Blow-up and energy decay for a class of wave equations with nonlocal Kirchhoff-type diffusion and weak damping

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Abstract: The purpose of this paper is to study the following equation driven by a nonlocal integro-differential operator $\mathcal{L}_K$:

$$u_{tt}+[u]_s^{2(\theta-1)}\mathcal{L}_K u+a|u_t|^{m-1}u_t=b|u|^{p-1}u$$

with homogeneous Dirichlet boundary condition and initial data, where $[u]_s^2$ is the Gagliardo seminorm, $a\geq 0,\ b>0,\ 0<s<1$, and $\theta\in[1,2_s^*/2)$ with $2_s^*=2N/(N-2s)$, N is the space dimension. By virtue of a differential inequality technique, an upper bound of the blow-up time is obtained with a bounded initial energy if $m<p$ and some additional conditions are satisfied. For $m\equiv 1$, in particular, the blow-up result with high initial energy also is showed by constructing a new control functional and combining energy inequalities with the concavity argument. Moreover, an estimate for the lower bound of the blow-up time is investigated. Finally, the energy decay estimate is proved as well. These results improve and complement some recent works obtained by Pan, Pucci and Zhang(*J. Evol. Equ.* **18** (2018) 385–409) and by Lin and Tian et al(*Discrete Contin. Dyn. Syst. Ser. S* **13** (7) (2020) 2095–2107).

Keywords: Kirchhoff-type diffusion; fractional Laplacian; weak damping; blow-up; energy decay estimate.

MSC(2010): 35L05; 35B40; 35B44; 47G20.

1 Introduction

In this paper, we are concerned with the following Kirchhoff-type hyperbolic equations involving the fractional Laplacian and the weak damping

$$\begin{cases} u_{tt}+[u]_s^{2(\theta-1)}\mathcal{L}_K u+a|u_t|^{m-1}u_t=b|u|^{p-1}u & \text{in } \Omega\times(0,T),\\ u(x,t)=0 & \text{in } \mathbb{R}^N\backslash\Omega\times(0,T),\\ u(x,0)=u_0(x),u_t(x,0)=u_1(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $[u]_s^2=\iint_{\mathbb{R}^{2N}}|u(x,t)-u(y,t)|^2K(x-y)dxdy$ is the Gagliardo seminorm, $a\geq 0,\ b>0$, $T\in(0,+\infty]$ is the maximal existence time of the solution, Ω is a bounded domain in $\mathbb{R}^N(N>2s)$ with smooth boundary $\partial\Omega$, $s\in(0,1)$ is fixed and $\mathcal{L}_K$ is the integro-differential operator, which (up to normalization factors) may be defined as

$$\mathcal{L}_K u(x)=2\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\backslash B_\varepsilon(x)}(u(x)-u(y))K(x-y)dy,\quad x\in\mathbb{R}^N \tag{1.2}$$

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for $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ is a ball with x as the center and ε as the radius, $x \in \mathbb{R}^N,\ \varepsilon > 0$, the kernel $K : \mathbb{R}^N\backslash\{0\} \to (0, +\infty)$ is a function satisfying the properties below:

(a) $\gamma K \in L^1(\mathbb{R}^N)$, where $\gamma(x) = \min\{|x|^2, 1\}$;

(b) there exists $\lambda > 0$ such that $K(x) \geq \lambda |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N\backslash\{0\}$;

(c) $K(x) = K(-x)$ for any $x \in \mathbb{R}^N\backslash\{0\}$.

Throughout this paper, let $u_0(x) \in X_0,\ u_1(x) \in L^2(\Omega)$ and the exponents satisfy the condition

$$(\mathbf{H}) \quad 1 \leq m,\ 1 \leq \theta < \frac{2_s^*}{2},\ 2\theta - 1 < p \leq 2_s^* - 1.$$

where $2_s^* := \frac{2N}{N-2s}$, X_0 will be showed later.

In 1883, Kirchhoff [10] first introduced the following hyperbolic equation:

$$\rho h u_{tt} + \delta u_t = \left\{p_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f, \quad 0 \leq x \leq L,\ t \geq 0,$$

to describe the nonlinear vibrations of an elastic string, where $u = u(x,t)$ is the lateral deflection, E is the Young's modulus, ρ is the mass density, h is the cross-section area, L is the length, p_0 is the initial axial tension, δ is the resistance modulus, and f is the external force. Lions [12] proposed a functional analysis framework to study the following higher dimension problem in presence of an external force term f:

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u).$$

Subsequently, the study of the mathematical theory to Kirchhoff type equations has been well developing by various authors. The interested readers can refer to references [8, 9, 16, 17] to be familiar with the latest results. Specifically, Han and Li [8] discussed the following initial boundary value problem for a class of Kirchhoff type parabolic equations with a nonlinear term

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = |u|^{q-1}u & \text{in } \Omega \times (0,T),\\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T),\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where the diffusion coefficient $M(s) = a + bs$ with the parameters $a,\ b$ being positive, $3 < q \leq 2^* - 1$ with 2^* being the Sobolev conjugate of 2. They applied the modified potential well method and variational method to give a threshold result for the solutions to exist globally or to blow up in finite time when the initial energy is subcritical and critical, respectively. The decay rate of the $L^2(\Omega)$ norm was also obtained for global solutions in these cases. Moreover, some sufficient conditions for the existence of global and blow-up solutions were also derived when the initial energy is supercritical.

In recent years, more and more attentions have been paid to various models involving fractional Laplacian and nonlocal operators. This type of operators arises in a lot of applications, such as, continuum mechanics, phase transition phenomena, population dynamics, image process, game theory and Lévy processes and so on. Servadei and Valdinoci [30] studied the following equation

$$\begin{cases} \mathcal{L}_K u + f(x,u) = 0 & \text{in } \Omega,\\ u = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

with

$$\mathcal{L}_K u(x) = \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))K(y)dy, \quad x \in \mathbb{R}^N.$$

They proved that the equation above admits a Mountain Pass type solution $u \in X_0$ which is not identically zero if f is a Carathéodory function satisfying suitable conditions. For more results with non-local operators of elliptic type, one may refer to [1, 29, 31, 33]. And there exist some papers on the study of fractional p-Laplacian evolution equations, the interested readers can refer to [18, 26, 27, 34]. For Kirchhoff type problem involving a nonlocal operator, many results have been showed. For example, Fiscella and Valdinoci [3] investigated

$$\begin{cases} -M([u]_s^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

where λ is a positive parameter, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and continuous function, and there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for any $t \in \mathbb{R}^+$. They proved that there exists $\lambda^* > 0$ such that the above problem has a nontrivial solution u_λ for all $\lambda \geq \lambda^*$. Xiang, Rădulescu and Zhang [35] studied the following problem

$$\begin{cases} u_t + M([u]_s^2)\mathcal{L}_K u = |u|^{p-2}u & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } \mathbb{R}^N \backslash \Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is continuous and there exists $m_0 > 0$ and $\theta > 1$ such that $M(\sigma) \geq m_0 \sigma^{\theta-1}$ for all $\sigma \in \mathbb{R}_0^+$. Under the suitable conditions, the local existence of nonnegative solutions was obtained by employing the Galerkin method. In addition, supposed that there exists a constant $\mu \geq 1$ such that $\mu \mathcal{M}(\sigma) \geq M(\sigma)\sigma$ for all $\sigma \in \mathbb{R}_0^+$ with $\mathcal{M}(\sigma) = \int_0^\sigma M(\tau)d\tau$, they proved that the local nonnegative solutions blows up in finite time with arbitrary negative initial energy and suitable initial values by virtue of a differential inequality technique. Moreover, an estimate for the lower and upper bounds of the blow-up time was obtained. These results were improved by Ding and Zhou [2].

To our best knowledge, there are many papers dedicated to the blow-up phenomenon of the solution for hyperbolic problems involving the Laplacian, one may see [4–7, 13, 15, 24, 25, 32] and the reference therein. However, there is hardly any work on the study of hyperbolic problems involving the fractional Laplacian, in particular, on the blow-up study with high initial energy. In this paper, we are devoted to the study of problem (1.1) on the blow-up phenomenon of the solution with the subcritical and high initial energy, respectively, and will give the energy decay estimate when the solution of problem (1.1) exists globally. Our results improve and complement some recent works in [21, 28], where for $a = 0$ and $b = 1$ in problem (1.1), Pan, Pucci and Zhang [28] obtained the global existence, vacuum isolating and blow-up of solutions by combining the Galerkin method with potential wells theory for subcritical initial energy under some appropriate assumptions, and investigated the existence of global solutions with the critical initial energy. It is noted that the blow-up phenomenon with high initial energy is not discussed in [28]. Hereafter, Lin and Tian et al [21] estimated the upper bound of the blow-up time with arbitrary positive energy under some assumptions, and gave the lower bound of the blow-up time for $p \in (2_s^* - 1, 2_s^*/2]$. In this paper, we investigate the the lower bound of the blow-up time for $p \in \left[2_s^* - 1, \frac{2_s^*}{2} - 2\theta\left(\frac{1}{2_s^*} - \frac{1}{2}\right)\right)$. Obviously, the range is extended due to $2_s^* > 2$.

