Global Existence and Temporal Decay of Large Solutions for the Poisson–Nernst–Planck Equations in Low Regularity Spaces*

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Abstract

We are concerned with the global existence and decay rates of large solutions for the Poisson–Nernst–Planck equations. Based on careful observation of algebraic structure of the equations and using the weighted Chemin–Lerner type norm, we obtain the global existence and optimal decay rates of large solutions without requiring the summation of initial densities of a negatively and positively charged species is small enough. Moreover, the large solution is obtained for initial data belonging to the low regularity Besov spaces with different regularity and integral indices for the different charged species, which indicates more specific coupling relations between the negatively and positively charged species.

Keywords: Poisson–Nernst–Planck equations; large solution; global existence; decay rates.

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1 Introduction

For the simplest model in semiconductor device simulation, the dynamics of the negatively and positively charged species are governed by the drift-diffusion system of bipolar type, which is described by the following elliptic-parabolic coupled system of Poisson–Nernst–Planck equations (cf. [10]):

$$\begin{cases} \partial_t n - \Delta n = -\nabla\cdot(n\nabla\phi) & \text{in } \mathbb{R}^d\times(0,\infty), \\ \partial_t p - \Delta p = \nabla\cdot(p\nabla\phi) & \text{in } \mathbb{R}^d\times(0,\infty), \\ -\varepsilon\Delta\phi = p - n & \text{in } \mathbb{R}^d\times(0,\infty), \\ n(x,0)=n_0(x), \quad p(x,0)=p_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $n = n(x,t)$ and $p = p(x,t)$ denote the densities of a negatively and positively charged species, respectively, ϕ stands for the electric potential, and the positive constant ε is the so-called Debye length that stands for the screening of the charged species.

System (1.1) was formulated by W. Nernst and M. Planck at the end of the nineteenth century as a basic model for the diffusion of ions in an electrolytes. It is also referred as the van Roosbroeck system in semiconductor devices, as the drift-diffusion Poisson system in plasma physics and as a basic model in chemotaxis (see for example [9, 15, 26]). Since finer structures of the semiconductor devices is now demanded, further mathematical discussion for the dynamics of the charged species is now required and

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is getting more important. Mathematical analysis of this system was first focused on the initial boundary value problems in 1980's, and some analytical results related to the existence, uniqueness and regularity of solutions and the asymptotic stability of stationary solutions were obtained by using the Green's function, the Poincaré inequality and the maximum principle of equations of parabolic type, see [2, 4, 5, 12–14, 23] and references therein for more details.

For the Cauchy problem of the system (1.1), the solvability of solutions has been relatively well-developed in various classes of functions and distributions. Kozono–Sugiyama [21] proved local existence of solutions in the Lebesgue space $L^p(\mathbb{R}^d)$ with $\frac{d}{2} < p < d$ and the Sobolev space $W^{2,p}(\mathbb{R}^d)$ with $1 < p \leq 2$. Biler–Cannone–Guerra–Karch [3] established global existence of solutions for small initial data in critical pseudomeasure space $\mathcal{PM}^{d-2}(\mathbb{R}^d)$. Ogawa–Shimizu [24, 25] established global existence of solutions for two dimensional system (1.1) with small initial data in the critical Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and the homogeneous Besov space $\dot{B}^0_{1,2}(\mathbb{R}^2)$, respectively. Considering the well-posedness and ill-posedness issues of the system (1.1) in largest low regularity Besov spaces, Karch [19] proved global existence for small initial data and local existence for large initial data in critical Besov space $\dot{B}^{-2+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)$ with $\frac{d}{2} < p < d$. Zhao–Liu–Cui [31] further extended these results in [19] to more extensive Besov spaces $\dot{B}^{-2+\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ with $1 < p < 2d$ and $1 \leq q \leq \infty$. Recently, Deng–Li [11] established a dichotomy for well-posedness and ill-posedness issues of the two dimensional system (1.1), more precisely, they proved that the system (1.1) is well-posed in $\dot{B}^{-\frac{3}{2}}_{4,2}(\mathbb{R}^2)$, and ill-posed in $\dot{B}^{-\frac{3}{2}}_{4,q}(\mathbb{R}^2)$ for $2 < q \leq \infty$. Iwabuchi–Ogawa [18] further proved that the system (1.1) is ill-posed in $\dot{B}^{-2+\frac{d}{p}}_{p,q}(\mathbb{R}^d)$ with $2d < p \leq \infty$ and $1 \leq q \leq \infty$, or $p = 2d$ and $2 < q \leq \infty$.

Notice that the above well-posed results for the drift-diffusion system (1.1) is quite different from unipolar type one ($p = 0$), or the Keller–Segel system of chemotaxis. Iwabuchi and Nakamura [17] showed the global existence of solutions of the Keller–Segel system with small initial data in $\dot{F}^{-2}_{\infty,2}(\mathbb{R}^d) = BMO^{-2}$ through the Triebel–Lizorkin spaces, which combining the well-posed results in [16] tell us that the Keller–Segel system is well-posed in $\dot{B}^{-2+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)$ ($\max\{1, \frac{d}{2}\} < p < \infty$) and $\dot{F}^{-2}_{\infty,2}(\mathbb{R}^d) = BMO^{-2}$, but ill-posed in $\dot{B}^{-2+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)$ ($2 < q \leq \infty$). These results are parallel to the well-posed and ill-posed results for the Navier–Stokes equations, see [7, 20, 27] in details.

In this paper, we are concerned with the global existence and optimal decay rates of large solutions for the system (1.1) in negative Besov spaces. For simplicity, we assume that $\varepsilon = 1$ and denote $v := n - p$ and $w := n + p$, then the system (1.1) is reduced into the following equations:

$$\begin{cases} \partial_t v - \Delta v = \nabla \cdot (w \nabla (-\Delta)^{-1} v) & \text{in } \mathbb{R}^d \times (0, \infty), \\ \partial_t w - \Delta w = \nabla \cdot (v \nabla (-\Delta)^{-1} v) & \text{in } \mathbb{R}^d \times (0, \infty), \\ v(x, 0) = v_0(x), \quad w(x, 0) = w_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1.2}$$

where

$$v_0(x) = n_0(x) - p_0(x), \quad w_0(x) = n_0(x) + p_0(x),$$

and the electric potential ϕ, which is determined by the Poisson equation (the third equation of (1.1)), has been represented as the volume potential of v:

$$\phi(x, t) = -(-\Delta)^{-1} v(x, t) = \begin{cases} -\frac{1}{d(d-2)\Omega_d} \int_{\mathbb{R}^d} \frac{v(y,t)}{|x-y|^{d-2}} dy, & d \geq 3, \\ \frac{1}{2\pi} \int_{\mathbb{R}^2} v(y, t) \log |x - y| dy, & d = 2, \end{cases}$$

where Ω_d denotes the surface area of the unit sphere in $\mathbb{R}^d$.

It has fundamental importance to observe that the second equation of system (1.2) is a linear equation for w, which has been observed by Deng–Li [11] and Iwabuchi–Ogawa [18] in analyzing the well-posedness and ill-posedness issues of (1.1) in the largest critical Besov spaces. Moreover, considering the algebraic structures of the nonlinear coupling terms in (1.2), by [16], the nonlinear term $v\nabla(-\Delta)^{-1}v$ has a nice structure as

$$\begin{aligned}\partial_{x_i}v\partial_{x_i}(-\Delta)^{-1}v =&\frac{1}{2}\partial_{x_i}(-\Delta)\{((-\Delta)^{-1}v)(\partial_{x_i}(-\Delta)^{-1}v)\}\\&+\partial_{x_i}\nabla\cdot\{((-\Delta)^{-1}v)(\nabla\partial_{x_i}(-\Delta)^{-1}v)\}\\&+\frac{1}{2}\partial_{x_i}^2\{((-\Delta)^{-1}v)v\}.\end{aligned}\tag{1.3}$$

This enables us to treat the equation of w in a weaker Besov space $\dot{B}_{p,q}^{-2+\frac{d}{p}}$ with $1\leq p<\infty$ and $1\leq q\leq\infty$. However, the nonlinear term $w\nabla(-\Delta)^{-1}v$ has lack of such a symmetric structure, which can not exhibit such a good expression as (1.3) and prevents us to obtain good estimates for the equation v in a weaker Besov spaces. These careful observations essentially indicate that the difference of charged species v plays more important role than the summation of charged species w in mathematical analysis of the system (1.1). Based on these careful observations, by using analytical methods in [22,32], we aim at proving the global existence of large solutions without smallness condition imposed on initial data w_0. Moreover, we consider the functional space of solutions of the system (1.2) with initial datum v_0 and w_0 belonging to the different low regularity Besov spaces with different regularity and integral indices, which can indicate more specific coupling relations between the negatively and positively charged species.

Before we state the main results, let us first introduce the definition of the homogeneous Besov spaces. We denote the Schwartz class of rapidly decreasing function by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$.

