal{H}_1, \mathcal{H}_2)$, *such that* $L_1 = L_2X$.

Moreover, if (i), (ii) or (iii) are valid, then there exists a unique operator X so that

(a) $\|X\|^2 = \inf\{\alpha > 0, L_1L_1^* \leq \alpha L_2L_2^*\}$;
(b) $N(L_1) = N(X)$;
(c) $R(X) \subset \overline{R(L_2^*)}$.

For every K-frame $F = \{f_i\}_{i\in I_m}$ of $\mathcal{H}_n$ using the Douglas' theorem, there exists a unique operator $X_F \in B(\mathcal{H}_n, \mathbb{C}^m)$ so that $\theta_F X_F = K$ and

$$\|X_F\|^2 = \inf\{\alpha > 0, \|K^*f\|^2 \leq \alpha\|\theta_F^* f\|^2; f \in \mathcal{H}_n\}. \tag{2.3}$$

Moreover, $\{X_F^*\delta_i\}_{i\in I_m}$ is a K-dual of F which its analysis operator obtains the minimal norm and is called the canonical K-dual. See [17]. For further information in K-frame theory we refer the reader to [5, 13, 15, 16, 32].

Throughout this paper, we suppose that $\mathcal{H}_n$ is an n-dimensional Hilbert space, $I_m = \{1, 2, ..., m\}$ and $\{\delta_i\}_{i\in I_m}$ is the standard orthonormal basis of $l^2(I_m)$. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the collection of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$, and we abbreviate $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. Matrix representation associated with an operator T is denoted by M_T and the operator associated with a matrix M is denoted by T_M. Also, we denote the range of $K \in B(\mathcal{H}_n)$ by $R(K)$ and pseudo inverse of K by $K^\dagger$. For a subspace $V \subseteq \mathcal{H}_n$ the identity operator on V and the orthogonal projection of $\mathcal{H}$ onto V are denoted by I_V and π_V, respectively.

3. Identification of the canonical K-dual

In this section, we are going to obtain more structures of K-duals and particularly the canonical K-dual of K-frames. We note that the definition used here as the canonical K-dual is completely different and more general than [5]. For convenience, we denote the set of all K-dual frames of $F = \{f_i\}_{i\in I_m}$ by KD_F. Obviously, KD_F is a closed convex subset of $\mathcal{H}_n^m$, the set of all m-tuples of vectors in $\mathcal{H}_n$. In the following, we obtain the canonical K-dual in a new form which is more useful in the proof of our results.

Lemma 3.1. *Let $F = \{f_i\}_{i\in I_m}$ be a K-frame of $\mathcal{H}_n$. With the above notations, there exists a unique bounded operator $\Gamma_F \in B(\mathcal{H}_n)$ so that $\{\Gamma_F^* f_i\}_{i\in I_m} = \{X_F^* \delta_i\}_{i\in I_m}$.*

Proof. Using Douglas' theorem, there is a unique operator $X_F \in B(\mathcal{H}_n, l^2(I_m))$ so that $\theta_F X_F = K$ and $R(X_F) \subseteq \overline{R(\theta_F^*)} = R(\theta_F^*)$. So by reusing Douglas' theorem there exists a unique bounded operator $\Gamma_F \in B(\mathcal{H}_n)$ so that $X_F = \theta_F^* \Gamma_F$ and

$$\|\Gamma_F\|^2 = \inf\{\alpha > 0, \|X_F^* f\|^2 \leq \alpha \|\theta_F f\|^2; f \in \mathcal{H}_n\}.$$

Moreover, we have

$$\Gamma_F^* f_i = \Gamma_F^* \theta_F \delta_i = X_F^* \delta_i,$$

for all $i \in I_m$. Hence, $\{\Gamma_F^* f_i\}_{i\in I_m}$ is exactly the canonical K-dual of F. □

Easily, it can be checked that a sequence $G = \{g_i\}_{i\in I_m}$ is a K-dual of F if and only if $g_i = \Gamma_F^* f_i + u_i$, for all $i \in I_m$ where $U = \{u_i\}_{i\in I_m}$ satisfies $\theta_F \theta_U^* = 0$.

Lemma 3.2. *Let $F = \{f_i\}_{i\in I_m}$ be a K-frame of $\mathcal{H}_n$ and $G = \{g_i\}_{i\in I_m}$ be a K-dual of F. Then G is the canonical K-dual if and only if $S_G = \theta_G \theta_Z^*$ for every K-dual Z of F.*

Proof. Suppose G is the canonical K-dual and Z is a K-dual of F, so by lemma 1 there exists a unique bounded operator $\Gamma_F \in B(\mathcal{H}_n)$, so that $G = \{\Gamma_F^* f_i\}_{i\in I_m}$ so,

$$\theta_G(\theta_G^* - \theta_Z^*) = \Gamma_F^* \theta_F (\theta_G^* - \theta_Z^*) = 0.$$

Thus, $S_G = \theta_G \theta_Z^*$ for every K-dual Z of F. Conversely, let for every K-dual Z of F we have $S_G = \theta_G \theta_Z^*$. Then

$$\|\theta_G^*\|^2 = \|\theta_G \theta_G^*\| = \|\theta_G \theta_Z^*\| \leq \|\theta_G^*\| \|\theta_Z^*\|.$$

This immediately implies that $\|\theta_G^*\| \leq \|\theta_Z^*\|$, i.e., the analysis operator of G has minimal norm and the proof is complete. □

In the case that F is a Parseval K-frame Lemma 4 can be reduced to a result of [31]. The next result presents the canonical K-dual of some classes of K-frames. The computations are relatively straightforward, so we provide a sketch of the proof for convenience of the reader.

Proposition 3.3. *Let $F = \{f_i\}_{i \in I_m}$ be a K-frame of $\mathcal{H}_n$. Then the following statements hold;*

(i) *If $F \subseteq R(K)$ then $\{K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\}_{i\in I_m}$ is the canonical K-dual of F.*

(ii) *If $R(K) \subseteq S_F(R(K))$ or $F \subseteq S_F(R(K))$, then $\{K^*((S_F|_{R(K)})^{-1})^*\pi_{R(K)}f_i\}_{i\in I_m}$ is the canonical K-dual of F.*

Proof. First we note that by the assumption $F \subseteq R(K)$ is a K-frame of $\mathcal{H}_n$ and so $G := \{K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}f_i\}_{i\in I_m}$ is a K-dual of $\pi_{R(K)}F = F$ by using Proposition 2.3 of [5]. Moreover,

$$\begin{aligned} S_G &= K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}S_F((S_F|_{R(K)})^{-1})^*K \\ &= K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}\theta_F\theta_Z^* \\ &= \theta_G\theta_Z^*, \end{aligned}$$

for every K-dual Z of F. Thus, by Lemma 3.2 we implies (i). Now, assume that $R(K) \subseteq S_F(R(K))$. The fact that the operator $S_F|_{R(K)} : R(K) \to S_F(R(K))$ is invertible implies that $R(K) = S_F(R(K))$. Thus

$$Kf = \sum_{i\in I_m} \langle f, K^*((S_F|_{R(K)})^{-1})^*\pi_{(R(K))}f_i\rangle f_i,$$

for every $f \in \mathcal{H}_n$. Hence, $G := \{K^*((S_F|_{R(K)})^{-1})^*\pi_{R(K)}f_i\}_{i\in I_m}$ is a K-dual of F. Again, we obtain $S_G = \theta_G\theta_Z^*$, for every K-dual Z of F. Again, the above computations along with Lemma 3.2, follows the desired result. The case of $F \subseteq S_F(R(K))$ is similar. $\square$

The converse of Proposition 3.3, does not hold in general. To see this and also the importance of the sufficiency conditions in Proposition 3.3, see Example 6.1 and Example 6.2 in Section 6.