The outline of the present paper is as follows: In Section 2, we will give the corresponding function space and some necessary lemmas. Section 3 is devoted to give a blow-up result with a bounded initial energy. In Section 4, a blow-up criterion with high initial energy is proved for $m \equiv 1$. The lower bound of the blow-up time will be discussed in Section 5. In Section 6, we prove that the solution exists globally, and give a energy decay estimate.

2 Preliminaries

We begin with introducing some spaces and lemmas, which can be found in [30]. In the sequel, we denote $Q = \mathbb{R}^{2N}\backslash\mathcal{O}$, where $\mathcal{O} = \mathcal{C}\Omega \times \mathcal{C}\Omega \subset \mathbb{R}^{2N}$ and $\mathcal{C}\Omega = \mathbb{R}^N\backslash\Omega$. X is the linear space of Lebesgue measurable functions from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in X belongs to $L^2(\Omega)$ and

$$\text{the mapping } (x, y) \to (u(x) - u(y))\sqrt{K(x-y)} \text{ is in } L^2(Q, dxdy).$$

The space X is endowed with the norm defined as

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \left(\iint_Q |u(x) - u(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}$$

Define

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathcal{C}\Omega\}.$$

Lemma 2.1. *[30] Let $K : \mathbb{R}^N\backslash\{0\} \to (0, +\infty)$ satisfy assumptions $(a) - (c)$.*

(1) For any $r \in [1, 2_s^*]$, there exists a positive constant c, depending only on N, s such that for any $u \in X_0$

$$\|u\|^2_{L^r(\Omega)} = \|u\|^2_{L^r(\mathbb{R}^N)} \le c \iint_Q |u(x) - u(y)|^2 K(x-y)dxdy;$$

(2) There exists a constant $C > 1$ depending only on N, s, θ and Ω such that for any $u \in X_0$

$$\iint_Q |u(x) - u(y)|^2 K(x-y)dxdy \le \|u\|^2_X \le C \iint_Q |u(x) - u(y)|^2 K(x-y)dxdy,$$

that is,

$$\|u\|_{X_0} = \left(\iint_Q |u(x) - u(y)|^2 K(x-y)dxdy\right)^{\frac{1}{2}}$$

is a norm on X_0 equivalent to the usual one.

Definition 2.1. (***Weak solution***) *A function $u = u(x,t)$ over $\Omega \times [0,T]$ is said to be a weak solution of problem (1.1) if $u \in C([0,T]; X_0)$ with $u_t \in C([0,T]; L^2(\Omega)) \cap L^{m+1}(0,T;\Omega)$, $u(x,0) = u_0(x) \in X_0$, $u_t(x,0) = u_1(x) \in L^2(\Omega)$ and*

$$\int_\Omega u_{tt}\varphi dx + \langle u(\cdot,t), \varphi(\cdot,t)\rangle_{X_0} + \int_\Omega a|u_t|^{m-1}u_t\varphi dx = \int_\Omega b|u|^{p-1}u\varphi dx \tag{2.1}$$

for any $\varphi \in X_0$, where

$$\langle u(\cdot,t), \varphi(\cdot,t)\rangle_{X_0} = \|u(\cdot,t)\|^{2(\theta-1)}_{X_0} \iint_Q (u(x,t) - u(y,t))(\varphi(x,t) - \varphi(y,t))K(x-y)dxdy.$$

The local existence of the solution to problem (1.1) may be proved by exploiting the Galerkin method in [35].

Definition 2.2. (***Blow-up in finite time***) *We say that the weak solution u of problem (1.1) blows up in finite time T^* if*

$$\lim_{t\to T^*}\|u\|_{p+1}^{p+1}=+\infty.$$

Define the energy functional

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{1}{2\theta}\|u\|_{X_0}^{2\theta}-\frac{b}{p+1}\|u\|_{p+1}^{p+1}. \tag{2.2}$$

Lemma 2.2. *If u is a solution for problem (1.1), the energy functional $E(t)$ is nonincreasing with respect to t, that is $E'(t)=-a\|u_t\|_{m+1}^{m+1}\le 0$.*

The proof of this lemma is pretty direct if we choose $\varphi=u_t$ in (2.1). Here, we omit the process.

3 Blow-up with a bounded initial energy

In what follows, we are committing to discussing the blow-up phenomenon with a bounded initial energy. Our results is as follows:

Theorem 3.1. *Let $m<p$ and (**H**) hold. Supposed that*

$$\|u_0\|_{X_0}^{2\theta}>\alpha_1:=(bc^{\frac{p+1}{2}})^{-\frac{2\theta}{(p+1-2\theta)}},\quad E(0)<E_1:=\Big(\frac{1}{2\theta}-\frac{1}{p+1}\Big)\alpha_1,$$

the solution of problem (1.1) blows up at finite time T^* and

$$T^*\le F^{-\frac{\sigma}{1-\sigma}}(0)\frac{M_2}{M_1}\frac{1-\sigma}{\sigma},$$

where $F(0),\ \sigma,\ M_1,\ M_2$ will be discussed below.

Let us first give two critical lemmas to help prove this theorem.

Lemma 3.1. *If u is a solution for problem (1.1) and all conditions in Theorem 3.1 are satisfied, there exists a positive constant $\alpha_2>\alpha_1$ such that*

$$\|u\|_{X_0}^{2\theta}\ge\alpha_2,\quad \forall t\ge 0. \tag{3.1}$$

Proof. It follows from (2.2) and Lemma 2.1(1) that

$$\begin{aligned}E(t)&\ge\frac{1}{2\theta}\|u\|_{X_0}^{2\theta}-\frac{b}{p+1}\|u\|_{p+1}^{p+1}\ge\frac{1}{2\theta}\|u\|_{X_0}^{2\theta}-\frac{bc^{\frac{p+1}{2}}}{p+1}\|u\|_{X_0}^{p+1}\\&=\frac{1}{2\theta}\alpha-\frac{bc^{\frac{p+1}{2}}}{p+1}\alpha^{\frac{p+1}{2\theta}}:=h(\alpha),\end{aligned} \tag{3.2}$$

where $\alpha:=\alpha(t)=\|u\|_{X_0}^{2\theta}$. It is easily verified that $h(\alpha)$ is increasing for $0<\alpha<\alpha_1$, decreasing for $\alpha_1<\alpha$, $h(\alpha)\to-\infty$ as $\alpha\to+\infty$, and $h(\alpha_1)=E_1$. Since $E(0)<E_1$, there exists an $\alpha_2>\alpha_1$ such that $h(\alpha_2)=E(0)$. Recall (3.2), then $h(\alpha(0))\le E(0)=h(\alpha_2)$, which implies $\alpha(0)\ge\alpha_2$

since the condition $\alpha(0) > \alpha_1$. To prove (3.1), we suppose by contradiction that for some $t_0 > 0$, $\alpha(t_0) < \alpha_2$. The continuity of $\alpha(t)$ illustrates that we could choose t_0 such that $\alpha_1 < \alpha(t_0) < \alpha_2$, and then we have

$$E(0) = h(\alpha_2) < h(\alpha(t_0)) \leq E(t_0).$$

This contradicts Lemma 2.2. □

Lemma 3.2. *Set $H(t) = E_2 - E(t)$ for all $t \geq 0$ with $E_2 \in (E(0), E_1)$. If all the conditions of Theorem 3.1 hold, for all $t \geq 0$,*

$$0 < H(0) \leq H(t) \leq \frac{b}{p+1}\|u\|_{p+1}^{p+1}. \tag{3.3}$$

Proof. Lemma 2.2 implies that $H(t)$ is nondecreasing with respect to t, thus for $t \geq 0$, $H(t) \geq H(0) = E_2 - E(0) > 0$. (2.2) and (3.1) illustrate

$$H(t) \leq E_1 - \frac{1}{2\theta}\alpha_2 + \frac{b}{p+1}\|u\|_{p+1}^{p+1} \leq E_1 - \frac{1}{2\theta}\alpha_1 + \frac{b}{p+1}\|u\|_{p+1}^{p+1} = \frac{b}{p+1}\|u\|_{p+1}^{p+1}.$$

