CRITICAL EXPONENT OF FUJITA-TYPE FOR SEMILINEAR WAVE EQUATIONS IN FRIEDMANN-LEMAÎTRE-ROBERTSON-WALKER SPACETIME

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ABSTRACT. We consider the nonlinear massless wave equation belonging to some family of the Friedmann–Lemaître–Robertson–Walker (FLRW) spacetime. We prove the global in time small data solutions for supercritical powers in the case of decelerating expansion universe.

1. INTRODUCTION

In this paper, we prove the global existence (in time) of small data solutions to the Cauchy problem for the semilinear wave equation with scale-invariant damping and decreasing in time propagation speed

$$
\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t}u_t(t,x) = f(u(t,x)), & t \geq 0,\ x \in \mathbf{R}^n, \\ u(0,x) = 0 = u_0(x), & x \in \mathbf{R}^n, \\ u_t(0,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{1}
$$

with $\ell \in (0,1)$ and $\beta > 0$. We assume that $f(u) = |u|^p$ for some $p > 1$ or, more in general, f verifies the following local Lipschitz-type condition

$$
|f(u) - f(v)| \leq C\,|u - v|\,\big(|u|^{p-1} + |v|^{p-1}\big). \tag{2}
$$

The case $\beta = 2$ in (1) is well known as FLRW spacetime model for the decelerating expansion universe, whereas in the particular case $\ell = \frac{2}{3}$, (1) is the nonsingular covariant massless field in the Einstein- de Sitter spacetime (see [15]).

Let us start with the state of the art in the case $\ell = 0$. If $\beta \geq \frac{5}{3}$ for $n = 1$, $\beta \geq 3$ for $n = 2$, or $\beta \geq n+2$ for $n \geq 3$, by assuming data in the energy spaces with additional regularity $L^1(R^n)$, the global (in time) existence result for (1) was proved in [3] for $p > p_F(n) \doteq 1 + \frac{2}{n}$, the well known Fujita index [14]. The exponent $p_F(n)$ is critical for this model, that is, for $p \leq p_F(n)$ and suitable, arbitrarily small data, there exists no global weak solution [8]. As conjectured in [7] and [9], if β becomes smaller with respect to the space dimension n, the critical exponent increase to $\max\{p_S(n+\beta), p_F(n)\}$, where p_S is the Strauss exponent for the semilinear undamped wave equation [18], [24]. In [20] the authors proved a blow-up result and gave the upper bound for the lifespan of solutions to (1) for $1 < p \leq p_S(n+\beta)$ and $\beta \in [0, \beta_\star)$, with $\beta_\star = \frac{n^2+n+2}{n+2}$. It is worth noticing that if $\beta \in [0, \beta_\star)$, then $p_F(n) < p_S(n+\beta)$ and, $p_F(n) = p_S(n+\beta_\star)$.
Recently, in [4] it is proved, in the case $\ell = 0$, that the critical exponent to (1) is equal to $\max\{p_S(n+\beta), p_F(n)\}$ for $n = 1$ and, in [5] it is proved the global existence

2020 *Mathematics Subject Classification.* 35A01, 35B33, 35L05, 35L71.
Key words and phrases. semilinear wave equations, critical exponent, global existence, small data solutions.

of small data solutions for $p > p_F(n)$ and $\beta \geq n$ in space dimension $2 \leq n \leq 5$. As far as we know, it is still a open problem to prove global existence of small data solutions for $p > p_F(n)$ in the cases $\beta_\star < \beta < n$ for $n \geq 3$ and for $p > p_S(n+\beta)$ for $0 < \beta < \beta_\star$ for $n \geq 2$.

For $\ell \in [0,1)$, $\beta \geq 0$ and $n \geq 2$, let $p_S(n,\ell,\beta)$ be the positive root of the quadratic equation

$$\left(n-1+\frac{\beta-\ell}{1-\ell}\right)p^2-\left(n+1+\frac{\beta+3\ell}{1-\ell}\right)p-2=0.$$

Recently, in [21] and, independently in [25] and [26], the authors have proved blow-up in a finite time and upper estimates of the lifespan for solutions to (1) for

$$1 < p \leq \max\{p_F(n(1-\ell)), p_S(n,\ell,\beta)\}.$$

A blow-up result for $\beta = 2$ and $\ell \in (0,1)$ in (1) was also proved in [15].
It is worth noticing that if $p_F(n(1-\ell)) = p_S(n,\ell,\beta_c(n,\ell))$, where

$$\beta_c(n,\ell) \doteq \ell+(1-\ell)\left(n+1-\frac{2}{p_F(n(1-\ell))}\right)=\frac{n^2(1-\ell)^2+n(1-\ell)(1+2\ell)+2}{2+n(1-\ell)}.$$

In particular, if $\beta \geq \beta_c(n,\ell)$, then $p_S(n,\ell,\beta) \leq p_F(n(1-\ell))$.

In [2], the authors proposed a classification of non-effective and effective dissipation, respectively, for the damped wave equation

$$u_{tt}(t,x)-a^2(t)\Delta u(t,x)+b(t)u_t(t,x)=0$$

with increasing speed of propagation. The authors derived sharp estimates for solutions to the Cauchy problem and, in the case of effective dissipation, i.e.,

$$b(t)\frac{A(t)}{a(t)} \to \infty,\ as\ t \to \infty, \qquad A(t)=1+\int_0^t a(\tau)\,d\tau,$$

derived global existence (in time) results for the semilinear problem with power nonlinearities. A similar classification was introduced in [12] in the case $a \in L^1$. A natural generalization for the model (1) is to consider a positive and decreasing speed of propagation $a(t)$, with $a \notin L^1$. But in this paper we restrict ourselves to the case in which a is a irrational function, since it includes interesting models by itself, for instance, if $\ell = \frac{2}{3}$ in (1), the considered model coincides with the non-singular wave equation in the Einstein de Sitter space-time ([16], [17]).

The main goal in this paper is to prove, under the assumption of small initial data in $L^1(\mathbf{R}^n) \cap H^{k-1}(\mathbf{R}^n)$, $k > 1$, the global existence (in time) of solutions to (1) for supercritical powers $p > p_F(n(1-\ell))$, by supposing that $\beta \geq \beta_c(n,\ell)$. Combine the obtained results in this paper with the blow-up results derived in [26] we conclude that $p_F(n,\ell) = 1+\frac{2}{n(1-\ell)}$ is the critical exponent for the global in time existence of solutions for $\beta \geq \beta_c(n,\ell)$.
As far as we know, it is still a open problem to prove global existence of small data solutions to (1) for $p > p_S(n,\ell,\beta)$ and $0 < \beta \leq \beta_c(n,\ell)$. It is expected that a similar approach to those used for the semilinear free wave equation may be appropriate to decrease values of β and to overcame some gaps that appear in this paper.

2. Main results

To simplify the writing, from now we consider

$$p_c(n,\ell) \doteq p_F(n(1-\ell)) = 1 + \frac{2}{n(1-\ell)}. \tag{3}$$

In the next theorems, due to the fact that $p_c(n,\ell) \to \infty$ as $\ell \to 1$, the choice of the spaces of solutions is related to fixed ranges for $\ell \in [0,1)$ and the space dimensions $n \geq 2$. To state our first result, let us define the following parameters

$$\bar{q} \doteq \frac{2(np_c(n,\ell)-1)}{n+1}, \qquad q_\sharp \doteq \frac{2(n+1)}{n-1}. \tag{4}$$

Theorem 2.1. *Let ℓ be such that*

$$\begin{cases} 0 \leq \ell < 1 - \frac{n-1}{2n}, & \text{if } 2 \leq n \leq 5 \\ 1 - \frac{2(n+1)}{n(n-3)} \leq \ell < 1 - \frac{n-1}{2n}, & \text{if } 6 \leq n \leq 8 \end{cases}$$

and

$$\beta \geq \ell + (n+1)(1-\ell) - \frac{2}{\bar{q}}(1-\ell),$$

with $\bar{q} \in [p_c(n,\ell), q_\sharp]$, where $p_c(n,\ell)$, $q_\sharp$ and $\bar{q}$ are given by (3) *and* (4)*. If*

$$p_c(n,\ell) < p \leq \frac{4p_c(n,\ell)}{n+3} + 1,$$

then there exists $\delta > 0$ such that for any initial data

$$u_1 \in \mathcal{D} = L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n), \qquad ||u_1||_{\mathcal{D}} \leq \delta,$$

there exists a unique weak solution $u \in C([0,\infty), L^{p_c}(\mathbf{R}^n) \cap L^{q_\sharp}(\mathbf{R}^n))$ to (1). Moreover, the solution satisfies the following estimates for $p_c \leq q \leq q_\sharp$:
If $\beta > \ell + (n+1)(1-\ell) - \frac{2}{q}(1-\ell)$ then [1]

$$||u(t,\cdot)||_{L^q} \lesssim (1+t)^{-n\left(1-\frac{1}{q}\right)(1-\ell)} ||u_1||_{\mathcal{D}} \tag{5}$$

whereas if $\ell + (n+1)(1-\ell) - \frac{2}{\bar{q}}(1-\ell) \leq \beta \leq \ell + (n+1)(1-\ell) - \frac{2}{q}(1-\ell)$, then for any $\varepsilon > 0$

$$||u(t,\cdot)||_{L^q} \lesssim (1+t)^{[\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)](1-\ell)-\frac{\beta-\ell}{2(1-\ell)}} ||u_1||_{\mathcal{D}}. \tag{6}$$

Remark 2.1. One of the crucial property in the proof of Theorem 2.1 is that $r(q)p_c(n,\ell) < q_\sharp$, for all $p_c(n,\ell) \leq q \leq q_\sharp$, with

$$\frac{1}{r(q)} \doteq \frac{1}{2n} + \frac{1}{2} + \frac{1}{nq}. \tag{7}$$

This condition is satisfied under some condition on ℓ, namely,

$$r(q_\sharp)p_c(n,\ell) < q_\sharp \Leftrightarrow \ell < 1 - \frac{4}{q_\sharp(n+1) - 2(n-1)} = 1 - \frac{n-1}{2n}.$$

Since $r(q) \leq r(q_\sharp)$ for all $p_c(n,\ell) \leq q \leq q_\sharp$, we also have

$$r(q)p_c(n,\ell) < q_\sharp \Leftrightarrow \ell < 1 - \frac{n-1}{2n}.$$

In particular, it implies the existence of $\bar{q}$ satisfying $\bar{q} < q_\sharp$.
For instance, for $\ell \in \left[0, \frac{3}{4}\right)$ if $n = 2$, and for $\ell \in \left[0, \frac{2}{3}\right)$ if $n = 3$.

[1] Let $f, g : \Omega \subset \mathbf{R}^n \to \mathbf{R}$ be two functions. From now one we use the notation $f \lesssim g$ if there exists a constant $C > 0$ such that $f(y) \leq Cg(y)$ for all $y \in \Omega$.

Remark 2.2. By using that

$$r(q_\sharp)p_c(n,\ell) < q_\sharp \Longleftrightarrow p_c(n,\ell)\left(1-\frac{r(q_\sharp)}{q_\sharp}\right)+1 = \frac{4p_c(n,\ell)}{n+3}+1 < \frac{q_\sharp}{r(q_\sharp)} = \frac{n+3}{n-1},$$

with $r(q)$ given by (7), we conclude that the upper bound for p in Theorem 2.1 satisfies

$$p \le \min\left\{p_c(n,\ell)\left(1-\frac{r(q_\sharp)}{q_\sharp}\right)+1, \frac{q_\sharp}{r(q_\sharp)}\right\} = \frac{4p_c(n,\ell)}{n+3}+1.$$

Remark 2.3. Taking into account that L^1-L^q linear estimates in Corollary 2 of [5] hold only for $\frac{2(n-1)}{n+1}\le q\le q_\sharp$, in the proof of Theorem 2.1 we have to assume

$$p_c(n,\ell) \ge \frac{2(n-1)}{n+1}.$$

Hence a restriction from below in ℓ is also needed, namely, $\ell \ge 1-\frac{2(n+1)}{n(n-3)}$, $n\ge 4$. This condition is true for $\ell=0$ under the assumption $2\le n\le 5$ in Theorem 2.1.