Definition 1.1 *Let $\varphi\in\mathcal{S}(\mathbb{R}^d)$ be a positive radial function such that φ is supported in the shell $\mathcal{C}=\{\xi\in\mathbb{R}^d,\ \frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$, and*

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1\quad\text{for}\quad\xi\in\mathbb{R}^d\backslash\{0\}.$$

Let $h=\mathcal{F}^{-1}\varphi$. Then for any $f\in\mathcal{S}'(\mathbb{R}^d)$, we define the homogeneous dyadic block Δ_j and the partial summation operator S_j as follows:

$$\Delta_jf(x):=2^{dj}\int_{\mathbb{R}^d}h(2^jy)f(x-y)dy,\qquad S_jf(x):=\sum_{k\leq j-1}\Delta_kf(x).$$

Let $\mathcal{S}'_h(\mathbb{R}^d)$ be the space of tempered distribution $f\in\mathcal{S}'(\mathbb{R}^d)$ such that

$$\lim_{j\to-\infty}S_jf(x)=0.$$

Then for any $s\in\mathbb{R}$, $1\leq p,r\leq\infty$ and $f\in\mathcal{S}'(\mathbb{R}^d)$, we set

$$\|f\|_{\dot{B}^s_{p,r}}:=\begin{cases}\Big(\sum_{j\in\mathbb{Z}}2^{srj}\|\Delta_jf\|_{L^p}^r\Big)^{\frac{1}{r}} & \text{for}\ \ 1\leq r<\infty,\\ \sup_{j\in\mathbb{Z}}2^{sj}\|\Delta_jf\|_{L^p} & \text{for}\ \ r=\infty,\end{cases}$$

and the homogeneous Besov space $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is defined by

- *For $s<\frac{d}{p}$ (or $s=\frac{d}{p}$ if $r=1$), we define*

$$\dot{B}^s_{p,r}(\mathbb{R}^d):=\Big\{f\in\mathcal{S}'_h(\mathbb{R}^d):\ \ \|f\|_{\dot{B}^s_{p,r}}<\infty\Big\}.$$

- *If $k\in\mathbb{N}$ and $\frac{d}{p}+k\leq s<\frac{d}{p}+k+1$ (or $s=\frac{d}{p}+k+1$ if $r=1$), then $\dot{B}^s_{p,r}(\mathbb{R}^d)$ is defined as the subset of distributions $f\in\mathcal{S}'(\mathbb{R}^d)$ such that $\partial^\beta f\in\dot{B}^{s-k}_{p,r}(\mathbb{R}^d)$ whenever $|\beta|=k$.*

The above homogeneous dyadic block Δ_j and the partial summation operator S_j satisfy the following quasi-orthogonal properties: for any $f, g \in \mathcal{S}'(\mathbb{R}^d)$, one has

$$\Delta_i\Delta_j f \equiv 0 \quad \text{if} \quad |i-j| \geq 2 \quad \text{and} \quad \Delta_i(S_{j-1}f\Delta_j g) \equiv 0 \quad \text{if} \quad |i-j| \geq 5. \tag{1.4}$$

Moreover, using Bony's homogeneous paraproduct decomposition (cf. [1, 6]), one can formally split the product of two temperate distributions f and g as follows:

$$fg = T_f g + T_g f + R(f,g), \tag{1.5}$$

where the paraproduct between f and g is defined by

$$T_f g := \sum_{j\in\mathbb{Z}} S_{j-1}f\Delta_j g = \sum_{j\in\mathbb{Z}}\sum_{k\leq j-2} \Delta_k f\Delta_j g,$$

and the remaining term is defined by

$$R(f,g) := \sum_{j\in\mathbb{Z}} \Delta_j f\widetilde{\Delta_j} g \quad \text{and} \quad \widetilde{\Delta_j} := \Delta_{j-1} + \Delta_j + \Delta_{j+1}.$$

Next, we recall the definition of the so-called Chemin–Lerner mixed time-space spaces.

Definition 1.2 *For $0 < T \leq \infty$, $s \in \mathbb{R}$ and $1 \leq p, r, \rho \leq \infty$. We define the mixed time-space $\mathcal{L}^\rho(0,T;\dot{B}^s_{p,r}(\mathbb{R}^d))$ as the completion of $\mathcal{C}([0,T];\mathcal{S}(\mathbb{R}^d))$ by the norm*

$$\|f\|_{\mathcal{L}^\rho_T(\dot{B}^s_{p,r})} := \left(\sum_{j\in\mathbb{Z}} 2^{srj}\left(\int_0^T \|\Delta_j f(\cdot,t)\|^\rho_{L^p}dt\right)^{\frac{r}{\rho}}\right)^{\frac{1}{r}} < \infty$$

with the usual change if $\rho = \infty$ or $r = \infty$.

Now we are ready to state our main results. The first one is the global existence of large solutions for the system (1.1).

Theorem 1.3 *Let p, q be two positive numbers such that*

$$1 \leq p, q < \infty \quad \textit{and} \quad \max\{\frac{1}{p}-\frac{1}{q}, \frac{1}{q}-\frac{1}{p}\} < \frac{1}{d} < \frac{1}{p}+\frac{1}{q}. \tag{1.6}$$

There exist two constants c_0, C_0 such that if the initial data $(v_0, w_0) \in \dot{B}^{-2+\frac{d}{p}}_{p,1}(\mathbb{R}^d) \times \dot{B}^{-2+\frac{d}{q}}_{q,1}(\mathbb{R}^d)$ satisfies

$$C_0\|v_0\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} \exp\{C_0\|w_0\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}\} \leq c_0, \tag{1.7}$$

then the system (1.2) admits a unique global solution (v, w) satisfying

$$\begin{cases} v \in C([0,\infty), \dot{B}^{-2+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap \mathcal{L}^\infty(0,\infty;\dot{B}^{-2+\frac{d}{p}}_{p,1}(\mathbb{R}^d)) \cap \mathcal{L}^1(0,\infty;\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)), \\ w \in C([0,\infty), \dot{B}^{-2+\frac{d}{q}}_{q,1}(\mathbb{R}^d)) \cap \mathcal{L}^\infty(0,\infty;\dot{B}^{-2+\frac{d}{q}}_{q,1}(\mathbb{R}^d)) \cap \mathcal{L}^1(0,\infty;\dot{B}^{\frac{d}{q}}_{q,1}(\mathbb{R}^d)). \end{cases}$$

Remark 1.1 The initial condition (1.7) exhibits that the initial data w_0 can be taken large as long as we take the initial data v_0 small enough compared with the size of w_0, which we can still get the global existence of solutions to the system (1.2). Hence, Theorem 1.3 implies global existence of solutions for the system (1.1) with only requiring the difference of initial charge densities of a negatively and positively charged species is small enough. Indeed, back to the original system (1.1), if $n_0 \equiv p_0$, then the above condition (1.7) implies that we can get global large solutions of system (1.1) without any smallness assumption imposed on n_0+p_0, which means that the difference of a negatively and positively charged species plays more important role in mathematical analysis of system (1.1).

Remark 1.2 The specific coupling relation between v and w was indicated by the condition (1.6), which tells us that the regularity of solution v or w can be taken beyond the regularity index $-\frac{3}{2}$, but one can not take both of them less than $-\frac{3}{2}$ at the same time. Indeed, the regularity of v can be taken much weaker as long as the regularity of w is not that much weaker, i.e., p can be taken large enough as long as we take q closing to d such that the condition (1.6) holds. Hence, Theorem 1.3 can be regarded as an extension of global existence results in [11, 18, 19, 31], where the global existence of solutions with small initial data was proved in critical Besov spaces with the same regularity and integral indices for v and w.

The second purpose of this paper is that we are attempt to establish the optimal decay rates of global large solutions obtained in Theorem 1.3. In our previous paper [30], we have established the regularizing-decay estimates of solutions to the system (1.2), which showed the analyticity of spatial variables as well as temporal decay estimates on spatial derivatives of solutions for large time. More specifically, we proved that for $(v_0,w_0)\in L^{\frac{d}{2}}(\mathbb{R}^d)$, let (v,w) be the corresponding solution of the system (1.2) satisfying $(v,w)\in\mathcal{X}_p$ for some $p\in(\frac{d}{2},d)$, where

$$\mathcal{X}_p=C([0,\infty),L^{\frac{d}{2}}(\mathbb{R}^d))\cap\Big\{u:\ u\in C((0,\infty),L^p(\mathbb{R}^d))\ \text{ and }\ \sup_{t>0}t^{1-\frac{d}{2p}}\|u(t)\|_{L^p}<\infty\Big\},$$

besides, assume further that there exist two finite constants M_1 and M_2 such that

$$\sup_{0<t<\infty}\|(v(t),w(t))\|_{L^{\frac{d}{2}}}\le M_1\quad\text{and}\quad\sup_{0<t<\infty}t^{\frac{d}{2}(\frac{1}{d}-\frac{1}{p})}\|(v(t),w(t))\|_{L^p}\le M_2. \tag{1.8}$$

Then there exist two positive constants K_1 and K_2 (depending only on d, p, M_1 and M_2) such that

$$\|(\partial_x^\beta v(t),\partial_x^\beta w(t))\|_{L^q}\le K_1(K_2|\beta|)^{|\beta|}t^{-\frac{|\beta|}{2}-1+\frac{d}{2q}} \tag{1.9}$$

for all $\frac{d}{2}\le q\le\infty$ and $\beta\in\mathbb{N}_0^d$, where $|\beta|:=\sum_{i=1}^d\beta_i$ and $\partial_x^\beta:=\partial_{x_1}^{\beta_1}\cdots\partial_{x_d}^{\beta_d}$ for multi-index $\beta:=(\beta_1,\cdots,\beta_d)$, $\mathbb{N}_0$ denotes the set of non-negative integers. Notice that if the initial data (v_0,w_0) is small enough in $L^{\frac{d}{2}}(\mathbb{R}^d)$, then the corresponding solution (v,w) naturally satisfies the condition (1.8). Moreover, similar decay estimates still hold for the system (1.2) in critical Besov spaces $\dot{B}^{-2+\frac{d}{p}}_{p,\infty}(\mathbb{R}^d)$ with $\frac{d}{2}<p<d$.

Motivated by the above decay estimates of solutions, we aim at improving the above decay rates of solutions for initial data belonging to the negative Besov spaces and obtain the following two decay results by using the general weighted energy approach and the interpolation techniques.

Theorem 1.4 *Let the assumptions of Theorem 1.3 be in force, and if we assume further that* $(v_0,w_0)\in\dot{B}^{-s}_{r,1}(\mathbb{R}^d)\cap\dot{B}^{N}_{r,1}(\mathbb{R}^d)$ *for an integer* N*, a real number* $s>0$ *and* $1\le r<\infty$ *such that*

$$-s+1+\frac{d}{p}>d\max\{0,\frac{1}{p}+\frac{1}{r}-1\}\quad\text{and}\quad-s+1+\frac{d}{q}>d\max\{0,\frac{1}{q}+\frac{1}{r}-1\},$$

then for any $\ell\in[-s,N]$*, there exists a constant* C_0 *such that for all* $t\ge0$*,*

$$\|(v(t),w(t))\|_{\dot{B}^{\ell}_{r,1}}\le C_0. \tag{1.10}$$

Moreover, we obtain the following decay estimate of the solution $(v(t),w(t))$*:*

$$\|(v(t),w(t))\|_{\dot{B}^{\ell}_{r,1}}\le C_0(1+t)^{-\frac{\ell+s}{2}}. \tag{1.11}$$

If we relax the high regularity condition imposed on the initial data in Theorem 1.4, then we can obtain the following decay result for the lower-order derivative of solutions.