□

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1: Define an auxiliary function

$$F(t) = H^{1-\sigma}(t) + \varepsilon \int_\Omega u_t u dx,$$

where $0 < \sigma \leq \min\left\{\frac{p-m}{(p+1)m}, \frac{p-1}{2(p+1)}\right\}$, $\varepsilon > 0$. The rest of this proof will be divided into three steps:

Step 1: Estimate for $F'(t)$ Differentiating directly $F(t)$, recalling (1.1), adding and subtracting $\varepsilon(p+1)(1-\varepsilon_1)H(t)$ with $0 < \varepsilon_1 < 1$, we obtain

$$\begin{aligned} F'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|_2^2 - \varepsilon\|u\|_{X_0}^{2\theta} - \varepsilon a \int_\Omega |u_t|^{m-1}u_t u dx + \varepsilon b\|u\|_{p+1}^{p+1} \\ &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\Big[1 + \frac{(p+1)(1-\varepsilon_1)}{2}\Big]\|u_t\|_2^2 + \varepsilon\Big[\frac{(p+1)(1-\varepsilon_1)}{2\theta} - 1\Big]\|u\|_{X_0}^{2\theta} \\ &\quad - \varepsilon a \int_\Omega |u_t|^{m-1}u_t u dx + \varepsilon(p+1)(1-\varepsilon_1)H(t) - \varepsilon(p+1)(1-\varepsilon_1)E_2 + \varepsilon\varepsilon_1 b\|u\|_{p+1}^{p+1}. \end{aligned} \tag{3.4}$$

Applying Young's inequality with $\varepsilon_2 > 1$ and $H'(t) = -E'(t)$, Lemma 2.2, the embedding $L^{p+1}(\Omega) \hookrightarrow L^{m+1}(\Omega)$, we easily get

$$\begin{aligned} \Big|\int_\Omega |u_t|^{m-1}u_t u dx\Big| &\leq \varepsilon_2 H^{-\sigma}(t)\|u_t\|_{m+1}^{m+1} + \frac{1}{\varepsilon_2^m}H^{\sigma m}(t)\|u\|_{m+1}^{m+1} \\ &\leq \frac{\varepsilon_2}{a}H^{-\sigma}(t)H'(t) + \frac{1}{\varepsilon_2^m}H^{\sigma m}(t)(|\Omega|+1)^{m+1}\|u\|_{p+1}^{m+1}. \end{aligned} \tag{3.5}$$

Recalling $0<\sigma\le\frac{p-m}{(p+1)m}$ and Lemma 3.2, apparently,

$$\begin{aligned}
&\frac{1}{\varepsilon_2^m}H^{\sigma m}(t)(|\Omega|+1)^{m+1}\|u\|_{p+1}^{m+1}\\
&\quad=\frac{1}{\varepsilon_2^m}(|\Omega|+1)^{m+1}\frac{H^{\sigma m+\frac{m+1}{p+1}-1}(t)}{H^{\sigma m+\frac{m+1}{p+1}-1}(0)}H^{1-\frac{m+1}{p+1}}(t)H^{\sigma m+\frac{m+1}{p+1}-1}(0)\|u\|_{p+1}^{m+1}\\
&\quad\le\frac{1}{\varepsilon_2^m}(|\Omega|+1)^{m+1}\Big(\frac{b}{p+1}\Big)^{\frac{p-m}{p+1}}(\|u\|_{p+1}^{p+1})^{1-\frac{m+1}{p+1}}H^{\sigma m+\frac{m+1}{p+1}-1}(0)\|u\|_{p+1}^{m+1}\\
&\quad=\frac{1}{\varepsilon_2^m}(|\Omega|+1)^{m+1}\Big(\frac{b}{p+1}\Big)^{\frac{p-m}{p+1}}H^{\sigma m+\frac{m+1}{p+1}-1}(0)\|u\|_{p+1}^{p+1}.
\end{aligned}\tag{3.6}$$

It follows from (3.4) (3.5) and (3.6) that

$$\begin{aligned}
F'(t)&\ge(1-\sigma-\varepsilon\varepsilon_2)H^{-\sigma}(t)H'(t)++\varepsilon\Big[1+\frac{(p+1)(1-\varepsilon_1)}{2}\Big]\|u_t\|_2^2\\
&+\varepsilon\Big[\frac{(p+1)(1-\varepsilon_1)}{2\theta}-1\Big]\|u\|_{X_0}^2+\varepsilon(p+1)(1-\varepsilon_1)H(t)-\varepsilon(p+1)(1-\varepsilon_1)E_2\\
&+\varepsilon\Big[\varepsilon_1b-\frac{a}{\varepsilon_2^m}(|\Omega|+1)^{m+1}\Big(\frac{b}{p+1}\Big)^{\frac{p-m}{p+1}}H^{\sigma m+\frac{m+1}{p+1}-1}(0)\Big]\|u\|_{p+1}^{p+1}.
\end{aligned}\tag{3.7}$$

Let us choose $0<\varepsilon_1<\frac{p+1-2\theta}{p+1}$ sufficiently small and choose $E_2\in(E(0),E_1)$, sufficiently close to $E(0)$ such that

$$E_2\le\alpha_1\Big(\frac{1}{2\theta}-\frac{1}{(p+1)(1-\varepsilon_1)}\Big)<E_1,$$

therefore, Lemma 3.1 implies

$$\begin{aligned}
&\varepsilon\Big[\frac{(p+1)(1-\varepsilon_1)}{2}-1\Big]\|u\|_{X_0}^{2\theta}-\varepsilon(p+1)(1-\varepsilon_1)E_2\\
&\quad\ge\varepsilon\Big[\frac{(p+1)(1-\varepsilon_1)}{2}-1\Big]\alpha_1-\varepsilon(p+1)(1-\varepsilon_1)E_2\ge0.
\end{aligned}$$

Let us fix the constant ε_2 so that

$$\varepsilon_1b>\frac{a}{\varepsilon_2^m}(|\Omega|+1)^{m+1}\Big(\frac{b}{p+1}\Big)^{\frac{p-m}{p+1}}H^{\sigma m+\frac{m+1}{p+1}-1}(0),$$

and then choose ε so small that $1-\sigma>\varepsilon\varepsilon_2$. Therefore, (3.7) can be written as

$$F'(t)\ge M_1\Big(\|u_t\|_2^2+H(t)+\|u\|_{p+1}^{p+1}\Big),\tag{3.8}$$

where

$$\begin{aligned}
M_1=\varepsilon\min\Big\{&1+\frac{(p+1)(1-\varepsilon_1)}{2},(p+1)(1-\varepsilon_1),\\
&\varepsilon_1b-\frac{a}{\varepsilon_2^m}(|\Omega|+1)^{m+1}\Big(\frac{b}{p+1}\Big)^{\frac{p-m}{p+1}}H^{\sigma m+\frac{m+1}{p+1}-1}(0)\Big\}.
\end{aligned}$$

Step 2: Estimate for $F^{\frac{1}{1-\sigma}}(t)$ We now consider

$$F^{\frac{1}{1-\sigma}}(t)=\Big(H^{1-\sigma}(t)+\varepsilon\int_\Omega u_tu\,dx\Big)^{\frac{1}{1-\sigma}}.\tag{3.9}$$

On the one hand, applying Cauchy-Schwarz inequality, embedding $L^{p+1}(\Omega) \hookrightarrow L^2(\Omega)$ and Young's inequality to show

$$\Big|\int_\Omega u_t u dx\Big|^{\frac{1}{1-\sigma}} \leq (1+|\Omega|)^{\frac{1}{1-\sigma}} \|u_t\|_2^{\frac{1}{1-\sigma}} \|u\|_{p+1}^{\frac{1}{1-\sigma}} \leq C_1 \|u_t\|_2^2 + C_2 \|u\|_{p+1}^{\frac{2}{2(1-\sigma)-1}}, \tag{3.10}$$

where

$$C_1 = \frac{(1+|\Omega|)^{\frac{1}{1-\sigma}}}{2(1-\sigma)}, \quad C_2 = \frac{(1+|\Omega|)^{\frac{1}{1-\sigma}}[2(1-\sigma)-1]}{2(1-\sigma)}.$$