Remark 2.4. Let $\frac{n}{r(q)} = \frac{1}{2}+\frac{n}{2}+\frac{1}{q}$. Condition (4) means that $\bar q$ is defined by $p_c(n,\ell)r(\bar q)=\bar q$. In particular, thanks to $(n-1)p_c(n,\ell)\ge 1$ for $n\ge 2$ it holds

$$\begin{aligned}
\frac{1}{\bar q}-\frac{n}{p_c(n,\ell)r(q_\sharp)}+\frac{n-1}{q_\sharp} &= \frac{n}{p_c(n,\ell)r(\bar q)}-\frac{n-1}{\bar q}-\frac{n}{p_c(n,\ell)r(q_\sharp)}+\frac{n-1}{q_\sharp}\\
&= \frac{1}{p_c(n,\ell)}\left(\frac{n}{r(\bar q)}-\frac{n}{r(q_\sharp)}\right)-(n-1)\left(\frac{1}{\bar q}-\frac{1}{q_\sharp}\right)\\
&= \left(\frac{1}{p_c(n,\ell)}-n+1\right)\left(\frac{1}{\bar q}-\frac{1}{q_\sharp}\right)\le 0.
\end{aligned}$$

In the case $\bar q = p_c(n,\ell)$, Theorem 2.1 yields the threshold value

$$\beta \ge \ell+(1-\ell)\left(n+1-\frac{2}{p_c(n,\ell)}\right) = \beta_c(n,\ell).$$

If $\ell=0$ then $p_c(n,0)=1+\frac{2}{n}$ and $\bar q=2$, so we have to assume $\beta\ge n+1-\frac{2}{\bar q}=n$. In particular for $\ell=0$ and $n=2$, this condition coincides with the threshold value $\beta\ge\beta_c(2,0)=2$.

In our results, the novelty is to use higher regularity $H^k(\mathbf{R}^n), k>\frac{n}{2}$, in order to consider larger values on the parameter ℓ and to relax the condition in the upper bound for p in Theorem 2.1. In particular, it is possible to include for $n=3$ the speed of propagation $a(t)=(1+t)^{-\frac{2}{3}}$, that appears in the well known Einstein de Sitter model for decelerating expanding universe [16].
In the next two theorems we restrict our analysis to the case of small values for β, whereas the simple case of large values for β is treated in Theorem 2.4.

Theorem 2.2. *Let $\ell\in\left(1-\frac{2}{n},\frac{2}{n}\right)$ for $n=2$ or $n=3$, and $k\doteq 1+\frac{n\ell}{2}$ such that either $\frac{n}{2}<k<2$ or $k=2$, i.e., $\ell=\frac{2}{n}$, for $n=3$. If $\ell+n(1-\ell)(1+\ell)\le\beta<2-\ell+n(1-\ell)(1+\ell)$ and $p>p_c(n,\ell)$, with $p_c(n,\ell)$ given by* (3)*, then there exists $\delta>0$ such that for any initial data*

$$u_1\in\mathcal{D}=H^{k-1}(\mathbf{R}^n)\cap L^1(\mathbf{R}^n), \qquad ||u_1||_{\mathcal{D}}\le\delta,$$

there exists a unique energy solution $u \in C([0,\infty), H^k(\mathbf{R}^n))$ to (1), which satisfies the following estimates

$$||u(t,\cdot)||_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}||u_1||_{\mathcal{D}},$$

$$||u(t,\cdot)||_{\dot{H}^{k-1}} \lesssim h(t)||u_1||_{\mathcal{D}},$$

with

$$h(t) = \begin{cases} (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}, & \ell + n(1-\ell)(1+\ell) < \beta < 2-\ell+n(1-\ell)(1+\ell), \\ (1+t)^{\frac{\ell-\beta}{2}}(\ln(e+t))^{\frac{1}{2}}, & \beta = \ell + n(1-\ell)(1+\ell), \end{cases}$$

and

$$||u(t,\cdot)||_{\dot{H}^k} \lesssim (1+t)^{\frac{\ell-\beta}{2}}||u_1||_{\mathcal{D}}.$$

Remark 2.5. We point out that

$$\ell > 1 - \frac{2}{n} \iff k = 1 + \frac{n\ell}{2} > \frac{n}{2} \iff p_c(n,\ell) > 2$$

for $n \geq 2$. Moreover, $k \leq p_c(n,\ell)$ iff $\ell(1-\ell)n^2 \leq 4$, in particular, this is true if $\ell \in \left(1-\frac{2}{n}, 1\right)$ for $n = 2,3$.

Example 2.1. If $\ell = \frac{2}{3}$, the conclusion of Theorem 2.2 holds for $n = 2,3$ with $\beta \geq \frac{1}{3}\left(2 + \frac{5n}{3}\right)$.

In the following result we may consider the case $\ell \in \left(\frac{2}{3}, 1\right)$ for $n = 3$, by looking for solutions with additional regularity $H^{\kappa(r_1)-1,r_2}(\mathbf{R}^3)$, with $\kappa(r_1) = 3\left(\frac{1}{2} - \frac{1}{r_1}\right)$ and r_1, r_2 satisfying

$$r_1 > \frac{2(3\ell-1)}{1-\ell}, \qquad 2 < r_2 < \frac{6}{2\kappa(r_1)-1}. \tag{8}$$

Theorem 2.3. *Let $n = 3$, $\ell \in \left(\frac{2}{3}, 1\right)$, r_1, r_2 satisfying (8) with $\kappa(r_1) = 3\left(\frac{1}{2} - \frac{1}{r_1}\right)$. If $\ell + 6(1-\ell)\left(1 - \frac{1}{r_1}\right) \leq \beta < 2 - \ell + 3(1-\ell)(1+\ell)$ and*

$$p_c(3,\ell) < p < 1 + \frac{r_1(2-\kappa(r_1))}{3}, \tag{9}$$

with $p_c(3,\ell)$ given by (3), then there exists $\delta > 0$ such that for any initial data

$$u_1 \in \mathcal{D} = H^{\kappa(r_1)-1}(\mathbf{R}^3) \cap L^1(\mathbf{R}^3), \qquad ||u_1||_{\mathcal{D}} \leq \delta,$$

there exists a unique energy solution $u \in C([0,\infty), H^{\kappa(r_1)}(\mathbf{R}^3) \cap \dot{H}^{\kappa(r_1)-1,r_2}(\mathbf{R}^3))$ to (1), which satisfies the following estimates

$$||u(t,\cdot)||_{\dot{H}^{j\kappa(r_1)}} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+j\kappa(r_1)\right)}||u_1||_{\mathcal{D}}, \quad j = 0,1;$$

and

$$\|u(t,\cdot)\|_{\dot{H}^{\kappa(r_1)-1,r_2}} \lesssim (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}||u_1||_{\mathcal{D}}.$$

Remark 2.6. We remark that (9) is not empty due to

$$r_1 > \frac{2(3\ell-1)}{1-\ell} \iff r_1(2-\kappa(r_1)) > \frac{2}{1-\ell}.$$

Since we are interested into consider small values of β, we take the smallest possible value for r_1.

Remark 2.7. From $r_1 > \frac{2(3\ell-1)}{1-\ell}$ it follows that $\kappa(r_1) > \frac{6\ell-3}{3\ell-1}$ and for $\ell \in \left(\frac{2}{3}, 1\right)$ we have

$$\frac{6\ell-3}{3\ell-1} > \frac{3\ell}{2} \Longleftrightarrow 3\ell^2 - 5\ell + 2 < 0.$$

Therefore, for $\ell \in \left(\frac{2}{3}, 1\right)$ it holds that $\kappa(r_1) > k-1$, with k given by Theorem 2.2, in particular,

$$\ell + 6(1-\ell)\left(1 - \frac{1}{r_1}\right) = \ell + 3(1-\ell) + 2(1-\ell)\kappa(r_1) > \ell + 3(1-\ell)(1+\ell).$$

In the last result we also consider higher space dimension, but due to the technique some additional lower bound for p and β come into play:

Theorem 2.4. *Let $\ell \in \left(1 - \frac{2}{n}, 1\right)$ for $n \geq 2$ and $k \doteq 1 + \frac{n\ell}{2}$.*
If $\beta \geq 2 - \ell + n(1-\ell)(1+\ell)$ and $p > \max\{p_c(n,\ell), k\}$, with $p_c(n,\ell)$ given by (3), then there exists $\delta > 0$ such that for any initial data

$$u_1 \in \mathcal{D} = H^{k-1}(\mathbf{R}^n) \cap L^1(\mathbf{R}^n), \qquad ||u_1||_{\mathcal{D}} \leq \delta,$$

there exists a unique energy solution $u \in C([0,\infty), H^k(\mathbf{R}^n))$ to (1), which satisfies the following estimates

$$||u(t,\cdot)||_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)} ||u_1||_{\mathcal{D}}; \tag{10}$$

and

(11)
$$||u(t,\cdot)||_{\dot{H}^k} \lesssim ||u_1||_{\mathcal{D}} \begin{cases} (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}, & \beta > \ell + n(1-\ell) + 2k(1-\ell) \\ (1+t)^{\frac{\ell-\beta}{2}} (\ln(e+t))^{\frac{1}{2}}, & \beta = \ell + n(1-\ell) + 2k(1-\ell). \end{cases}$$

3. Representation of solutions to the linear Cauchy problem

Let $s \geq 0$ be a parameter. We need to solve a family of parameter dependent linear ($f(u) = 0$) Cauchy problems corresponding to (1):

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t} u_t(t,x) = 0, \ t \geq s \\ u(s,x) = g_1(s,x) \\ u_t(s,x) = g_2(s,x). \end{cases} \tag{12}$$

We begin by applying Fourier transform to the solution of the problem (12). We denote the partial Fourier transform of a tempered distribution or of a function $u : \mathbf{R_0^+} \times \mathbf{R^n} \to \mathbf{C}$ with respect to x, by $\hat{u} = \mathscr{F}u$ or $\hat{u}(t,\cdot) = \mathscr{F}u(t,\cdot)$. The notation $\mathscr{F}^{-1}$ denotes the inverse Fourier transform, in the appropriate sense.
Following as in [12], we make the change of variables $\tau = \frac{(1+t)^{1-\ell}}{1-\ell}|\xi|$ and $v(\tau, s) = \hat{u}(t,s,\xi)$. If $u(t,s,x)$ is the solution of (12) then $v(\tau,s)$ satisfies

$$\begin{cases} v''(\tau) + \frac{\beta-\ell}{(1-\ell)\tau} v'(\tau) + v(\tau) = 0 \\ v\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) = \hat{g_1}(s,\xi) \\ v'\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) = \frac{\hat{g_2}(s,\xi)}{|\xi|}. \end{cases} \tag{13}$$

Moreover, if we are looking for a solution in the product form $v(\tau,s) = \tau^\rho w(\tau,s)$, then $w(\tau,s)$ is a solution of the Bessel's differential equation of order $\pm\rho$:

$$\tau^2 w''(\tau) + \tau w'(\tau) + (\tau^2 - \rho^2) w(\tau) = 0 \tag{14}$$

where $\rho = \frac{1-\beta}{2(1-\ell)}$. We will use the set of Hankel functions, $\{H^+_\rho(\tau), H^-_\rho(\tau)\}$ to write the general solution of the ODE (14). First, according to [27] we introduce an auxiliary function

$$\psi_{j,\gamma,\delta}(t,s,\xi) = |\xi|^j \begin{vmatrix} H^-_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & H^-_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ H^+_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & H^+_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix} \tag{15}$$

where j, γ, δ, s are real parameters. Since $H^\pm_\gamma = J_\gamma \pm iY_\gamma$, we can rewrite it in the form

$$\psi_{j,\gamma,\delta}(t,s,\xi) = 2i|\xi|^j \begin{vmatrix} J_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ Y_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & Y_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix} \tag{16}$$

if $\gamma, \gamma+\delta \in \mathbf{Z}$, or

$$\psi_{j,\gamma,\delta}(t,s,\xi) = 2i\csc(\gamma\pi)|\xi|^j \begin{vmatrix} J_{-\gamma}\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{-\gamma-\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \\ (-1)^\delta J_\gamma\left(\frac{(1+s)^{1-\ell}|\xi|}{1-\ell}\right) & J_{\gamma+\delta}\left(\frac{(1+t)^{1-\ell}|\xi|}{1-\ell}\right) \end{vmatrix} \tag{17}$$

if $\gamma, \gamma+\delta \notin \mathbf{Z}$, where J_γ, Y_γ denote the Bessel functions of the first and second kind, respectively. We then determine the Fourier multipliers and the first order partial derivatives with respect to t to represent $\hat{u}$ and $\hat{u}_t$ in the explicit form.

Lemma 3.1. *(see [10]) Let $u(t,s,x)$ be the solution of (12). Then the partial Fourier transform of u with respect to x, $\hat{u}$, is represented by*

$$\hat{u}(t,s,\xi) = m_0(t,s,\xi)\hat{g}_1(s,\xi) + m_1(t,s,\xi)\hat{g}_2(s,\xi) \tag{18}$$

with Fourier multipliers and the first order partial derivatives with respect to t given by

$$\partial_t^j m_k = \frac{(-1)^k \pi i}{4(1-\ell)}(1+s)^{1+(\beta-1)/2}(1+t)^{(1-\beta)/2-j\ell}\psi_{1+j-k,\rho+k-1,1-j-k} \tag{19}$$

where $\rho = \frac{1-\beta}{2(1-\ell)}$, $k, j = 0, 1$.