Remark 3.4. The structure of the canonical K-dual of a Parseval K-frame F is $K^\dagger F$. See [27]. Indeed, in this case $\Gamma_F = (K^\dagger)^*$. Also, in this regard, for a K-frame $F \subseteq R(K)$ we have that $\Gamma_F = ((S_F|_{R(K)})^{-1})^*K$.

4. Minimal redundancy condition

In this section, we provide the concept of minimal redundancy condition and maximal robust for K-frames and give some necessary conditions for a finite set of indices which satisfies minimal redundancy condition. Then, we discuss the problem of robustness under operator perturbation of K-frame, particularly when the erasure set satisfies the minimal redundancy condition or the K-frame is maximal robust. For more information of these concepts on classical frames we refer the reader to [1, 2, 22, 23]

Suppose F denotes the associated matrix of a K-frame $\{f_i\}_{i\in I_m}$ in Hilbert space $\mathcal{H}_n$. A finite set of indices $\sigma \subset I_m$ satisfies the minimal redundancy condition (MRC) for F whenever $\{f_i\}_{i\in\sigma^c}$ is a K-frame for $\mathcal{H}_n$. Furthermore, we say F satisfies MRC for r-erasures if every subset $\sigma \subset I_m$, $|\sigma| = r$ satisfies MRC for F. Also, F is said to be of uniform excess r if it is an exact K-frame when r columns of F

are removed and F is called maximal robust (MR) if every r_k columns of F is an exact K-frame, where $r_k := rankK$. Note that, for a K-frame that is MR, every submatrix $n \times r_k$ has a left inverse. However, the converse does not hold, in general. For instance, in Example 6.2, $rankK = 2$ and every 2 columns of F are linearly independent so every submatrix of F containing 2 columns has a left inverse, but $\{f_3, f_4\}$ is not a K-frame. In what follows, we give some necessary conditions for a finite set of indices $\sigma \subset I_m$ which satisfies MRC. To be convenient, we use θ_σ, S_σ and ξ_σ^* to denote the synthesis operator, frame operator of a K-frame and the analysis operator of the canonical K-dual whenever the index set is limited to σ.

Theorem 4.1. *Suppose $F = \{f_i\}_{i\in I_m}$ is a K-frame of $\mathcal{H}_n$ and $\sigma \subset I_m$ satisfies MRC for F. Then*

(i) $R(\theta_F^* K) \cap span\{\delta_i\}_{i\in\sigma} = \{0\}$.

(ii) *If F is a Parseval K-frame then $(K - \theta_\sigma \xi_\sigma^*)|_{R(K^\dagger)}$ is an invertible operator from $R(K^\dagger)$ onto $S_{\sigma^c}(R(K))$.*

Proof. To show (i), on the contrary, assume that there exists a non-zero element $\alpha \in R(\theta_F^* K) \cap span\{\delta_i\}_{i\in\sigma}$. Then, there exists $f \in \mathcal{H}_n$, $\{c_i\}_{i\in\sigma} \subseteq \mathbb{C}$ so that

$$\alpha = \theta_F^* K f = \sum_{i\in\sigma} c_i \delta_i.$$

Thus, $\theta_F^* K f \perp \delta_i$ for every $i \in \sigma^c$ and so

$$\langle Kf, f_i\rangle = \langle Kf, \theta_F \delta_i\rangle = \langle \theta_F^* K f, \delta_i\rangle = 0,$$

for every $i \in \sigma^c$. Hence $Kf \perp \{f_i\}_{i\in\sigma^c}$ that is a contradiction. This implies the desired result.

Now, let F be a Parseval K-frame then

$$K - \theta_\sigma \xi_\sigma^* = \theta_{\sigma^c} \xi_{\sigma^c}^* = S_{\sigma^c}(K^\dagger)^*,$$

where the last equality is obtained by Remark 3.4 so it is sufficient to prove that $S_{\sigma^c}(K^\dagger)^*|_{R(K^\dagger)}$ is an invertible operator. Since σ satisfies MRC the operator $S_{\sigma^c}|_{R(K)}$ is invertible from $R(K)$ onto $S_{\sigma^c}(R(K))$. Consider $\Gamma_{\sigma^c} := K^*(S_{\sigma^c}|_{R(K)})^{-1}$, we show that Γ_{σ^c} is the inverse of the operator $S_{\sigma^c}(K^\dagger)^*|_{R(K^\dagger)}$. Indeed

$$\begin{aligned} & \Gamma_{\sigma^c} S_{\sigma^c}(K^\dagger)^* f \\ = \;& K^*(S_{\sigma^c}|_{R(K)})^{-1} S_{\sigma^c}|_{R(K)} (K^\dagger)^* f \\ = \;& K^*(K^\dagger)^* f \\ = \;& (K^\dagger K)^* f \\ = \;& K^\dagger K f = f, \end{aligned}$$

for every $f \in R(K^\dagger)$. Thus $\Gamma_{\sigma^c} S_{\sigma^c}(K^\dagger)^*|_{R(K^\dagger)} = I_{R(K^\dagger)}$. On the other hand,

$$\begin{aligned} & S_{\sigma^c}(K^\dagger)^* \Gamma_{\sigma^c} f \\ = \;& S_{\sigma^c}(K^\dagger)^* K^* (S_{\sigma^c}|_{R(K)})^{-1} f \\ = \;& S_{\sigma^c} K K^\dagger (S_{\sigma^c}|_{R(K)})^{-1} f \\ = \;& S_{\sigma^c}|_{R(K)} (S_{\sigma^c}|_{R(K)})^{-1} f = f, \end{aligned}$$

for every $f \in S_{\sigma^c}(R(K))$. Hence, $S_{\sigma^c}(K^\dagger)^*\Gamma_{\sigma^c}|_{S_{\sigma^c}(R(K))} = I_{S_{\sigma^c}(R(K))}$. This implies the desired result. $\square$

It is worth noting that the condition (i) in the above theorem, unlike ordinary frames [2], is not sufficient for a subset σ to satisfy MRC. See Example 6.3; moreover, by applying Theorem 4.1, if $\sigma \subset I_m$ satisfies MRC, we get some K-frames and $K^\dagger$-frame with $K^\dagger$-dual on the remained index set σ^c.

