Validity for a phase-field model on the martensitic phase transformations

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Abstract

We show the existence of global-in-time solutions to an initial-boundary value problem in one space dimension of a phase-field model on the martensitic phase transformations, which consists of the equations of degenerate and a non-degenerate nonlinear parabolic equation of second order.

Keywords: Phase-field model; Coupled parabolic equations; Degenerate; Weak solution; Banach fixed-point theorem.

1 Introduction

The transformations from austenite to martensite in material are called the Martensitic phase transformations. And the phase transformations are often accompanied by twinning via reducing the energy associated with internal elastic stresses [1]. Twinning is a common mechanism of metal plastic deformation, which is related to stacking faults [23]. In the process of plastic deformation, an area of the crystal lattice is sheared into a new orientation, and it is most obvious at low temperatures or high strain rates [3].

Phase-field approach is widely utilized for modeling microstructure evolution processes in materials. For the common phase field models, they are mainly the mathematical models used to solve interface problems which incorporate the corresponding features of stress-strain curves, large strain formulation, surface tension, and they are applied to solidification dynamics [4], fracture dynamics [5] and vesicle dynamics [6]. On the phase-field model, the Cahn-Hilliard and/or Allen-Cahn equations are the two well-known models for temporal evolution of microstructures, which have the conserved order parameter or not conserved order parameter during the phase separation respectively [7-10]. These two kinds of order parameter are also presented in numerous articles [9-12], and the properties of solutions to the parabolic problems are investigated in the phase-field models [13-16].

In this article, we investigate a phase field model of transformations between martensitic variants and multiple twinning within martensitic variants, which is developed for lattice rotations and large strains [7]. For the model, we just utilize one angular order parameter $\phi(t, x)$ to describe variant-variant transformation and multiple twinnings within every martensitic variant, and use $u(t,x)$ as the radial coordinate of the austenite-martensite transformations. The Helmholtz free energy is given [3]

$$
\begin{aligned}
F[u,\ \phi] &= \int_\Omega f\ dV \\
&= \int_\Omega \Big(f^c(u,\ \phi) + f^i(u,\ \nabla u,\ \nabla\phi)\Big) dV,
\end{aligned}
\tag{1.1}
$$

where

$$
\begin{aligned}
f^c(u,\ \phi) &= \frac{1}{3}\eta_1 q(u) + a_1 m g(\phi) q(u) + a_0 m \eta_0 h(u), \\
f^i(u,\ \nabla u,\ \nabla\phi) &= \frac{m\beta_u}{2}|\nabla u|^2 + \frac{m\beta_\phi}{2} q(u)|\nabla\phi|^2.
\end{aligned}
$$

Taking the variation of both sides of (1.1) with respect to t, it is easy to obtain

$$
\begin{aligned}
\frac{dF}{dt} &= \int_\Omega \Big(\frac{\partial}{\partial t}(\frac{m\beta_u}{2}|\nabla u|^2) + \frac{\partial}{\partial t}(\frac{m\beta_\phi}{2} q(u)|\nabla\phi|^2) + \frac{\partial f^c}{\partial u} u_t + \frac{\partial f^c}{\partial \phi}\phi_t\Big)\ dV \\
&= \int_\Omega \Big(\big(-m\beta_u \Delta u + \frac{m\beta_\phi}{2} q'(u)|\nabla\phi|^2 + \frac{\partial f^c}{\partial u}\big) u_t + \big(-m\beta_\phi \mathrm{div}\big(q(u)\nabla\phi\big) + \frac{\partial f^c}{\partial\phi}\big)\phi_t\Big)\ dV.
\end{aligned}
$$

To satisfy the second law of thermodynamics $\frac{dF}{dt} \le 0$ [17], we set

$$
u_t = \frac{k_u}{m}\big(m\beta_u \Delta u - \frac{m\beta_\phi}{2} q'(u)|\nabla\phi|^2 - \frac{\partial f^c}{\partial u}\big), \tag{1.2}
$$

$$
\phi_t = \frac{k_\phi}{m}\big(m\beta_\phi \mathrm{div}\big(q(u)\nabla\phi\big) - \frac{\partial f^c}{\partial\phi}\big). \tag{1.3}
$$

Considering the equations (1.2)-(1.3) in one space dimension as

$$
\phi_t = \alpha_2\big(q(u)\phi_x\big)_x - \rho g'(\phi) q(u), \tag{1.4}
$$

$$
u_t = \alpha_1 u_{xx} - \alpha_3 q'(u)|\phi_x|^2 - \frac{\partial f^j}{\partial u}, \tag{1.5}
$$

for $(t,\ x) \in (0, +\infty) \times \Omega,\ \Omega = (a, b)$.

The boundary and initial conditions are

$$
\frac{\partial u}{\partial n} = 0,\quad \frac{\partial\phi}{\partial n} = 0,\quad (t,\ x) \in (0, +\infty) \times \partial\Omega, \tag{1.6}
$$

$$
u(0,\ x) = u_0(x),\quad \phi(0,\ x) = \phi_0(x),\quad x \in \Omega. \tag{1.7}
$$

Notations and function specifications: The positive constants β_i, k_i $(i = u, \phi)$, a_j, η_j $(j = 0, 1)$, m are the gradient energy coefficients, kinetic coefficients, barriers for variant-variant transformations, the temperature at the relevant states, the ratio of mass densities respectively. We note $\alpha_1 = \beta_u k_u,\ \alpha_2 = \beta_\phi k_\phi,\ \alpha_3 = \frac{1}{2}\beta_\phi k_u,\ \rho_1 = a_0\eta_0 k_u,\ \rho_2 = \frac{\eta_1 k_u}{3m},\ \rho_3 = \frac{3a_1 m}{\eta_1}$, $\rho = a_1 k_\phi$.

The function $\frac{\partial f^j}{\partial u} = \rho_1 h'(u) + \rho_2 q'(u)(1 + \rho_3 g(\phi))$ in (1.5) with the double-well potential with minima at $\phi = 0$ and $\phi = 1$, and the other double-well potential with minima at $u = 0$ and $u = 1$ are given by

$$
g(\phi) = \phi^2(1-\phi)^2,\ h(u) = u^2(1-u)^2,
$$

and the interpolation function

$$q(u) = u^2(3 - 2u).$$

Here $q(u)$ depends only on the radial coordinate u, so one technical difficulty is that, when $q(u) = 0$, equation (1.4) degenerates, which will lose its parabolic character. Otherwise, for the case of $q(u) > 0$, we have the consult of $\phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$ in [18].