Recalling $0<\sigma\leq \frac{p-1}{2(p+1)}$, Lemma 3.2, we obtain

$$\|u\|_{p+1}^{\frac{2}{2(1-\sigma)-1}} \leq \Big[\frac{(p+1)H(t)}{b}\Big]^{\frac{2-(p+1)[2(1-\sigma)-1]}{(p+1)[2(1-\sigma)-1]}} \|u\|_{p+1}^{p+1} \leq C_3 \|u\|_{p+1}^{p+1} \tag{3.11}$$

with $C_3 = \Big(\min\Big\{\frac{(p+1)H(0)}{b}, 1\Big\}\Big)^{\frac{2-(p+1)[2(1-\sigma)-1]}{(p+1)[2(1-\sigma)-1]}}$. Inserting (3.11) into (3.10), and combining (3.9), we obtain

$$F^{\frac{1}{1-\sigma}}(t) \leq M_2\Big(H(t) + \|u_t\|_2^2 + \|u\|_{p+1}^{p+1}\Big), \tag{3.12}$$

where

$$M_2 = 2^{\frac{\sigma}{1-\sigma}} \max\Big\{1, \varepsilon^{\frac{1}{1-\sigma}} C_1, \varepsilon^{\frac{1}{1-\sigma}} C_2 C_3\Big\}.$$

Step 3: Blow-up result Combining (3.8) and (3.12) arrives at $F^{\frac{1}{1-\sigma}}(t) \leq \frac{M_2}{M_1} F'(t)$, which implies by Gronwall's inequality

$$F^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{F^{\frac{\sigma}{1-\sigma}}(0) - \frac{M_2}{M_1}\frac{\sigma}{1-\sigma} t},$$

with $F(0) = H^{1-\sigma}(0) + \varepsilon \int_\Omega u_1(x) u_0(x) dx > 0$ by choosing suitable ε, which further yields $F(t) \to +\infty$ at finite time T^* and $T^* \leq F^{\frac{\sigma}{1-\sigma}}(0) \frac{M_2}{M_1} \frac{1-\sigma}{\sigma}$.

In what follows, we will prove $\lim_{t\to T^*} F(t) \to +\infty \Longrightarrow \lim_{t\to T^*} \|u\|_{p+1}^{p+1} = +\infty$. Let us consider the following two cases based on the definition of $F(t)$ as $t \to T^*$:
Case 1: $H(t) \to +\infty$. In this case, Lemma 3.2 yields $\|u\|_{p+1}^{p+1} \to +\infty$.
Case 2: $\int_\Omega u_t u dx \to +\infty$. Cauchy's inequality and Lemma 2.1(1) illustrate

$$\int_\Omega u_t u dx \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\|u\|_2^2 \leq \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2} c \|u\|_{X_0}^2. \tag{3.13}$$

Recalling (2.2) and $E(t) \leq E(0)$, we have

$$\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2\theta}\|u\|_{X_0}^{2\theta} = E(t) + \frac{b}{p+1}\|u\|_{p+1}^{p+1} \leq E(0) + \frac{b}{p+1}\|u\|_{p+1}^{p+1} \tag{3.14}$$

It is direct by combining (3.13) with (3.14) that if there exists $\int_\Omega u_t u dx \to +\infty$, we obtain $\|u\|_{p+1}^{p+1} \to +\infty$. This completes the proof of this theorem. $\square$

4 Blow-up with high initial energy

In Section 3, we show a blow-up result with a bounded initial energy. The aim of this section is to give a blow-up criterion with high initial energy for $m \equiv 1$. Before moving further, let us first give a lemma, which comes from soma ideas in [7, 15, 22].

Lemma 4.1. *Let* (**H**) *hold. If u is a weak solution of problem* (1.1)*, there exists the following inequality:*

$$\begin{aligned}&\frac{d}{dt}\Big[\int_\Omega uu_t dx - \frac{p+1}{M_0}E(t) - \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0}\Big]\\&\quad\geq M_0\left[\int_\Omega uu_t dx - \frac{p+1}{M_0}E(t) - \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0}\right].\end{aligned}\tag{4.1}$$

where $M_0 = \dfrac{2(p+1-2\theta)(p+1)}{\tilde{c}^\theta[2a+4(p+1)]} > 0,\ \tilde{c}^\theta = \max\left\{1, c^\theta, \frac{4(p+1-2\theta)(p+1)}{(p+3)[2a+4(p+1)]}\right\}$.

Proof. Define an auxiliary function $G(t) = \int_\Omega uu_t dx$. The first equality in problem (1.1), (2.2) and Cauchy's inequality with $\delta > 0$ yield

$$\begin{aligned}G'(t) &= \|u_t\|_2^2 + \int_\Omega uu_{tt}dx = \|u_t\|_2^2 - \|u\|_{X_0}^{2\theta} - a\int_\Omega uu_t dx + b\|u\|_{p+1}^{p+1}\\&\geq \frac{p+3}{2}\|u_t\|^2 + \frac{p+1-2\theta}{2\theta}\|u\|_{X_0}^{2\theta} - \frac{\delta a}{2}\|u\|_2^2 - \frac{a}{2\delta}\|u_t\|_2^2 - (p+1)E(t).\end{aligned}$$

By choosing $\delta = \dfrac{p+1-2\theta}{\tilde{c}^\theta[2a+4(p+1)]}$ and combining Lemma 2.2, we have

$$\begin{aligned}&\frac{d}{dt}\Big[G(t) - \frac{p+1}{M_0}E(t)\Big]\\&\geq \frac{p+3}{2}\|u_t\|_2^2 + \frac{p+1-2\theta}{2\theta}\|u\|_{X_0}^{2\theta} - \frac{\delta a}{2}\|u\|_2^2 + \Big(\frac{(p+1)a}{M_0} - \frac{a}{2\delta}\Big)\|u_t\|_2^2 - (p+1)E(t)\\&= \frac{p+3}{2}\|u_t\|_2^2 + \frac{p+1-2\theta}{2\theta}\|u\|_{X_0}^{2\theta} - \frac{\delta a}{2}\|u\|_2^2 - (p+1)E(t).\end{aligned}\tag{4.2}$$

Moreover, Lemma 2.1(1) and Cauchy's inequality yield

$$\frac{p+1-2\theta}{2\theta}\|u\|_{X_0}^{2\theta} \geq \frac{p+1-2\theta}{2\theta\tilde{c}^\theta}\|u\|_2^{2\theta} \geq \frac{p+1-2\theta}{2\theta\tilde{c}^\theta}[\theta\|u\|_2^2 - (\theta-1)].\tag{4.3}$$

Substituting (4.3) into (4.2) yields

$$\begin{aligned}&\frac{d}{dt}\Big[G(t) - \frac{p+1}{M_0}E(t) - \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0}\Big]\\&\geq M_0\left(\|u_t\|_2^2 - \frac{p+1}{M_0}E(t) - \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0} + \|u\|_2^2\right)\\&\geq M_0\left(G(t) - \frac{p+1}{M_0}E(t) - \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0}\right).\end{aligned}\tag{4.4}$$

(4.1) obviously follows from (4.4). $\square$

Lemma 4.2. [11, 14] *Suppose a positive, twice-differentiable function $\psi(t)$ satisfies the inequality*

$$\psi''(t)\psi(t) - (1+\theta)(\psi'(t))^2 \geq 0,$$

where $\theta > 0$. If $\psi(0) > 0,\ \psi'(0) > 0$, then $\psi(t) \to \infty$ as $t \to t_1 \leq t_2 = \frac{\psi(0)}{\theta\psi'(0)}$.

Based on the two lemmas above, we give our result as follows:

Theorem 4.1. *Let* (**H**) *hold. Assume that the following conditions are fulfilled:*

$$(\mathbf{H_1}) \quad \int_\Omega u_0 u_1 dx > \frac{p+1}{M_0}E(0) + \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0};$$

$$(\mathbf{H_2}) \quad \frac{p+1-2\theta}{2\theta\tilde{c}^\theta(p+1)}[\theta\|u_0\|_2^2 - (\theta-1)] > E(0),$$

then any weak solution u blows up at finite time T^*. And T_{max} can be estimate from above as follows:

$$T^* \leq \frac{2(\|u_0\|_2^2 + \rho\sigma^2)}{(p-1)[\int_\Omega u_1 u_0 dx + \rho\sigma] - 2\|u_0\|_2^2}.$$

with $\rho = \frac{p+1-2\theta}{\theta\tilde{c}^\theta(p+1)}[\theta\|u_0\|_2^2 - (\theta-1)] - 2E(0) > 0$ and $\sigma > \max\Big\{0, \frac{2\|u_0\|_2^2}{(p-1)\rho} - \frac{\int_\Omega u_1 u_0 dx}{\rho}\Big\}$.

Proof. In what follows, the proof will be divided into two steps.