4. $L^p - L^q$ estimates

In order to obtain an estimate of (18) we have to distinguish between large and small τ values. We divide the extended phase space $\mathbf{R_0^+} \times \mathbf{R_0^+} \times \mathbf{R^+}$ into three zones. We define the zone of high frequencies

$$Z_1 = \{(t,s,|\xi|) : |\xi| \geq (1+s)^{\ell-1}\},$$

and the zones of low frequencies

$$Z_2 = \{(t,s,|\xi|) : (1+t)^{\ell-1} \leq |\xi| \leq (1+s)^{\ell-1}\},$$
$$Z_3 = \{(t,s,|\xi|) : |\xi| \leq (1+t)^{\ell-1}\},$$

separated by the boundary $\{(t,s,|\xi|) : (1+t)^{1-\ell}|\xi| = (1-\ell)\}$.
Given a cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ satisfying $\begin{cases} 1 & \text{if } r \leq \frac{1}{2} \\ 0 & \text{if } r \geq 1 \end{cases}$ we define

$$\chi_1(s,\xi) = 1 - \chi((1+s)^{1-\ell}|\xi|),$$
$$\chi_2(t,s,\xi) = \chi((1+s)^{1-\ell}|\xi|)\left(1 - \chi((1+t)^{1-\ell}|\xi|)\right),$$
$$\chi_3(t,s,\xi) = \chi((1+s)^{1-\ell}|\xi|)\chi((1+t)^{1-\ell}|\xi|),$$

such that $\chi_1+\chi_2+\chi_3=1$.

Lemma 4.1. *Let $\ell\in(0,1)$, $\gamma\neq 0$, and $k\geq 0$. It holds*

$$|\xi|^k|\psi_{0,\gamma,0}(t,s,\xi)|\lesssim \left\{\begin{array}{ll} |\xi|^{k-1}(1+s)^{(\ell-1)/2}(1+t)^{(\ell-1)/2} & \text{if } (t,s,\xi)\in Z_1\\ |\xi|^{k-|\gamma|-1/2}(1+s)^{(\ell-1)|\gamma|}(1+t)^{(\ell-1)/2} & \text{if } (t,s,\xi)\in Z_2\\ |\xi|^k(1+s)^{(\ell-1)|\gamma|}(1+t)^{(1-\ell)|\gamma|} & \text{if } (t,s,\xi)\in Z_3,\end{array}\right. \tag{20}$$

for all $s\geq 0$ and $t\geq s$.

Proof. For any $N\in(0,1)$, the following properties hold:

$$|H^{\pm}_\gamma(\tau)| \lesssim \tau^{-\frac{1}{2}},\ \tau\in[N,\infty); \tag{21}$$

$$|H^{\pm}_\gamma(\tau)| \lesssim \tau^{-|\gamma|},\ \tau\in(0,N),\ \gamma\neq 0; \tag{22}$$

$$|J_\gamma(\tau)| \lesssim \tau^{\gamma},\ \tau\in(0,N); \tag{23}$$

$$|Y_\gamma(\tau)| \lesssim \tau^{-\gamma},\ \tau\in(0,N),\ \gamma\neq 0. \tag{24}$$

To conclude the estimates in zones Z_1 and Z_2 we may use the representation (15), estimates (21) and (22), whereas in the zone Z_3 we use (16)-(17) and (23)-(24). □

Proposition 4.1. *Let $n\geq 2$, $q\geq 2$ and $\ell\in(0,1)$.*
Assume that $g_2\in\left\{\begin{array}{ll} L^1(\mathbf{R^n})\cap L^m(\mathbf{R^n}) & \text{if } 0\leq k<1\\ L^1(\mathbf{R^n})\cap \dot H^{k-1}(\mathbf{R}^n) & \text{if } k\geq 1\end{array}\right.$, *with $m\in[1,2]$ such that*

$$m=m(k,n,q)>\frac{nq}{n+q(1-k)},\quad k\in[0,1).$$

The solution u of the problem (12) satisfies the following a priori *estimates.*

- *For $k\in[0,1)$ and $2\leq q<\frac{nm}{[n-m+mk]_+}$:*
  (i): *If $1<\beta\leq \ell+2n(1-\ell)\left(1-\frac{1}{q}\right)+2k(1-\ell)$ then*

$$|||D|^k u(t,s,\cdot)||_{L^q}\lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\phi(g_2); \tag{25}$$

  (ii): *If $\beta>\ell+2n(1-\ell)\left(1-\frac{1}{q}\right)+2k(1-\ell)$ then*

$$|||D|^k u(t,s,\cdot)||_{L^q}\lesssim (1+s)(1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\phi(g_2); \tag{26}$$

  where $\phi(g_2)=d_q(t,s)||g_2||_{L^1}+(1+s)^{n(1-\ell)\left(1-\frac{1}{m}\right)}||g_2||_{L^m}$ with

$$d_q(t,s)=\left\{\begin{array}{ll}\left(\ln\left(\frac{e+t}{e+s}\right)\right)^{1-\frac{1}{q}} & \text{if } \beta=\ell+2n(1-\ell)\left(1-\frac{1}{q}\right)+2k(1-\ell)\\ 1 & \text{otherwise}\end{array}\right.;$$

- *For $k\geq 1$:*
  (i): *If $1<\beta\leq\ell+n(1-\ell)+2k(1-\ell)$ then*

$$||u(t,s,\cdot)||_{\dot H^k}\lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)}\eta(g_2); \tag{27}$$

  (ii): *If $\beta>\ell+n(1-\ell)+2k(1-\ell)$ then*

$$||u(t,s,\cdot)||_{\dot H^k}\lesssim (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\eta(g_2); \tag{28}$$

where $\eta(g_2) = d_2(t,s)||g_2||_{L^1} + (1+s)^{(1-\ell)\left(\frac{n}{2}+k-1\right)}\|g_2\|_{\dot{H}^{k-1}}$ *with*

$$d_2(t,s) = \begin{cases} \left(\ln\left(\frac{e+t}{e+s}\right)\right)^{\frac{1}{2}} & \text{if } \beta = \ell + n(1-\ell) + 2k(1-\ell) \\ 1 & \text{otherwise} \end{cases}.$$

Proof. It is showed the estimates for the three zones.
Considerations in Z_3: In the zone Z_3, by Lemma 4.1 we may estimate

$$|\xi|^k |m_1(t,s,\xi)| \lesssim |\xi|^k(1+s).$$

By using Haussdorff-Young inequality and Hölder inequality, setting

$$\frac{1}{r} = 1 - \frac{1}{q},$$

for $q \geq 2$, one may estimate

$$\begin{aligned}\|\mathscr{F}^{-1}(\chi_3(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^q} &\lesssim \|\chi_3(s,\xi)|\xi|^k m_1(t,s,\xi)\hat{g}_2\|_{L^{q'}} \\ &\lesssim \|\chi_3(s,\xi)|\xi|^k m_1(t,s,\xi)\|_{L^r}\|\hat{g}_2\|_{L^\infty} \\ &\lesssim (1+s)(1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\|g_2\|_{L^1}\end{aligned}$$

thanks to

$$\|\chi_3(s,\xi)|\xi|^k\|^r_{L^r(Z_3)} = \int_{Z_3} |\xi|^{rk}\, d\xi \lesssim (1+t)^{(kr+n)(\ell-1)}.$$

However, if $\beta < \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell)$, by using that

$$(1+s)(1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)} \leq (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)} \tag{29}$$

we obtain

$$\|\mathscr{F}^{-1}(\chi_3(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^q} \lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}\|g_2\|_{L^1}.$$

Considerations in Z_1: In the zone Z_1 by Lemma 4.1 we may estimate

$$|\xi|^k|m_1(t,s,\xi)| \lesssim |\xi|^{k-1}(1+s)^{(\beta+\ell)/2}(1+t)^{(\ell-\beta)/2}.$$

By using Haussdorff-Young inequality and Hölder inequality, setting

$$\frac{1}{r} = \frac{1}{q'} - \frac{1}{m'} = \frac{1}{m} - \frac{1}{q}, \quad m \in [1,2)$$

for $2 \leq q < \frac{nm}{(n-m+mk)_+}$ and $k \in [0,1)$, one may estimate

$$\begin{aligned}&\|\mathscr{F}^{-1}(\chi_1(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^q} \lesssim \|\chi_1(s,\xi)|\xi|^k m_1(t,s,\xi)\hat{g}_2\|_{L^{q'}} \\ &\lesssim \|\chi_1(s,\xi)|\xi|^k m_1(t,s,\xi)\|_{L^r}\|\hat{g}_2\|_{L^{m'}} \\ &\lesssim (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{\frac{\ell+\beta}{2}}(1+s)^{\left(n\left(\frac{1}{m}-\frac{1}{q}\right)+k-1\right)(\ell-1)}\|g_2\|_{L^m}\end{aligned}$$

thanks to

$$\|\chi_1(s,\xi)|\xi|^{k-1}\|^r_{L^r(Z_1)} = \int_{Z_1} |\xi|^{r(k-1)} d\xi \lesssim (1+s)^{(n+r(k-1))(\ell-1)}, \quad r(k-1)+n < 0.$$

For $k \geq 1$ it is clear that

$$\|\mathscr{F}^{-1}(\chi_1(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^2} \lesssim (1+s)^{\frac{\ell+\beta}{2}}(1+t)^{\frac{\ell-\beta}{2}}\|g_2\|_{\dot{H}^{k-1}}.$$

However, if $\beta \geq \ell + 2n(1-\ell)\left(1-\frac{1}{q}\right) + 2k(1-\ell)$, by using that

$$(1+s)^{\frac{\ell+\beta}{2}}(1+t)^{\frac{\ell-\beta}{2}} \lesssim (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}(1+s)^{\ell+(1-\ell)\left(n\left(1-\frac{1}{q}\right)+k\right)} \tag{30}$$

we may gluing the estimates in zones Z_1 and Z_3, namely, for $2 \leq q < \frac{nm}{(n-m+mk)_+}$ and $k \in [0,1)$, we get

$$\|\mathscr{F}^{-1}(\chi_1(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^q} \lesssim (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)}(1+s)^{1+n(1-\ell)\left(1-\frac{1}{m}\right)}\|g_2\|_{L^m},$$

whereas for $k \geq 1$ and $q = 2$ we get

$$\|\mathscr{F}^{-1}(\chi_1(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^2} \lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}(1+s)^{\ell+(1-\ell)\left(\frac{n}{2}+k\right)}\|g_2\|_{\dot{H}^{k-1}}.$$

Considerations in Z_2: In the zone Z_2, by Lemma 4.1 we may estimate

$$|\xi|^k|m_1(t,s,\xi)| \lesssim |\xi|^{k-\alpha}(1+s)(1+t)^{(\ell-\beta)/2},$$

where $\alpha = \frac{\beta-\ell}{2(1-\ell)}$. Setting

$$\frac{1}{r} = 1 - \frac{1}{q},$$

for $q \geq 2$ then thanks to

$$\begin{aligned}
\|\chi_2(s,\xi)|\xi|^{k-\alpha}\|^r_{L^r(Z_2)} &= \int_{Z_2} |\xi|^{(k-\alpha)r}d\xi \\
&\lesssim \begin{cases} (1+s)^{(\ell-1)(n+r(k-\alpha))} & \text{if } \alpha < k + n\left(1-\frac{1}{q}\right) \\ \ln\left(\frac{e+t}{e+s}\right) & \text{if } \alpha = k + n\left(1-\frac{1}{q}\right) \\ (1+t)^{(\ell-1)(n+r(k-\alpha))} & \text{if } \alpha > k + n\left(1-\frac{1}{q}\right) \end{cases}
\end{aligned}$$

one may estimate

$$\begin{aligned}
&\|\mathscr{F}^{-1}(\chi_2(s,\xi)|\xi|^k m_1(t,s,\xi)) * g_2\|_{L^q} \lesssim \|\chi_2(s,\xi)|\xi|^k m_1(t,s,\xi)\hat{g}_2\|_{L^{q'}} \\
&\lesssim \|\chi_2(s,\xi)|\xi|^k m_1(t,s,\xi)\|_{L^r}\|\hat{g}_2\|_{L^\infty} \lesssim (1+s)\|g_2\|_{L^1} \\
&\times \begin{cases} (1+t)^{(\ell-\beta)/2}(1+s)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)+(\beta-\ell)/2} & \text{if } 1 < \beta < \ell + (1-\ell)[2n\left(1-\frac{1}{q}\right)+2k] \\ (1+t)^{(\ell-\beta)/2}\left(\ln\left(\frac{e+t}{e+s}\right)\right)^{1-\frac{1}{q}} & \text{if } \beta = \ell + (1-\ell)[2n\left(1-\frac{1}{q}\right)+2k] \\ (1+t)^{(\ell-1)\left(n\left(1-\frac{1}{q}\right)+k\right)} & \text{if } \beta > \ell + (1-\ell)[2n\left(1-\frac{1}{q}\right)+2k]. \end{cases}
\end{aligned}$$