Corollary 4.2. *Let $F = \{f_i\}_{i\in I_m}$ be a K-frame of $\mathcal{H}_n$ and $\sigma \subset I_m$ satisfies MRC then*

(i) $\{K^*(S_{\sigma^c}|_{R(K)})^{-1}\pi_{S_{\sigma^c}(R(K))}f_i\}_{i\in\sigma^c}$ *is a $K^\dagger$-frame with $K^\dagger$-dual $\{(K^\dagger)^*K^\dagger f_i\}_{i\in\sigma^c}$.*
(ii) $\{(K^\dagger)^*K^\dagger f_i\}_{i\in\sigma^c}$ *is also a K-frame for $\mathcal{H}_n$.*

Proof. Since $S_{\sigma^c}|_{R(K)}$ is invertible we have that

$$\begin{aligned} K^\dagger f &= K^*(S_{\sigma^c}|_{R(K)})^{-1}S_{\sigma^c}|_{R(K)}(K^\dagger)^*K^\dagger f \\ &= K^*(S_{\sigma^c}|_{R(K)})^{-1}\pi_{S_{\sigma^c}(R(K))}\sum_{i\in\sigma^c}\langle (K^\dagger)^*K^\dagger f, f_i\rangle f_i \\ &= \sum_{i\in\sigma^c}\langle (K^\dagger)^*K^\dagger f, f_i\rangle K^*(S_{\sigma^c}|_{R(K)})^{-1}\pi_{S_{\sigma^c}(R(K))}f_i, \end{aligned}$$

for every $f \in \mathcal{H}_n$. Hence, (i) is obtained by Lemma 2.2 of [5]. Using the above computations and the fact that

$$R(K) = R(K^\dagger)^* \subseteq span\{(K^\dagger)^*K^\dagger f_i\}_{i\in\sigma^c},$$

we get (ii). $\square$

Theorem 4.3. *Let F be the associated matrix of a K-frame for $\mathcal{H}_n$. Then the following assertions hold, where in all matrix products below, we let the sizes be compatible.*

(i) *AFU is T_AK-frame for any matrix A and a unitary matrix U. In particular FU is a K-frame and $GU \in KD_{AFU}$ for every $G \in KD_F$.*
(ii) *If A is invertible and U is a unitary matrix then $G \in KD_F$ if and only if $GU \in KD_{AFU}$.*
(iii) *If F is α-tight K-frame then AFU is α-tight T_AK-frame for any matrix A and a unitary matrix U. Moreover, FU is an α-tight K-frame.*
(iv) *If F is EN then UFD is also EN for any unitary matrix U and unitary diagonal matrix D.*
(v) *If F is MR K-frame then AFD as an T_AK-frame is MR for any invertible matrix A and unitary diagonal matrix D.*
(vi) *If F satisfies MRC for r-erasures then AFD as a T_AK-frame satisfies MRC for r-erasures for any unitary diagonal matrix D and square matrix A.*

Proof. Suppose γ is a lower K-frame bound of F. Then by the assumption in (i) we obtain

$$\begin{aligned} AFU(AFU)^* &= AFUU^*F^*A^* \\ &= AFF^*A^* \\ &\geq \gamma AM_KM_K^*A^* \\ &= \gamma AM_K(AM_K)^*. \end{aligned}$$

The existence of the upper bound is clear, so AFU is T_AK-frame of $\mathcal{H}_n$. Moreover, if A is the identity matrix, FU is a K-frame of $\mathcal{H}_n$. On the other hand, for every $G \in KD_F$ we have that

$$AFUU^*G^* = AFG^* = AM_K,$$

so $GU \in KD_{AFU}$ and (i) is proved. The cases (ii), (iii) and (iv) are proved by definitions and some straightforward computations. For (v), we note that AFD is T_AK-frame by (i), so we only show that AFD is MR. Indeed, let $F_{n\times m}$ be MR K-frame, moreover $A_{n\times n}$ and $D_{m\times m}$ be invertible and diagonal unitary matrices, respectively. Then a submatrix $n \times r_k$ of AFD is as $A\mathcal{M}\mathcal{N}$ where $\mathcal{M}_{n\times r_k}$ is a submatrix of F and $\mathcal{N}_{r_k\times r_k}$ is a diagonal submatrix of D. Hence, the columns of $\mathcal{M}$ constitute an exact K-frame and so $\mathcal{M}$ has a left inverse. This implies that $A\mathcal{M}\mathcal{N}$ also has a left inverse i.e., its columns are linearly independent and generate $R(T_AK)$. Moreover, this vector columns constitute an exact T_AK-frame. Thus, AFD is MR.

Finally, let F satisfies MRC for r-erasures, A and D be arbitrary $n \times n$ matrix and $m \times m$ unitary diagonal matrix, respectively. A submatrix $n \times (m-r)$ of AFD is as $A\mathcal{M}\mathcal{N}$ where $\mathcal{M}_{n\times(m-r)}$ is a submatrix of F and $\mathcal{N}_{(m-r)\times(m-r)}$ is a diagonal submatrix of D. Since $\mathcal{M}$ is a K-frame, applying the assumption, one immediately obtains that $A\mathcal{M}\mathcal{N}$ is also a T_AK-frame by (i). This completes the proof. □

5. Perfect reconstructions by (r, k)-matrices

In what follows, we present some matrix methods which lead to fewer errors if K-frame is of uniform excess or even we have perfect reconstruction under erasures. To this end, we present two approaches, one of which is motivated by [18, 19], however unlike ordinary frames, for K-frames it does not work very well. Hence, we set a new concept so called (r, k)-matrix to get perfect reconstruction in this case. Also, we show this approach works for frames sometimes better than the previous methods.

Let $F = \{f_i\}_{i\in I_m}$ be a K-frame of $\mathcal{H}_n$ with uniform excess r and $G = \{g_i\}_{i\in I_m}$ be a K-dual of F. Since $\{f_i\}_{i=r+1}^m$ is an exact K-frame then for any g_i, $(1 \le i \le r)$ there exist unique coefficients $\{\alpha_{i,j}\}_{j=r+1}^m \subset \mathbb{C}$ so that

$$\pi_{R(K)} g_i = \sum_{j=r+1}^{m} \alpha_{i,j} f_j, \quad (1 \le i \le r).$$

Consider

$$\mathcal{M} = \begin{bmatrix} 1 & 0 & . & . & . & 0 & -\alpha^*_{1,r+1} & . & . & . & -\alpha^*_{1,m} \\ 0 & 1 & . & . & . & 0 & -\alpha^*_{2,r+1} & . & . & . & -\alpha^*_{2,m} \\ \\ \\ 0 & 0 & . & . & . & 1 & -\alpha^*_{r,r+1} & . & . & . & -\alpha^*_{r,m} \end{bmatrix}.$$

Then

$$\mathcal{M}\begin{bmatrix} \langle f, \pi_{R(K)}g_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, \pi_{R(K)}g_r\rangle \\ \langle f, f_{r+1}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, f_m\rangle \end{bmatrix} = 0,$$

for every $f \in \mathcal{H}_n$ and consequently

$$\mathcal{M}_1\begin{bmatrix} \langle f, \pi_{R(K)}g_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, \pi_{R(K)}g_r\rangle \end{bmatrix} + \mathcal{M}_2\begin{bmatrix} \langle f, f_{r+1}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, f_m\rangle \end{bmatrix} = 0, \tag{5.1}$$

where $\mathcal{M}_1$ is the submatrix consisting of the first r columns of $\mathcal{M}$, and $\mathcal{M}_2$ is the submatrix consisting of the remaining columns. This assures that for any r-erasures of K-dual coefficients $\{\langle f, g_i\rangle\}_{i\in\Lambda}$, $|\Lambda| = r$ we may recover the coefficients $\{\langle f, \pi_{R(K)}g_i\rangle\}_{i\in\Lambda}$ by solving the equation (5.1) as follows

$$\begin{bmatrix} \langle f, \pi_{R(K)}g_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, \pi_{R(K)}g_r\rangle \end{bmatrix} = -M_2\begin{bmatrix} \langle f, f_{r+1}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, f_m\rangle \end{bmatrix}. \tag{5.2}$$