Theorem 1.5 *Let the assumptions of Theorem 1.3 be in force, and if we assume further that* $(v_0, w_0) \in \dot{B}^{-s}_{r,1}(\mathbb{R}^d)$ *with* $1 \le r \le \min\{p, q\}$, $s > \max\{0, 2 - \frac{d}{r}\}$, *and*

$$-s + 1 + \frac{d}{p} > d \max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad -s + 1 + \frac{d}{q} > d \max\{0, \frac{1}{q} + \frac{1}{r} - 1\},$$

then for any $\ell \in [-s - d(\frac{1}{r} - \frac{1}{p}), -2 + \frac{d}{p}]$, *there exists a constant* C_0 *such that for all* $t \ge 0$,

$$\|v(t)\|_{\dot{B}^{\ell}_{p,1}} \le C_0(1+t)^{-\frac{\ell+s}{2} - \frac{d}{2}(\frac{1}{r} - \frac{1}{p})}; \tag{1.12}$$

and for any $\ell \in [-s - d(\frac{1}{r} - \frac{1}{q}), -2 + \frac{d}{q}]$, *there exists a constant* C_0 *such that for all* $t \ge 0$,

$$\|w(t)\|_{\dot{B}^{\ell}_{q,1}} \le C_0(1+t)^{-\frac{\ell+s}{2} - \frac{d}{2}(\frac{1}{r} - \frac{1}{q})}. \tag{1.13}$$

Remark 1.3 It is clear that (1.12) and (1.13) improve (1.9) since $L^q(\mathbb{R}^d) \hookrightarrow \dot{B}^{-s}_{r,\infty}(\mathbb{R}^d)$ with $q = \frac{dr}{d+sr}$. Moreover, we do not assume that the $\dot{B}^{-s}_{r,1}$ norm of initial data is small enough, and this norm enhances the time decay rates of the solution with the factor $\frac{s}{2}$.

Remark 1.4 The general L^q temporal decay rates of solutions can be obtained by the standard embedding theory, for instance, by (1.12), we know that for any $1 \le q < \infty$,

$$\|v(t)\|_{L^q} \le C\|v(t)\|_{\dot{H}^{\frac{d}{2} - \frac{d}{q}}} \le C\|v(t)\|_{\dot{B}^{\frac{d}{2} - \frac{d}{q}}_{2,1}} \le C(1+t)^{-\frac{s}{2} - \frac{d}{2}(\frac{1}{r} - \frac{1}{q})},$$

which is faster than (1.9) when $|\beta| = 0$.

This paper is organized as follows. In section 2, we first establish two crucial bilinear estimates in the Chemin–Lerner type spaces, then give the proof of Theorem 1.3 by using weighted Chemin–Lerner type norm. In section 3, applying the elementary Fourier splitting argument, we intend to establish two weighted energy inequalities in terms of the lower-order and higher-order derivative of solutions, then we complete the proof of Theorems 1.4 and 1.5 in section 4. Throughout the paper, C stands for a generic constant, and we use the notation $\mathcal{A} \lesssim \mathcal{B}$ to denote the relation $\mathcal{A} \le C\mathcal{B}$ and the notation $\mathcal{A} \approx \mathcal{B}$ to denote the relations $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$.

2 Global existence of large solutions

Before going to the proof, we recall the following two interpolation inequalities in stationary/time dependent Besov spaces (cf. [1]). For $s_1, s_2 \in \mathbb{R}$ such that $s_1 < s_2$ and $\theta \in [0, 1]$, there exists a constant C such that

$$\|u\|_{\dot{B}^{s_1\theta + s_2(1-\theta)}_{p,r}} \le C\|u\|^{\theta}_{\dot{B}^{s_1}_{p,r}} \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,r}}. \tag{2.1}$$

Moreover, for any $0 < T \le \infty$, $1 \le \rho, \rho_1, \rho_2 \le \infty$ such that $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}$, one has

$$\|u\|_{\mathcal{L}^{\rho}_T(\dot{B}^{s_1\theta + s_2(1-\theta)}_{p,r})} \le C\|u\|^{\theta}_{\mathcal{L}^{\rho_1}_T(\dot{B}^{s_1}_{p,r})} \|u\|^{1-\theta}_{\mathcal{L}^{\rho_2}_T(\dot{B}^{s_2}_{p,r})}. \tag{2.2}$$

The essential parts in the proof of Theorem 1.3 are the following two bilinear estimates in the Chemin–Lerner spaces. The first one corresponds to the nonlinear term $\nabla \cdot (w\nabla(-\Delta)^{-1}v)$ in the first equation of system (1.2), and one can easily see that

$$\|\nabla \cdot (w\nabla(-\Delta)^{-1}v)\|_{\mathcal{L}^1_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \approx \|w\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{p}}_{p,1})}. \tag{2.3}$$

For the right-hand side of (2.3), we get the following bilinear estimates.

Lemma 2.1 *Let p, q be two positive numbers such that $1 \le p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{d} < \frac{1}{p} + \frac{1}{q}$. Then we have*

$$
\begin{aligned}
\|w\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{p}}_{p,1})} \lesssim{}& \|w\|^{\theta_1}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{q}}_{q,1})} \|w\|^{1-\theta_1}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{q}}_{q,1})} \|v\|^{\theta_2}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \|v\|^{1-\theta_2}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} \\
&+ \|w\|^{\theta_2}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{q}}_{q,1})} \|w\|^{1-\theta_2}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{q}}_{q,1})} \|v\|^{\theta_1}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \|v\|^{1-\theta_1}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})},
\end{aligned}
\tag{2.4}
$$

where $\frac{1}{2} < \theta_1 \le 1$, $\theta_2 = 1 - \theta_1$ are two given constants.

Proof. Notice that the particular case $1 \le p = q < 2d$ has been established in [32]. Here we address the case $p \ne q$. Thanks to the Bony's paraproduct decomposition (1.5), one has

$$
w\nabla(-\Delta)^{-1}v = T_w\nabla(-\Delta)^{-1}v + T_{\nabla(-\Delta)^{-1}v}w + R(w, \nabla(-\Delta)^{-1}v). \tag{2.5}
$$

To estimate three terms on the right-hand side of (2.5), let $2 < \rho_1 \le \infty$ be large enough such that $1 - \frac{2}{\rho_1} > 0$ and $1 + \frac{d}{q} - \frac{d}{p} - \frac{2}{\rho_1} > 0$, and let $1 \le \rho_2 < 2$ be the conjugate of ρ_1, i.e., $\frac{1}{\rho_1} + \frac{1}{\rho_2} = 1$, then one can apply (1.4) and the Hölder's inequality to get

$$
\begin{aligned}
&\|\Delta_j(T_w\nabla(-\Delta)^{-1}v)\|_{L^1_T(L^p)} \lesssim \sum_{|j'-j|\le 4} \|S_{j'-1}w\|_{L^{\rho_1}_T(L^\infty)} \|\Delta_{j'}\nabla(-\Delta)^{-1}v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim \sum_{|j'-j|\le 4} \sum_{k\le j'-2} 2^{\frac{dk}{q}} \|\Delta_k w\|_{L^{\rho_1}_T(L^q)} 2^{-j'} \|\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim \sum_{|j'-j|\le 4} 2^{(1-\frac{d}{p}-\frac{2}{\rho_2})j'} \sum_{k\le j'-2} 2^{(2-\frac{2}{\rho_1})k} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_1})k} \|\Delta_k w\|_{L^{\rho_1}_T(L^q)} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'} \|\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim 2^{(1-\frac{d}{p})j} \sum_{|j'-j|\le 4} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'} \|\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)} \|w\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{q}+\frac{2}{\rho_1}}_{q,1})}.
\end{aligned}
\tag{2.6}
$$

Next we estimate the symmetric term $T_{\nabla(-\Delta)^{-1}v}w$. The case $1 \le q \le p$ is simple, it follows that

$$
\begin{aligned}
&\|\Delta_j(T_{\nabla(-\Delta)^{-1}v}w)\|_{L^1_T(L^p)} \lesssim 2^{(\frac{d}{q}-\frac{d}{p})j} \|\Delta_j(T_{\nabla(-\Delta)^{-1}v}w)\|_{L^1_T(L^q)} \\
&\lesssim 2^{(\frac{d}{q}-\frac{d}{p})j} \sum_{|j'-j|\le 4} \|S_{j'-1}\nabla(-\Delta)^{-1}v\|_{L^{\rho_1}_T(L^\infty)} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \\
&\lesssim 2^{(\frac{d}{q}-\frac{d}{p})j} \sum_{|j'-j|\le 4} \sum_{k\le j'-2} 2^{(-1+\frac{d}{p})k} \|\Delta_k v\|_{L^{\rho_1}_T(L^p)} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \\
&\lesssim 2^{(\frac{d}{q}-\frac{d}{p})j} \sum_{|j'-j|\le 4} 2^{(2-\frac{d}{q}-\frac{2}{\rho_2})j'} \sum_{k\le j'-2} 2^{(1-\frac{2}{\rho_1})k} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_1})k} \|\Delta_k v\|_{L^{\rho_1}_T(L^p)} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_2})j'} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \\
&\lesssim 2^{(1-\frac{d}{p})j} \sum_{|j'-j|\le 4} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_2})j'} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \|v\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})},
\end{aligned}
\tag{2.7}
$$

while for the case $q > p$, one can derive from the fact $\frac{1}{p} = \frac{1}{q} + \frac{q-p}{pq}$ that

$$
\begin{aligned}
&\|\Delta_j(T_{\nabla(-\Delta)^{-1}v}w)\|_{L^1_T(L^p)} \lesssim \sum_{|j'-j|\le 4} \|S_{j'-1}\nabla(-\Delta)^{-1}v\|_{L^{\rho_1}_T(L^{\frac{pq}{q-p}})} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \\
&\lesssim \sum_{|j'-j|\le 4} \sum_{k\le j'-2} 2^{-k+d(\frac{1}{p}-\frac{q-p}{pq})k} \|\Delta_k v\|_{L^{\rho_1}_T(L^p)} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \\
&\lesssim \sum_{|j'-j|\le 4} 2^{(2-\frac{d}{q}-\frac{2}{\rho_2})j'} \sum_{k\le j'-2} 2^{(1+\frac{d}{q}-\frac{d}{p}-\frac{2}{\rho_1})k} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_1})k} \|\Delta_k v\|_{L^{\rho_1}_T(L^p)} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_2})j'} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \\
&\lesssim 2^{(1-\frac{d}{p})j} \sum_{|j'-j|\le 4} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_2})j'} \|\Delta_{j'}w\|_{L^{\rho_2}_T(L^q)} \|v\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}.
\end{aligned}
\tag{2.8}
$$