Step 1: Finite time blow-up Suppose, on the contrary, that the solutions u exist globally, i.e. $T_{max} = \infty$. Using Hölder's inequality and Lemma 2.2, then for all $t \in [0, \infty)$,

$$\begin{aligned} \|u(t)\|_2 &= \Big\|u_0 + \int_0^t u_\tau d\tau\Big\|_2 \leq \|u_0\|_2 + \int_0^t \|u_\tau(\tau)\|_2 d\tau \\ &\leq \|u_0\|_2 + \sqrt{t}\Big(\int_0^t \|u_\tau(\tau)\|_2^2 d\tau\Big)^{\frac{1}{2}} \\ &= \|u_0\|_2 + \sqrt{\frac{t}{a}}(E(0) - E(t))^{\frac{1}{2}}. \end{aligned} \tag{4.5}$$

Since u is a global solution of problem (1.1), we have $E(t) \geq 0$ for all $t \in [0, \infty)$. Otherwise, there exists $t_0 \in [0, \infty)$ such that $E(t_0) < 0$. Choosing $u(x, t_0)$ as the new initial data, then it is clear from Theorem 3.1 that the solution blows up in finite time, which is a contradiction. It follows from Lemma 2.2 that $0 \leq E(t) \leq E(0)$. Thus, (4.5) can be rewritten as

$$\|u(t)\|_2 \leq \|u_0\|_2 + \sqrt{\frac{t}{a}}(E(0))^{\frac{1}{2}} \tag{4.6}$$

for all $t \in [0, \infty)$.

Let

$$\mathcal{F}(t) = \int_\Omega u u_t dx - \frac{p+1}{M_0}E(t) - \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0},$$

then it follows from Lemma 4.1 and $(\mathbf{H_1})$ that

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_2^2 = 2\int_\Omega u u_t dx &\geq 2\Big[\mathcal{F}(0)e^{M_0 t} + \frac{p+1}{M_0}E(t) + \frac{(p+1-2\theta)(\theta-1)}{2\theta\tilde{c}^\theta M_0}\Big] \\ &\geq 2\mathcal{F}(0)e^{M_0 t} > 0. \end{aligned} \tag{4.7}$$

Integrating (4.7) from 0 to t yields

$$\begin{aligned} \|u(t)\|_2^2 &= \|u_0\|_2^2 + 2\int_0^t \int_\Omega u u_\tau dx d\tau \\ &\geq \|u_0\|_2^2 + 2\int_0^t \mathcal{F}(0) e^{M_0 \tau} d\tau \\ &= \|u_0\|_2^2 + \frac{2}{M_0}\mathcal{F}(0)(e^{M_0 t} - 1), \end{aligned} \tag{4.8}$$

which contradicts (4.6) for t sufficiently large. Thus, the solution u for problem (1.1) blows up in finite time.

Step 2: An upper bound of the blow-up time. In what follows, we are devoted to get the upper bound of the blow-up time. We still denote by T^* the maximal existence time, and $T^* < \infty$ due to step 1. For any $T < T^*$, define an auxiliary function

$$\Psi(t) = \|u(t)\|_2^2 + a\int_0^t \|u(\tau)\|_2^2 d\tau + a(T^* - t)\|u_0\|_2^2 + \rho(t+\sigma)^2 \quad \text{for } t \in [0, T]. \tag{4.9}$$

It follows from a direct computation that

$$\begin{aligned} \Psi'(t) &= 2\int_\Omega u_t u dx + a\|u(t)\|_2^2 - a\|u_0\|_2^2 + 2\rho(t+\sigma) \\ &= 2\int_\Omega u_t u dx + 2a\int_0^t \int_\Omega u_\tau u dx d\tau + 2\rho(t+\sigma) \quad \text{for } t \in [0, T]. \end{aligned}$$

Let us combine problem (1.1) and the above one, then

$$\begin{aligned} \Psi''(t) &= 2\frac{d}{dt}\int_\Omega u_t u dx + 2a\int_\Omega u_t u dx + 2\rho \\ &= 2\|u_t(t)\|_2^2 - 2\|u\|_{X_0}^{2\theta} + 2b\|u\|_{p+1}^{p+1} + 2\rho \quad \text{for } t \in [0, T]. \end{aligned}$$

By making full use of Hölder's inequality and Cauchy's inequality, one has

$$\begin{aligned} \xi(t) : &= \Big[\|u(t)\|_2^2 + a\int_0^t \|u(\tau)\|_2^2 d\tau + \rho(t+\sigma)^2\Big]\Big[\|u_t(t)\|_2^2 + a\int_0^t \|u_\tau(\tau)\|_2^2 d\tau + \rho\Big] \\ &\quad - \Big[\int_\Omega u_t u dx + a\int_0^t \int_\Omega u_\tau u dx d\tau + \rho(t+\sigma)\Big]^2 \\ &= \|u(t)\|_2^2 \|u_t(t)\|_2^2 - \Big(\int_\Omega u_t u dx\Big)^2 \\ &\quad + \rho\|u(t)\|_2^2 + \rho(t+\sigma)^2\|u_t(t)\|_2^2 - 2\rho(t+\sigma)\int_\Omega u_t u dx \\ &\quad + \rho a\int_0^t \|u(\tau)\|_2^2 d\tau + \rho(t+\sigma)^2 a\int_0^t \|u_\tau(\tau)\|_2^2 d\tau - 2\rho(t+\sigma)a\int_0^t \int_\Omega u_\tau u dx d\tau \\ &\quad + a^2\int_0^t \|u(\tau)\|_2^2 d\tau \int_0^t \|u_\tau(\tau)\|_2^2 d\tau - \Big(a\int_0^t \int_\Omega u_\tau u dx d\tau\Big)^2 \\ &\quad + a\|u(t)\|_2^2 \int_0^t \|u_\tau(\tau)\|_2^2 d\tau + a\|u_t(t)\|_2^2 \int_0^t \|u(\tau)\|_2^2 d\tau \\ &\quad - 2a\int_\Omega u_t u dx \int_0^t \int_\Omega u_\tau u dx d\tau \geq 0 \quad \text{for } t \in [0, T]. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\Psi(t)\Psi''(t)-\frac{p+3}{4}(\Psi'(t))^2\\
&=\Psi(t)\Psi''(t)-(p+3)\Big[\int_\Omega u_t u dx + a\int_0^t\int_\Omega u_\tau u dx d\tau+\rho(t+\sigma)\Big]^2\\
&=\Psi(t)\Psi''(t)+(p+3)\Big\{-\Big[\|u(t)\|_2^2+a\int_0^t\|u(\tau)\|_2^2d\tau+\rho(t+\sigma)^2\Big]\\
&\quad\times\Big[\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big]+\xi(t)\Big\}\\
&=\Psi(t)\Psi''(t)+(p+3)\Big\{-\Big[\Psi(t)-a(T^*-t)\|u_0\|_2^2\Big]\\
&\quad\times\Big[\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big]\Big\}+(p+3)\xi(t)\\
&=\Psi(t)\Psi''(t)-(p+3)\Psi(t)\Big(\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big)+(p+3)\xi(t)\\
&\quad+(p+3)(T^*-t)\|u_0\|_2^2\Big[\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big]\\
&\geq\Psi(t)\eta(t)\quad\text{for } t\in[0,T],
\end{aligned}\tag{4.10}$$

where

$$\eta(t):=\Psi''(t)-(p+3)\Big(\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big).$$

It follows from the definition of $E(t)$ in (2.2) and (4.3) that

$$\begin{aligned}
\eta(t)&\geq 2\|u_t(t)\|_2^2-2\|u\|_{X_0}^{2\theta}+2b\|u\|_{p+1}^{p+1}+2\rho-(p+3)\Big(\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big)\\
&\geq 2\|u_t(t)\|_2^2-2\|u\|_{X_0}^{2\theta}+2\Big[\frac{p+1}{2}\|u_t\|_2^2+\frac{p+1}{2\theta}\|u\|_{X_0}^{2\theta}-(p+1)E(t)\Big]+2\rho\\
&\quad-(p+3)\Big(\|u_t(t)\|_2^2+a\int_0^t\|u_\tau(\tau)\|_2^2d\tau+\rho\Big)\\
&=\frac{p+1-2\theta}{\theta}\|u(t)\|_{X_0}^{2\theta}-2(p+1)E(t)-a(p+3)\int_0^t\|u_\tau(\tau)\|_2^2d\tau-(p+1)\rho\\
&\geq\frac{p+1-2\theta}{\theta\tilde{c}^\theta}[\theta\|u\|_2^2-(\theta-1)]-2(p+1)E(0)+a(p-1)\int_0^t\|u_\tau(\tau)\|_2^2d\tau-(p+1)\rho\\
&\geq\frac{p+1-2\theta}{\theta\tilde{c}^\theta}[\theta\|u_0\|_2^2-(\theta-1)]-2(p+1)E(0)-(p+1)\rho\quad\text{for } t\in[0,T].
\end{aligned}\tag{4.11}$$