Replacing the coefficients $\{\langle f, \pi_{R(K)}g_i\rangle\}_{i\in\Lambda}$ by $\{\langle f, \sum_{j\in\Lambda^c}\alpha_{i,j}f_i\rangle\}_{i\in\Lambda}$ and using the fact that the error operator is obtained by

$$E_\Lambda = \sum_{i\in\Lambda} f_i \otimes g_i = \sum_{i\in\Lambda} f_i \otimes \pi_{R(K)}g_i + \sum_{i\in\Lambda} f_i \otimes \pi_{R(K)^\perp}g_i,$$

we get a reduced error operator as

$$\tilde{E}_\Lambda = E_\Lambda - \Delta_\Lambda,$$

where $\Delta_\Lambda = \sum_{i\in\Lambda} f_i \otimes \pi_{R(K)}g_i$. Equivalently, we have $\tilde{E}_\Lambda f = \sum_{i\in\Lambda}\langle f, \pi_{R(K)^\perp}g_i\rangle f_i$, for every $f \in \mathcal{H}_n$. Hence, for computing of the error operator one needs only find a K-dual frame G which satisfies

$$\max_{|\Lambda|=r}\left\|\sum_{i\in\Lambda} f_i \otimes \pi_{R(K)^\perp}g_i\right\| = \min\left\{\max_{|\Lambda|=r}\left\|\sum_{i\in\Lambda} f_i \otimes \pi_{R(K)^\perp}h_i\right\|; \quad \{h_i\}_{i\in I_m} \in KD_F\right\}. \tag{5.3}$$

From this point of view, by a K-frame with uniform excess property which has a K-dual $\{g_i\}_{i\in I_m} \subseteq R(K)$, we will have the perfect reconstruction. Otherwise, for every K-dual of F which satisfies (5.3) the error rate is reduced.

Now, we present a new method which allows a perfect reconstruction. Moreover, by this approach K-frames with uniform excess under some erasures of K-dual coefficients make a complete reconstruction and this process is independent of the choice of K-dual. We recall the spark of a matrix [1] is the size of the smallest linearly dependent subset of the columns and the spark of a collection of vectors in a finite dimensional Hilbert space is considered as the spark of its synthesis matrix. Moreover, for any $m \times n$ matrix A

$$sparkA = \min\{\|x\|_0 : Ax = 0, x \neq 0\}, \tag{5.4}$$

where $\|x\|_0$, the Humming weight of a vector $x = \{x_i\}_{i \in I_n}$, is defined as follows

$$\|x\|_0 = |\{j \in I_n : x_j \neq 0\}| .$$

See [1, 11] for more information. Let $F = \{f_i\}_{i \in I_m}$ be a K-frame of $\mathcal{H}_n$ with a K-dual $G = \{g_i\}_{i \in I_m}$. Then we have that

$$\sum_{i \in I_m} \langle f_i, f_j \rangle \langle f, g_i \rangle = \langle Kf, f_j \rangle,$$

for all $j \in I_m$. Equivalently

$$\begin{bmatrix} \langle f_1, f_1 \rangle & \langle f_2, f_1 \rangle & . & . & . & \langle f_m, f_1 \rangle \\ \langle f_1, f_2 \rangle & \langle f_2, f_2 \rangle & . & . & . & \langle f_m, f_2 \rangle \\ & . & & & . & \\ & . & & & . & \\ & & & & & \\ \langle f_1, f_m \rangle & \langle f_2, f_m \rangle & . & . & . & \langle f_m, f_m \rangle \end{bmatrix} \begin{bmatrix} \langle f, g_1 \rangle \\ \\ . \\ . \\ \\ \langle f, g_m \rangle \end{bmatrix} = \begin{bmatrix} \langle Kf, f_1 \rangle \\ \\ . \\ . \\ \\ \langle Kf, f_m \rangle \end{bmatrix},$$

subsequently we get

$$\mathcal{G}_F G^* = F^* M_K. \tag{5.5}$$

This motivates the following definition.

Definition 5.1. Suppose that $F = \{f_i\}_{i \in I_m}$ is a K-frame of $\mathcal{H}_n$ with a K-dual G. Then an $m \times m$ matrix $M_{F,G}$ with spark $r + 1$ is called an (r, k)-matrix associated with F and G whenever

$$(M_{F,G} - \mathcal{G}_F)G^* = 0. \tag{5.6}$$

Remark 5.2. Note that by Definition 5.1, every K-frame F with non-zero vectors has at least a $(1, k)$-matrix $M_{F,G} = \mathcal{G}_F$ associated with F and an arbitrary $G \in KD_F$.

The next result shows that for a K-frame F, the existence of an (r, k)-matrix associated with F and $G \in KD_F$ assures the unknown r-erasures of K-dual coefficients can be completely recovered.

Theorem 5.3. *Let $F = \{f_i\}_{i \in I_m}$ be a K-frame of $\mathcal{H}_n$ with a K-dual G and $c = \{c_i\}_{i \in I_m}$ be a sequence of K-dual frame coefficients.*

(i) *If there exists an (r, k)-matrix $M_{F,G}$ associated with F and G then any r-erasures of K-dual coefficients can be recovered by solving the equation*

$$(M_{F,G} - \mathcal{G}_F)c = 0, \tag{5.7}$$

(ii) *If* $sparkF = r+1$ *then any* r*-erasures of* K*-dual frame coefficients can be recovered by solving the equation* $\mathcal{G}_F c = \theta_F^* K$*.*

Proof. To show (i), without losing the generality, we suppose for an original vector f the erasure coefficients are $c_1 := \{c_i\}_{i=1}^r = \{\langle f, g_i\rangle\}_{i=1}^r$ and the remaining coefficients are $c_2 := \{\langle f, g_i\rangle\}_{i=r+1}^m$. Furthermore, let M_1 and M_2 denote submatrices of $M_{F,G}$ containing of the first r columns and the rest, respectively. Then using (5.7) we get

$$M_1 c_1 + M_2 c_2 = \mathcal{G}_F c = \theta_F^* \theta_F c.$$

Equivalently,

$$M_1 \begin{bmatrix} \langle f, g_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, g_r\rangle \end{bmatrix} = \begin{bmatrix} \langle Kf, f_1\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle Kf, f_m\rangle \end{bmatrix} - M_2 \begin{bmatrix} \langle f, g_{r+1}\rangle \\ \cdot \\ \cdot \\ \cdot \\ \langle f, g_m\rangle \end{bmatrix}. \tag{5.8}$$

Using the assumption that the columns of M_1 are linearly independent and so the pseudo inverse $M_1^\dagger = (M_1^T M_1)^{-1} M_1^T$ there exists [7]. Hence by (5.8) we obtain

$$[c_i]_{i=1}^r = [\langle f, g_i\rangle]_{i=1}^r = M_1^\dagger \left([\langle Kf, f_i\rangle]_{i=1}^m - M_2 [\langle f, g_i\rangle]_{i=r+1}^m \right).$$

Thus, the missing coefficients are obtained completely and we have the perfect reconstruction.