In this article, we use the notation $\|\cdot\|_{L^2(\Omega)} = \|\cdot\|$, write $Q_T := (0,\ T) \times \Omega$ for any given constant $T > 0$, denote $<\cdot,\ \cdot>$ as the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$.

Before stating our main results, we give the definition of a weak solution to problem (1.4)-(1.7).

Definition 1.1. Assume that $\phi_0 \in L^2(\Omega)$, and $u_0 \in L^2(\Omega)$. A function $(\phi,\ u)$ with

$$\phi \in L^\infty(0, T; H^1(\Omega)), \tag{1.8}$$
$$\phi_t \in L^2(0, T; (H^1(\Omega))'), \tag{1.9}$$
$$u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \cap L^4(Q_T), \tag{1.10}$$
$$u_t \in L^2(Q_T), \tag{1.11}$$

is a weak solution to problem (1.4)-(1.7), if for each $\varphi \in C_0^\infty((-\infty, T) \times \Omega)$,

$$(\phi, \varphi_t)_{Q_T} - \alpha_2(q(u)\phi_x, \varphi_x)_{Q_T} - \rho(g'(\phi)q(u), \varphi)_{Q_T} + (\phi_0, \varphi(0))_\Omega = 0, \tag{1.12}$$
$$(u, \varphi_t)_{Q_T} - \alpha_1(u_x, \varphi_x)_{Q_T} - \alpha_3(q'(u)\phi_x^2, \varphi)_{Q_T} - (\frac{\partial f^j}{\partial u}, \varphi)_{Q_T} + (u_0, \varphi(0))_\Omega = 0. \tag{1.13}$$

The main result of this article is the following:

Theorem 1.1 *For all $\phi_0 \in L^2(\Omega)$, and $u_0 \in H^1(\Omega)$, there exists a weak solution $(\phi,\ u)$ to the problem* (1.4)-(1.7) *in the sense of Definition 1.1.*

Our proof is based on the following modified problem which depend on a small accessory positive parameter ϵ. And we can utilize the Banach fixed-point theorem to show its existence of weak solutions by avoiding the above mentioned possibility of degeneracy of parabolicity in equation (1.4). The modified problem is given

$$\phi_t^\epsilon = \alpha_2\big(q(\epsilon_u)\phi_x^\epsilon\big)_x - \rho g'(\phi^\epsilon)q(u^\epsilon) \quad in \ \ Q_T, \tag{1.14}$$
$$u_t^\epsilon - \alpha_1 u_{xx}^\epsilon = -\alpha_3 q'(u^\epsilon)|\phi_x^\epsilon|^2 - \frac{\partial (f^j)^\epsilon}{\partial u^\epsilon} \quad in \ \ Q_T, \tag{1.15}$$
$$\frac{\partial u^\epsilon}{\partial n} = 0, \quad \frac{\partial \phi^\epsilon}{\partial n} = 0, \quad (t,\ x) \in (0, +\infty) \times \partial\Omega, \tag{1.16}$$
$$u^\epsilon(0,\ x) = u_0^\epsilon(x), \quad \phi^\epsilon(0,\ x) = \phi_0^\epsilon(x), \quad x \in \Omega, \tag{1.17}$$

where we define

$$u_\epsilon = \begin{cases} 1, & \text{if } \ u > 1, \\ u, & \text{if } \ u \in [\epsilon,\ 1], \\ \epsilon, & \text{if } \ u < \epsilon, \end{cases} \tag{1.18}$$

and $\epsilon_u = \eta_\epsilon * u_\epsilon$, here $*$ is a convolution operation, $\eta_\epsilon \in C_0^\infty(\Omega)$ is the positive mollifier function and it converges to the Dirac delta function, ϵ is a positive constant.

The rest of this paper is organized as follows: In Section 2, we study the modified problem (1.14)-(1.17) depending on a small positive parameter ϵ, and prove the existence of approximate solution by employing the Banach fixed-point theorem. In Section 3, we derive the bounds, uniform with respect to ϵ for the approximate solution, and utilize the Aubin-Lions lemma to show that the limit of a subsequence of approximate solution solve the original problem (1.4)-(1.7).

2 Existence of solutions to the modified problems

We first introduce a Banach space $\mathcal{X}$, that is

$$\mathcal{X} = \{(\phi,\ u)\ \mid\ \phi \in (L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \\ u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\ \epsilon \le u \le 1 \text{ a.e. in } Q_T\}.$$

To simplify the calculation on the constant coefficients, we set $\rho_2 = \alpha_3$ in equation (1.5), then it can be rewritten as

$$u_t = \alpha_1 u_{xx} - \alpha_3 q'(u)[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1 - 2u)].$$

Then to ease the notation, we will omit the superscript ϵ of the variables ϕ^ϵ and u^ϵ for the problem (1.14)-(1.17) in this section.

2.1 Existence results to the fixed problem

With $u \in \mathcal{X}$ fixed, we consider the parabolic problem of ϕ

$$\begin{cases} \phi_t - \alpha_2\big(q(\epsilon_u)\phi_x\big)_x = -\rho g'(\phi)q(u) \qquad in \quad Q_T, \\ \phi(0,\ x) = \phi_0^\epsilon(x), \qquad x \in \Omega; \quad \frac{\partial \phi}{\partial n} = 0, \qquad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.1}$$

We have the following lemma:

Lemma 2.1 *If $\phi_0^\epsilon \in H^1(\Omega)$, then (2.1) has a unique solution $\phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, and*

$$\|\phi\|_{L^2(0,T;H^2(\Omega))} + \|\phi\|_{L^\infty(0,T;H^1(\Omega))} \le C. \tag{2.2}$$

Proof: From Them.7.1 of [19] and Them.6.7 of [20], it is easy to obtain the proof of this lemma.