Finally, we deal with the remaining term $R(w, \nabla(-\Delta)^{-1}v)$, we also need to consider two cases: In the case $1 \le p < d$, applying (1.4) yields that

$$\begin{aligned}
&\|\Delta_j R(w,\nabla(-\Delta)^{-1}v)\|_{L^1_T(L^p)} \lesssim \sum_{j'\ge j-2} \|\Delta_{j'} w\|_{L^{\rho_1}_T(L^\infty)} \|\widetilde{\Delta}_{j'}\nabla(-\Delta)^{-1}v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim \sum_{j'\ge j-2} 2^{(\frac{d}{q}-1)j'} \|\Delta_{j'} w\|_{L^{\rho_1}_T(L^q)} \|\widetilde{\Delta}_{j'} v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim \sum_{j'\ge j-2} 2^{(1-\frac{d}{p})j'} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_1})j'} \|\Delta_{j'} w\|_{L^{\rho_1}_T(L^q)} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'} \|\widetilde{\Delta}_{j'} v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim 2^{(1-\frac{d}{p})j} \sum_{j'\ge j-2} 2^{(1-\frac{d}{p})(j'-j)} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'} \|\widetilde{\Delta}_{j'} v\|_{L^{\rho_2}_T(L^p)} \|w\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{q}+\frac{2}{\rho_1}}_{q,1})}; \qquad (2.9)
\end{aligned}$$

while in the case $p \ge d$, it can be bounded that

$$\begin{aligned}
&\|\Delta_j R(w,\nabla(-\Delta)^{-1}v)\|_{L^1_T(L^p)} \lesssim 2^{\frac{dj}{q}} \sum_{j'\ge j-2} \|\Delta_{j'} w\|_{L^{\rho_1}_T(L^q)} \|\widetilde{\Delta}_{j'}\nabla(-\Delta)^{-1}v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim 2^{\frac{dj}{q}} \sum_{j'\ge j-2} 2^{(1-\frac{d}{p}-\frac{d}{q})j'} 2^{(-2+\frac{d}{q}+\frac{2}{\rho_1})j'} \|\Delta_{j'} w\|_{L^{\rho_1}_T(L^q)} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'} \|\widetilde{\Delta}_{j'} v\|_{L^{\rho_2}_T(L^p)} \\
&\lesssim 2^{(1-\frac{d}{p})j} \sum_{j'\ge j-2} 2^{(1-\frac{d}{p}-\frac{d}{q})(j'-j)} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'} \|\widetilde{\Delta}_{j'} v\|_{L^{\rho_2}_T(L^p)} \|w\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{q}+\frac{2}{\rho_1}}_{q,1})}. \qquad (2.10)
\end{aligned}$$

Combining all above estimates (2.6)–(2.10) and using Definition 1.2 we get

$$\begin{aligned}
\|w\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{p}}_{p,1})} &\lesssim \|w\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{q}+\frac{2}{\rho_1}}_{q,1})} \|v\|_{\mathcal{L}^{\rho_2}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_2}}_{p,1})} \\
&\quad + \|w\|_{\mathcal{L}^{\rho_2}_T(\dot{B}^{-2+\frac{d}{q}+\frac{2}{\rho_2}}_{q,1})} \|v\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}. \qquad (2.11)
\end{aligned}$$

By using the interpolation result (2.2) with $\theta_1 = 1 - \frac{1}{\rho_1}$ and $\theta_2 = 1 - \frac{1}{\rho_2}$, we finally obtain (2.4). The proof of Lemma 2.1 is achieved. □

The second one corresponds to the nonlinear term $\nabla\cdot(v\nabla(-\Delta)^{-1}v)$ in the second equation of system (1.2), and one can see that

$$\|\nabla\cdot(v\nabla(-\Delta)^{-1}v)\|_{\mathcal{L}^1_T(\dot{B}^{-2+\frac{d}{q}}_{q,1})} \approx \|v\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{q}}_{q,1})}. \qquad (2.12)$$

For the right-hand side of (2.12), we use the symmetric structure of this nonlinear term to get the following bilinear estimate.

Lemma 2.2 *Let p, q be two positive numbers such that $1 \le p, q < \infty$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{d}$. Then we have*

$$\begin{aligned}
\|u\nabla(-\Delta)^{-1}v + v\nabla(-\Delta)^{-1}u\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{q}}_{q,1})} &\lesssim \|u\|^{\theta_1}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \|u\|^{1-\theta_1}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} \|v\|^{\theta_2}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \|v\|^{1-\theta_2}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} \\
&\quad + \|u\|^{\theta_2}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \|u\|^{1-\theta_2}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})} \|v\|^{\theta_1}_{\mathcal{L}^\infty_T(\dot{B}^{-2+\frac{d}{p}}_{p,1})} \|v\|^{1-\theta_1}_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}, \qquad (2.13)
\end{aligned}$$

where $\frac{1}{2} < \theta_1 \le 1$, $\theta_2 = 1 - \theta_1$ are two corresponding constants in Lemma 2.1.

Proof. The particular case $1 \le p = q < \infty$ has been established in [16, 29]. Here we address the case $p \ne q$. We divide the proof of Lemma 2.2 into the following two cases.

Case 1: $1 \le p \le q$**.** In this case, it is clear that the following identity holds:

$$u\nabla(-\Delta)^{-1}v + v\nabla(-\Delta)^{-1}u = -\nabla\cdot\big(\nabla(-\Delta)^{-1}u\nabla(-\Delta)^{-1}v\big). \qquad (2.14)$$

Then the imbedding relation $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{q}}_{q,1}(\mathbb{R}^d)$ yields that

$$\|-\nabla\cdot\big(\nabla(-\Delta)^{-1}u\nabla(-\Delta)^{-1}v\big)\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{q}}_{q,1})} \approx \|\nabla(-\Delta)^{-1}u\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{q}}_{q,1})}$$

$$\lesssim \|\nabla(-\Delta)^{-1}u\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_T(\dot{B}^{\frac{d}{p}}_{p,1})}. \tag{2.15}$$

Based on this fact, we can strictly follow the argument in [16, 29] to complete the proof of (2.13), here we omit the details.

Case 2, $1 \le q < p$**.** The case $q < p$ is a little tricky, here we resort to the Bony's decomposition (1.5) to split the left-hand side of (3.12) into the following three terms:

$$u\nabla(-\Delta)^{-1}v + v\nabla(-\Delta)^{-1}u := I_1 + I_2 + I_3, \tag{2.16}$$

where

$$\begin{aligned}
I_1 :&= \sum_{j'\in\mathbb{Z}} S_{j'-1}u\nabla(-\Delta)^{-1}\Delta_{j'}v + S_{j'-1}v\nabla(-\Delta)^{-1}\Delta_{j'}u,\\
I_2 :&= \sum_{j'\in\mathbb{Z}} \Delta_{j'}u\nabla(-\Delta)^{-1}S_{j'-1}v + \Delta_{j'}v\nabla(-\Delta)^{-1}S_{j'-1}u,\\
I_3 :&= \sum_{j'\in\mathbb{Z}} \Delta_{j'}u\nabla(-\Delta)^{-1}\widetilde{\Delta}_{j'}v + \Delta_{j'}v\nabla(-\Delta)^{-1}\widetilde{\Delta}_{j'}u.
\end{aligned}$$

Since u and v play the same roles in our derivation, it suffices to deal with the first two terms in I_1 and I_2, and the left two terms can be analogously handled. Based on the conditions $1 \le q < p$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{d} < \frac{2}{d}$, one can derive that

$$\begin{aligned}
&\|\Delta_j \sum_{j'\in\mathbb{Z}} S_{j'-1}u\nabla(-\Delta)^{-1}\Delta_{j'}v\|_{L^1_T(L^q)} \lesssim \sum_{|j-j'|\le 4} \|S_{j'-1}u\|_{L^{\rho_1}_T(L^{\frac{pq}{p-q}})}\|\nabla(-\Delta)^{-1}\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim \sum_{|j-j'|\le 4} 2^{(1-\frac{d}{p}-\frac{2}{\rho_2})j'} \sum_{k\le j'-2} 2^{(2+\frac{d}{p}-\frac{d}{q}-\frac{2}{\rho_1})k}2^{(-2+\frac{d}{p}+\frac{2}{\rho_1})k}\|\Delta_k u\|_{L^p}2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim 2^{(1-\frac{d}{q})j} \sum_{|j-j'|\le 4} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)}\|u\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})},
\end{aligned}$$

$$\begin{aligned}
&\|\Delta_j \sum_{j'\in\mathbb{Z}} \Delta_{j'}u\nabla(-\Delta)^{-1}S_{j'-1}v\|_{L^1_T(L^q)} \lesssim \sum_{|j-j'|\le 4} \|\Delta_{j'}u\|_{L^{\rho_2}_T(L^p)}\|\nabla(-\Delta)^{-1}S_{j'}v\|_{L^{\rho_1}_T(L^{\frac{pq}{p-q}})}\\
&\lesssim \sum_{|j-j'|\le 4} \|\Delta_{j'}u\|_{L^{\rho_2}_T(L^p)} \sum_{k\le j'-2} 2^{(1+\frac{d}{p}-\frac{d}{q}-\frac{2}{\rho_1})k}2^{(-2+\frac{d}{p}+\frac{2}{\rho_1})j'}\|\Delta_{j'}v\|_{L^{\rho_1}_T(L^p)}\\
&\lesssim 2^{(1-\frac{d}{q})j} \sum_{|j-j'|\le 4} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\Delta_{j'}u\|_{L^{\rho_2}_T(L^p)}\|v\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}.
\end{aligned}$$