On the other hand, it is known that

$$KerF = KerF^*F = Ker\mathcal{G}_F.$$

Therefore by (5.4) we have that $sparkF = spark\mathcal{G}_F$. Hence $M_{F,G} = \mathcal{G}_F$ is an (r,k)-matrix associated with F and G. Now, the proof of (ii) is complete by using (i). Note that this (r,k)-matrix is independent of K-dual G. □

Corollary 5.4. *Let* $F = \{f_i\}_{i \in I_m}$ *be a* K*-frame of* $\mathcal{H}_n$ *with uniform excess* $r > 0$*. Then every* $m-r+1$ *columns of* F *is linearly dependent. Moreover, any* $(m-r)$*-erasures of* K*-dual coefficients can be exactly recovered for every* K*-dual of* F*.*

Proof. Since F is with uniform excess r so any $m-r$ columns of associate matrix F constitutes an exact K-frame for $\mathcal{H}_n$. Without loss of generality, let the first $m-r+1$ columns of F be linearly independent. Moreover, assume that $R(K) = span\{\eta_i\}_{i \in I_l}$. Then for every η_i there exist unique coefficients $\{\alpha_{i,j}\}_{i=1}^{m-r}$ and $\{\alpha'_{i,j}\}_{i=2}^{m-r+1}$ so that

$$\eta_i = \sum_{j=1}^{m-r} \alpha_{i,j} f_j = \sum_{j=2}^{m-r+1} \alpha'_{i,j} f_j, \quad (i \in I_l).$$

By these equalities, and the assumption that $\{f_i\}_{i=1}^{m-r+1}$ is linearly independent, we conclude that

$$\alpha_{i,1} = \alpha'_{i,m-r+1} = 0,$$
$$\alpha_{i,j} = \alpha'_{i,j} \quad (2 \le j \le m-r).$$

Consequently,

$$\eta_i = \sum_{j=2}^{m-r} \alpha_{i,j} f_j$$

for all $i \in I_l$. So $\{f_i\}_{i=2}^{m-r}$ is also a K-frame of $\mathcal{H}_n$ that is a contradiction. This follows that every collection of $m-r+1$ columns of F is linearly dependent. The moreover part follows from Theorem 5.3 (ii) and the fact that $sparkF = m - r + 1$. □

It is worth noticing that, Corollary 5.4 for $r = 0$ fails. Indeed, if F is a K-frame with uniform excess 0 then F is an exact K-frame and so $sparkF = +\infty$. Also, Theorem 5.3 leads to a new approach for erasure recovery by using ordinary frames in finite dimensional Hilbert spaces by changing encode and decode frames.

Corollary 5.5. *Let $F = \{f_i\}_{i\in I_m}$ be a frame of $\mathcal{H}_n$ with $m > n$ and a dual frame G so that $sparkF = r + 1$. Then any r-erasures of dual frame coefficients can be recovered by solving the equation $\mathcal{G}_F c = \theta_F^*$, where c is a dual frame coefficient with unknowns $\{c_i\}_{i\in\Lambda}$, $|\Lambda| = r$.*

Corollary 5.6. *Let $F = \{f_i\}_{i\in I_m}$ be a frame of $\mathcal{H}_n$ with $m > n$ and a dual frame G. Then any 1-erasure of dual frame coefficients can be recovered by $\mathcal{G}_F c = \theta_F^*$, for unknown erasure c_j, $j \in I_m$.*

The above corollaries illustrate the advantage and difference of using (r, k)-matrix and erasure recovery matrix [18]. Indeed, if (F, G) is a pair of dual frames for $\mathcal{H}_n$. Unlike the customary approach, we consider dual frame G to encode a signal and F to decode the measurements. Then every erasure of encoding frame coefficients as $\{\langle f, g_i\rangle\}_{i\in\sigma}$, $|\sigma| \le sparkF - 1$ can be exactly recovered by Corollary 5.5. So, frames with large spark are resilient against more erasures of associated dual frame coefficients; since for frames $K = I_{\mathcal{H}_n}$ we call (r, k)-matrix associated to F and G an r-matrix for convenience. In this case, if $M_{F,G}$ is an r-matrix and N is an r-erasure recovery matrix for F, i.e., $NF^* = 0$ and $sparkN = r + 1$ then

$$NM_{F,G}G^* = NF^* = 0.$$

Thus, $NM_{F,G}$ is an ρ-erasure recovery matrix for G with $\rho = sparkNM_{F,G} \ge r+1$ since $KerM_{F,G} \subseteq KerNM_{F,G}$.

5.1. The existence of (r, k)-matrices

In the following, we show the relation between the existence of (r, k)-matrices with MRC. The following result gives a necessary condition for the existence of (r, k)-matrices and a sufficient condition for a K-dual to satisfy MRC.

Theorem 5.7. *Let $F = \{f_i\}_{i\in I_m}$ be a K-frame of $\mathcal{H}_n$ and $G \in KD_F$. If there exists a matrix $M_{F,G}$ which satisfies* (5.6)*, then G satisfies MRC for $(spark(M_{F,G}-\mathcal{G}_F)-1)$-erasures.*

Proof. If $M_{F,G} - \mathcal{G}_F$ has a zero column then $spark(M_{F,G} - \mathcal{G}_F) = 1$ so the result clearly holds. Now, let all columns of $M_{F,G} - \mathcal{G}_F$ be non-zero. Hence, $spark(M_{F,G} - \mathcal{G}_F) \ge 2$. Also, by the assumption, we have $(M_{F,G} - \mathcal{G}_F)G^* = 0$. On the other hand, if all columns of $M_{F,G} - \mathcal{G}_F$ are linearly independent, then it is invertible which implies $\theta_G^* = 0$ that is a contradiction. Therefore,

$$2 \leq spark(M_{F,G} - \mathcal{G}_F) < \infty.$$

Now, consider

$$\rho = spark(M_{F,G} - \mathcal{G}_F) - 1$$

implies that every ρ columns of $M_{F,G} - \mathcal{G}_F$ is linearly independent and so every $Kf \in R(K)$ can be recovered from the coefficients $\{\langle f, g_i\rangle\}_{i\in\Lambda^c}$ for every $\Lambda \subset I_m$, $|\Lambda| \leq \rho$. Without loss of the generality, we discuss the first ρ columns. More precisely, for every $f \in \mathcal{H}_n$ there exists $\{\alpha_{i,j}\}_{j=\rho+1}^{m}$ so that

$$\langle f, g_i\rangle = \sum_{j=\rho+1}^{m} \alpha_{i,j}\langle f, g_j\rangle, \quad (i \in I_\rho).$$