Here we have used $\|u(t)\|_2^2\geq\|u_0\|_2^2$ obtained by (4.7). By recalling

$$\rho=\frac{p+1-2\theta}{\theta\tilde{c}^\theta(p+1)}[\theta\|u_0\|_2^2-(\theta-1)]-2E(0),$$

then we directly obtain $\eta(t)\geq 0$. Therefore, it follows from (4.10) that

$$\Psi(t)\Psi''(t)-\frac{p+3}{4}(\Psi'(t))^2\geq 0\quad\text{for } t\in[0,T].$$

It is noted that

$$\Psi(0)=\|u_0\|_2^2+T^*\|u_0\|_2^2+\rho\sigma^2>0,$$

$$\Psi'(0) = 2\int_\Omega u_1u_0dx + 2\rho\sigma > 0,$$

and

$$T \le \frac{2(\|u_0\|_2^2 + T^*\|u_0\|_2^2 + \rho\sigma^2)}{(p-1)[\int_\Omega u_1u_0dx + \rho\sigma]}.$$

Here we have used Lemma 4.2. Since the arbitrariness of $T < T^*$, we obtain

$$T^* \le \frac{2(\|u_0\|_2^2 + T^*\|u_0\|_2^2 + \rho\sigma^2)}{(p-1)[\int_\Omega u_1u_0dx + \rho\sigma]},$$

or equivalently

$$T^* \le \frac{2(\|u_0\|_2^2 + \rho\sigma^2)}{(p-1)[\int_\Omega u_1u_0dx + \rho\sigma] - 2\|u_0\|_2^2}. \tag{4.12}$$

□

Remark 4.1. For $m > 1$, we find this method is not applicable. It is necessary to propose a new method to discuss the blow-up phenomenon with high initial energy.

5 Lower bounds for blow-up time

In what follows, we will discuss lower bounds for blow-up time in this section based on some ideas in [5, 6, 32]. The following is the main result.

Theorem 5.1. *If all conditions of Theorem 3.1(or Theorem 4.1) hold, the blow-up time T^* satisfies the following estimate:*

(1) for $p \in (2\theta - 1, \frac{2_s^*}{2}]$,

$$\int_{\Phi(0)}^{+\infty} \frac{1}{B_1y^{\frac{p}{\theta}} + B_2y + B_3}dy \le T^*, \tag{5.1}$$

where $\Phi(0) = \|u_0\|_{p+1}^{p+1},\ B_1,\ B_2,\ B_3$ will be discussed later.

(2) for $p \in \left[\frac{2_s^*}{2}, \frac{2_s^*}{2} - 2\theta\left(\frac{1}{2_s^*} - \frac{1}{2}\right)\right)$,

$$\int_{\Psi(0)}^{+\infty} \frac{1}{B_8y^{\eta}(t) + B_9y^{\zeta}(t) + B_{10}}dy \le T^*, \tag{5.2}$$

where $\Psi(0) = \|u_0\|_{\frac{2_s^*}{2}+1}^{\frac{2_s^*}{2}+1},\ B_8,\ B_9,\ B_{10},\ \eta,\ \zeta$ will be discussed later.

Proof. (1) Denote by $\Phi(t)$ the norm $\|u\|_{p+1}^{p+1}$ for simplicity. It is direct that

$$\Phi'(t) = (p+1)\int_\Omega |u|^{p-1}uu_tdx \le \frac{p+1}{2}\left(\|u\|_{2p}^{2p} + \|u_t\|_2^2\right) \le \frac{p+1}{2}\left(c^p\|u\|_{X_0}^{2p} + \|u_t\|_2^2\right) \tag{5.3}$$

due to Cauchy's inequality. Recalling (3.14), we directly get

$$\begin{aligned}\Phi'(t) &\le \frac{p+1}{2}\Big[c^p(2\theta)^{\frac{p}{\theta}}\Big(E(0) + \frac{b}{p+1}\|u\|_{p+1}^{p+1}\Big)^{\frac{p}{\theta}} + 2\Big(E(0) + \frac{b}{p+1}\|u\|_{p+1}^{p+1}\Big)\Big] \\ &\le B_1\Phi^{\frac{p}{\theta}}(t) + B_2\Phi(t) + B_3,\end{aligned} \tag{5.4}$$

where

$$B_1=\frac{p+1}{2}c^p2^{\frac{p-\theta}{\theta}}(2\theta)^{\frac{p}{\theta}}\Big(\frac{b}{p+1}\Big)^{\frac{p}{\theta}},\ B_2=b,\ B_3=\frac{p+1}{2}c^p2^{\frac{p-\theta}{\theta}}(2\theta)^{\frac{p}{\theta}}(E(0))^{\frac{p}{\theta}}+(p+1)E(0).$$

Inequality (5.4) yields (5.1) due to Theorem 3.1 or Theorem 4.1.

(2) Define $\Psi(t)=\|u\|_\delta^\delta$ with $\delta=\frac{2_s^*}{2}+1$. Clearly,

$$\Psi'(t)\le\frac{\delta}{2}\Big(\|u\|_{2(\delta-1)}^{2(\delta-1)}+\|u_t\|_2^2\Big)\le\frac{\delta}{2}\Big(c^{\frac{2_s^*}{2}}\|u\|_{X_0}^{2(\delta-1)}+\|u_t\|_2^2\Big)\tag{5.5}$$

since $2(\delta-1)=2_s^*$. (3.14) implies

$$\begin{aligned}\Psi'(t)&\le\frac{\delta}{2}\Big[c^{\frac{2_s^*}{2}}(2\theta)^{\frac{\delta-1}{\theta}}\Big(E(0)+\frac{b}{p+1}\|u\|_{p+1}^{p+1}\Big)^{\frac{\delta-1}{\theta}}+2\Big(E(0)+\frac{b}{p+1}\|u\|_{p+1}^{p+1}\Big)\Big]\\&\le B_3\Phi^{\frac{\delta-1}{\theta}}(t)+B_4\Phi(t)+B_5,\end{aligned}\tag{5.6}$$

where

$$B_3=\frac{\delta}{2}c^{\frac{2_s^*}{2}}2^{\frac{\delta-1-\theta}{\theta}}(2\theta)^{\frac{\delta-1}{\theta}}\Big(\frac{b}{p+1}\Big)^{\frac{\delta-1}{\theta}},\ B_4=\frac{b\delta}{p+1},\ B_5=\frac{\delta}{2}c^{\frac{2_s^*}{2}}2^{\frac{\delta-1-\theta}{\theta}}(2\theta)^{\frac{\delta-1}{\theta}}(E(0))^{\frac{\delta-1}{\theta}}+\delta E(0).$$

By the interpolation inequality, we get

$$\Phi(t)\le\|u\|_\delta^{(1-\gamma)(p+1)}\|u\|_{2_s^*}^{\gamma(p+1)}\le c^{\frac{\gamma(p+1)}{2}}\|u\|_\delta^{(1-\gamma)p+1}\|u\|_{X_0}^{\gamma(p+1)},\tag{5.7}$$

where $\frac{1}{p+1}=\frac{1-\gamma}{\delta}+\frac{\gamma}{2_s^*}$. Noticing that $\frac{2\theta}{\gamma(p+1)}>1$, and then using Young's inequality with $\varepsilon>0$, (5.7) can be written as

$$\begin{aligned}\Phi(t)&\le c^{\frac{\gamma(p+1)}{2}}\Big(\frac{2\theta-\gamma(p+1)}{2\theta}\varepsilon^{-\frac{2\theta}{2\theta-\gamma(p+1)}}\|u\|_\delta^{\frac{2\theta(1-\gamma)(p+1)}{2\theta-\gamma(p+1)}}+\frac{\gamma(p+1)}{2\theta}\varepsilon^{\frac{2\theta}{\gamma(p+1)}}\|u\|_{X_0}^{2\theta}\Big)\\&\le c^{\frac{\gamma(p+1)}{2}}\frac{2\theta-\gamma(p+1)}{2\theta}\varepsilon^{-\frac{2\theta}{2\theta-\gamma(p+1)}}\Psi^{\frac{2\theta(1-\gamma)(p+1)}{\delta[2\theta-\gamma(p+1)]}}(t)+c^{\frac{\gamma(p+1)}{2}}\gamma(p+1)\varepsilon^{\frac{2\theta}{\gamma(p+1)}}\Big(E(0)+\frac{b\Phi(t)}{p+1}\Big).\end{aligned}$$

Let us now choose ε such that $1-c^{\frac{\gamma(p+1)}{2}}\gamma\varepsilon^{\frac{2\theta}{\gamma(p+1)}}b>0$, it follows form the inequality above

$$\Phi(t)\le B_6\Psi^{\frac{2\theta(1-\gamma)(p+1)}{\delta[2\theta-\gamma(p+1)]}}(t)+B_7\tag{5.8}$$

where

$$B_6=\frac{c^{\frac{\gamma(p+1)}{2}}\frac{2\theta-\gamma(p+1)}{2\theta}\varepsilon^{-\frac{2\theta}{2\theta-\gamma(p+1)}}}{1-c^{\frac{\gamma(p+1)}{2}}\gamma\varepsilon^{\frac{2\theta}{\gamma(p+1)}}b},\ B_7=\frac{c^{\frac{\gamma(p+1)}{2}}\gamma(p+1)\varepsilon^{\frac{2\theta}{\gamma(p+1)}}E(0)}{1-c^{\frac{\gamma(p+1)}{2}}\gamma\varepsilon^{\frac{2\theta}{\gamma(p+1)}}b},$$

which implies $\lim_{t\to T^*}\Psi(t)=+\infty$ due to Theorem 3.1 or Theorem 4.1.