Given the function $\phi \in \mathcal{X}$, we consider the quasilinear parabolic problem of u

$$\begin{cases} u_t - \alpha_1 u_{xx} = -\alpha_3 q'(u)[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1 - 2u)] \quad in \quad Q_T, \\ u(0,\ t) = u_0^\epsilon(x), \qquad x \in \Omega; \quad \frac{\partial u}{\partial n} = 0, \qquad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.3}$$

For the problem (2.3), we also have

Lemma 2.2 *Let $\epsilon \le u_0^\epsilon \le 1$ a.e. $x \in \Omega$. If $u_0^\epsilon \in H^1(\Omega)$, then (2.3) has a unique solution $\epsilon \le u \le 1$ a.e. in Q_T, and*

$$\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \le C, \quad \|u_t\|_{L^2(Q_T)} \le C. \tag{2.4}$$

To prove the existence of solutions to (2.3), we first consider the problem

$$\begin{cases} u_t - \alpha_1 u_{xx} = -\alpha_3 \tilde{q}'(u)\big[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1-2u)\big] \quad in \quad Q_T, \\ \\ u(0,x) = u_0^\epsilon(x), \qquad x \in \Omega; \qquad \frac{\partial u}{\partial n} = 0, \qquad (t,\ x) \in (0,\ T) \times \partial\Omega, \end{cases} \tag{2.5}$$

where $\phi \in \mathcal{X}$, and

$$\tilde{q}'(u) = \begin{cases} q'(\epsilon), & \text{if } u \in (-\infty,\ \epsilon), \\ q'(u), & \text{if } u \in [\epsilon,\ 1], \\ q'(1), & \text{if } u \in (1,\ \infty). \end{cases} \tag{2.6}$$

Lemma 2.3 *Let $\epsilon \le u_0^\epsilon \le 1$ a.e. $x \in \Omega$. If $u_0^\epsilon \in H^1(\Omega)$, then (2.5) has a solution $\epsilon \le u \le 1$ a.e. in Q_T, and*

$$\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \le C. \tag{2.7}$$

Proof: The existence of solution u in (2.5) can be proved from Them.7.1 of [19] and Them.6.7 of [20], and we get

$$\|u\|^2_{L^\infty(0,T;H^1(\Omega))} + \|u\|^2_{L^2(0,T;H^2(\Omega))} \le C.$$

Next, we prove that if $\epsilon \le u_0^\epsilon \le 1$ a.e. $x \in \Omega$, then $\epsilon \le u \le 1$ a.e. in Q_T.

Multiplying the equation in (2.5) by $(u-\epsilon)^-$, where $(u-\epsilon)^- = \max(0,\ \epsilon - u)$, and integrating over Ω, we get, for a.e. $t \in (0,T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |(u-\epsilon)^-|^2 dx + \alpha_1 \int_\Omega |(u-\epsilon)^-_x|^2 dx = \ & -\alpha_3 \int_\Omega \tilde{q}'(u)(u-\epsilon)^-[\frac{\rho_1}{3\alpha_3}(1-2u) + 1 + |\phi_x|^2 \\ & + \rho_3 g(\phi)]dx. \end{aligned}$$

It follows from the definitions of $(u-\epsilon)^-$ and $\tilde{q}'(u)$ that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u-\epsilon)^-|^2 dx + \alpha_1 \int_\Omega |(u-\epsilon)^-_x|^2 dx = 0. \tag{2.8}$$

Due to the fact that the second term in (2.8) is positive, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u-\epsilon)^-|^2 dx \le 0.$$

From the data $(u_0-\epsilon)^- = 0$, it is easy to verify $(u-\epsilon)^- = 0$. Hence, $u \ge \epsilon$ a.e. in Q_T.

We next multiply the equation in (2.5) by $(u-1)^+$, integrate over Ω, and obtain, for a.e. $t \in (0,T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |(u-1)^+|^2 dx + \alpha_1 \int_\Omega |(u-1)^+_x|^2 dx = & - \alpha_3 \int_\Omega \tilde{q}'(u)(u-1)^+[\frac{\rho_1}{3\alpha_3}(1-2u) + 1 \\ & + |\phi_x|^2 + \rho_3 g(\phi)]dx. \end{aligned}$$

Making use of the definitions of $\tilde{q}'(u)$ and $(u-1)^+$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u-1)^+|^2 dx + \alpha_1 \int_\Omega |(u-1)_x^+|^2 dx = 0. \tag{2.9}$$

Similarly, we deduce $u \leq 1$ a.e. in Q_T. The proof of Lemma 2.3 is complete.

The proof of Lemma 2.2: From Lemma 2.3 and the hypothesis of $\epsilon \leq u \leq 1$, it is easy to obtain (2.4). For the uniqueness of solution, we consider u_1 and u_2 two solutions to the problem (2.3), and note $\hat{u} = u_1 - u_2$, then it satisfies

$$\begin{cases} \hat{u}_t - \alpha_1 \hat{u}_{xx} = -\rho_1\big(h'(u_1) - h'(u_2)\big) - \alpha_3\big(q'(u_1) - q'(u_2)\big)\big[1 + |\phi_x|^2 + \rho_3 g(\phi)\big], \\ \hat{u}(0,\ x) = 0, \quad x \in \Omega; \quad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0, \quad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.10}$$

Multiplying the equation in (2.10) by $\hat{u}$, integrating over Ω, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\hat{u}\|^2 + \alpha_1\|\hat{u}_x\|^2 &= -\int_\Omega \big\{\rho_1\big(h'(u_1) - h'(u_2)\big) + \alpha_3(q'(u_1) - q'(u_2))[1 + |\phi_x|^2 + \rho_3 g(\phi)\big\}\hat{u}dx \\ &\leq C\big(1 + \|\phi_x\|^2_{L^\infty(\Omega)} + \|g(\phi)\|_{L^\infty(\Omega)}\big)\|\hat{u}\|^2 \\ &\leq C\big(1 + \|\phi_{xx}\|^{\frac{3}{2}}\|\phi\|^{\frac{1}{2}} + \|\phi\|^4_{H^1(\Omega)}\big)\|\hat{u}\|^2. \end{aligned}$$

From Gronwall's lemma [21], it is easy to obtain

$$\|\hat{u}\|^2 \leq e^{2C\int_0^t \left(1 + \|\phi_{xx}\|^{\frac{3}{2}}\|\phi\|^{\frac{1}{2}} + \|\phi\|^4_{H^1(\Omega)}\right)ds}\|\hat{u}(0,\ x)\|^2 = 0,$$

which ensures the uniqueness. The proof of Lemma 2.2 is complete.