Thus, we can write

$$\begin{aligned}
\|Kf\|^4 &= |\langle Kf, Kf\rangle|^2 \\
&= \left|\left\langle \sum_{i=1}^{\rho}\sum_{j=\rho+1}^{m} \alpha_{i,j}\langle f, g_j\rangle f_i + \sum_{i=\rho+1}^{m} \langle f, g_i\rangle f_i, Kf \right\rangle\right|^2 \\
&= \left|\left\langle \sum_{j=\rho+1}^{m} \langle f, g_j\rangle \left(f_j + \sum_{i=1}^{\rho} \alpha_{i,j} f_i\right), Kf \right\rangle\right|^2 \\
&\leq \sum_{j=\rho+1}^{m} |\langle f, g_j\rangle|^2 \sum_{j=\rho+1}^{m} \left|\left\langle \left(f_j + \sum_{i=1}^{\rho} \alpha_{i,j} f_i\right), Kf \right\rangle\right|^2 \\
&\leq B\|Kf\|^2 \sum_{j=\rho+1}^{m} |\langle f, g_j\rangle|^2,
\end{aligned}$$

where the existence of the upper bound B in the last inequality is assured by the assumption that F is a K-frame. Therefore,

$$\|Kf\|^2/B \leq \sum_{j=\rho+1}^{m} |\langle f, g_j\rangle|^2 \leq D\|f\|^2,$$

for an upper bound D of G. Hence, $\{g_i\}_{i=\rho+1}^{m}$ is also a K^*-frame for $\mathcal{H}_n$. This implies the desired result. □

6. Examples

In this section, we present several examples to show not only the importance of the necessary or sufficient conditions in our main results, but also the advantages and significant differences of (r, k)-matrices with respect to r-erasure recovery matrices [18]. In particular, some examples of some K-frames (frames) are given for which there does not exist any appropriate erasure recovery matrix, but infinitely many (r, k)-matrices. In this section, we consider $\{e_i\}_{i\in I_n}$ as the standard orthonormal basis of $\mathbb{R}^n$. The first example shows that the converse of Proposition 3.3 does not hold in general.

Example 6.1. Consider, $\mathcal{H} = \mathbb{R}^3$ and $F = \{e_1, e_2\}$. Also, let $K \in B(\mathcal{H})$ so that $Kf = \left(c_1 + c_2 + \frac{1}{2}c_3\right) e_1$, for every $f = \sum_{i\in I_3} c_i e_i$. Then F is a K-frame for $\mathcal{H}$ and $S_F|_{R(K)} = I_{R(K)}$. Hence

$$G := K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}F = K^*((S_F|_{R(K)})^{-1})^*\pi_{R(K)}F = \{e_1 + e_2 + \frac{1}{2}e_3, 0\}.$$

A straightforward computation reveals that

$$\theta_F\theta_G^* = K,$$

so $G \in KD_F$. Moreover $S_G = \theta_G\theta_Z^*$ for every K-dual Z of F, i.e., G is the canonical K-dual of F, however F is not a subset of $R(K)$ or $S_F(R(K))$.

Also, the following example shows the importance of the sufficient conditions in Proposition 3.3.

Example 6.2. Let $\mathcal{H} = \mathbb{R}^4$. Define $F = \{e_1, e_2, e_3, e_1 + e_3\}$ and $K \in B(\mathcal{H})$ as $Kf = (c_1 + c_3)e_1 + (c_2 + \frac{1}{2}c_4)e_2$, for every $f = \sum_{i\in I_4} c_i e_i$. Then F is a K-frame for $\mathcal{H}$ and

$$\pi_{R(K)}F = \{e_1, e_2, 0, e_1\}.$$

Furthermore, the restriction of K-frame operator

$$S_F|_{R(K)} : span\{e_1, e_2\} \to span\{2e_1 + e_3, e_2\}$$

is given by

$$S_F|_{R(K)}(e_i) = \begin{cases} 2e_1 + e_3 & i = 1, \\ e_2 & i = 2 \end{cases}$$

that is an invertible operator. So, we obtain

$$G := K^*(S_F|_{R(K)})^{-1}\pi_{S_F(R(K))}F = \left\{ \begin{bmatrix} \frac{2}{5} \\ 0 \\ \frac{2}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{1}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{3}{5} \\ 0 \end{bmatrix} \right\}.$$

The Bessel sequence G is a K-dual of $\pi_{R(K)}F$. However, $G = \{g_i\}_{i=1}^4$ is neither a K-dual of F, nor the canonical K-dual of $\pi_{R(K)}F$. Indeed, for every $f = \sum_{i\in I_4} c_i e_i$

$$\theta_F\theta_G^* f = e_1(c_1 + c_3) + e_2(c_2 + c_4/2) + 4/5e_3(c_1 + c_3),$$

and consequently,

$$\theta_F\theta_G^* e_3 = e_1 + \frac{4}{5}e_3 \neq Ke_3.$$

Thus, G is not a K-dual of F. Also, consider

$$H = \{g_1, g_2, 0, g_4\}.$$

Then H is a K-dual of $\pi_{R(K)}F$. Moreover, $\|\theta_H^*\| < \|\theta_G^*\|$ and this implies that G cannot be the canonical K-dual of $\pi_{R(K)}F$.

The next example shows that the condition (i) in Theorem 4.1 is not sufficient for a subset $\sigma \subset I_m$ to satisfies MRC.

Example 6.3. Let $\mathcal{H}$, K and F be as in Example 6.2. Take $\sigma = \{1,3\}$ then the sequence $\{f_i\}_{i\in\sigma^c} = \{e_2, e_1 + e_3\}$ clearly is not a K-frame and so σ does not satisfy MRC. However,

$$R(\theta_F^* K) = span\{(a, b, 0, a) : a, b \in \mathbb{R}\},$$

which implies that $R(\theta_F^* K) \cap span\{\delta_i\}_{i\in\sigma} = \{0\}$.

In the sequel, we observe the advantages of using (r, k)-matrices with respect to r-erasure recovery matrices. In fact, we present some K-frames (frames) for which there does not exist any appropriate erasure recovery matrix, but infinitely many (r, k)-matrices.

Example 6.4. Suppose that $F = \left\{e_1 - e_3, e_3, 2e_4 - e_3, \frac{1}{2}(e_1 + e_3)\right\}$ and $K \in B(\mathbb{R}^4)$ is defined by $Ke_1 = Ke_2 = Ke_3 = e_1$, $Ke_4 = e_4 + e_1$. Then F is a K-frame for $\mathbb{R}^4$ and the Gramian matrix is obtained by

$$\mathcal{G}_F = \begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & -1 & \frac{1}{2} \\ 1 & -1 & 5 & \frac{-1}{2} \\ 0 & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \end{bmatrix}.$$

Since $spark\mathcal{G}_F = 3$ by Theorem 5.3 (ii), we can derive any 2-erasures of K-dual frame coefficients for every K-dual of F. Moreover, there is not any appropriate erasure recovery matrix for F. Indeed, if N is an $l \times 4$ matrix so that $NF^* = 0$ then the third column of N is zero, i.e., $sparkN = 1$. Thus N can only preserves one erasure of K-frame coefficients. However, there exist infinitely many (r, k)-matrix. Put

$$G = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

One may check that $FG^* = K$ so the sequence G is a K-dual of F and every matrix $M_{F,G}$ satisfies (5.6) is as follows

$$M_{F,G} = \begin{bmatrix} \alpha_1 & 1-\alpha_1 & \alpha_2 & \frac{1}{2} - \frac{1}{2}\alpha_2 \\ \beta_1 & -\beta_1 & \beta_2 & -\frac{1}{2}\beta_2 \\ \gamma_1 & -\gamma_1 & \gamma_2 & 2 - \frac{1}{2}\gamma_2 \\ \eta_1 & \frac{1}{2} - \eta_1 & \eta_2 & \frac{1}{4} - \frac{1}{2}\eta_2 \end{bmatrix}$$

Therefore, we can find so many (r, k)-matrices associated with F and G, for $1 \leq r \leq 3$. For example set $\alpha_1 = \alpha_2 = 1, \beta_1 = \beta_2 = 1, \gamma_1 = -1, \gamma_2 = 2$ and $\eta_1 = \eta_2 = 1/2$. Then we obtain an $(3, k)$-matrix.