□

5. Global existence results

By Duhamel's principle, a function $u \in X$, where X is a suitable space, is a solution to (1) if, and only if, it satisfies the equality

$$u(t,x) = u^0(t,x) + \int_0^t K(t,s,x) *_{(x)} f(u(s,x))\,ds\,, \qquad \text{in } X, \tag{31}$$

where $u^0(t,x)$ is the solution to the linear Cauchy problem

$$\begin{cases} u_{tt}(t,x) - (1+t)^{-2\ell}\Delta u(t,x) + \frac{\beta}{1+t}u_t(t,x) = 0,\ t \geq 0 \\ u(0,x) = 0 \\ u_t(0,x) = u_1(x) \end{cases} \tag{32}$$

and $K(t,s,x)\ast_{(x)} f(u(s,x))$ is the solution to the linear Cauchy problem (12) with $g_1\equiv 0$ and $g_2\equiv f(u)$, being $K(t,s,x)=\mathscr{F}^{-1}(m_1)(t,s,x)$, i.e.

$$K(t,s,x)=-\frac{\pi i}{4(1-\ell)}(1+s)^{1+(\beta-1)/2}(1+t)^{(1-\beta)/2}\mathscr{F}^{-1}\left(\psi_{0,\rho,0}\right)(t,s,x).$$

The proof of our global existence results is based on the following scheme: We define an appropriate data function space $\mathcal{D}$ and an evolution space for solutions X(T) equipped with a norm relate to the estimates of solutions to the linear Cauchy problem (32) such that

$$\|u^0\|_X\leq C\,\|u_1\|_{\mathcal{D}}.$$

For any $u\in X$, we define the operator P by

$$P: u\in X(T)\to Pu(t,x):=u^0(t,x)+Fu(t,x),$$

with

$$Fu(t,x)\doteq\int_0^t K(t,s,x)\ast_{(x)}\, f(u(s,x))\,ds\,,$$

then we prove the estimates

$$\|Pu\|_X\leq C\,\|u_1\|_{\mathcal{D}}+C_1(t)\|u\|_X^p\,,$$
$$\|Pu-Pv\|_X\leq C_2(t)\|u-v\|_X\left(\|u\|_X^{p-1}+\|v\|_X^{p-1}\right).$$

The estimates for the image Pu allow us to apply Banach's fixed point theorem. In this way we get simultaneously a unique solution to $Pu=u$ locally in time for large data and globally in time for small data [11]. To prove the local (in time) existence we use that $C_1(t), C_2(t)$ tend to zero as t goes to zero, whereas to prove the global (in time) existence we use $C_1(t)\leq C$ and $C_2(t)\leq C$ for all $t\geq 0$.

5.1. **Proof of Theorem 2.4.**

Proof. (Theorem 2.4) We define the space

$$X(T)\doteq C([0,\infty),H^k(\mathbf{R}^n)),\quad k\doteq\frac{n\ell}{2}+1,$$

equipped with the norm

$$\|u\|_{X(T)}\doteq\sup_{t\in[0,T]}(1+t)^{(1-\ell)\frac{n}{2}}\left(\|u(t,\cdot)\|_{L^2}+g(t)(1+t)^{(1-\ell)k}\|u(t,\cdot)\|_{\dot H^k}\right),$$

with

$$g(t)=\begin{cases}1 & \bar k>k\\ (\ln(e+t))^{-\frac12} & \bar k=k,\end{cases}$$

where $\bar k\doteq\frac{\beta-\ell}{2(1-\ell)}-\frac{n}{2}$. We have to prove the global existence in time of the solution u assuming that there exists $\delta>0$ such that

$$u_1\in\mathcal{D}\doteq H^{k-1}(\mathbf{R}^n)\cap L^1(\mathbf{R}^n),\qquad ||u_1||_{\mathcal{D}}\leq\delta.$$

Thanks to Proposition 4.1, $u^0\in X(T)$ and it satisfies

$$\|u^0\|_X\leq C\,\|u_1\|_{\mathcal{D}}.$$

It remains to show the estimates

$$\|Fu\|_X\leq C\|u\|_X^p\,,\tag{33}$$

$$\|Fu-Fv\|_X\leq C\|u-v\|_X\left(\|u\|_X^{p-1}+\|v\|_X^{p-1}\right).\tag{34}$$

Let us begin by prove (33). Taking into account the definition of the norm in the function space $X(T)$, we split the proof accordingly to size of β:

Let $\bar{k} > k$, i.e., $\beta > \ell + n(1-\ell) + 2k(1-\ell)$. Applying Proposition 4.1 we have

$$\|Fu(t,\cdot)\|_{L^2} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\frac{n}{2}} \Big(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{(1-\ell)n\left(1-\frac{1}{m}\right)} \||u(s,\cdot)|^p\|_{L^m} \Big)\, ds\,,$$

where $\frac{2n}{n+2} < m \leq 2$ and

$$\|Fu(t,\cdot)\|_{\dot{H}^k} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)} \Big(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{(1-\ell)\left(\frac{n}{2}+k-1\right)} \||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} \Big)\, ds.$$

First, we use Gagliardo-Nirenberg inequality

$$\|u(s,\cdot)\|_{L^q} \lesssim \|u(s,\cdot)\|_{L^2}^{1-\theta} \|u(s,\cdot)\|_{\dot{H}^k}^{\theta}, \quad \theta(q) = \frac{n}{k}\left(\frac{1}{2} - \frac{1}{q}\right),$$

for $q = jp$ with $j = 1, m$. We point out that $\theta(jp) < 1$ for all $p > 1$ provided that $\ell > 1 - \frac{2}{n}$ and $\theta(jp) > 0$ for $p > p_c(n,\ell) > 2$. Since $u \in X(T)$ we may estimate

$$\begin{aligned}
\||u(s,\cdot)|^p\|_{L^{\frac{q}{p}}} &= \|u(s,\cdot)\|_{L^q}^p \lesssim \|u(s,\cdot)\|_{\dot{H}^k}^{p\theta} \|u(s,\cdot)\|_{L^2}^{(1-\theta)p} \\
&\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k\right)\theta p + (\ell-1)\frac{n(1-\theta)p}{2}} \|u\|_{X(T)}^p \lesssim (1+s)^{n(\ell-1)(p-\frac{p}{q})} \|u\|_{X(T)}^p,
\end{aligned}$$

$q = p$ and $q = mp$ with $\frac{2n}{n+2} < m \leq 2$, for $p > p_c(n,\ell)$. Therefore, we obtain

$$\begin{aligned}
\|Fu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{(\ell-1)\frac{n}{2}} \int_0^t (1+s)^{1+n(\ell-1)(p-1)} ds \|u\|_{X(T)}^p \\
&+ (1+t)^{(\ell-1)\frac{n}{2}} \int_0^t (1+s)^{1+(1-\ell)n\left(1-\frac{1}{m}\right)+n(\ell-1)\left(p-\frac{1}{m}\right)} ds \|u\|_{X(T)}^p \\
&\lesssim (1+t)^{(\ell-1)\frac{n}{2}} \|u\|_{X(T)}^p,
\end{aligned}$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

Then, in order to estimate $\|Fu(t,\cdot)\|_{\dot{H}^k}$, we may use that $H^k(\mathbf{R}^\mathbf{n})$, with $k > \frac{n}{2}$, is imbedded into $L^\infty(\mathbf{R}^\mathbf{n})$. Indeed, thanks to Corollary 5.2, for $p > \max\{1, k-1\}$ we may estimate

$$\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} \leq C\|u(s,\cdot)\|_{\dot{H}^{k-1}} \|u(s,\cdot)\|_{L^\infty}^{p-1}.$$

Since $u \in X(T)$ we have

$$\|u(s,\cdot)\|_{\dot{H}^{k-1}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)} \|u\|_{X(T)},$$

and thanks to Lemma 5.1 for $\tilde{k} < \frac{n}{2} < k$ it follows

$$\|u(s,\cdot)\|_{L^\infty} \lesssim \|u(s,\cdot)\|_{\dot{H}^{\tilde{k}}} + \|u(s,\cdot)\|_{\dot{H}^k} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde{k}\right)} \|u\|_{X(T)}.$$

If we choose $\tilde{k} = \frac{n}{2} - \varepsilon_0$, with ε_0 sufficiently small, then

$$\||u(s,\cdot)|^p\|_{\dot{H}^{k-1}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)+(\ell-1)(n-\varepsilon_0)(p-1)} \|u\|_{X(T)}^p,$$

hence

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot H^k} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t (1+s)^{1+n(\ell-1)(p-1)}ds\|u\|^p_{X(T)}\\ &+ (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t (1+s)^{1+(n-\varepsilon_0)(\ell-1)(p-1)}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\|u\|^p_{X(T)},\end{aligned}$$

for $p>1+\frac{2}{n(1-\ell)}$.
The case $\bar k=k$, i.e., $\beta=\ell+n(1-\ell)+2k(1-\ell)$:
In this case one may conclude that

$$\|Fu(t,\cdot)\|_{L^2}\lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},$$

and

$$\|Fu(t,\cdot)\|_{\dot H^{\bar k}}\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+\bar k\right)}(\ln(e+t))^{\frac{1}{2}}\|u\|^p_{X(T)}$$

for $p>1+\frac{2}{n(1-\ell)}$.

Finally, let us discuss the proof of (34) only in the case $\beta>\ell+n(1-\ell)+2k(1-\ell)$. Applying Proposition 4.1 we have

$$\begin{aligned}\|Fu(t,\cdot)-Fv(t,\cdot)\|_{L^2} &\lesssim (1+t)^{(\ell-1)\frac{n}{2}}\int_0^t (1+s)\|(f(u)-f(v))(s,\cdot)\|_{L^1}ds\\ &+ (1+t)^{(\ell-1)\frac{n}{2}}\int_0^t (1+s)^{1+(1-\ell)n\left(1-\frac{1}{m}\right)}\|(f(u)-f(v))(s,\cdot)\|_{L^m}ds.\end{aligned}$$

Here, we may take $m\in[1,2]$ such that $m>\frac{2n}{n+2}$.
By using (2) and Hölder inequality, we find that

$$\begin{aligned}&\|(f(u)-f(v))(s,\cdot)\|_{L^\alpha}\\ &\quad\le C_1\,\|(u-v)(|u|^{p-1}+|v|^{p-1})(s,\cdot)\|_{L^\alpha}\\ &\quad\le C_1\,\|(u-v)(s,\cdot)\|_{L^{p\alpha}}\left(\|u(s,\cdot)\|^{p-1}_{L^{p\alpha}}+\|v(s,\cdot)\|^{p-1}_{L^{p\alpha}}\right)\\ &\quad\le C_2\,(1+s)^{n(\ell-1)\left(p-\frac{1}{\alpha}\right)}\,\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right),\end{aligned}\tag{35}$$

for any $1\le\alpha\le m$. Therefore

$$\begin{aligned}&\|Fu(t,\cdot)-Fv(t,\cdot)\|_{L^2}\\ \lesssim\;& (1+t)^{(\ell-1)\frac{n}{2}}\int_0^t(1+s)^{1+n(\ell-1)(p-1)}ds\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right)\\ +\;& (1+t)^{(\ell-1)\frac{n}{2}}\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right),\end{aligned}$$

for $p>1+\frac{2}{n(1-\ell)}$.
Applying again Proposition 4.1 we have

$$\begin{aligned}&\|Fu(t,\cdot)-Fv(t,\cdot)\|_{\dot H^k}\\ \lesssim\;& (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t(1+s)\|(f(u)-f(v))(s,\cdot)\|_{L^1}\,ds\\ +\;& (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t(1+s)^{1+(1-\ell)\left(\frac{n}{2}+k-1\right)}\|(f(u)-f(v))(s,\cdot)\|_{\dot H^{k-1}}\,ds.\end{aligned}$$

From now we assume that $f(u) = |u|^p$, without lose of generality. In order to estimate $\|(f(u)-f(v))(s,\cdot)\|_{\dot H^{k-1}}$ we use

$$|u(s,x)|^p - |v(s,x)|^p = p\int_0^1 |v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,x)d\tau(u-v)(s,x).$$

Hence, applying Proposition 5.3 gives

$$\begin{aligned} & \||u(s,x)|^p - |v(s,x)|^p\|_{\dot H^{k-1}} \\ \lesssim\ & \|(u-v)(s,\cdot)\|_{\dot H^{k-1}} \int_0^1 \||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_\infty\, d\tau \\ +\ & \|(u-v)(s,\cdot)\|_\infty \int_0^1 \||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_{\dot H^{k-1}}\, d\tau. \end{aligned}$$

Now, since $u,v\in X(T)$ we have

$$\|(u-v)(s,\cdot)\|_{\dot H^{k-1}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\|u-v\|_{X(T)}.$$