2.2 The Banach fixed-point method

Below we introduce the Banach fixed-point theorem [19], that is

Theorem 2.2 (Banach Fixed-Point Theorem) *$\mathcal{A}$ is a Banach space. Assume $\tau :\ \mathcal{A} \to \mathcal{A}$ is a nonlinear mapping, and suppose that*

$$\|\ \tau(u_1) - \tau(u_2)\ \| \leq \gamma\ \|\ u_1 - u_2\ \|,\ (u_1,\ u_2 \in \mathcal{A}) \tag{2.11}$$

for some constant $0 \leq \gamma < 1$. Then τ has a unique fixed point.

Now, we begin to show the existence of local solution to problem (1.14)-(1.17). First, from the proofs of Lemma 2.1 and Lemma 2.2, we can define a map $A : \mathcal{X} \to \mathcal{X}$ by setting $A(\bar{\phi},\ \bar{u}) = (\phi,\ u)$, where $(\phi,\ u)$ and $(\bar{\phi},\ \bar{u})$ satisfy

$$\begin{cases} \phi_t - \alpha_2\big(q(\epsilon_{\bar{u}})\phi_x\big)_x = -\rho g'(\bar{\phi})q(\bar{u}) \quad in\ \ Q_T, \\ u_t - \alpha_1 u_{xx} = -\alpha_3 q'(u)[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1 - 2u)] \quad in\ \ Q_T, \\ \phi(0, x) = \phi_0^\epsilon(x),\ u(0, x) = u_0^\epsilon(x),\ x \in \Omega; \\ \frac{\partial \phi}{\partial n} = 0, \frac{\partial u}{\partial n} = 0,\ (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.12}$$

Next we need to claim that A is a compressed mapping if $T > 0$ is small enough. To prove this, we choose $(\bar{\phi}_i,\ \bar{u}_i) \in \mathcal{X}$ so that $A(\bar{\phi}_i,\ \bar{u}_i) = (\phi_i,\ u_i)$ with the norm

$$\|(\bar{\phi}_i,\ \bar{u}_i)\|_{\mathcal{X}} = \|\bar{\phi}_i\|_{L^\infty(0,T;H^1(\Omega))} + \|\bar{\phi}_i\|_{L^2(0,T;H^2(\Omega))} + \|\bar{u}_i\|_{L^\infty(0,T;H^1(\Omega))} + \|\bar{u}_i\|_{L^2(0,T;H^2(\Omega))}$$

where $(\phi_i,\ u_i),\ i = 1, 2$ are the solutions to the syestem (2.12).

Lemma 2.4 *Assume $u_0^\epsilon \in H^1(\Omega)$, $\phi_0^\epsilon \in H^1(\Omega)$, and $\|(\bar{\phi}_i,\ \bar{u}_i)\|_{\mathcal{X}} \leq C$. Suppose $A:\ \mathcal{X} \to \mathcal{X}$ is a nonlinear mapping, if the mapping satisfies*

$$\|A(\bar{\phi}_1,\ \bar{u}_1) - A(\bar{\phi}_2,\ \bar{u}_2)\|_{\mathcal{X}} \leq \gamma\|(\bar{\phi}_1 - \bar{\phi}_2,\ \bar{u}_1 - \bar{u}_2)\|_{\mathcal{X}},$$

for some constant $0 \leq \gamma < 1$. Then A is a contraction.

Proof: We set $\phi = \phi_1 - \phi_2$, $U = u_1 - u_2$, then (ϕ, U) satisfy

$$\begin{cases} \phi_t - \alpha_2\big(q(\epsilon_{\bar{u}_1})\phi_{1x} - q(\epsilon_{\bar{u}_2})\phi_{2x}\big)_x = -\rho g'(\bar{\phi}_1)q(\bar{u}_1) + \rho g'(\bar{\phi}_2)q(\bar{u}_2) \quad in \ \ Q_T, \\ U_t - \alpha_1 U_{xx} = -\alpha_3 q'(u_1)\big(1 + |\phi_{1x}|^2 + \rho_3 g(\phi_1)\big) + \alpha_3 q'(u_2)\big(1 + |\phi_{2x}|^2 + \rho_3 g(\phi_2)\big) \\ \qquad\qquad - \rho_1\big(h'(u_1) - h'(u_2)\big) \qquad\qquad in \ \ Q_T, \\ \phi(0, x) = 0,\ U(0, x) = 0, \quad x \in \Omega; \quad \frac{\partial \phi}{\partial n} = 0, \frac{\partial U}{\partial n} = 0, \quad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.13}$$

Multiplying the first equation in (2.13) by ϕ, integrating over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \alpha_2 m\|\phi_x\|^2 \\ \leq\ & \alpha_2 \int_\Omega |q(\epsilon_{\bar{u}_1}) - q(\epsilon_{\bar{u}_2})||\phi_{2x}||\phi_x|dx + \rho\|q(\bar{u}_1)\|_{L^\infty(\Omega)} \int_\Omega |g'(\bar{\phi}_1) - g'(\bar{\phi}_2)||\phi|dx \\ & + \rho\int_\Omega |g'(\bar{\phi}_2)||q(\bar{u}_1) - q(\bar{u}_2)||\phi|dx \\ \leq\ & \frac{\alpha_2 m}{2}\|\phi_x\|^2 + \frac{1}{2}\|\phi\|^2(\|\bar{\phi}_1\|^4_{L^\infty(\Omega)} + \|\bar{\phi}_2\|^4_{L^\infty(\Omega)} + \|g'(\bar{\phi}_2)\|^2_{L^\infty(\Omega)}) + C(\|\phi_{2x}\|^2 + \|\bar{\phi}\|^2 + \|\bar{u}\|^2) \\ \leq\ & \frac{\alpha_2 m}{2}\|\phi_x\|^2 + \frac{1}{2}\|\phi\|^2(\|\bar{\phi}_1\|^4_{H^1(\Omega)} + \|\bar{\phi}_2\|^4_{H^1(\Omega)} + \|\bar{\phi}_2\|^6_{H^1(\Omega)}) + C(\|\phi_2\|^2_{H^1(\Omega)} + \|\bar{\phi}\|^2 + \|\bar{u}\|^2), \end{aligned} \tag{2.14}$$

where we set $0 < m \leq q(\bar{u}_i) \leq M,\ i = 1,\ 2$.