In the last example, we survey the case that a K-frame F satisfies $F \subseteq R(K)$, i.e., it can be considered as a frame for $R(K)$. Moreover, we observe that unlike r-erasure recovery matrices for ordinary frames [18] the existence of (r, k)-matrices is independent of the fact that a K-frame or its dual satisfies MRC.

Example 6.5. Let $\mathcal{H} = \mathbb{R}^4$ and $F = \{-e_4, e_2, 2e_2 - e_4, e_1\}$. Also, assume that $Kf = c_1e_1 + c_2e_2 + (c_3 + c_4)e_4$, for every $f = \sum_{i \in I_4} c_ie_i$. Then F is a K-frame for $\mathcal{H}$ and

$$\mathcal{G}_F = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus $spark\mathcal{G}_F = 3$ and we can consider $\mathcal{G}_F$ as a $(2, k)$-matrix associated with F and each one of its K-duals. We observe that, none of 2-columns in F produce $R(K)$. Put

$$G = \left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

we have that

$$\theta_F \theta_G^* = K,$$

i.e., $G \in KD_F$. However, $R(K^*) = span\{e_1, e_2, e_3 + e_4\}$ and so G does not satisfy MRC even for 1-erasures. Moreover, none of 2-columns in G remain K^*-frame for $\mathbb{R}^4$. It is worth to note that by taking

$$M_{F,G} = \begin{bmatrix} 1 & 0 & a_1 & 0 \\ 0 & 1 & a_2 & 0 \\ 1 & 2 & a_3 & 0 \\ 0 & 0 & a_4 & 1 \end{bmatrix},$$

for all $a_i \in \mathbb{R}$, $i \in I_4$, we obtain a family of (r,k)-matrices with respect to F and G which r is dependent on the choice of a_i, $i \in I_4$. In this case 3 columns of $M_{F,G} - \mathcal{G}_F$ are zero but with appropriate choices of a_i, we get $sparkM_{F,G} = 4$.

It is worth noticing that F in Example 6.5 is also a frame for $R(K)$. From this point of view every $m \times 4$ matrix N so that $NF^* = 0$ has a zero column. Hence $sparkN = 1$ and so there is no appropriate erasure recovery matrix for F, however we obtain infinitely (r,k)-matrices with respect to F and G with $r > 1$.

Example 6.6. Consider $\mathcal{H} = \mathbb{R}^5$ and

$$F = \left\{ e_1, e_2, e_1 + e_5, \frac{3}{2}e_1 + e_2, -e_2, 2(e_5 - e_1), \frac{-5}{2}e_5 \right\}.$$

Also let $Ke_1 = Ke_2 = e_1$, $Ke_3 = Ke_4) = e_2$ and $Ke_5 = e_5 + e_1$. Then F is a K-frame for $\mathcal{H}$. A K-dual of F is as follows

$$G = \left\{ \begin{bmatrix} 1 \\ 1 \\ -3/2 \\ -3/2 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

A class of (r,k)-matrices associated to F, G are as follows

$$M_{F,G} = \begin{bmatrix} 1 & a_{12} & \frac{-1}{3} & -a_{12} & \frac{3}{2} & a_{16} & -a_{16} \\ 0 & a_{22} & \frac{2}{3} & -a_{22} & 1 & a_{26} & (-1 - a_{26}) \\ 1 & a_{32} & 2 & -a_{32} & \frac{3}{2} & a_{36} & (\frac{-5}{2} - a_{36}) \\ \frac{3}{2} & a_{42} & \frac{3}{2} & -a_{42} & \frac{13}{4} & a_{46} & (-3 - a_{46}) \\ 0 & -1 & \frac{-2}{3} & a_{54} & -a_{54} & a_{56} & (1 - a_{56}) \\ -2 & a_{62} & 2 & -a_{62} & -3 & a_{66} & -a_{66} \\ 0 & a_{72} & \frac{5}{2} & 3 & (-3 - a_{72}) & a_{76} & (-a_{76} - \frac{25}{4}) \end{bmatrix},$$

where parameters a_{ij} in matrix are arbitrary real numbers. A straightforward computation shows that $(M_{F,G} - \mathcal{G}_F)G^* = 0$, where $\mathcal{G}_F$ is the Gramian matrix of F. Moreover, every matrix $M_{F,G}$ as above is invertible and so any 6-erasure (or less) of K-dual coefficients we obtain perfect recovery by Theorem 5.3.

7. Numerical results

In this section, we provide numerical results for the recovery of signals from erased and noisy K-dual frame coefficients. To this end, we use unit norm signals that are generated with uniform distributed random numbers. Also, we produce randomly

Trial	Vectors	Error of $\sigma = 0.002$	Error of $\sigma = 0.02$	Error of $\sigma = 0.05$
1	100	0.0085724	0.098009	0.22036
2	100	0.0091058	0.092629	0.23210
3	100	0.0094286	0.095462	0.23652
4	100	0.0092686	0.087105	0.25751
5	100	0.0103040	0.091419	0.22461
6	100	0.0094113	0.089906	0.22679
7	100	0.0092544	0.097622	0.22807
8	100	0.0097604	0.095072	0.24331
9	100	0.0100730	0.089506	0.24264
10	100	0.0097157	0.087446	0.23899

TABLE 1

Error rate of recovery where $n = 10$, $m = 12$, $r = 1$, $spark\mathcal{G}_F = 2$

generated matrices K and K-frames . It is worth noticing that, the numerical results confirmed our method for perfect recovery based on the results in section 5, in case the coefficients are noise-free. Hence, we consider several cases of noisy coefficients to survey the sensitivity and resistance of our approach under noises generated by uniformly distributed random.

First, we apply Example 6.6 to compute error rate under noise and different erasures. Indeed, taking different choices for a_{ij} in $M_{F,G}$ effects on the errors under erasures. Figure 1 illustrates the error rate of erased coefficients under randomly generated noises in $[0, 0.5]$ by using (r, k)-matrix $M_{F,G}$ in case $a_{ij} = 1$. Also, we observe the situation of errors in Figure 2 with different choices of a_{ij}.