The above two estimates tell us that for $i = 1, 2$,

$$\begin{aligned}
\|\Delta_j I_i\|_{L^1_T(L^q)} \lesssim 2^{(1-\frac{d}{q})j} &\sum_{|j-j'|\le 4} 2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\big(\|\Delta_{j'}u\|_{L^{\rho_2}_T(L^p)} + \|\Delta_{j'}v\|_{L^{\rho_2}_T(L^p)}\big)\\
&\times\big(\|u\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})} + \|v\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}\big).
\end{aligned} \tag{2.17}$$

Finally we tackle with the most difficult term I_3. Inspired by observation (1.3), one can further split I_3 into the following three terms for $k = 1, 2, \cdots, d$:

$$I_3 := I_{31} + I_{32} + I_{33}, \tag{2.18}$$

where

$$I_{31} := \sum_{j'\in\mathbb{Z}} (-\Delta)\Big\{\big((-\Delta)^{-1}\Delta_{j'}u\big)\big(\partial_{x_k}(-\Delta)^{-1}\widetilde{\Delta}_{j'}v\big)\Big\},$$

$$
\begin{aligned}
I_{32} &:= \sum_{j'\in\mathbb{Z}} 2\nabla\cdot\Big\{\big((-\Delta)^{-1}\Delta_{j'}u\big)\big(\partial_{x_k}\nabla(-\Delta)^{-1}\widetilde{\Delta}_{j'}v\big)\Big\},\\
I_{33} &:= \sum_{j'\in\mathbb{Z}} \partial_{x_k}\Big\{\big((-\Delta)^{-1}\Delta_{j'}u\big)\widetilde{\Delta}_{j'}v\Big\}.
\end{aligned}
$$

Based on this decomposition, one can treat I_{3i} $(i=1,2,3)$ as follows:

$$
\begin{aligned}
\|\Delta_j I_{31}\|_{L^1_T(L^q)} &\lesssim 2^{2j}\sum_{j'\geq j-2}\|(-\Delta)^{-1}\Delta_{j'}u\|_{L^{\rho_1}_T(L^{\frac{pq}{p-q}})}\|\partial_{x_k}(-\Delta)^{-1}\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim 2^{2j}\sum_{j'\geq j-2}2^{(-1-\frac{d}{q})j'}2^{(-2+\frac{d}{p}+\frac{2}{\rho_1})j'}\|\Delta_{j'}u\|_{L^{\rho_1}_T(L^p)}2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim 2^{(1-\frac{d}{q})j}\sum_{j'\geq j-2}2^{(-1-\frac{d}{q})(j'-j)}2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\|u\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})},
\end{aligned}
$$

$$
\begin{aligned}
\|\Delta_j I_{32}(\Delta_j I_{33})\|_{L^1_T(L^q)} &\lesssim 2^{j}\sum_{j'\geq j-2}\|(-\Delta)^{-1}\Delta_{j'}u\|_{L^{\rho_1}_T(L^{\frac{pq}{p-q}})}\|\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim 2^{j}\sum_{j'\geq j-2}2^{-\frac{dj'}{q}}2^{(-2+\frac{d}{p}+\frac{2}{\rho_1})j'}\|\Delta_{j'}u\|_{L^{\rho_1}_T(L^p)}2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\\
&\lesssim 2^{(1-\frac{d}{q})j}\sum_{j'\geq j-2}2^{-\frac{d}{q}(j'-j)}2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\|u\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}.
\end{aligned}
$$

As a consequence of these estimates, we obtain from (2.18) that

$$
\|\Delta_j I_3\|_{L^1_T(L^q)} \lesssim 2^{(1-\frac{d}{q})j}\sum_{j'\geq j-2}2^{-\frac{d}{q}(j'-j)}2^{(-2+\frac{d}{p}+\frac{2}{\rho_2})j'}\|\widetilde{\Delta}_{j'}v\|_{L^{\rho_2}_T(L^p)}\|u\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}. \tag{2.19}
$$

Hence, plugging (2.17) and (2.19) into (2.16), we obtain

$$
\begin{aligned}
\|u\nabla(-\Delta)^{-1}v+v\nabla(-\Delta)^{-1}u\|_{\mathcal{L}^1_T(\dot{B}^{-1+\frac{d}{q}}_{q,1})} &\lesssim \|u\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}\|v\|_{\mathcal{L}^{\rho_2}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_2}}_{p,1})}\\
&\quad+\|u\|_{\mathcal{L}^{\rho_2}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_2}}_{p,1})}\|v\|_{\mathcal{L}^{\rho_1}_T(\dot{B}^{-2+\frac{d}{p}+\frac{2}{\rho_1}}_{p,1})}.
\end{aligned} \tag{2.20}
$$

Again, we get (2.13) by using the interpolation result (2.2). The proof of Lemma 2.2 is achieved. □

Once we get the above two desired bilinear estimates, one can apply the Banach contraction mapping theorem to obtain that there exists $T>0$ such that the system (1.2) has a unique solution (v,w) on $[0,T)$ satisfying

$$
\begin{cases}
v\in C([0,T),\dot{B}^{-2+\frac{d}{p}}_{p,1}(\mathbb{R}^d))\cap\mathcal{L}^\infty(0,T;\dot{B}^{-2+\frac{d}{p}}_{p,1}(\mathbb{R}^d))\cap\mathcal{L}^1(0,T;\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d)),\\
w\in C([0,T),\dot{B}^{-2+\frac{d}{q}}_{q,1}(\mathbb{R}^d))\cap\mathcal{L}^\infty(0,T;\dot{B}^{-2+\frac{d}{q}}_{q,1}(\mathbb{R}^d))\cap\mathcal{L}^1(0,T;\dot{B}^{\frac{d}{q}}_{q,1}(\mathbb{R}^d)).
\end{cases} \tag{2.21}
$$

Moreover, the solution is global if the initial data is small enough. The proof is more or less a standard procedure, thus we safely omit it here, and for more details, we refer the readers to see [32].

Next, we intend to drop the smallness assumption imposed on w_0 to still ensure the global existence of solutions, which need us to employ the following product estimates in Besov spaces, for details, see [1,29].

Lemma 2.3 *Let* $1\leq p_1,p_2\leq\infty$, $s_1\leq\frac{d}{p_1}$, $s_2\leq\min\{\frac{d}{p_1},\frac{d}{p_2}\}$, *and* $s_1+s_2>d\max\{0,\frac{1}{p_1}+\frac{1}{p_2}-1\}$. *Assume that* $f\in\dot{B}^{s_1}_{p_1,1}(\mathbb{R}^d)$, $g\in\dot{B}^{s_2}_{p_2,1}(\mathbb{R}^d)$. *Then we have* $fg\in\dot{B}^{s_1+s_2-\frac{d}{p_1}}_{p_2,1}(\mathbb{R}^d)$, *and there exists a positive constant* C *such that*

$$
\|fg\|_{\dot{B}^{s_1+s_2-\frac{d}{p_1}}_{p_2,1}}\leq C\|f\|_{\dot{B}^{s_1}_{p_1,1}}\|g\|_{\dot{B}^{s_2}_{p_2,1}}. \tag{2.22}
$$

Now we mainly use the ideas in [22,32] to complete the proof of Theorem 1.3. Let us denote by T^* the maximal existence time of local solution (v,w) satisfying (2.21). Then to prove Theorem 1.3, it suffices to prove $T^*=\infty$ under the initial condition (1.7).

We first estimate w. Let $\eta>0$ be a small positive constant which the exact value will be determined later, and denote

$$T_\eta := \sup\big\{t\in[0,T^*): \|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{d}{p}}_{p,1})}+\kappa\|v\|_{\mathcal{L}^1_t(\dot B^{\frac{d}{p}}_{p,1})}\le \eta\big\}. \tag{2.23}$$

Applying the dyadic operator Δ_j to the second equation of (1.2) and taking L^2 inner product of the resulting equation with $|\Delta_j w|^{q-2}\Delta_j w$, then using the following lower bound for the integral involving the Laplace operator $-\Delta$ (see for example [8,28]):

$$-\int_{\mathbb{R}^d}\Delta\Delta_j w\cdot|\Delta_j w|^{p-2}\Delta_j w dx\ge \kappa 2^{2j}\|\Delta_j w\|^p_{L^p}\quad\text{for any}\quad p\in[1,\infty), \tag{2.24}$$

where κ is a positive constant depending only on d and p, we derive from the Hölder's inequality that

$$\frac{d}{dt}\|\Delta_j w\|_{L^q}+\kappa 2^{2j}\|\Delta_j w\|_{L^q}\le 2^j\|\Delta_j(v\nabla(-\Delta)^{-1}v)\|_{L^q}. \tag{2.25}$$

Integrating the above inequality (2.25) in time interval $[0,t]$ for any $0<t<T_\eta$, then multiplying by $2^{(-2+\frac{d}{q})j}$ and taking l^1 norm to the resultant inequality, we can deduce that

$$\|w\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{d}{q}}_{q,1})}+\kappa\|w\|_{\mathcal{L}^1_t(\dot B^{\frac{d}{q}}_{q,1})}\le\|w_0\|_{\dot B^{-2+\frac{d}{q}}_{q,1}}+\|v\nabla(-\Delta)^{-1}v\|_{\mathcal{L}^1_t(\dot B^{-1+\frac{d}{q}}_{q,1})}. \tag{2.26}$$