Inserting (5.8) into (5.6) is to get

$$\Psi'(t)\le B_8\Psi^\eta(t)+B_9\Psi^\zeta(t)+B_{10},\tag{5.9}$$

where

$$\eta=\frac{2\theta(1-\gamma)(p+1)}{\delta[2\theta-\gamma(p+1)]}\frac{\delta-1}{\theta},\ \zeta=\frac{2\theta(1-\gamma)(p+1)}{\delta[2\theta-\gamma(p+1)]},$$

$$B_8=B_3B_6^{\frac{\delta-1}{\theta}}2^{\frac{\delta-1}{\theta}-1},\ B_9=B_4B_6,\ B_{10}=B_32^{\frac{\delta-1}{\theta}-1}B_7^{\frac{\delta-1}{\theta}}+B_4B_7+B_5.$$

Clearly, (5.2) follows (5.9). $\square$

Remark 5.1. For $a \equiv 0$, i.e. in the absence of the weak damping, Theorem 5.1 still holds. And this result improves that one in Theorem 4.3 of [21], which only discussed the lower bound of the blow-up time for $p \in (2\theta - 1, \frac{2_s^*}{2}]$.

Remark 5.2. For $p \in \left[\frac{2_s^*}{2} - 2\theta\left(\frac{1}{2_s^*} - \frac{1}{2}\right), 2_s^* - 1\right]$, we do not discuss the corresponding lower bound of the blow-up time.

6 Energy decay estimates

In this section, we show the global existence of the solution and the following decay estimate:

Theorem 6.1. *Let $2\theta - 1 < m < p$ and $(\mathbf{H})$ hold. Supposed that*

$$0 < E(0) < E_1, \ \|u_0\|_{X_0}^{2\theta} < \alpha_1,$$

then the solution of problem (1.1) exists globally. Moreover, the energy functional satisfies the following estimate

$$E(t) \leq E(0)\Big[\frac{K(m+1)}{2K + (m-1)t}\Big]^{\frac{2}{m-1}} \quad \text{for all } t \geq 0, \tag{6.1}$$

where the constant K may be defined in (6.13).

Let us give some lemmas to prove Theorem 6.1.

Lemma 6.1. [23]*Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing function of class C^1such that*

$$\phi(0) = 0 \text{ and } \phi(t) \to +\infty \text{ as } t \to +\infty.$$

Assume that there exist $\sigma \geq 0$, and $\omega > 0$ such that

$$\forall s \geq 0, \int_s^{+\infty} E(t)^{1+\sigma}\phi'(t)dt \leq \frac{1}{\omega}E^\sigma(0)E(s).$$

Then $E(t)$ has the following decay property:

$$\text{if } \sigma = 0, \text{ then } E(t) \leq E(0)e^{1-\omega\phi(t)}, \quad \forall t \geq 0,$$
$$\text{if } \sigma > 0, \text{ then } E(t) \leq E(0)\left(\frac{1+\sigma}{1+\omega\sigma\phi(t)}\right)^{\frac{1}{\sigma}}, \quad \forall t \geq 0.$$

Lemma 6.2. *Let all conditions of Theorem 6.1 hold, then there exists a positive constant $0 < \tilde{\alpha_2} < \alpha_1$ such that*

$$\|u\|_{X_0}^{2\theta} \leq \tilde{\alpha_2}, \quad \forall t \geq 0. \tag{6.2}$$

Proof. With the proof of Lemma 3.1 in mind. This proof is clear. Since $E(0) < E_1$, there exists an $\tilde{\alpha_2} < \alpha_1$ such that $h(\tilde{\alpha_2}) = E(0)$. Recall (3.2), then $h(\alpha(0)) \leq E(0) = h(\tilde{\alpha_2})$, which implies $\alpha(0) \leq \tilde{\alpha_2}$ since the condition $\alpha(0) < \alpha_1$. To prove (6.2), we suppose by contradiction that for some $t_0 > 0$, $\alpha(t_0) > \tilde{\alpha_2}$. The continuity of $\alpha(t)$ illustrates that we could choose t_0 such that $\alpha_1 > \alpha(t_0) > \tilde{\alpha_2}$, then we have

$$E(0) = h(\tilde{\alpha_2}) < h(\alpha(t_0)) \leq E(t_0).$$

This is a contradiction since Lemma 2.2. $\square$

Lemma 6.3. *Let all conditions of Theorem 6.1 hold. The following inequalities hold:*

$$\|u\|_{p+1}^{p+1} \le \frac{2\theta c^{\frac{p+1}{2}} \tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}}{1-\frac{b}{p+1}2\theta c^{\frac{p+1}{2}} \tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}} E(t), \quad \forall t \ge 0. \tag{6.3}$$

$$\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2\theta}\|u\|_{X_0}^{2\theta} \le \frac{p+1}{p+1-2\theta}E(t) \le \frac{p+1}{p+1-2\theta}E(0), \quad \forall t \ge 0. \tag{6.4}$$

Proof. Lemma 2.1(1), (2.2) and (6.2) implies

$$\begin{aligned}\|u\|_{p+1}^{p+1} &\le c^{\frac{p+1}{2}}\|u\|_{X_0}^{p+1} \le 2\theta c^{\frac{p+1}{2}}\|u\|_{X_0}^{p+1-2\theta}\Big(E(t)+\frac{b}{p+1}\|u\|_{p+1}^{p+1}\Big)\\ &\le 2\theta c^{\frac{p+1}{2}}\tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}\Big(E(t)+\frac{b}{p+1}\|u\|_{p+1}^{p+1}\Big)\end{aligned}$$

The above inequality directly implies (6.3). And combining (6.3) with (2.2), we have (6.4). □

Remark 6.1. Lemma 6.3 yields that the solutions of problem (1.1) exist globally.

Proof of Theorem 6.1: Some ideas in [19, 20] are used here. Multiplying the first identity of problem (1.1) by $E^{\beta}(t)u(\beta > 0)$ and integrating over $\Omega \times (s,T)(s<T)$ give

$$\begin{aligned}&\int_s^T E^{\beta}(t)\frac{d}{dt}\int_\Omega uu_t dxdt + \int_s^T E^{\beta}(t)\|u\|_{X_0}^{2\theta}dt\\ &\quad + \int_s^T E^{\beta}(t)\Big(-\|u_t\|_2^2 - b\|u\|_{p+1}^{p+1} + a\int_\Omega |u_t|^{m-1}u_t u dx\Big)dt = 0.\end{aligned} \tag{6.5}$$

Inserting (2.2) into (6.5) yields

$$\begin{aligned}2\theta\int_s^T E^{\beta+1}(t)dt &\le -\int_s^T \frac{d}{dt}\Big[E^{\beta}(t)\int_\Omega uu_t dx\Big]dt + \beta\int_s^T E^{\beta-1}(t)E'(t)\int_\Omega uu_t dxdt\\ &\quad + a\int_s^T E^{\beta}(t)\int_\Omega |u_t|^{m-1}u_t u dxdt + \frac{2\theta+2}{2}\int_s^T E^{\beta}(t)\|u_t\|_2^2 dt\\ &\quad + b\Big(1-\frac{2\theta}{p+1}\Big)\int_s^T E^{\beta}(t)\|u\|_{p+1}^{p+1}dt := J_1+J_2+J_3+J_4+J_5.\end{aligned} \tag{6.6}$$