We get from Gronwall's lemma and Hölder inequality that

$$\begin{aligned} \|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} &\leq C(1 + \int_0^t \|\bar{u}\|^2 ds + \int_0^t \|\bar{\phi}\|^2_{H^1(\Omega)}\ ds) \\ &\leq C(\|\bar{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\bar{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))})(\int_0^t ds) \\ &\leq CT(\|\bar{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\bar{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))}). \end{aligned} \tag{2.15}$$

The estimates (2.14)-(2.15) together yield

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;H^1(\Omega))} \leq CT(\|\bar{u}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\bar{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))}).$$

We multiply the first equation in (2.13) by $-\phi_{xx}$, integrate over Ω, and get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2+\alpha_2 m\|\phi_{xx}\|^2\\
\le\ & \alpha_2\int_\Omega |q'(\bar u_1)\bar u_{1x}\phi_{1x}-q'(\bar u_2)\bar u_{2x}\phi_{2x}||\phi_{xx}|dx+\rho\int_\Omega |q(\bar u_1)||g'(\bar\phi_1)-g'(\bar\phi_2)||\phi_{xx}|dx\\
&+\rho\int_\Omega |g'(\bar\phi_2)(q(\bar u_1)-q(\bar u_2))||\phi_{xx}|dx\\
=:\ & I_1+I_2+I_3,
\end{aligned}\tag{2.16}$$

where we set $b=\max(|q'(\bar u_1)|,\ |q'(\bar u_2)|)$, and

$$\begin{aligned}
I_1 &= \alpha_2\int_\Omega |q'(\bar u_1)\bar u_{1x}\phi_{1x}-q'(\bar u_2)\bar u_{2x}\phi_{2x}||\phi_{xx}|dx\\
&\le \alpha_2 m\int_\Omega |\bar u_x||\phi_{1x}||\phi_{xx}|dx+b\alpha_2\int_\Omega |\bar u_{2x}||\phi_x||\phi_{xx}|dx\\
&\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2+C\big(\|\bar u_x\|^2\|\phi_{1x}\|^2_{L^\infty(\Omega)}+\|\bar u_{2x}\|^2_{L^\infty(\Omega)}\|\phi_x\|^2\big),
\end{aligned}\tag{2.17}$$

$$\begin{aligned}
I_2 &= \rho\int_\Omega |q(\bar u_1)||g'(\bar\phi_1)-g'(\bar\phi_2)||\phi_{xx}|dx\\
&\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2+\|q(\bar u_1)\|^2_{L^\infty(\Omega)}\|g'(\bar\phi_1)-g'(\bar\phi_2)\|^2\\
&\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2+C(\|\bar\phi_1\|^4_{L^\infty(\Omega)}+\|\bar\phi_2\|^4_{L^\infty(\Omega)})\|\bar\phi\|^2,
\end{aligned}\tag{2.18}$$

$$\begin{aligned}
I_3 &= \rho\int_\Omega |g'(\bar\phi_2)(q(\bar u_1)-q(\bar u_2))||\phi_{xx}|dx\\
&\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2+C\int_\Omega |g'(\bar\phi_2)|^2|\bar u|^2dx\\
&\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2+C\|\bar u\|^2_{L^2(\Omega)}\|\bar\phi_2\|^6_{H^1(\Omega)}.
\end{aligned}\tag{2.19}$$

It follows from (2.16)–(2.19) that

$$\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2+\frac{\alpha_2 m}{4}\|\phi_{xx}\|^2\le C\|\bar u_{2x}\|^2_{L^\infty(\Omega)}\|\phi_x\|^2+C_1\big(\|\bar u_x\|^2\|\phi_{1x}\|^2_{L^\infty(\Omega)}+\|\bar\phi_1+\bar\phi_2\|^4_{L^\infty(\Omega)}\|\bar\phi\|^2\big).\tag{2.20}$$

Applying Gronwall's lemma, the estimates (2.16) and (2.20) imply

$$\|\phi_x\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\phi_{xx}\|^2_{L^2(0,T;L^2(\Omega))}\le CT(\|\bar\phi\|^2_{L^\infty(0,T;H^1(\Omega))}+\|\bar u\|^2_{L^\infty(0,T;L^2(\Omega))}).\tag{2.21}$$

We multiply the second equation in (2.13) by U, and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2+\alpha_1\|U_x\|^2\le C_1(1+\|\phi_{1x}\|^2_{L^\infty(\Omega)}+\|\phi_{2x}\|^2_{L^\infty(\Omega)})\|U\|^2+C_2\int_\Omega \|g(\phi_1)-g(\phi_2)\|^2dx.$$

Applying Gronwall's lemma, we get

$$\|U\|^2_{L^\infty(0,T;L^2(\Omega))} + \|U\|^2_{L^2(0,T;H^1(\Omega))} \le CT(\|\bar{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\bar{u}\|^2_{L^\infty(0,T;L^2(\Omega))}). \quad (2.22)$$

Similarly, we multiply the second equation in (2.13) by $-U_{xx}$ and integrate over Ω to get

$$\|U_x\|^2_{L^\infty(0,T;L^2(\Omega))} + \|U_{xx}\|^2_{L^2(0,T;L^2(\Omega))} \le CT(\|\bar{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\bar{u}\|^2_{L^\infty(0,T;L^2(\Omega))}). \quad (2.23)$$

Finally, the estimates (2.14)-(2.23) yield

$$\|(\phi, U)\|_{\mathcal{X}} \le (CT)^{\frac{1}{2}} \|(\bar{\phi},\ \bar{u})\|_{\mathcal{X}}. \quad (2.24)$$

Thus, A is a contractive operator if $(CT)^{\frac{1}{2}} < 1$ holds. Then we can utilize the Banach fixed-point theorem to get a solution $(\phi^\epsilon,\ u^\epsilon)$ to problem (1.14)-(1.17) on a small interval $[0,\ T_1]$, where $0 < T_1 < T$ satisfies $(CT_1)^{\frac{1}{2}} < 1$.

3 Existence of weak solutions

In section 2, we proved the problem (1.14)-(1.17) has a solution $(\phi^\epsilon,\ u^\epsilon)$. Thus, our goal in this section is to choose a subsequence of $(\phi^\epsilon,\ u^\epsilon)$ and send the parameter ϵ to zero to obtain a weak solution of the original problem (1.4)-(1.7).