Applying Lemma 2.2, the second term of the right hand side of (2.26) can be bounded by choosing $u=v$, $\theta_1=1$ and $\theta_2=0$,

$$\|w\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{d}{q}}_{q,1})}+\kappa\|w\|_{\mathcal{L}^1_t(\dot B^{\frac{d}{q}}_{q,1})}\le\|w_0\|_{\dot B^{-2+\frac{d}{q}}_{q,1}}+C_1\|v\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{d}{p}}_{p,1})}\|v\|_{\mathcal{L}^1_t(\dot B^{\frac{d}{p}}_{p,1})},$$

where C_1 is a constant. Therefore, we can choose η small enough such that $\frac{C_1\eta}{\kappa}<1$, it follows from (2.23) that

$$\|w\|_{\mathcal{L}^\infty_t(\dot B^{-2+\frac{d}{q}}_{q,1})}+\kappa\|w\|_{\mathcal{L}^1_t(\dot B^{\frac{d}{q}}_{q,1})}\le\|w_0\|_{\dot B^{-\alpha+\frac{d}{q}}_{q,1}}+\frac{C_1\eta^2}{\kappa}\le\|w_0\|_{\dot B^{-\alpha+\frac{d}{q}}_{q,1}}+\eta. \tag{2.27}$$

Next we turn to bound v. Since the second equation of (1.2) is a linear equation for w, we intend to use some weighted function $f(t)$ to eliminate the difficulties caused by the nonlinear term $\nabla\cdot(w\nabla(-\Delta)^{-1}v)$ on the right-hand side of the first equation of (1.2) . Based on this idea, let us introduce the following weighted Chemin–Lerner type norm: for $f(t)\in L^1_{\text{loc}}(0,+\infty)$, $f(t)\ge0$, define

$$\|u\|_{\mathcal{L}^\rho_{t,f}(\dot B^s_{p,r})}:=\Big\{\sum_{j\in\mathbb{Z}}2^{srj}\big(\int_0^t f(\tau)\|\Delta_j u(\tau)\|^\rho_{L^p}d\tau\big)^{\frac{r}{\rho}}\Big\}^{\frac{1}{r}}$$

for $s\in\mathbb{R}$, $p\in[1,\infty]$, $\rho,r\in[1,\infty)$ with the standard modification if $\rho=\infty$ or $r=\infty$. Let $\lambda>0$ be a positive constant which the exact value will be specified later, we set

$$f(t):=\|w(\cdot,t)\|_{\dot B^{\frac{d}{q}}_{q,1}}\quad\text{and}\quad v_{\lambda,f}(x,t):=v(x,t)\exp(-\lambda\int_0^t f(\tau)d\tau).$$

It is easy to verify that $v_{\lambda,f}$ satisfies the following equation:

$$\partial_t v_{\lambda,f}+\lambda f(t)v_{\lambda,f}-\Delta v_{\lambda,f}=\nabla\cdot(w\nabla(-\Delta)^{-1}v_{\lambda,f}). \tag{2.28}$$

Applying the dyadic operator Δ_j to (2.28) and taking L^2 inner product of the resulting equation with $|\Delta_j v_{\lambda,f}|^{p-2}\Delta_j v_{\lambda,f}$, one has

$$\frac{1}{p}\frac{d}{dt}\|\Delta_j v_{\lambda,f}\|^p_{L^p}+\lambda f(t)\|\Delta_j v_{\lambda,f}\|^p_{L^p}-\int_{\mathbb{R}^d}\Delta\Delta_j v_{\lambda,f}|\Delta_j v_{\lambda,f}|^{p-2}\Delta_j v_{\lambda,f}dx$$

$$= \int_{\mathbb{R}^d} \Delta_j \nabla \cdot (w \nabla (-\Delta)^{-1} v_{\lambda,f}) |\Delta_j v_{\lambda,f}|^{p-2} \Delta_j v_{\lambda,f} dx.$$

Arguing like the derivation of (2.26) yields that

$$\begin{aligned} &\|v_{\lambda,f}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \lambda \|v_{\lambda,f}\|_{\mathcal{L}^1_{t,f}(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \kappa \|v_{\lambda,f}\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\ &\qquad \le \|v_0\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|w \nabla (-\Delta)^{-1} v_{\lambda,f}\|_{\mathcal{L}^1_t(\dot{B}^{-1+\frac{d}{p}}_{p,1})}. \end{aligned} \tag{2.29}$$

According to the Minkowski's inequality, it is readily to see that

$$\|w \nabla (-\Delta)^{-1} v_{\lambda,f}\|_{\mathcal{L}^1_t(\dot{B}^{-1+\frac{d}{p}}_{p,1})} \approx \int_0^t \|w(\tau) \nabla (-\Delta)^{-1} v_{\lambda,f}(\tau)\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} d\tau.$$

Then we can apply Lemma 2.3 by setting $s_1 = \frac{d}{q}$, $s_2 = -1 + \frac{d}{p}$, $p_1 = q$ and $p_2 = p$ to obtain that there exists a constant C_2 such that

$$\|w \nabla (-\Delta)^{-1} v_{\lambda,f}\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} \le C_2 \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}} \|\nabla (-\Delta)^{-1} v_{\lambda,f}\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} \le C_2 \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}} \|v_{\lambda,f}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}},$$

which implies that

$$\|w \nabla (-\Delta)^{-1} v_{\lambda,f}\|_{\mathcal{L}^1_t(\dot{B}^{-1+\frac{d}{p}}_{p,1})} \le C_2 \|v_{\lambda,f}\|_{\mathcal{L}^1_{t,f}(\dot{B}^{-2+\frac{d}{p}}_{p,1})}. \tag{2.30}$$

Taking (2.30) into (2.29) gives us to

$$\begin{aligned} &\|v_{\lambda,f}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \lambda \|v_{\lambda,f}\|_{\mathcal{L}^1_{t,f}(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \kappa \|v_{\lambda,f}\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \\ &\qquad \le \|v_0\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + C_2 \|v_{\lambda,f}\|_{\mathcal{L}^1_{t,f}(\dot{B}^{-2+\frac{d}{p}}_{p,1})}. \end{aligned} \tag{2.31}$$

Therefore, we can choose λ large enough, e.g., $\lambda = 2C_2$, to eliminate the second term on the right-hand side of (2.31):

$$\|v_{\lambda,f}\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + (\lambda - C_2) \|v_{\lambda,f}\|_{\mathcal{L}^1_{t,f}(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \kappa \|v_{\lambda,f}\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \le \|v_0\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}},$$

which combining (2.27) gives us to

$$\begin{aligned} \|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \kappa \|v\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} &\le \|v_0\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} \exp\big\{2C_2 \int_0^t \|w(\tau)\|_{\dot{B}^{\frac{d}{q}}_{q,1}} d\tau\big\} \\ &\le \|v_0\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} \exp\big\{\frac{2C_2}{\kappa} (\|w_0\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}} + \eta)\big\}. \end{aligned} \tag{2.32}$$

Finally we conclude that if we take C_0 large enough and c_0 small enough in (1.7), then it follows from (2.32) that

$$\|v\|_{\mathcal{L}^\infty_t(\dot{B}^{-2+\frac{d}{p}}_{p,1})} + \kappa \|v\|_{\mathcal{L}^1_t(\dot{B}^{\frac{d}{p}}_{p,1})} \le \frac{\eta}{2}$$

for all $t < T_\eta$, which contradicts with the maximality of T_η, thus $T^* = \infty$. We complete the proof of Theorem 1.3.

3 Weighted energy inequalities

In this section, we intend to establish two weighted energy inequalities in terms of the lower-order and higher-order derivative of solutions to the system (1.2). Denote

$$\mathcal{E}(t) := \|v(t)\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|w(t)\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}, \quad Y(t) := \int_0^t \big(\|v(\tau)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|w(\tau)\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\big) d\tau,$$

and let us introduce the following two weighted functions:

$$\widetilde{v}(x,t) := e^{-KY(t)}v(x,t), \quad \widetilde{w}(x,t) := e^{-KY(t)}w(x,t),$$

where $K > 0$ is a positive constant which the exact value will be designated later. Then we see that $(\widetilde{v}, \widetilde{w})$ satisfies the following equations:

$$\begin{cases} \partial_t \widetilde{v} - \Delta \widetilde{v} = \nabla \cdot (w\nabla(-\Delta)^{-1}\widetilde{v}) - KY'(t)\widetilde{v}, \\ \partial_t \widetilde{w} - \Delta \widetilde{w} = \nabla \cdot (v\nabla(-\Delta)^{-1}\widetilde{v}) - KY'(t)\widetilde{w}, \\ \widetilde{v}(x,0) = v_0(x), \quad \widetilde{w}(x,0) = w_0(x). \end{cases} \tag{3.1}$$

Applying Lemmas 2.1 and 2.2 by only neglecting the time variable and taking $\theta_1 = 1$ and $\theta_2 = 0$, one can easily derive the following two weighted bilinear estimates.

Lemma 3.1 *Let p, q be two positive numbers such that $1 \leq p, q < \infty$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{d} < \frac{1}{p} + \frac{1}{q}$. Then we have*

$$\|w\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} \lesssim Y'(t)\big(\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}\big). \tag{3.2}$$

Lemma 3.2 *Let $1 \leq p, q < \infty$ and $\frac{1}{q} - \frac{1}{p} < \frac{1}{d}$. Then we have*

$$\|u\nabla(-\Delta)^{-1}\widetilde{v} + v\nabla(-\Delta)^{-1}\widetilde{u}\|_{\dot{B}^{-1+\frac{d}{q}}_{q,1}} \lesssim Y'(t)\big(\|\widetilde{u}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}}\big). \tag{3.3}$$

Based on these two lemmas, one can get the following weighted energy estimates of the system (1.2).

Proposition 3.3 *Let the assumptions of Theorem 1.3 be in force. Then there exists a constant $K > 0$ such that for all $t \geq 0$, the unique solution (v, w) of the system (1.2) satisfies the following weighted energy inequality:*

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{E}(t)) + \kappa e^{-KY(t)}\big(\|v(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|w(t)\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\big) \leq 0, \tag{3.4}$$

where κ is a constant appeared in (2.24).