Applying Lemma 5.1, for $\tilde k < \frac{n}{2} < k$ it follows

$$\|(u-v)(s,\cdot)\|_\infty \lesssim \|(u-v)(s,\cdot)\|_{\dot H^{\tilde k}} + \|(u-v)(s,\cdot)\|_{\dot H^{k}} \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde k\right)}\|u-v\|_{X(T)},$$

and

$$\||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_\infty \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde k\right)(p-1)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right),$$

with $\tilde k = \frac{n}{2}-\varepsilon_0$ and ε_0 sufficiently small.
For $p>k$ Corollary 5.2 implies

$$\begin{aligned} & \||v+\tau(u-v)|^{p-2}(v+\tau(u-v))(s,\cdot)\|_{\dot H^{k-1}} \\ & \leq C\|(v+\tau(u-v))(s,\cdot)\|_{\dot H^{k-1}}\|(v+\tau(u-v))(s,\cdot)\|^{p-2}_{L^\infty} \\ & \lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}(1+s)^{(\ell-1)\left(\frac{n}{2}+\tilde k\right)(p-2)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} & \|Fu(t,\cdot)-Fv(t,\cdot)\|_{\dot H^k} \\ \lesssim\ & (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t (1+s)^{1+n(\ell-1)(p-1)}ds\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right) \\ +\ & (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\int_0^t (1+s)^{(\ell-1)(n-\varepsilon_0)(p-1)}ds\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right) \\ \leq\ & (1+t)^{(\ell-1)\left(\frac{n}{2}+k\right)}\|u-v\|_{X(T)}\left(\|u\|^{p-1}_{X(T)}+\|v\|^{p-1}_{X(T)}\right), \end{aligned}$$

for $p > 1+\frac{2}{n(1-\ell)}$. □

5.2. **Proof of Theorem 2.2.**

Proof. (Theorem 2.2) Let $n=2$ or $n=3$ and $k \doteq \frac{n\ell}{2}+1$ such that $\frac{n}{2}<k\leq 2$. For $\bar k \doteq \frac{\beta-\ell}{2(1-\ell)}-\frac{n}{2}$ satisfying $\frac{n\ell}{2}\leq \bar k < k$, we define the space

$$X(T) \doteq C([0,\infty), H^k(\mathbf{R}^n)),$$

equipped with the norm

$$\|u\|_{X(T)} \doteq \sup_{t\in[0,T]}\left((1+t)^{(1-\ell)\frac{n}{2}}\|u(t,\cdot)\|_{L^2}+h(t)\|u(t,\cdot)\|_{\dot H^{k-1}}+(1+t)^{\frac{\beta-\ell}{2}}\|u(t,\cdot)\|_{\dot H^k}\right),$$

with

$$h(t)=\begin{cases}(1+t)^{(1-\ell)\left(\frac{n}{2}+k-1\right)}, & \ell+n(1-\ell)(1+\ell)<\beta<2-\ell+n(1-\ell)(1+\ell),\\ (1+t)^{\frac{\beta-\ell}{2}}(\ln(e+t))^{\frac12}, & \beta=\ell+n(1-\ell)(1+\ell).\end{cases}$$

In the following we only prove (33). Let $k-1<\bar{k}<k$, i.e.,

$$\ell+n(1-\ell)+2(k-1)(1-\ell)<\beta<\ell+n(1-\ell)+2k(1-\ell).$$

Applying again Proposition 4.1 we have

$$\|Fu(t,\cdot)\|_{L^2}\lesssim\int_0^t(1+s)(1+t)^{(\ell-1)\frac{n}{2}}\left(\||u(s,\cdot)|^p\|_{L^1}+(1+s)^{(1-\ell)n\left(1-\frac{1}{m}\right)}\||u(s,\cdot)|^p\|_{L^m}\right)ds,$$

with $\frac{2n}{n+2}<m\le 2$. We will use now the fractional Sobolev embedding (for instance, see [1])

$$\|u(s,\cdot)\|_{L^q}\lesssim\|u(s,\cdot)\|_{\dot{H}^{\kappa(q)}},\quad \kappa(q)=n\left(\frac12-\frac1q\right),\quad 2\le q<\infty,$$

by taking $q=jp$ with $j=1,m$. We consider three possibilities for $\kappa_j=\kappa(jp)=n\left(\frac12-\frac{1}{jp}\right)$, $j=1,m$, which satisfying $\kappa_1<\kappa_m<k$ for $\ell>1-\frac{2}{n}$. In the first one, suppose that $k_m\le k-1$ with $\kappa_m=\kappa(mp)=n\left(\frac12-\frac{1}{mp}\right)$ we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^j} &= \|u(s,\cdot)\|^p_{L^{jp}}\lesssim\|u(s,\cdot)\|^p_{\dot{H}^{\kappa_j}}\\ &\lesssim (1+s)^{n(\ell-1)\left(p-\frac1j\right)}\|u\|^p_{X(T)},\quad j=1,m,\end{aligned}$$

hence, as in the proof of Theorem 2.4 we conclude

$$\|Fu(t,\cdot)\|_{L^2}\lesssim(1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},$$

for $p>1+\frac{2}{n(1-\ell)}$. In the second one, suppose that $\kappa_1\le k-1<k_m$ we may estimate

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^1} &= \|u(s,\cdot)\|^p_{L^p}\lesssim\|u(s,\cdot)\|^p_{\dot{H}^{\kappa_1}}\\ &\lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+\kappa_1\right)}\|u\|^p_{X(T)},\end{aligned}$$

with $\kappa_1=\kappa(p)=n\left(\frac12-\frac1p\right)$, whereas

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^m} &= \|u(s,\cdot)\|^p_{L^{mp}}\lesssim\|u(s,\cdot)\|^p_{\dot{H}^{\kappa_m}}\\ &\lesssim \|u(s,\cdot)\|^{(1-\theta)p}_{\dot{H}^{k-1}}\|u(s,\cdot)\|^{p\theta}_{\dot{H}^k}\lesssim(1+s)^{p(\ell-1)\left(\frac{n}{2}+k-1\right)}\|u\|^p_{X(T)},\end{aligned}$$

with $\theta=k_m-k+1$ since $k-1<k_m<k$. Therefore, if $m>\frac{2n}{n+2}$ is chosen sufficiently small we conclude that

$$\begin{aligned}\|Fu(t,\cdot)\|_{L^2} &\lesssim (1+t)^{\frac{n}{2}(\ell-1)}\int_0^t(1+s)^{1+n(\ell-1)(p-1)}ds\|u\|^p_{X(T)}\\ &+ (1+t)^{\frac{n}{2}(\ell-1)}\int_0^t(1+s)^{1+(1-\ell)n\left(1-\frac1m\right)+p(\ell-1)\left(\frac{n}{2}+k-1\right)}ds\|u\|^p_{X(T)}\\ &\lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},\end{aligned}$$

for

$$p>1+\frac{2}{n(1-\ell)}>\frac{1}{1+\ell}+\frac{2}{n(1-\ell)}.$$

In the last one, suppose that $k-1<\kappa_1<k_m<k$ we may estimate

$$\||u(s,\cdot)|^p\|_{L^j} = \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot H^{\kappa_j}} \lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+k-1\right)}\|u\|^p_{X(T)}, \ \ j=1,m$$

and we can conclude as in the previous one that

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)}$$

for $p>1+\frac{2}{n(1-\ell)}$.
If $\ell=\frac{2}{n}$ for $n=3$, i.e., $k=2$, applying again Proposition 4.1, we have

$$\|Fu(t,\cdot)\|_{\dot H^1} \lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\left(\||u(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{n(1-\ell)}{2}}\||u(s,\cdot)|^p\|_{L^2}\right)ds.$$

Using the fractional Sobolev embedding we may estimate

$$\||u(s,\cdot)|^p\|_{L^2} = \|u(s,\cdot)\|^p_{L^{2p}} \lesssim \|u(s,\cdot)\|^p_{\dot H^{\kappa(2p)}} \lesssim (1+s)^{p(\ell-1)\left(\frac{n}{2}+1\right)}\|u\|^p_{X(T)},$$

where $\kappa(2p)=\frac{3}{2}\left(1-\frac{1}{p}\right)>1$ for $p>3$, i.e., $p>p_c\left(3,\frac{2}{3}\right)$. Therefore, if $\ell=\frac{2}{n}$ and $n=3$ we obtain

$$\begin{aligned}
\|Fu(t,\cdot)\|_{\dot H^1} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\int_0^t (1+s)^{1+\max\{n(\ell-1)(p-1)\,;\,p(\ell-1)\left(\frac{n}{2}+1\right)\}}ds\|u\|^p_{X(T)}\\
&+ (1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\int_0^t (1+s)^{1+\frac{n(1-\ell)}{2}+p(\ell-1)\left(\frac{n}{2}+1\right)}ds\|u\|^p_{X(T)}\\
&\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+1\right)}\|u\|^p_{X(T)},
\end{aligned}$$

for $p>3$, i.e., $p>p_c\left(3,\frac{2}{3}\right)$.
However, if $\ell<\frac{2}{n}$, i.e., $k<2$ applying again Proposition 4.1, we obtain

$$\begin{aligned}
\|Fu(t,\cdot)\|_{\dot H^{k-1}} &\lesssim \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds\\
&+ \int_0^t (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}(1+s)^{1+n(1-\ell)\left(1-\frac{1}{m}\right)}\||u(s,\cdot)|^p\|_{L^m}\,ds,
\end{aligned}$$

where $m>\frac{2n}{n+2(2-k)}$, i.e., $m>\frac{2n}{n(1-\ell)+2}$. Using the fractional Sobolev embedding we may estimate

$$\begin{aligned}
\||u(s,\cdot)|^p\|_{L^j} &= \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot H^{\kappa_j}}\\
&\lesssim (1+s)^{\max\{n(\ell-1)\left(p-\frac{1}{j}\right)\,;\,p(\ell-1)\left(\frac{n}{2}+k-1\right)\}}\|u\|^p_{X(T)}, \ \ j=1,m
\end{aligned}$$

for $\kappa_j<k$, where $\kappa_j=n\left(\frac{1}{2}-\frac{1}{jp}\right)$. As we seen we have to consider three possibilities for κ_j. Suppose that $\kappa_j>k-1$ (otherwise we can prove as before), if $m>\frac{2n}{n+2}$ is chosen sufficiently small we conclude that

$$\begin{aligned}
\|Fu(t,\cdot)\|_{\dot H^{k-1}} &\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\int_0^t (1+s)^{1+p(\ell-1)\left(\frac{n}{2}+k-1\right)}ds\|u\|^p_{X(T)}\\
&+ (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\int_0^t (1+s)^{1+n(1-\ell)\left(1-\frac{1}{m}\right)+p(\ell-1)\left(\frac{n}{2}+k-1\right)}ds\|u\|^p_{X(T)}\\
&\lesssim (1+t)^{(\ell-1)\left(\frac{n}{2}+k-1\right)}\|u\|^p_{X(T)},
\end{aligned}$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

Moreover, for $\ell + n(1-\ell)(1+\ell) < \beta < \ell + n(1-\ell) + 2k(1-\ell)$ applying again Proposition 4.1 we have

$$\begin{aligned} \|Fu(t,\cdot)\|_{\dot H^k} &\lesssim \int_0^t (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds \\ &+ \int_0^t (1+t)^{\frac{\ell-\beta}{2}}(1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)+(1-\ell)\left(\frac{n}{2}+k-1\right)}\||u(s,\cdot)|^p\|_{\dot H^{k-1}}\,ds. \end{aligned}$$

As before we may estimate

$$\begin{aligned} \||u(s,\cdot)|^p\|_{L^1} &= \|u(s,\cdot)\|^p_{L^p} \lesssim \|u(s,\cdot)\|^p_{\dot H^{\kappa_1}} \\ &\lesssim \|u\|^p_{X(T)} \begin{cases} (1+s)^{(\ell-1)n(p-1)} & \text{if } \kappa_1 \le k-1 \\ (1+s)^{p(\ell-1)\left(\frac{n}{2}+k-1\right)} & \text{if } k-1 < \kappa_1 < k \end{cases} \end{aligned}$$

with $\kappa_1 = n\left(\frac{1}{2} - \frac{1}{p}\right)$ and

$$\int_0^t (1+s)^{1+\frac{\beta-\ell}{2}+(\ell-1)\left(\frac{n}{2}+k\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds \lesssim \|u\|^p_{X(T)}$$

for $p > 1 + \frac{2}{n(1-\ell)}$ and $\beta < \ell + n(1-\ell) + 2k(1-\ell)$.