First a general version of the Aubin-Lions lemma under the weak assumption $\frac{\partial v_i}{\partial t} \in L^1(0,T;B_1)$ is given [22], [23]:

Theorem 3.3 *Let B_0 be a normed linear space imbedded compactly into another normed linear space B, which is continuously imbedded into a Hausdorff locally convex space B_1, and $1 \le p < +\infty$. If $v,\ v_i \in L^p(0,T;B_0)$, $i \in \mathbb{N}$, the sequence $\{v_i\}_{i\in\mathbb{N}}$ converges weakly to v in $L^p(0,T;B_0)$, and $\{\frac{\partial v_i}{\partial t}\}_{i\in\mathbb{N}}$ is bounded in $L^1(0,T;B_1)$, then v_i converges to v strongly in $L^p(0,T;B)$.*

The other two convergence arguments are as follows:

Lemma 3.5 *Let B_0, B and B_1 be three Banach spaces with*

$$B_0 \subset\subset B \subset B_1,$$

where B_0, B_1 are reflexive spaces. Suppose that B_0 is compactly embedded in B and B is continuously embedded in B_1. For $1 < p_0,\ p_1 < +\infty$, let

$$W = \Big\{ f \Big| f \in L^{p_0}(0,T;B_0),\ f' = \frac{\partial f}{\partial t} \in L^{p_1}(0,T;B_1) \Big\}.$$

Then the embedding of W into $L^{p_0}(0,T;B)$ is compact.

Lemma 3.6 *Let $Q_T = (0,T) \times \Omega$ be a bounded open set in $\mathbb{R}_t \times \mathbb{R}^n_x$. Assume that u_m, u are functions which belong to $L^q(Q_T)$ for any given $1 < q < +\infty$, and which satisfy*

$$u_m \to u \ \ a.e. \ \ in \ \ Q_T, \ \ \|u_m\|_{L^q(Q_T)} \le C. \quad (3.1)$$

Then u_m converges weakly to u in $L^q(Q_T)$.

The proof of the above lemmas can be found in [24], [25], [26].

3.1 Existence of local solutions

Lemma 3.7 *There exists a constant C independent of ϵ, such that*

$$\|\phi^\epsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|q(\epsilon_u)\phi^\epsilon_x\|_{L^2(0,T;H^1(\Omega))} \le C, \tag{3.2}$$

$$\|(q(\epsilon_u)\phi^\epsilon_x)_t\|_{L^1(0,T;H^{-1}(\Omega))} \le C, \qquad \|\phi^\epsilon_t\|_{L^2(0,T;(H^1(\Omega))')} \le C, \tag{3.3}$$

$$\|u^\epsilon\|_{L^\infty(0,T;H^1(\Omega))} + \|u^\epsilon\|_{L^2(0,T;H^2(\Omega))} \le C, \qquad \|u^\epsilon_t\|_{L^2(Q_T)} \le C. \tag{3.4}$$

Proof: We multiply equation (1.14) by ϕ^ϵ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi^\epsilon\|^2 + \alpha_2\int_\Omega q(\epsilon_u)(\phi^\epsilon_x)^2dx + \rho\int_\Omega g'(\phi^\epsilon)\phi^\epsilon q(u^\epsilon)dx = 0.$$

Using the Hölder inequality and the boundedness of $q(u^\epsilon)$, we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\phi^\epsilon\|^2 + \alpha_2\int_\Omega q(\epsilon_u)(\phi^\epsilon_x)^2dx + \rho\int_\Omega q(u^\epsilon)|\phi^\epsilon|^4dx \le C\|\phi^\epsilon\|^2. \tag{3.5}$$

Applying Gronwall's lemma, the estimate (3.5) implies

$$\|\phi^\epsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|q(\epsilon_u)\phi^\epsilon_x\|^2_{L^2(0,T;L^2(\Omega))} \le C\|\phi^\epsilon_0\|_{L^2(\Omega)} \le C. \tag{3.6}$$

Then multiplying equation (1.14) by $-\phi^\epsilon_{xx}$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi^\epsilon_x\|^2 + \alpha_2\int_\Omega q(\epsilon_u)(\phi^\epsilon_{xx})^2dx &\le C\int_\Omega \big(|q'(\epsilon_u)(\epsilon_u)_x\phi^\epsilon_x\phi^\epsilon_{xx}| + |g'(\phi^\epsilon)\phi^\epsilon_{xx}q(u^\epsilon)|\big)dx\\ &\le \frac{\alpha_2}{2}\int_\Omega q(\epsilon_u)(\phi^\epsilon_{xx})^2dx + C(\|(\epsilon_u)_x\|^2_{L^\infty(\Omega)}\|\phi^\epsilon_x\|^2 + \|g'(\phi^\epsilon)\|^2)\\ &\le \frac{\alpha_2}{2}\int_\Omega q(\epsilon_u)(\phi^\epsilon_{xx})^2dx + C(\|u\|^2_{H^1(\Omega)}\|\phi^\epsilon_x\|^2 + \|\phi^\epsilon_x\|^2\|\phi^\epsilon\|^4),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\|(\epsilon_u)_x\|^2_{L^\infty(\Omega)} &= \|u_x * \eta_\epsilon\|^2_{L^\infty(\Omega)}\\ &\le \|u_x\|^2_{L^2(\Omega)}\|\eta_\epsilon\|^2_{L^2(\Omega)}\\ &\le C\|u\|^2_{H^1(\Omega)}.\end{aligned} \tag{3.8}$$

Combination Gronwall's lemma and the estimate (3.7) yield

$$\|\phi^\epsilon_x\|^2_{L^\infty(0,T;L^2(\Omega))} + \|q(\epsilon_u)\phi^\epsilon_{xx}\|^2_{L^2(0,T;L^2(\Omega))} \le C. \tag{3.9}$$