Proof. Firstly, for the first equation of (3.1), one can proceed as the derivation of (2.25) to obtain the estimate for v as

$$\frac{d}{dt}\|\Delta_j\widetilde{v}\|_{L^p} + \kappa 2^{2j}\|\Delta_j\widetilde{v}\|_{L^p} \lesssim 2^j\|\Delta_j(w\nabla(-\Delta)^{-1}\widetilde{v})\|_{L^p} - KY'(t)\|\Delta_j\widetilde{v}\|_{L^p}.$$

Multiplying the above inequality by $2^{(-2+\frac{d}{p})j}$ and taking l^1 norm to the resultant inequality, one can infer from Lemma 3.1 that

$$\frac{d}{dt}\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \kappa\|\widetilde{v}\|_{\dot{B}^{\frac{d}{p}}_{p,1}} \leq CY'(t)\big(\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}\big) - KY'(t)\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}}. \tag{3.5}$$

Secondly, similar to the derivation of (3.5), one can resort Lemma 3.2 to obtain the estimate for w as

$$\frac{d}{dt}\|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}} + \kappa\|\widetilde{w}\|_{\dot{B}^{\frac{d}{q}}_{q,1}} \leq CY'(t)\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} - KY'(t)\|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}. \tag{3.6}$$

Adding up (3.5) and (3.6) together implies that

$$\frac{d}{dt}(\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}) + \kappa\big(\|\widetilde{v}(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|\widetilde{w}(t)\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\big) \leq (2C - K)Y'(t)(\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}).$$

Finally, we can choose the constant K sufficiently large such that $K > 2C$, which directly leads to

$$\frac{d}{dt}(\|\widetilde{v}\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|\widetilde{w}\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}) + \kappa\big(\|\widetilde{v}(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|\widetilde{w}(t)\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\big) \leq 0.$$

This is exactly (3.4). We complete the proof of Proposition 3.3. □

Now we derive the lower-order and higher-order spatial derivatives of solutions to the system (1.2). Let ℓ be a real number and $1 \le r < \infty$, and let us denote

$$\mathcal{F}(t) := \|(v(t), w(t))\|_{\dot{B}^{\ell}_{r,1}}.$$

We obtain the following weighted energy inequality.

Proposition 3.4 *Let the assumptions of Theorem 1.3 be in force, and let us further assume that* $(v_0, w_0) \in \dot{B}^{\ell}_{r,1}(\mathbb{R}^d)$ *with* $1 \le r < \infty$, *and*

$$\ell + 1 + \frac{d}{p} > d\max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \textit{and} \quad \ell + 1 + \frac{d}{q} > d\max\{0, \frac{1}{q} + \frac{1}{r} - 1\}.$$

Then there exists a positive constant K *such that for all* $t \ge 0$, *the unique solution* (v, w) *of the system* (1.2) *satisfies the following weighted energy inequality:*

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{F}(t)) + \frac{\kappa}{2}e^{-KY(t)}\|(v(t), w(t))\|_{\dot{B}^{\ell+2}_{r,1}} \le 0. \tag{3.7}$$

Proof. Applying the operator $\Delta_j\Lambda^\ell$ to the first and second equation of (3.1), then taking L^2 inner product with $|\Delta_j\Lambda^\ell\widetilde{v}|^{r-2}\Delta_j\Lambda^\ell\widetilde{v}$ to the first resultant, and $|\Delta_j\Lambda^{\ell-1}\widetilde{w}|^{r-2}\Delta_j\Lambda^{\ell-1}\widetilde{w}$ to the second resultant, respectively, we obtain that

$$\begin{aligned}
&\frac{1}{r}\frac{d}{dt}\|(\Delta_j\Lambda^\ell\widetilde{v}, \Delta_j\Lambda^\ell\widetilde{w})\|_{L^r}^r - \big(\Delta\Delta_j\Lambda^\ell\widetilde{v}, |\Delta_j\Lambda^\ell\widetilde{v}|^{r-2}\Delta_j\Lambda^\ell\widetilde{v}\big) - \big(\Delta\Delta_j\Lambda^\ell\widetilde{w}, |\Delta_j\Lambda^\ell\widetilde{w}|^{r-2}\Delta_j\Lambda^\ell\widetilde{w}\big)\\
&= \big(\Delta_j\Lambda^\ell\nabla\cdot(w\nabla(-\Delta)^{-1}\widetilde{v}), |\Delta_j\Lambda^\ell\widetilde{v}|^{r-2}\Delta_j\Lambda^\ell\widetilde{v}\big) + \big(\Delta_j\Lambda^\ell\nabla\cdot(v\nabla(-\Delta)^{-1}\widetilde{v}), |\Delta_j\Lambda^\ell\widetilde{w}|^{r-2}\Delta_j\Lambda^\ell\widetilde{w}\big)\\
&\le \|\Delta_j\Lambda^\ell\nabla\cdot(w\nabla(-\Delta)^{-1}\widetilde{v})\|_{L^r}\|\Delta_j\Lambda^\ell\widetilde{v}\|_{L^r}^{r-1} + \|\Delta_j\Lambda^\ell\nabla\cdot(v\nabla(-\Delta)^{-1}\widetilde{v})\|_{L^r}\|\Delta_j\Lambda^\ell\widetilde{w}\|_{L^r}^{r-1}\\
&\quad - KY'(t)\|\Delta_j\Lambda^\ell\widetilde{v}\|_{L^r}^r - KY'(t)\|\Delta_j\Lambda^\ell\widetilde{w}\|_{L^r}^r,
\end{aligned}$$

which one can easily obtain the following inequality:

$$\frac{d}{dt}\|(\widetilde{v}, \widetilde{w})\|_{\dot{B}^{\ell}_{r,1}} + \kappa\|(\widetilde{v}, \widetilde{w})\|_{\dot{B}^{\ell+2}_{r,1}} \lesssim \|w\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} + \|v\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} - KY'(t)\|(\widetilde{v}, \widetilde{w})\|_{\dot{B}^{\ell}_{r,1}}. \tag{3.8}$$

To prove Proposition 3.4, the case $\ell > -1$ is simple due to the fact that $\dot{B}^{\ell+1}_{r,1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a Banach algebra and $\dot{B}^{\frac{d}{p}}_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{0}_{\infty,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d)$ for all $1 \le p < \infty$. Therefore, for the first term in the right-hand side of (3.8), since the solution (v, w) is bounded in $\dot{B}^{-2+\frac{d}{p}}_{p,1}(\mathbb{R}^d) \times \dot{B}^{-2+\frac{d}{q}}_{q,1}(\mathbb{R}^d)$ with respect to the time variable, we obtain

$$\begin{aligned}
\|w\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} &= \|w\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}}\\
&\lesssim \|w\|_{L^\infty}\|\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} + \|\widetilde{w}\|_{\dot{B}^{\ell+1}_{r,1}}\|\nabla(-\Delta)^{-1}v\|_{L^\infty}\\
&\lesssim \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\|\widetilde{v}\|_{\dot{B}^{\ell}_{r,1}} + \|\widetilde{w}\|_{\dot{B}^{\ell+1}_{r,1}}\|v\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}}\\
&\lesssim \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\|\widetilde{v}\|_{\dot{B}^{\ell}_{r,1}} + \|v\|^{\frac{1}{2}}_{\dot{B}^{-2+\frac{d}{p}}_{p,1}}\|v\|^{\frac{1}{2}}_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{w}\|^{\frac{1}{2}}_{\dot{B}^{\ell}_{r,1}}\|\widetilde{w}\|^{\frac{1}{2}}_{\dot{B}^{\ell+2}_{r,1}}\\
&\le \frac{\kappa}{2}\|\widetilde{w}\|_{\dot{B}^{\ell+2}_{r,1}} + C(\|v\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}})\|(\widetilde{v}, \widetilde{w})\|_{\dot{B}^{\ell}_{r,1}},
\end{aligned} \tag{3.9}$$

where we have used the following two interpolation inequalities according to (2.1):

$$\|v\|_{\dot{B}^{-1+\frac{d}{p}}_{p,1}} \lesssim \|v\|^{\frac{1}{2}}_{\dot{B}^{-2+\frac{d}{p}}_{p,1}}\|v\|^{\frac{1}{2}}_{\dot{B}^{\frac{d}{p}}_{p,1}}, \quad \|\widetilde{w}\|_{\dot{B}^{\ell+1}_{r,1}} \lesssim \|\widetilde{w}\|^{\frac{1}{2}}_{\dot{B}^{\ell}_{r,1}}\|\widetilde{w}\|^{\frac{1}{2}}_{\dot{B}^{\ell+2}_{r,1}}.$$

Similarly, one can derive that

$$\|v\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} \le \frac{\kappa}{2}\|\widetilde{v}\|_{\dot{B}^{\ell+2}_{r,1}} + C\|v\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{v}\|_{\dot{B}^{\ell}_{r,1}}. \tag{3.10}$$

On the other hand, in the case $\ell \le -1$, one can apply Lemma 2.3 to estimate the term $w\nabla(-\Delta)^{-1}\widetilde{v}$ by choosing $f = w$, $g = \nabla(-\Delta)^{-1}\widetilde{v}$, $s_1 = \frac{d}{q}$, $s_2 = \ell + 1$, $p_1 = q$ and $p_2 = r$:

$$\|w\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} \lesssim \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\|\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} \approx \|w\|_{\dot{B}^{\frac{d}{q}}_{q,1}}\|\widetilde{v}\|_{\dot{B}^{\ell}_{r,1}}; \tag{3.11}$$

while the term $v\nabla(-\Delta)^{-1}\widetilde{v}$ can be bounded by choosing $f = v$, $g = \nabla(-\Delta)^{-1}\widetilde{v}$, $s_1 = \frac{d}{p}$, $s_2 = \ell + 1$, $p_1 = p$ and $p_2 = r$:

$$\|v\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} \lesssim \|v\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\nabla(-\Delta)^{-1}\widetilde{v}\|_{\dot{B}^{\ell+1}_{r,1}} \approx \|v\|_{\dot{B}^{\frac{d}{p}}_{p,1}}\|\widetilde{v}\|_{\dot{B}^{\ell}_{r,1}}. \tag{3.12}$$

Therefore, plugging (3.9)–(3.12) into (3.8), we conclude that

$$\frac{d}{dt}\|(\widetilde{v},\widetilde{w})\|_{\dot{B}^{\ell}_{r,1}} + \frac{\kappa}{2}\|(\widetilde{v},\widetilde{w})\|_{\dot{B}^{\ell+2}_{r,1}} \le (2C-K)Y'(t)\|(\widetilde{v},\widetilde{w})\|_{\dot{B}^{\ell}_{r,1}}.$$