Using Lemma 5.1 for $k-1 < \tilde k < \frac{n}{2} < k$ it follows

$$\begin{aligned} \|u(s,\cdot)\|_{L^\infty} &\lesssim \|u(s,\cdot)\|_{\dot H^{\tilde k}} + \|u(s,\cdot)\|_{\dot H^k} \\ &\lesssim \|u(s,\cdot)\|^{(1-\theta)}_{\dot H^{k-1}}\|u(s,\cdot)\|^{\theta}_{\dot H^k} \\ &\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)+\theta\left[\frac{\ell-\beta}{2}+(1-\ell)\left(\frac{n}{2}+k-1\right)\right]}\|u\|_{X(T)}, \end{aligned}$$

with $\theta = \tilde k - k + 1$, and we may estimate

$$\begin{aligned} \||u(s,\cdot)|^p\|_{\dot H^{k-1}} &\lesssim \|u(s,\cdot)\|_{\dot H^{k-1}}\|u(s,\cdot)\|^{p-1}_{L^\infty} \\ &\lesssim (1+s)^{(\ell-1)\left(\frac{n}{2}+k-1\right)p+\theta\left[\frac{\ell-\beta}{2}+(1-\ell)\left(\frac{n}{2}+k-1\right)\right](p-1)}\|u\|^p_{X(T)}. \end{aligned}$$

If we choose $\tilde k = \frac{n}{2} - \varepsilon_0$, with ε_0 sufficiently small, then $\theta = \frac{n(1-\ell)}{2} - \varepsilon_0$ and we obtain

$$\int_0^t (1+s)^{\frac{\beta+\ell}{2}}\||u(s,\cdot)|^p\|_{\dot H^{k-1}}ds \lesssim \|u\|^p_{X(T)}$$

for $p > 1 + \frac{2}{n(1-\ell)}$, hence

$$\|Fu(t,\cdot)\|_{\dot H^k} \lesssim (1+t)^{\frac{\ell-\beta}{2}}\|u\|^p_{X(T)}.$$

Here we remark that $k = 1 + \frac{n\ell}{2}$, $\theta < 1$,

$$\ell + \frac{(\beta-\ell)}{2}(1-\theta(p-1)) + (\ell-1)\left(\frac{n}{2}+k-1\right) + (\theta-1)(1-\ell)\left(\frac{n}{2}+k-1\right)(p-1) < -1$$

and $\theta(p-1) > 1$ for $p > 1 + \frac{2}{n(1-\ell)}$.

The case $\bar k = k-1$, i.e., $\beta = \ell + n(1-\ell) + 2(k-1)(1-\ell)$:
In this case one may conclude that

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{n}{2}(\ell-1)}\|u\|^p_{X(T)},$$

$$\|Fu(t,\cdot)\|_{\dot H^k} \lesssim (1+t)^{\frac{\ell-\beta}{2}}\|u\|^p_{X(T)},$$

and

$$\|Fu(t,\cdot)\|_{\dot H^{\bar k}} \lesssim (1+t)^{\frac{\ell-\beta}{2}}(\ln(e+t))^{\frac{1}{2}}\|u\|^p_{X(T)}$$

for $p > 1 + \frac{2}{n(1-\ell)}$.

□

5.3. **The proof of Theorem 2.3.**

Proof. (Theorem 2.3) Let $n=3$ and $\ell \in \left(\frac{2}{3}, 1\right)$. We consider r_1, r_2 satisfying

$$r_1 > \frac{2(3\ell-1)}{1-\ell}, \quad 2 < r_2 < \frac{6}{2\kappa(r_1)-1}$$

with $\kappa(r_1) = 3\left(\frac{1}{2}-\frac{1}{r_1}\right)$. If $\ell + 6(1-\ell)\left(1-\frac{1}{r_1}\right) \le \beta < 2-\ell+3(1-\ell)(1+\ell)$ we define the following space

$$X(T) \doteq C([0,\infty), H^{\kappa(r_1)}(\mathbf{R}^3) \cap \dot H^{\kappa(r_1)-1,r_2}(\mathbf{R}^3)),$$

equipped with the norm

$$\begin{aligned}\|u\|_{X(T)} \doteq \sup_{t\in[0,T]} &\left((1+t)^{(1-\ell)\frac{3}{2}}\left(\|u(t,\cdot)\|_{L^2} + (1+t)^{(1-\ell)\kappa(r_1)}\|u(t,\cdot)\|_{\dot H^{\kappa(r_1)}}\right)\right)\\ &+ \sup_{t\in[0,T]}\left((1+t)^{(1-\ell)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\|u(t,\cdot)\|_{\dot H^{\kappa(r_1)-1,r_2}}\right).\end{aligned}$$

Taking into account the proof of Theorem 2.2, we can prove that

$$\|Fu(t,\cdot)\|_{L^2} \lesssim (1+t)^{\frac{3}{2}(\ell-1)}\|u\|^p_{X(T)}.$$

For

$$\beta > \ell + 6(1-\ell)\left(1-\frac{1}{r_2}\right) + 2(\kappa(r_1)-1)(1-\ell)$$

applying Proposition 4.1 we obtain

$$\begin{aligned}\|Fu(t,\cdot)\|_{\dot H^{\kappa(r_1)-1,r_2}} \lesssim & \int_0^t (1+s)(1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds\\ &+ \int_0^t (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}(1+s)^{1+\frac{3(1-\ell)}{2}}\||u(s,\cdot)|^p\|_{L^2}\,ds.\end{aligned}$$

We have

$$1 + \frac{r_1(2-\kappa(r_1))}{3} \le \frac{r_1}{2} \Longleftrightarrow r_1 \ge 6.$$

But $r_1 > \frac{2(3\ell-1)}{1-\ell} > 6$ for $\ell > \frac{2}{3}$. Hence, the assumption that $p < 1 + \frac{r_1(2-\kappa(r_1))}{3}$ implies $\kappa(2p) \le \kappa(r_1)$ and using the fractional Sobolev embedding we get

$$\begin{aligned}\||u(s,\cdot)|^p\|_{L^j} &= \|u(s,\cdot)\|^p_{L^{jp}} \lesssim \|u(s,\cdot)\|^p_{\dot H^{\kappa(jp)}}\\ &\lesssim (1+s)^{3(\ell-1)\left(p-\frac{1}{j}\right)}\|u\|^p_{X(T)},\ j=1,2.\end{aligned}$$

Hence

$$\begin{aligned}
&\|Fu(t,\cdot)\|_{\dot H^{\kappa(r_1)-1,r_2}}\\
\lesssim\ & (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\int_0^t (1+s)^{1+3(\ell-1)(p-1)}ds\|u\|^p_{X(T)}\\
+\ & (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\int_0^t (1+s)^{1+\frac{3(1-\ell)}{2}+3(\ell-1)\left(p-\frac{1}{2}\right)}ds\|u\|^p_{X(T)}\\
\lesssim\ & (1+t)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\|u\|^p_{X(T)},
\end{aligned}$$

for $p>1+\frac{2}{3(1-\ell)}$. Then applying again Proposition 4.1,

$$\begin{aligned}
\|Fu(t,\cdot)\|_{\dot H^{\kappa(r_1)}} \lesssim\ & \int_0^t (1+s)(1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}\||u(s,\cdot)|^p\|_{L^1}\,ds\\
&+\int_0^t (1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}(1+s)^{1+(1-\ell)\left(\frac{3}{2}+\kappa(r_1)-1\right)}\||u(s,\cdot)|^p\|_{\dot H^{\kappa(r_1)-1}}\,ds.
\end{aligned}$$

Using Proposition 5.2 we may estimate

$$\||u(s,\cdot)|^p\|_{\dot H^{\kappa(r_1)-1}}\lesssim\|u(s,\cdot)\|_{\dot H^{\kappa(r_1)-1,r_2}}\|u(s,\cdot)\|^{p-1}_{L^{r_1}} \tag{36}$$

with r_1 e r_2 satisfying $\dfrac{p-1}{r_1}+\dfrac{1}{r_2}=\dfrac{1}{2}$. In the admissible range of r_2 we claim that p is bounded above, i.e.,

$$p=1+r_1\left(\frac{1}{2}-\frac{1}{r_2}\right)<1+\frac{r_1(2-\kappa(r_1))}{3}.$$

Using the fractional Sobolev embedding and the definition of $u\in X(T)$ we obtain

$$\|u(s,\cdot)\|_{L^{r_1}}\lesssim\|u(s,\cdot)\|_{\dot H^{\kappa(r_1)}}\lesssim(1+s)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}\|u\|_{X(T)}; \tag{37}$$

$$\|u(s,\cdot)\|_{\dot H^{\kappa(r_1)-1,r_2}}\lesssim(1+s)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)}\|u\|_{X(T)}. \tag{38}$$

Therefore from (36), (37) and (38) it follows that

$$\||u(s,\cdot)|^p\|_{\dot H^{\kappa(r_1)-1}}\lesssim(1+s)^{(\ell-1)\left(3\left(1-\frac{1}{r_2}\right)+\kappa(r_1)-1\right)+(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)(p-1)}\|u\|^p_{X(T)},$$

so that

$$\begin{aligned}
(1+s)^{1+(1-\ell)\left(\frac{3}{2}+\kappa(r_1)-1\right)}\||u(s,\cdot)|^p\|_{\dot H^{\kappa(r_1)-1}} &\lesssim\\
\lesssim(1+s)^{1+(\ell-1)3\left(\frac{1}{2}-\frac{1}{r_2}\right)+(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)(p-1)}\|u\|^p_{X(T)}&\\
\lesssim(1+s)^{1+(\ell-1)\frac{3(p-1)}{r_1}+3(\ell-1)\left(1-\frac{1}{r_1}\right)(p-1)}\|u\|^p_{X(T)}.&
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
\|Fu(t,\cdot)\|_{\dot H^{\kappa(r_1)}} \lesssim\ & \int_0^t (1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}(1+s)^{1+3(\ell-1)(p-1)}\,ds\|u\|^p_{X(T)}\\
\lesssim\ & (1+t)^{(\ell-1)\left(\frac{3}{2}+\kappa(r_1)\right)}\|u\|^p_{X(T)}
\end{aligned}$$

for $1+\frac{2}{3(1-\ell)}<p<1+\frac{r_1(2-\kappa(r_1))}{3}$.

$\square$

5.4. **The proof of Theorem 2.1.**

Proof. (Theorem 2.1) By applying the change of variable
$$v(\tau,x)=u(t,x), \qquad 1+\tau=\frac{(1+t)^{1-\ell}}{1-\ell},$$
the Cauchy problem (1) takes the form
$$\begin{cases} v_{\tau\tau}-\Delta v+\frac{\mu}{1+\tau}v_\tau=g(v), & \tau\geq s,\ x\in\mathbf{R}^n,\\ v(s,x)=0, & x\in\mathbf{R}^n,\\ v_\tau(s,x)=u_1(x), & x\in\mathbf{R}^n,\end{cases} \tag{39}$$
with $s=\frac{\ell}{1-\ell}$, $g(v)=[(1-\ell)(1+\tau)]^{\frac{2\ell}{1-\ell}}|v|^p$ and
$$\mu=\frac{\beta-\ell}{1-\ell}.$$

We enunciate Corollary 2 from [5], which will be useful in the proof of the Theorem 2.1. There was introduced the following notation: For any $1\leq r\leq q\leq\infty$, let be
$$d(r,q)=\begin{cases} \frac{n}{r}-\frac{n-1}{2}-\frac{1}{q}, & \text{if } r\leq q'\\ \frac{1}{r}+\frac{n-1}{2}-\frac{n}{q}, & \text{if } r\geq q'.\end{cases}$$

Corollary 5.1. *(see [5]) Let $\mu\geq 2$. Let $n=2$ and $2<q\leq q_\sharp$, or $n=3$ and $q\in(1,q_\sharp]$ or $n\geq 4$ and $\frac{2(n-1)}{n+1}\leq q\leq q_\sharp$. Then there exists $r_2\in(1,\min\{q,q'\}]$ such that $d(r_2,q)=1$ and the solution to (39), with an arbitrary parameter s and $g\equiv 0$, verifies the following $(L^1\cap L^{r_2})-L^q$ decay estimate*
$$||v(\tau,\cdot)||_{L^q}\lesssim(1+s)(1+\tau)^{-n\left(1-\frac{1}{q}\right)}\left(||u_1||_{L^1}+(1+s)^{\frac{n-1}{2}-\frac{1}{q}}||u_1||_{L^{r_2}}\right)$$
if $\mu>n+1-\dfrac{2}{q}$, and for any $\varepsilon>0$ verifies the $(L^1\cap L^{r_2})-L^q$ estimate
$$||v(\tau,\cdot)||_{L^q}\lesssim(1+s)^{\frac{\mu}{2}-\varepsilon}(1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q}\right)-\frac{\mu}{2}}\left((1+s)^{\frac{1}{q}-\frac{n-1}{2}}||u_1||_{L^1}+||u_1||_{L^{r_2}}\right)$$
if $\mu\leq n+1-\dfrac{2}{q}$.

Remark 5.1. The condition $q\leq q_\sharp$ is equivalent to $d(q',q)=\frac{n}{q'}-\frac{n-1}{2}-\frac{1}{q}\leq 1$.