Then from (3.6)-(3.9) and the boundedness of ϵ_u, we infer

$$\begin{aligned}\|(q(\epsilon_u)\phi^\epsilon_x)_x\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^t \|q'(\epsilon_u)(\epsilon_u)_x\phi^\epsilon_x + q(\epsilon_u)\phi^\epsilon_{xx}\|^2ds\\ &\le C(\int_0^t \|(\epsilon_u)_x\|^2_{L^\infty(\Omega)}\|q(\epsilon_u)\phi^\epsilon_x\|^2ds + \int_0^t \|q(\epsilon_u)\phi^\epsilon_{xx}\|^2ds)\\ &\le C,\end{aligned} \tag{3.10}$$

and for all $\psi \in L^\infty(0,T;H_0^1(\Omega))$, we have

$$\begin{aligned}
\int_0^t \langle (q(\epsilon_u)\phi_x^\epsilon)_t, \psi\rangle ds &= \int_{Q_T} (q'(\epsilon_u)(\epsilon_u)_t\phi_x^\epsilon\psi + q(\epsilon_u)\phi_{xt}^\epsilon\psi)dxds \\
&= \int_{Q_T} (q'(\epsilon_u)(\epsilon_u)_t\phi_x^\epsilon\psi - q'(\epsilon_u)(\epsilon_u)_x\phi_t^\epsilon\psi - q(\epsilon_u)\phi_t^\epsilon\psi_x)dxds \\
&\le C\int_0^t (\|\psi\|_{L^\infty(\Omega)}\|q'(\epsilon_u)\phi_x^\epsilon\|\|u_t\| + \|\psi\|_{L^\infty(\Omega)}\|(\epsilon_u)_x\|\|\phi_t^\epsilon\| + \|\phi_t^\epsilon\|\|\psi_x\|)ds \\
&\le C\|\psi\|_{L^\infty(0,T;H_0^1(\Omega))}. \qquad (3.11)
\end{aligned}$$

Therefore (3.11) implies $(q(\epsilon_u)\phi_x^\epsilon)_t \in L^1(0,T;H^{-1}(\Omega))$. From Theorem 3.3 and (3.10), it is easy to claim that $q(\epsilon_u)\phi_x^\epsilon$ converges strongly in $L^2(Q_T)$ to a limit function $G \in L^2(Q_T)$.

For the estimate of ϕ_t^ϵ, we multiply equation (1.14) by $\eta \in L^2(0,T;H^1(\Omega))$ to get

$$\begin{aligned}
\left|\int_0^t \langle \phi_t^\epsilon, \eta\rangle ds\right| &\le C\big(\|q(\epsilon_u)\phi_x^\epsilon\|_{L^2(Q_T)}\|\eta\|_{L^2(0,T;H^1(\Omega))} + \|g'(\phi^\epsilon)\|_{L^2(Q_T)}\|\eta\|_{L^2(Q_T)}\big) \\
&\le C\|\eta\|_{L^2(0,T;H^1(\Omega))},
\end{aligned}$$

which verifies (3.3).

Now we estimate (3.2). Multiplying equation (1.15) by $-u_{xx}^\epsilon$ and integrating over Ω, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_x^\epsilon\|^2 + \alpha_1\|u_{xx}^\epsilon\|^2 &= \rho_1\int_\Omega h'(u^\epsilon)u_{xx}^\epsilon dx + \alpha_3\int_\Omega q'(u^\epsilon)[1+|\phi_x^\epsilon|^2 + \rho_3 g(\phi^\epsilon)]u_{xx}^\epsilon dx \\
&\le \frac{\alpha_1}{2}\|u_{xx}^\epsilon\|^2 + C\big(\|h'(u^\epsilon)\|^2 + \|q'(u^\epsilon)\phi_x^\epsilon\|_{L^\infty(\Omega)}^2\|\phi_x^\epsilon\|^2 + \|g(\phi^\epsilon)\|_{L^\infty(\Omega)}^2\big) \\
&\le \frac{\alpha_1}{2}\|u_{xx}^\epsilon\|^2 + C(1 + \|q(u^\epsilon)\phi_x^\epsilon\|_{H^1(\Omega)}^2\|\phi^\epsilon\|_{H^1(\Omega)}^2 + \|\phi^\epsilon\|_{H^1(\Omega)}^8). \qquad (3.12)
\end{aligned}$$

It follows from Gronwall's lemma and (3.12) that

$$\|u_x^\epsilon\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|u_{xx}^\epsilon\|_{L^2(Q_T)}^2 \le C,$$

hence (3.5) follows.

Since equations (1.4) and (1.5) are nonlinear, we must prove that ϕ^ϵ and u^ϵ converge pointwise almost everywhere.

Lemma 3.8 *For any $\epsilon \to 0$, we have*

$$u^\epsilon \to u, \quad \phi^\epsilon \to \phi, \quad a.e. \ \ in\ Q_T, \qquad (3.13)$$

$$q(\epsilon_u) \to q(u),\ h'(u^\epsilon) \to h'(u),\ g(\phi^\epsilon) \to g(\phi), \quad a.e.\ \ in\ Q_T, \qquad (3.14)$$

$$q(\epsilon_u) \to q(u), h'(u^\epsilon) \to h'(u)\,strongly\ in\ L^2(Q_T), g(\phi^\epsilon) \rightharpoonup g(\phi)\,weakly\ in\ L^2(Q_T), \qquad (3.15)$$

$$q(u^\epsilon)g'(\phi^\epsilon) \rightharpoonup q(u)g'(\phi),\ q'(u^\epsilon)g(\phi^\epsilon) \rightharpoonup q'(u)g(\phi) \quad weakly \quad in \quad L^1(Q_T), \qquad (3.16)$$

$$q(\epsilon_u)\phi_x^\epsilon \rightharpoonup q(u)\phi_x,\ q'(\epsilon_u)(\phi_x^\epsilon)^2 \rightharpoonup q'(u)\phi_x^2 \quad weakly \quad in \quad L^1(Q_T). \qquad (3.17)$$

proof: 1. We deduce from Lemma 3.7 that u^ϵ and ϕ^ϵ are bounded respectively in

$$W_1 = \{v \in L^2(0,T; H^2(\Omega));\ v_t \in L^2(Q_T)\},$$

and in

$$W_2 = \{v \in L^2(0,T; H^1(\Omega));\ v_t \in L^2(0,T; (H^1(\Omega))')\}.$$

Using Lemma 3.5, we conclude that there exist

$$u \in L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)),\ \phi \in L^2(0,T; L^2(\Omega)),$$

and a subsequence, that to ease the nation we still denote u^ϵ and ϕ^ϵ, satisfying as $\epsilon \to 0$ that

$$u^\epsilon \to u \quad \text{strongly in } L^2(0,T; H^1(\Omega)),\ \phi^\epsilon \to \phi \quad \text{strongly in } L^2(0,T; L^2(\Omega)), \tag{3.18}$$

which mean $u^\epsilon \to u,\ \phi^\epsilon \to \phi,\ a.e.\ in\ Q_T$.