This yields (3.7) immediately by choosing K sufficiently large such that $K > 2C$, and we complete the proof of Proposition 3.4. □

4 Optimal decay rates of large solutions

We shall prove Theorems 1.4 and 1.5 by using the analytic approach illustrated in [29]. To prove Theorem 1.4, we first observe that Proposition 3.4 implies (1.10) directly, so it suffices to prove (1.11). To this end, for any $s > 0$ such that

$$-s + 1 + \frac{d}{p} > d\max\{0, \frac{1}{p} + \frac{1}{r} - 1\} \quad \text{and} \quad -s + 1 + \frac{d}{q} > d\max\{0, \frac{1}{q} + \frac{1}{r} - 1\},$$

one can choose $\ell = -s$ in Proposition 3.4 to obtain that for all $t \ge 0$,

$$\|(v(t),w(t))\|_{\dot{B}^{-s}_{r,1}} \le C\|(v_0,w_0)\|_{\dot{B}^{-s}_{r,1}} \le C_0. \tag{4.1}$$

This particularly yields (1.11) with $\ell = -s$. On the other hand, for any $\ell \in (-s, N]$, applying interpolation inequality (2.1) yields that

$$\|(v(t),w(t))\|_{\dot{B}^{\ell}_{r,1}} \le C\|(v(t),w(t))\|^{\frac{2}{\ell+s+2}}_{\dot{B}^{-s}_{r,1}}\|(v(t),w(t))\|^{1-\frac{2}{\ell+s+2}}_{\dot{B}^{\ell+2}_{r,1}}.$$

This together with (4.1) implies that

$$\|(v(t),w(t))\|_{\dot{B}^{\ell+2}_{r,1}} \ge C\|(v(t),w(t))\|^{-\frac{2}{\ell+s}}_{\dot{B}^{-s}_{r,1}}\|(v(t),w(t))\|^{1+\frac{2}{\ell+s}}_{\dot{B}^{\ell}_{r,1}} \ge C\|(v(t),w(t))\|^{1+\frac{2}{\ell+s}}_{\dot{B}^{\ell}_{r,1}},$$

which leads further to

$$\|(v(t),w(t))\|_{\dot{B}^{\ell+2}_{r,1}} \ge C\|(v(t),w(t))\|^{1+\frac{2}{\ell+s}}_{\dot{B}^{\ell}_{r,1}} = C\mathcal{F}(t)^{1+\frac{2}{\ell+s}}. \tag{4.2}$$

Plugging (4.2) into (3.7), there exists a constant C such that

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{F}(t)) + Ce^{-KY(t)}\mathcal{F}(t)^{1+\frac{2}{\ell+s}} \le 0,$$

which combining the fact that the function $Y(t)$ is positive along time evolution yields that

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{F}(t)) + C(e^{-KY(t)}\mathcal{F}(t))^{1+\frac{2}{\ell+s}} \le 0. \tag{4.3}$$

Solving this ordinary differential inequality directly yields that

$$\mathcal{F}(t) \le e^{KY(t)}\left(\mathcal{F}(0)^{-\frac{2}{\ell+s}} + \frac{2Ct}{\ell+s}\right)^{-\frac{\ell+s}{2}}.$$

Since the function $Y(t)$ is bounded, we know that for all $t \ge 0$, there exists a constant C_0 such that

$$\|(v(t),w(t))\|_{\dot{B}^{\ell}_{r,1}} \le C_0\,(1+t)^{-\frac{\ell+s}{2}}. \tag{4.4}$$

The proof of Theorem 1.4 is achieved.

Finally, we give the proof of Theorem 1.5. Since $1 \le r \le \min\{p,q\}$, we can infer from the imbedding results in Besov spaces that

$$\dot{B}^{-s}_{r,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{-s-d(\frac{1}{r}-\frac{1}{p})}_{p,1}(\mathbb{R}^d) \quad \text{and} \quad \dot{B}^{-s}_{r,1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{-s-d(\frac{1}{r}-\frac{1}{q})}_{q,1}(\mathbb{R}^d),$$

which together with (4.1) yields that for all $t \ge 0$,

$$\|v(t)\|_{\dot{B}^{-s-d(\frac{1}{r}-\frac{1}{p})}_{p,1}} + \|w(t)\|_{\dot{B}^{-s-d(\frac{1}{r}-\frac{1}{q})}_{q,1}} \le C_0. \tag{4.5}$$

On the other hand, for any $s \ge \max\{0, 2-\frac{d}{r}\}$, applying the interpolation inequality (2.1) yields that

$$\begin{aligned}
\|v(t)\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} &\le C\|v(t)\|^{\frac{2}{s+\frac{d}{r}}}_{\dot{B}^{-s-d(\frac{1}{r}-\frac{1}{p})}_{p,1}}\|v(t)\|^{1-\frac{2}{s+\frac{d}{r}}}_{\dot{B}^{\frac{d}{p}}_{p,1}},\\
\|w(t)\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}} &\le C\|w(t)\|^{\frac{2}{s+\frac{d}{r}}}_{\dot{B}^{-s-d(\frac{1}{r}-\frac{1}{q})}_{q,1}}\|w(t)\|^{1-\frac{2}{s+\frac{d}{r}}}_{\dot{B}^{\frac{d}{q}}_{q,1}}.
\end{aligned}$$

This together with (4.5) implies that

$$\begin{aligned}
\|v(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} &\ge C\|v(t)\|^{-\frac{2}{s+\frac{d}{r}-2}}_{\dot{B}^{-s-1-d(\frac{1}{r}-\frac{1}{p})}_{p,1}}\|v(t)\|^{1+\frac{2}{s+\frac{d}{r}-2}}_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} \ge C\|v(t)\|^{1+\frac{2}{s+\frac{d}{r}-2}}_{\dot{B}^{-2+\frac{d}{p}}_{p,1}},\\
\|w(t)\|_{\dot{B}^{\frac{d}{q}}_{q,1}} &\ge C\|w(t)\|^{-\frac{2}{s+\frac{d}{r}-2}}_{\dot{B}^{-s-1-d(\frac{1}{r}-\frac{1}{q})}_{q,1}}\|w(t)\|^{1+\frac{2}{s+\frac{d}{r}-2}}_{\dot{B}^{-2+\frac{d}{q}}_{q,1}} \ge C\|w(t)\|^{1+\frac{2}{s+\frac{d}{r}-2}}_{\dot{B}^{-2+\frac{d}{q}}_{q,1}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|v(t)\|_{\dot{B}^{\frac{d}{p}}_{p,1}} + \|w(t)\|_{\dot{B}^{\frac{d}{q}}_{q,1}} &\ge C(\|v(t)\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|w(t)\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}})^{1+\frac{2}{s+\frac{d}{r}-2}}\\
&= C\mathcal{E}(t)^{1+\frac{2}{s+\frac{d}{r}-2}}.
\end{aligned} \tag{4.6}$$

Plugging (4.6) into (3.4), by using the function $Y(t)$ is positive along time evolution, we obtain

$$\frac{d}{dt}(e^{-KY(t)}\mathcal{E}(t)) + C(e^{-KY(t)}\mathcal{E}(t))^{1+\frac{2}{s+\frac{d}{r}-2}} \le 0. \tag{4.7}$$

Solving this ordinary differential inequality, we obtain

$$\mathcal{E}(t) \le e^{KY(t)}\left(\mathcal{E}(0)^{-\frac{2}{s+\frac{d}{r}-2}} + \frac{2Ct}{s+\frac{d}{r}-2}\right)^{-\frac{s+\frac{d}{r}-2}{2}}.$$

Since the function $Y(t)$ is bounded, there exists a constant C_0 such that for all $t \ge 0$,

$$\|v(t)\|_{\dot{B}^{-2+\frac{d}{p}}_{p,1}} + \|w(t)\|_{\dot{B}^{-2+\frac{d}{q}}_{q,1}} \le C_0\,(1+t)^{-\frac{s+\frac{d}{r}-2}{2}}. \tag{4.8}$$

Notice that (4.8) gives in particular (1.12) with $\ell = -2 + \frac{d}{p}$, and (1.13) with $\ell = -2 + \frac{d}{q}$, respectively. Finally, for any $\ell \in [-s - d(\frac{1}{r} - \frac{1}{p}), -2 + \frac{d}{p})$, by using the interpolation inequality (2.1) again, one obtains that

$$\|v(t)\|_{\dot{B}^{\ell}_{p,1}} \leq C\|v(t)\|^{\frac{\frac{d}{p}-\ell-2}{s+\frac{d}{r}-2}}_{\dot{B}^{-s-d(\frac{1}{r}-\frac{1}{p})}_{p,1}} \|v(t)\|^{\frac{\ell+s+d(\frac{1}{r}-\frac{1}{p})}{s+\frac{d}{r}-2}}_{\dot{B}^{-2+\frac{d}{p}}_{p,1}},$$

which combining (4.5) and (4.8) implies that

$$\|v(t)\|_{\dot{B}^{\ell}_{p,1}} \leq C_0(1+t)^{-(\frac{\ell+s}{2})-\frac{d}{2}(\frac{1}{r}-\frac{1}{p})}.$$

Similarly, for any $\ell \in [-s - d(\frac{1}{r} - \frac{1}{q}), -2 + \frac{d}{q})$, one has

$$\|w(t)\|_{\dot{B}^{\ell}_{q,1}} \leq C\|w(t)\|^{\frac{\frac{d}{q}-\ell-2}{s+\frac{d}{r}-2}}_{\dot{B}^{-s-d(\frac{1}{r}-\frac{1}{q})}_{q,1}} \|w(t)\|^{\frac{\ell+s+d(\frac{1}{r}-\frac{1}{q})}{s+\frac{d}{r}-2}}_{\dot{B}^{-2+\frac{d}{q}}_{q,1}},$$

which combining (4.5) and (4.8) again implies that

$$\|w(t)\|_{\dot{B}^{\ell}_{q,1}} \leq C_0(1+t)^{-(\frac{\ell+s}{2})-\frac{d}{2}(\frac{1}{r}-\frac{1}{q})}.$$

We complete the proof of Theorem 1.5.

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