It is enough to prove the global existence of small data solutions to (39). We define the space
$$X(T)\doteq C([0,\infty),L^{p_c}(\mathbf{R}^n)\cap L^{q_\sharp}(\mathbf{R}^n)),$$
equipped with the norm
$$\|v\|_{X(T)}\doteq\sup_{\tau\in[0,T]}\left\{(1+\tau)^{n\left(1-\frac{1}{p_c}\right)}\|v(\tau,\cdot)\|_{L^{p_c}}+(1+\tau)^{n\left(1-\frac{1}{q_\sharp}\right)}\|v(\tau,\cdot)\|_{L^{q_\sharp}}\right\}$$
if $\mu>n+1-\frac{2}{q_\sharp}$ and,
$$\begin{aligned}\|v\|_{X(T)} &\doteq \sup_{\tau\in[0,T]}\Big\{(1+\tau)^{n\left(1-\frac{1}{p_c}\right)}\|v(\tau,\cdot)\|_{L^{p_c}}+(1+\tau)^{n\left(1-\frac{1}{\bar q}\right)}\|v(\tau,\cdot)\|_{L^{\bar q}}\\ &+ (1+\tau)^{(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)+\frac{\mu}{2}-\varepsilon}\|v(\tau,\cdot)\|_{L^{q_\sharp}}\Big\}\end{aligned}$$

if $n+1-\frac{2}{\bar q}<\mu\leq n+1-\frac{2}{q_\sharp}$ and,

$$\begin{aligned}\|v\|_{X(T)} &\doteq \sup_{\tau\in[0,T]}\Big\{(1+\tau)^{n\left(1-\frac{1}{p_c}\right)}\|v(\tau,\cdot)\|_{L^{p_c}}+(1+\tau)^{(n-1)\left(\frac12-\frac{1}{\bar q}\right)+\frac{\mu}{2}-\varepsilon}\|v(\tau,\cdot)\|_{L^{\bar q}}\\ &+ (1+\tau)^{(n-1)\left(\frac12-\frac{1}{q_\sharp}\right)+\frac{\mu}{2}-\varepsilon}\|v(\tau,\cdot)\|_{L^{q_\sharp}}\Big\}\end{aligned}$$

if $\mu=n+1-\frac{2}{\bar q}$.

For any $v\in X$, we define the operator P by

$$P: v\in X(T)\to Pv(\tau,x):=v^0(\tau,x)+Fv(\tau,x),$$

where $v^0(\tau,x)$ is the solution to (39) with $g\equiv 0$ and

$$Fv(\tau,x)=\int_0^\tau K(\tau,s,x)\ast_{(x)}\ [(1-\ell)(1+s)]^{\frac{2\ell}{1-\ell}}|v|^p\,ds,$$

with $K(\tau,s,x)\ast_{(x)}h(v)$ is the solution to (39) with $g\equiv 0$ and $v_\tau(s,x)\equiv h(v)$. Then we prove the estimates

$$\begin{aligned}\|Pv\|_X &\leq C\,\|u_1\|_{\mathcal D}+C_1(t)\|v\|_X^p\,,\\ \|Pu-Pv\|_X &\leq C_2(t)\|u-v\|_X\left(\|u\|_X^{p-1}+\|v\|_X^{p-1}\right).\end{aligned}$$

By Corollary 5.1, if $\mu\geq 2$, then $v^0\in X(T)$ and it satisfies

$$\|v^0\|_X\leq C\,\|u_1\|_{\mathcal D}.$$

Let us prove the desire estimate for $\|Fv(\tau,\cdot)\|_X$. One may follow the steps of the proof of $\|Fv(\tau,\cdot)\|_X$ to conclude the Lipschitz property.
First, if $\mu>n+1-\frac{2}{q_\sharp}$ applying Corollary 5.1 we have

$$\|Fv(\tau,\cdot)\|_{L^q}\lesssim\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}}(1+\tau)^{-n\left(1-\frac1q\right)}\left(\||v(s,\cdot)|^p\|_{L^1}+(1+s)^{\frac{n-1}{2}-\frac1q}\||v(s,\cdot)|^p\|_{L^{r(q)}}\right)ds$$

with $r(q)\in[1,2[$ given by $\frac{n}{r(q)}=\frac12+\frac n2+\frac1q$, for all $p_c\leq q\leq q_\sharp$. Taking into account that $v\in X(T)$, thanks to $r(q_\sharp)p_c<q_\sharp$ we may estimate

$$\begin{aligned}\||v(s,\cdot)|^p\|_{L^{r(q)}} &= \|v(s,\cdot)\|^p_{L^{r(q)p}}\lesssim(1+s)^{-n\left(1-\frac{1}{pr(q)}\right)p}\|v\|^p_{X(T)}\\ &\lesssim (1+s)^{-n\left(p-\frac{1}{r(q)}\right)}\|v\|^p_{X(T)}\end{aligned}$$

for $p_c\leq q\leq q_\sharp$ and $p_c<p\leq\frac{q_\sharp}{r(q_\sharp)}$. Therefore, if $\mu>n+1-\frac{2}{q_\sharp}$ we have

$$\begin{aligned}\|Fv(\tau,\cdot)\|_{L^q} &\lesssim (1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}-n(p-1)}ds\|v\|^p_{X(T)}\\ &+ (1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}-n\left(p-\frac{1}{r(q)}\right)}(1+s)^{\frac{n-1}{2}-\frac1q}ds\|v\|^p_{X(T)}\\ &\lesssim (1+\tau)^{-n\left(1-\frac1q\right)}\|v\|^p_{X(T)}\end{aligned}$$

for $p_c\leq q\leq q_\sharp$ and $p>1+\frac{2}{n(1-\ell)}$.
Then, if $n+1-\frac{2}{\bar q}<\mu\leq n+1-\frac{2}{q_\sharp}$, applying again Corollary 5.1 we may estimate

$$\|Fv(\tau,\cdot)\|_{L^q}\lesssim(1+\tau)^{-n\left(1-\frac1q\right)}\int_0^\tau(1+s)^{1+\frac{2\ell}{1-\ell}}\left(\||v(s,\cdot)|^p\|_{L^1}+(1+s)^{\frac{n-1}{2}-\frac1q}\||v(s,\cdot)|^p\|_{L^{r(q)}}\right)ds$$

with $\frac{n}{r(q)} = \frac{1}{2} + \frac{n}{2} + \frac{1}{q}$, for $p_c \le q < \bar{q}$. We may estimate

$$\begin{aligned} \||v(s,\cdot)|^p\|_{L^p} &= \|v(s,\cdot)\|_{L^p}^p \lesssim \|v(s,\cdot)\|_{L^{p_c}}^{(1-\theta)p} \|v(s,\cdot)\|_{L^{q_\sharp}}^{\theta p} \\ &\lesssim (1+s)^{-n\left(1-\frac{1}{p_c}\right)(1-\theta)p+\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}\right)p\theta} \|v\|_{X(T)}^p \\ &\lesssim (1+s)^{-n\left(1-\frac{1}{p_c}\right)p} \|v\|_{X(T)}^p, \end{aligned}$$

thanks to

$$\left[\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{q_\sharp}\right) - \frac{\mu}{2} + n\left(1 - \frac{1}{p_c}\right)\right]\theta p \le \left[\varepsilon\theta + (1-n)\left(\frac{1}{p_c} - \frac{1}{p}\right)\right] p \le 0$$

for $\varepsilon > 0$ sufficiently small and $\theta = \left(\frac{1}{p_c} - \frac{1}{p}\right) / \left(\frac{1}{p_c} - \frac{1}{q_\sharp}\right)$. Moreover, thanks to $r(q)p_c < r(\bar{q})p_c = \bar{q}$ we may estimate

$$\begin{aligned} \||v(s,\cdot)|^p\|_{L^{r(q)}} &= \|v(s,\cdot)\|_{L^{r(q)p}}^p \lesssim \|v(s,\cdot)\|_{L^{r(q)p_c}}^{(1-\theta)p} \|v(s,\cdot)\|_{L^{r(\bar{q})p}}^{\theta p} \\ &\lesssim (1+s)^{-n\left(1-\frac{1}{r(q)p_c}\right)(1-\theta)p+\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{r(\bar{q})p}\right)-\frac{\mu}{2}\right)p\theta} \|v\|_{X(T)}^p, \end{aligned}$$

for all $p_c \le q < \bar{q}$, with $\theta = \left(\frac{1}{r(q)p_c} - \frac{1}{r(q)p}\right) / \left(\frac{1}{r(q)p_c} - \frac{1}{r(\bar{q})p}\right)$. Let

$$\begin{aligned} \gamma &= -n\left(1 - \frac{1}{r(q)p_c}\right)p + \left(\varepsilon - (n-1)\left(\frac{1}{2} - \frac{1}{r(\bar{q})p}\right) - \frac{\mu}{2} + n\left(1 - \frac{1}{r(q)p_c}\right)\right)p\theta \\ &\le -np + \frac{n}{r(q)} + \left(\varepsilon - \frac{(n-1)}{2} - \frac{1}{r(\bar{q})p} - \frac{n+1}{2} + \frac{1}{\bar{q}} + n\right)p\theta \\ &\le -np + \frac{n}{r(q)} + \left(\frac{1}{r(\bar{q})p_c} - \frac{1}{r(\bar{q})p}\right)p\theta + \varepsilon p\theta \\ &\le -np + \frac{n}{r(q)} + \left(\frac{1}{r(\bar{q})}\left(\frac{1}{p_c} - \frac{1}{p}\right)\right)p + \varepsilon p\theta \\ &\le -np\left(1 - \frac{1}{p_c}\right) - \frac{np}{p_c} + \frac{n}{r(q)} + \frac{p}{p_c} - 1 + \varepsilon p\theta. \\ &\le -np\left(1 - \frac{1}{p_c}\right) - n\left(1 - \frac{1}{r(q)}\right) + (n-1)\left(1 - \frac{p}{p_c}\right) + \varepsilon p\theta. \end{aligned}$$

Therefore, if $n + 1 - \frac{2}{\bar{q}} < \mu \le n + 1 - \frac{2}{q_\sharp}$ we conclude that

$$\begin{aligned} \|Fv(t,\cdot)\|_{L^q} &\lesssim (1+\tau)^{-n\left(1-\frac{1}{q}\right)} \int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}-n\left(1-\frac{1}{p_c}\right)p} ds \|v\|_{X(T)}^p \\ &+ (1+\tau)^{-n\left(1-\frac{1}{q}\right)} \int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}+\frac{n-1}{2}-\frac{1}{q}+\gamma} ds \|v\|_{X(T)}^p \\ &\lesssim (1+\tau)^{-n\left(1-\frac{1}{q}\right)} \int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}-n\left(1-\frac{1}{p_c}\right)p} ds \|v\|_{X(T)}^p \\ &+ (1+\tau)^{-n\left(1-\frac{1}{q}\right)} \int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}-n\left(1-\frac{1}{p_c}\right)p+\varepsilon p\theta} ds \|v\|_{X(T)}^p \\ &\lesssim (1+\tau)^{-n\left(1-\frac{1}{q}\right)} \|v\|_{X(T)}^p, \end{aligned}$$

for $p_c \le q < \bar{q}$ and

$$p > \frac{2}{n(1-\ell)} \frac{p_c}{p_c - 1} = 1 + \frac{2}{n(1-\ell)}.$$

Now, if $\mu = n+1-\frac{2}{\bar q}$, applying again Corollary 5.1, we may estimate $\|Fv(\tau,\cdot)\|_{L^{p_c}}$ as before, whereas for $q=\bar q$ or $q=q_\sharp$

$$\begin{aligned}\|Fv(\tau,\cdot)\|_{L^q} &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac12-\frac1q\right)-\frac{\mu}{2}}\\ &\quad\times\int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\left((1+s)^{-\frac{n-1}{2}+\frac1q}\||v(s,\cdot)|^p\|_{L^1}+\||v(s,\cdot)|^p\|_{L^{r(q)}}\right)ds\\ &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac12-\frac1q\right)-\frac{\mu}{2}}\\ &\quad\times\int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+1-\varepsilon}\||v(s,\cdot)|^p\|_{L^1}+(1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\||v(s,\cdot)|^p\|_{L^{r(q)}}ds,\end{aligned}$$

for any $\varepsilon>0$, with $\frac{n}{r(q)}=\frac12+\frac n2+\frac1q$.
Taking into account that $u\in X(T)$, as before, we may estimate

$$\||v(s,\cdot)|^p\|_{L^1}\lesssim (1+s)^{-n\left(1-\frac{1}{p_c}\right)p}\|v\|^p_{X(T)}$$

and thanks to $r(\bar q)p_c\le r(q_\sharp)p_c<q_\sharp$(see Remark 2.1), we may estimate

$$\begin{aligned}\||v(s,\cdot)|^p\|_{L^{r(q)}} &= \|v(s,\cdot)\|^p_{L^{pr(q)}}\\ &\lesssim (1+s)^{\left(\varepsilon-(n-1)\left(\frac12-\frac{1}{pr(q)}\right)-\frac{\mu}{2}\right)p}\|v\|^p_{X(T)}\end{aligned}$$

for any $\varepsilon>0$ and $p_c<p\le\frac{q_\sharp}{r(q_\sharp)}$.
We may write

$$\begin{aligned}&\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon+\left(\varepsilon-(n-1)\left(\frac12-\frac{1}{r(q)p}\right)-\frac{\mu}{2}\right)p\\ &=1+\frac{2\ell}{1-\ell}+\varepsilon(p-1)+n-\frac{1}{r(\bar q)}-(n-1+\mu)\frac p2+\gamma,\end{aligned}$$

with

$$\gamma=\frac{\mu}{2}-1+n\left(\frac{1}{r(q)}-1\right)+\frac{1}{r(\bar q)}-\frac{1}{r(q)}.$$