For more general discussions, we consider an n-dimensional Hilbert space $\mathcal{H}_n$ and choose a matrix K of order n that is generated by the identity matrix with zeros in some random rows. We try different choices of n, m, r and $spark\mathcal{G}_F$ for randomly generated K-frames F with m vectors in $\mathcal{H}_n$ and a K-dual frame G in case erasures occur in the first r coefficients constructed by K-dual G so that $r < spark\mathcal{G}_F$. We get 10 trail and in each trial we take 100 or 50 test vectors that are also generated with uniform distributed random numbers and compute the mean of errors. Then we use Theorem 5.3 for reconstruction of signals. We set out the numerical results in 4 tables and note that the algorithms are completely based on Theorem 5.3. The tables demonstrate the error rates in different noise situations. Due to the fact that our results are associated to perfect recovery of signals under erased coefficients without noise, so as we expect, the numerical results show that the method is sensitive with respect to noise. Although it completely depends on the noise level and we observe that the approach works well enough in low noise cases.

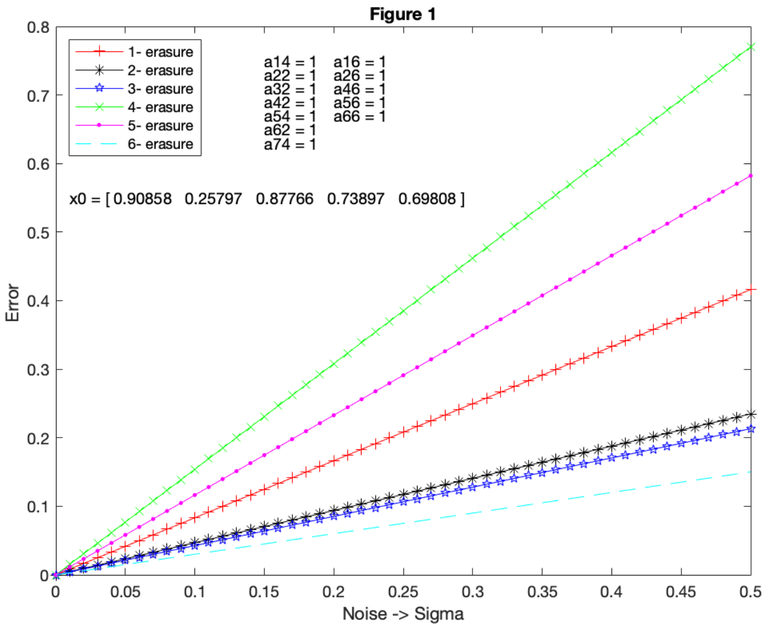
Figure 1
1- erasure
2- erasure
3- erasure
4- erasure
5- erasure
6- erasure
a14 = 1 a16 = 1
a22 = 1 a26 = 1
a32 = 1 a46 = 1
a42 = 1 a56 = 1
a54 = 1 a66 = 1
a62 = 1
a74 = 1
x0 = [0.90858 0.25797 0.87766 0.73897 0.69808]
Error
Noise -> Sigma
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8

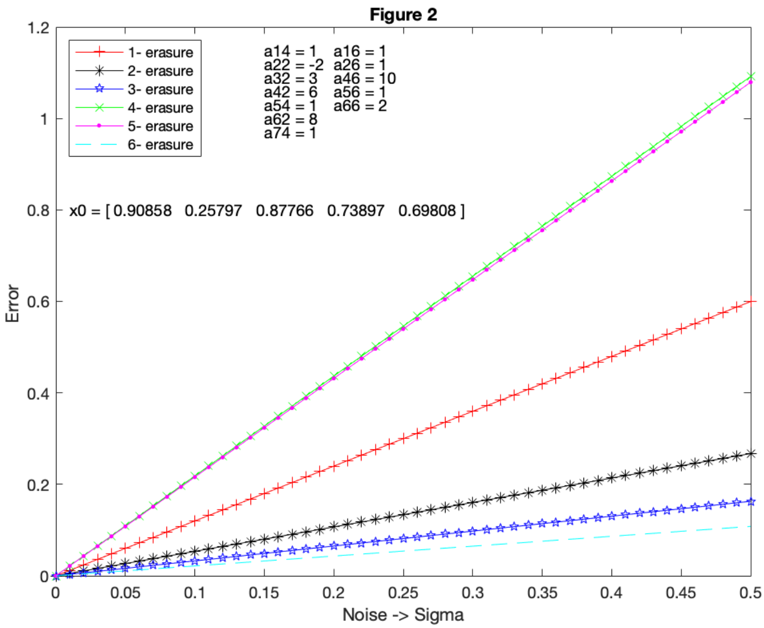
Figure 2
1- erasure
2- erasure
3- erasure
4- erasure
5- erasure
6- erasure
a14 = 1 a16 = 1
a22 = -2 a26 = 1
a32 = 3 a46 = 10
a42 = 6 a56 = 1
a54 = 1 a66 = 2
a62 = 8
a74 = 1
x0 = [0.90858 0.25797 0.87766 0.73897 0.69808]
Error
Noise -> Sigma
0 0.05 0.1 0.15 0.2 0.25 0.3 0.35 0.4 0.45 0.5
0 0.2 0.4 0.6 0.8 1 1.2

Trial	Vectors	Error of $\sigma = 0.002$	Error of $\sigma = 0.02$	Error of $\sigma = 0.05$
1	50	0.014189	0.15041	0.39533
2	50	0.014351	0.15213	0.37953
3	50	0.014878	0.15375	0.39598
4	50	0.015939	0.14863	0.3760
5	50	0.016308	0.15805	0.36498
6	50	0.015795	0.15551	0.38558
7	50	0.015823	0.15329	0.36468
8	50	0.014342	0.16171	0.36440
9	50	0.014095	0.15123	0.38340
10	50	0.014850	0.15101	0,38548

TABLE 2

Error rate of recovery where $n = 15$, $m = 23$, $r = 2$, $spark\mathcal{G}_F = 4$

Trial	Vectors	Error of $\sigma = 0.002$	Error of $\sigma = 0.02$	Error of $\sigma = 0.05$
1	100	0.014967	0.14517	0.39464
2	100	0.015608	0.15213	0.39415
3	100	0.015394	0.15783	0.37885
4	100	0.016159	0.14830	0.37944
5	100	0.015569	0.15320	0.38815
6	100	0.015294	0.15439	0.38101
7	100	0.015397	0.15421	0.38292
8	100	0.014943	0.15615	0.41209
9	100	0.014787	0.14919	0.37709
10	100	0.015713	0.15331	0.40505

TABLE 3

Error rate of recovery where $n = 21$, $m = 25$, $r = 3$, $spark\mathcal{G}_F = 4$

Trial	Vectors	Error of $\sigma = 0.002$	Error of $\sigma = 0.02$	Error of $\sigma = 0.05$
1	100	0.0085436	0.094801	0.19616
2	100	0.0087253	0.08523	0.2190
3	100	0.0088595	0.086994	0.20568
4	100	0.0088853	0.080176	0.23631
5	100	0.0086112	0.087957	0.22753
6	100	0.0078584	0.089711	0.22771
7	100	0.0088493	0.088402	0.22035
8	100	0.0084432	0.097616	0.23225
9	100	0.0094813	0.086268	0.21164
10	100	0.0086436	0.08513	0.2013

TABLE 4

Error rate of recovery where $n = 10$, $m = 16$, $r = 5$, $spark\mathcal{G}_F = 6$

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