It follows from Young's inequality and (6.3) and Lemma 2.2 that

$$\begin{aligned}|J_1| &= \Big|E^{\beta}(s)\int_\Omega uu_t(\cdot,s)dx - E^{\beta}(T)\int_\Omega uu_t(\cdot,T)dx\Big|\\ &\le \frac{E^{\beta}(s)}{2}\Big(\|u(\cdot,s)\|_2^2 + \|u_t(\cdot,s)\|_2^2 + \|u(\cdot,T)\|_2^2 + \|u_t(\cdot,T)\|_2^2\Big)\\ &\le \frac{E^{\beta}(s)}{2}\Big[c\Big(\|u(\cdot,s)\|_{X_0}^2 + \|u(\cdot,T)\|_{X_0}^2\Big) + \|u_t(\cdot,s)\|_2^2 + \|u_t(\cdot,T)\|_2^2\Big]\\ &\le \Big[c\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}} + \frac{2(p+1)}{p+1-2\theta}\Big]E^{\beta+\frac{1}{\theta}-1}(0)E(s);\end{aligned} \tag{6.7}$$

$$
\begin{aligned}
|J_2| &\le -\frac{\beta}{2}\int_s^T E^{\beta-1}(t)E'(t)(\|u\|_2^2+\|u_t\|_2^2)dt \le -\frac{\beta}{2}\int_s^T E^{\beta-1}(t)E'(t)(c\|u\|_{X_0}^2+\|u_t\|_2^2)dt \\
&\le -\frac{\beta c}{2}\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}}\int_s^T E^{\beta+\frac{1}{\theta}-1}(t)E'(t)dt - \frac{\beta(p+1)}{p+1-2\theta}\int_s^T E^{\beta}(t)E'(t)dt \\
&\le -\frac{\beta c}{2}\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}}\frac{\theta}{\theta\beta+1}\Big[E^{\beta+\frac{1}{\theta}}(s)-E^{\beta+\frac{1}{\theta}}(T)\Big] \\
&\quad + \frac{\beta(p+1)}{(p+1-2\theta)(\beta+1)}\Big[E^{\beta+1}(s)-E^{\beta+1}(T)\Big] \\
&\le \Big[\frac{\beta c}{2}\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}}\frac{\theta}{\theta\beta+1}E^{\beta+\frac{1}{\theta}-1}(0) + \frac{\beta(p+1)}{(p+1-2\theta)(\beta+1)}E^{\beta}(0)\Big]E(s).
\end{aligned}
\tag{6.8}
$$

Using Young's inequality with $0<\varepsilon<1$, Lemma 2.1(1) and (6.3), we obtain

$$
\begin{aligned}
|J_3| &\le a\int_s^T E^{\beta}(t)\Big(\frac{m}{m+1}\varepsilon^{-\frac{m+1}{m}}\|u_t\|_{m+1}^{m+1} + \frac{1}{m+1}\varepsilon^{m+1}\|u\|_{m+1}^{m+1}\Big)dt \\
&\le \frac{m}{m+1}\frac{\varepsilon^{-\frac{m+1}{m}}}{\beta+1}\Big[E^{\beta+1}(s)-E^{\beta+1}(T)\Big] + \frac{a\varepsilon^{m+1}}{m+1}c^{\frac{m+1}{2}}\int_s^T E^{\beta}(t)\|u\|_{X_0}^{m+1}dt \\
&\le \frac{m}{m+1}\frac{\varepsilon^{-\frac{m+1}{m}}}{\beta+1}E^{\beta}(0)E(s) + \frac{a\varepsilon^{m+1}}{m+1}\Big[\frac{2\theta(p+1)}{b(p+1-2\theta)}\Big]^{\frac{m+1}{2\theta}}E^{\frac{m+1}{2\theta}-1}(0)\int_s^T E^{\beta+1}(t)dt.
\end{aligned}
\tag{6.9}
$$

For $m+1>2$, one may apply Hölder's inequality and Young's inequality with $\varepsilon_1>0$ to establish the following inequalities

$$
\begin{aligned}
|J_4| &\le (1+|\Omega|)^2\int_s^T E^{\beta}(t)\|u_t\|_{m+1}^2dt \\
&\le \varepsilon_1^{\frac{m+1}{m-1}}\frac{m-1}{m+1}(1+|\Omega|)^2\int_s^T E^{\frac{(m+1)\beta}{m-1}}(t)dt + \frac{2}{m+1}\varepsilon_1^{-\frac{m+1}{2}}(1+|\Omega|)^2\int_s^T \|u_t\|_{m+1}^{m+1}dt \\
&\le \varepsilon_1^{\frac{m+1}{m-1}}\frac{m-1}{m+1}(1+|\Omega|)^2\int_s^T E^{\beta+1}(t)dt - \frac{2}{(m+1)a}\varepsilon_1^{-\frac{m+1}{2}}(1+|\Omega|)^2\int_s^T E'(t)dt \\
&\le \varepsilon_1^{\frac{m+1}{m-1}}\frac{m-1}{m+1}(1+|\Omega|)^2\int_s^T E^{\beta+1}(t)dt + \frac{2}{(m+1)a}\varepsilon_1^{-\frac{m+1}{2}}(1+|\Omega|)^2E(s),
\end{aligned}
\tag{6.10}
$$

where $\frac{(m+1)\beta}{m-1}=\beta+1$. (6.3) directly implies

$$
|J_5| \le b\Big(1-\frac{2\theta}{p+1}\Big)\frac{2\theta c^{\frac{p+1}{2}}\tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}}{1-\frac{b}{p+1}2\theta c^{\frac{p+1}{2}}\tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}}\int_s^T E^{\beta+1}(t)dt.
\tag{6.11}
$$

Obviously,

$$
\omega := b\Big(1-\frac{2\theta}{p+1}\Big)\frac{2\theta c^{\frac{p+1}{2}}\tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}}{1-\frac{b}{p+1}2\theta c^{\frac{p+1}{2}}\tilde{\alpha_2}^{\frac{p+1-2\theta}{2\theta}}} < 2\theta
$$

We choose $\varepsilon,\varepsilon_1$ satisfying

$$
\frac{a\varepsilon^{m+1}}{m+1}\Big[\frac{2\theta(p+1)}{b(p+1-2\theta)}\Big]^{\frac{m+1}{2\theta}}E^{\frac{m+1}{2\theta}-1}(0) = \frac{p+1-2\theta}{4},\quad \varepsilon_1^{\frac{m+1}{m-1}}\frac{m-1}{m+1}(1+|\Omega|)^2 = \frac{p+1-2\theta}{4},
$$

and utilizing (6.6) − (6.11), we arrive at the following inequalities

$$\begin{aligned}\int_s^T E^{\beta+1}(t)dt \leq & \frac{2}{p+1-2\theta}\Big[c\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}} + \frac{2(p+1)}{p+1-2\theta}\Big]E^{\beta+\frac{1}{\theta}-1}(0)E(s)\\ & + \frac{2}{p+1-2\theta}\Big[\frac{\beta c}{2}\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}}\frac{\theta E^{\beta+\frac{1}{\theta}-1}(0)}{\theta\beta+1} + \frac{\beta(p+1)E^{\beta}(0)}{(p+1-2\theta)(\beta+1)}\Big]E(s)\\ & + \frac{2}{p+1-2\theta}\Big[\frac{m}{m+1}\frac{\varepsilon^{-\frac{m+1}{m}}}{\beta+1}E^{\beta}(0)E(s) + \varepsilon_1^{\frac{m+1}{m-1}}\frac{m-1}{m+1}(1+|\Omega|)^2E(s)\Big]\\ & := KE^{\beta}(0)E(s),\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}K = & \frac{2}{p+1-2\theta}\Bigg\{\Big[c\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}} + \frac{2(p+1)}{p+1-2\theta}\Big]E^{\frac{1}{\theta}-1}(0)\\ & + \frac{\beta c}{2}\Big(\frac{2\theta(p+1)}{p+1-2\theta}\Big)^{\frac{1}{\theta}}\frac{\theta E^{\frac{1}{\theta}-1}(0)}{\theta\beta+1} + \frac{\beta(p+1)}{(p+1-2\theta)(\beta+1)}\\ & + \frac{m}{m+1}\frac{\varepsilon^{-\frac{m+1}{m}}}{\beta+1} + \varepsilon_1^{\frac{m+1}{m-1}}\frac{m-1}{m+1}(1+|\Omega|)^2E^{-\beta}(0)\Bigg\}.\end{aligned} \tag{6.13}$$

Obviously, (6.1) follows from (6.12) and Lemma 6.1. This completes the proof of Theorem 6.1. □

Acknowledgements

The first author wishes to express her sincere gratitude to Professor Wenjie Gao for his support and constant encouragement as well as to Professor Bin Guo for some pretty useful suggestions to improve the quality of the present paper. This work is supported by the National Natural Science Foundation of China(No. 12071391).

Competing Interests

The authors declare that they have no competing interests.

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