For $\mu=n+1-\frac{2}{\bar q}$ and $q=\bar q$ we have that $\gamma=0$, whereas for $q=q_\sharp$ we have

$$\begin{aligned}\gamma &= \frac{\mu}{2}-1+n\left(\frac{1}{r(q_\sharp)}-1\right)+\frac{1}{r(\bar q)}-\frac{1}{r(q_\sharp)}\\ &= \frac{(n+1)}{2}-\frac{1}{\bar q}-1+\frac{(1-n)}{2}+\frac{1}{q_\sharp}+\frac1n\left(\frac{1}{\bar q}-\frac{1}{q_\sharp}\right)\\ &= \left(\frac1n-1\right)\left(\frac{1}{\bar q}-\frac{1}{q_\sharp}\right)<0.\end{aligned}$$

We conclude that for $\mu=n+1-\frac{2}{\bar q}$ and $q=\bar q$ or $q=q_\sharp$

$$\begin{aligned}\|Fv(\tau,\cdot)\|_{L^q} &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac12-\frac1q\right)-\frac{\mu}{2}}\|v\|^p_{X(T)}\\ &\quad\times\int_0^\tau (1+s)^{1+\frac{2\ell}{1-\ell}+\varepsilon(p-1)+n-\frac{1}{r(\bar q)}-(n-1+\mu)\frac p2+\gamma}ds\\ &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac12-\frac1q\right)-\frac{\mu}{2}}\|v\|^p_{X(T)},\end{aligned}$$

for any $\varepsilon>0$, $p>p_c=1+\frac{2}{n(1-\ell)}$ and

$$1+\frac{2\ell}{1-\ell}+\varepsilon(p-1)+n-\frac{1}{r(\bar q)}-(n-1+\mu)\frac p2<-1$$

i.e.

$$(n-1+\mu)\frac{p_c}{2} \geq \frac{2}{1-\ell} + n - \frac{1}{r(\bar{q})}$$

is equivalent to

$$\mu \geq n+1 - \frac{2}{p_c r(\bar{q})} = n+1-\frac{2}{\bar{q}}.$$

Moreover, for $n+1-\frac{2}{\bar{q}} < \mu \leq n+1-\frac{2}{q_\sharp}$, we have

$$\begin{aligned}
\|Fv(\tau,\cdot)\|_{L^{q_\sharp}} &\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}} \\
&\quad \times \int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\left((1+s)^{-\frac{n-1}{2}+\frac{1}{q_\sharp}}\||v(s,\cdot)|^p\|_{L^1} + \||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}}\right) ds \\
&\lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}} \\
&\quad \times \int_0^\tau (1+s)^{\frac{2\ell}{1-\ell}+1-\varepsilon}\||v(s,\cdot)|^p\|_{L^1} + (1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} ds
\end{aligned}$$

for any $\varepsilon>0$, with $\frac{n}{r(q_\sharp)} = \frac{1}{2}+\frac{n}{2}+\frac{1}{q_\sharp}$.
If $n+1-\frac{2}{p_c r(q_\sharp)} < \mu \leq n+1-\frac{2}{q_\sharp}$ we may estimate

$$\||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} \lesssim (1+s)^{\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q_\sharp)}\right)-\frac{\mu}{2}\right)p}\|v\|^p_{X(T)},$$

hence

$$\begin{aligned}
(1+s)^{\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon}\||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} &\leq (1+s)^{\frac{2\ell}{1-\ell}+\varepsilon(p-1)-\frac{\mu(p-1)}{2}-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q_\sharp)}\right)p} \\
&\leq (1+s)^{\frac{2\ell}{1-\ell}-n(p-1)+1+\varepsilon(p-1)+\gamma} \leq (1+s)^{-1}
\end{aligned}$$

for $\varepsilon>0$ sufficiently small and

$$p_c < p \leq 1 + \left(\frac{1}{r(q_\sharp)}-\frac{1}{q_\sharp}\right)p_c r(q_\sharp) = 1+p_c - \frac{p_c r(q_\sharp)}{q_\sharp}$$

thanks to

$$\gamma = \frac{p-1}{p_c r(q_\sharp)} + \frac{1}{q_\sharp} - \frac{1}{r(q_\sharp)} \leq 0.$$

Finally, if $n+1-\frac{2}{\bar{q}} < \mu \leq n+1-\frac{2}{p_c r(q_\sharp)}$ we may estimate for $p_c < p \leq \frac{q_\sharp}{r(q_\sharp)}$

$$\begin{aligned}
\||v(s,\cdot)|^p\|_{L^{r(q_\sharp)}} &= \|v(s,\cdot)\|^p_{L^{pr(q_\sharp)}} \\
&\lesssim (1+s)^{\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{pr(q_\sharp)}\right)-\frac{\mu}{2}\right)p}\|v\|^p_{X(T)}
\end{aligned}$$

for any $\varepsilon>0$.
Now we may write

$$\begin{aligned}
&\frac{2\ell}{1-\ell}+\frac{\mu}{2}-\varepsilon+\left(\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{r(q_\sharp)p}\right)-\frac{\mu}{2}\right)p \\
&= 1+\frac{2\ell}{1-\ell}+\varepsilon(p-1)+n-\frac{1}{r(\bar{q})}-(n-1+\mu)\frac{p}{2}+\gamma,
\end{aligned}$$

with

$$\gamma = \frac{\mu}{2}-1+n\left(\frac{1}{r(q_\sharp)}-1\right)+\frac{1}{r(\bar{q})}-\frac{1}{r(q_\sharp)}.$$

It remains to prove that $\gamma \leq 0$. Indeed, for $\mu \leq n+1-\frac{2}{p_c r(q_\sharp)}$ and $\frac{1}{q_\sharp} \leq \frac{1}{n-1}\left(\frac{n}{p_c r(q_\sharp)} - \frac{1}{\bar{q}}\right)$ (see Remark 2.4) we have

$$\begin{aligned}
\gamma &= \frac{\mu}{2} - 1 + n\left(\frac{1}{r(q_\sharp)} - 1\right) + \frac{1}{r(\bar{q})} - \frac{1}{r(q_\sharp)} \\
&= \frac{\mu}{2} - 1 + \frac{1-n}{2} + \frac{1}{q_\sharp} + \frac{1}{n}\left(\frac{1}{\bar{q}} - \frac{1}{q_\sharp}\right) \\
&\leq \frac{\mu}{2} - \frac{1+n}{2} + \frac{1}{p_c r(q_\sharp)} \leq 0.
\end{aligned}$$

Therefore, if $n+1-\frac{2}{\bar{q}} < \mu \leq n+1-\frac{2}{q_\sharp}$ we have proved that

$$\|Fv(s,\cdot)\|_{L^{q_\sharp}} \lesssim (1+\tau)^{\varepsilon-(n-1)\left(\frac{1}{2}-\frac{1}{q_\sharp}\right)-\frac{\mu}{2}} \|v\|^p_{X(T)},$$

for any $\varepsilon > 0$ sufficiently small and $p > p_c = 1 + \frac{2}{n(1-\ell)}$. □

Appendix

In the Appendix we list some notations used through the paper and results of Harmonic Analysis which are important tools for proving results on the global existence of small data solutions for semi-linear models with power non-linearities. Through this paper, we use the following.

For $s \geq 0$, we denote by $|D|^s f = \mathcal{F}^{-1}(|\xi|^s \hat{f})$ and $\langle D\rangle^s f = \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{f})$, with $\langle\xi\rangle^s = (1+|\xi|^2)^{\frac{s}{2}}$.

For any $q \in [1,\infty]$, we denote by $L^q(\mathbf{R}^n)$ the usual Lebesgue space over $\mathbf{R}^n$. Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then

$$\begin{aligned}
H^{s,p}(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) : \|\langle D\rangle^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{H^s_p(\mathbb{R}^n)} < \infty\}, \\
\dot{H}^{s,p}(\mathbb{R}^n) &= \{u \in \mathcal{Z}'(\mathbb{R}^n) : \||D|^s u\|_{L^p(\mathbb{R}^n)} = \|u\|_{\dot{H}^s_p(\mathbb{R}^n)} < \infty\}
\end{aligned}$$

are called Bessel and Riesz potential spaces, respectively. If $p = 2$, then we use the notations $H^s(\mathbb{R}^n)$ and $\dot{H}^s(\mathbb{R}^n)$, respectively. In the definition of the Riesz potential spaces we use the space of distributions $\mathcal{Z}'(\mathbb{R}^n)$. This space of distributions can be identified with the factor space $\mathcal{S}'/\mathcal{P}$, where $\mathcal{S}'$ denotes the dual of Schwartz space and $\mathcal{P}$ denotes the set of all polynomials.

We recall that $H^{s,q}(\mathbf{R}^n) = W^{s,q}(\mathbf{R}^n)$, the usual Sobolev space, for any $q \in (1,\infty)$ and $s \in \mathbf{N}$.

The following inequality can be found in [13], Part 1, Theorem 9.3.

Proposition 5.1 (Fractional Gagliardo-Nirenberg inequality)**.** *Let $1 < p, p_0, p_1 < \infty$, $\sigma > 0$ and $s \in [0,\sigma)$. Then it holds the following fractional Gagliardo-Nirenberg inequality for all $u \in L^{p_0}(\mathbb{R}^n) \cap \dot{H}^{\sigma,p_1}(\mathbb{R}^n)$:*

$$\|u\|_{\dot{H}^{s,p}} \lesssim \|u\|^{1-\theta}_{L^{p_0}} \|u\|^{\theta}_{\dot{H}^{\sigma,p_1}},$$

where $\theta = \theta_{s,\sigma}(p,p_0,p_1) = \frac{\frac{1}{p_0}-\frac{1}{p}+\frac{s}{n}}{\frac{1}{p_0}-\frac{1}{p_1}+\frac{\sigma}{n}}$ and $\frac{s}{\sigma} \leq \theta \leq 1$.

In the following the symbol $\lceil s\rceil$ denotes the smallest integer greater than or equal to s. We present here two results for fractional powers (for instance, see [22] and [23]):

Proposition 5.2. *Let $f(u)=|u|^p$ or $f(u)=|u|^{p-1}u$, with $p>\max\{1,\lceil s\rceil\}$ and $1<r,r_1,r_2<\infty$ satisfying*
$$\frac{1}{r}=\frac{p-1}{r_1}+\frac{1}{r_2}.$$
Then the following estimate holds:
$$\||D|^s f(u)\|_{L^r}\leq C\|u\|^{p-1}_{L^{r_1}}\||D|^s u\|_{L^{r_2}},$$
for any $u\in L^{r_1}\cap \dot{H}^{s,r_2}$.

Corollary 5.2. *Let $f(u)=|u|^p$ or $f(u)=|u|^{p-1}u$, with $p>\max\{1,s\}$ and $u\in H^{s,m}\cap L^\infty$, $1<m<\infty$. Then the following estimate holds*:
$$\|f(u)\|_{\dot{H}^{s,m}}\leq C\|u\|_{\dot{H}^{s,m}}\|u\|^{p-1}_{L^\infty}.$$

We refer to [6] for the nex result:

Lemma 5.1. *Let $0<2s_1<n<2s_2$. Then for any function $f\in \dot{H}^{s_1}\cap\dot{H}^{s_2}$ one has*
$$\|f\|_\infty\lesssim\|f\|_{\dot{H}^{s_1}}+\|f\|_{\dot{H}^{s_2}}.$$

The next result combine in some sense some familiar results as Leibniz rule for the product of two function and Hölder's inequality for derivatives of fractional order (Theorem 7.6.1 in [19]):

Proposition 5.3. *Let us assume $s>0$ and $1\leq r\leq\infty, 1<p_1,p_2,q_1,q_2\leq\infty$ satisfying the relation*
$$\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}.$$
Then the following fractional Leibniz rules hold:
$$\|\,|D|^s(u\,v)\|_{L^r}\lesssim\|\,|D|^s u\|_{L^{p_1}}\|v\|_{L^{p_2}}+\|u\|_{L^{q_1}}\|\,|D|^s v\|_{L^{q_2}}$$
for any $u\in\dot{H}^{s,p_1}(\mathbf{R}^n)\cap L^{q_1}(\mathbf{R}^n)$ and $v\in\dot{H}^{s,q_2}(\mathbf{R}^n)\cap L^{p_2}(\mathbf{R}^n)$,
$$\|\langle D\rangle^s(u\,v)\|_{L^r}\lesssim\|\langle D\rangle^s u\|_{L^{p_1}}\|v\|_{L^{p_2}}+\|u\|_{L^{q_1}}\|\langle D\rangle^s v\|_{L^{q_2}}$$
for any $u\in H^{s,p_1}(\mathbf{R}^n)\cap L^{q_1}(\mathbf{R}^n)$ and $v\in H^{s,q_2}(\mathbf{R}^n)\cap L^{p_2}(\mathbf{R}^n)$.

Acknowledgments

The first author is partially supported by "Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP)" grant number 2020/08276-9 and "Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq)" grant number 304408/2020-4. The second author is partially supported by "FCT – Fundação para a Ciência e a Tecnologia, I.P., Project UIDB/05037/2020".

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