2. Recalling $g(\phi^\epsilon) = (\phi^\epsilon)^4 - 2(\phi^\epsilon)^3 + (\phi^\epsilon)^2$, combining (3.13) and the properties of strongly convergence, we arrive at

$$g(\phi^\epsilon) \to g(\phi) \quad a.e.\ \ in\ \ Q_T.$$

Then $q(\epsilon_u)$ and $h'(u^\epsilon)$ can be obtained the properties of the pointwise convergence by utilizing the similar method.

3. The convergence of $q(\epsilon_u)$ and $h'(u^\epsilon)$ are obtained using the embedding of Sobolev space $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and $u^\epsilon \to u$ in $L^2(Q_T)$ strongly. For the convergence of $g(\phi^\epsilon)$, it is proved from (3.14), the boundedness of $\|g(\phi^\epsilon)\|_{L^2(Q_T)}$ and Lemma 3.5.

4. The convergences in (3.16) and (3.17) required the strongly convergence of $q(\epsilon_u)$ and the boundedness of $\|\phi\|_{L^\infty(0,T;H^1(\Omega))}$, here we only give a detailed proof of the second term in (3.17). Let $\eta \in L^\infty(Q_T)$, we consider

$$\mathcal{Q} = \int_{Q_T} \big(q'(\epsilon_u)(\phi_x^\epsilon)^2 - q'(u)\phi_x^2\big)\eta\ dxds.$$

We write $\mathcal{Q}$ in the following form:

$$\mathcal{Q} = \int_{Q_T} \big(q'(\epsilon_u)\phi_x^\epsilon(\phi_x^\epsilon - \phi_x) + q'(\epsilon_u)\phi_x(\phi_x^\epsilon - \phi_x) + (q'(\epsilon_u) - q'(u))\phi_x^2\big)\eta\ dxds. \tag{3.19}$$

Denote $\mathcal{Q}_1$ and $\mathcal{Q}_2$ as the first term and second term in the expression of $\mathcal{Q}$, then we have

$$\begin{aligned}
\mathcal{Q}_1 =: &\int_{Q_T} \big((q'(\epsilon_u)\phi_x^\epsilon - G)(\phi_x^\epsilon - \phi_x)\eta + G\cdot(\phi_x^\epsilon - \phi_x)\eta\big)\ dxds \\
\le\ & C\int_{Q_T} \big((q(\epsilon_u)\phi_x^\epsilon - G)(\phi_x^\epsilon - \phi_x)\eta + G\cdot(\phi_x^\epsilon - \phi_x)\eta\big)\ dxds \\
\le\ & C\Big(\|q(\epsilon_u)\phi_x^\epsilon - G\|_{L^2(Q_T)}\|(\phi_x^\epsilon - \phi_x)\eta\|_{L^2(Q_T)} + \int_{Q_T}(\phi_x^\epsilon - \phi_x)G\cdot\eta\ dxds\Big),
\end{aligned}$$

and

$$\begin{aligned}\mathcal{Q}_2 =: &\int_{Q_T} \big((q'(\epsilon_u) - q'(u))(\phi_x^\epsilon - \phi_x)\phi_x\eta + q'(u)(\phi_x^\epsilon - \phi_x)\phi_x\eta\big)\ dxds \\ \leq\ & C\Big(\|q'(\epsilon_u) - q'(u)\|_{L^2(Q_T)}\|(\phi_x^\epsilon - \phi_x)\phi_x\eta\|_{L^2(Q_T)} + \int_{Q_T}(\phi_x^\epsilon - \phi_x)q'(u)\phi_x\eta\ dxds\Big).\end{aligned}$$

From the boundedness of $\|\phi_x^\epsilon\|_{L^\infty(0,T;L^2(\Omega))}$, the weakly convergence of ϕ_x^ϵ in $L^2(Q_T)$, the strongly convergence of $q(\epsilon_u)\phi_x^\epsilon$ in $L^2(Q_T)$ and $q'(\epsilon_u)$ in $L^2(Q_T)$, it is easy to obtain $\mathcal{Q}_1$ and $\mathcal{Q}_2$ tend to 0 as $\epsilon \to 0$ respectively. Otherwise we observe that the last term in (3.19), it also tends to 0 as $\epsilon \to 0$ owing to (3.15) and to the fact that $\phi_x^2\eta \in L^1(Q_T)$. Thus (3.17) follows.

Proof of Theorem 1.1: Now we show that $(\phi,\ u)$ is a weak solution to the problem (1.4)-(1.7). By Definition 1.1, the equations (1.4) and (1.5) are satisfied weakly if the relations (1.12) and (1.13) hold. Thus we multiply (1.14) and (1.15) by a test function $\varphi \in C_0^\infty((-\infty,T)\times\Omega)$ integrate over Q_T to obtain

$$(\phi^\epsilon, \varphi_t)_{Q_T} - \alpha_2(q(\epsilon_u)\phi_x^\epsilon, \varphi_x)_{Q_T} - \rho(g'(\phi^\epsilon)q(u^\epsilon), \varphi)_{Q_T} + (\phi_0^\epsilon, \varphi(0))_\Omega = 0, \tag{3.20}$$

$$(u^\epsilon, \varphi_t)_{Q_T} - \alpha_1(u_x^\epsilon, \varphi_x)_{Q_T} - \alpha_3(q'(u^\epsilon)|\phi_x^\epsilon|^2, \varphi)_{Q_T} - (\frac{\partial(f^j)^\epsilon}{\partial u^\epsilon}, \varphi)_{Q_T} + (u_0^\epsilon, \varphi(0))_\Omega = 0, \tag{3.21}$$

where $\frac{\partial(f^j)^\epsilon}{\partial u^\epsilon} = \rho_1 h'(u^\epsilon) + \rho_2 q'(u^\epsilon)(1+\rho_3 g(\phi^\epsilon))$. We infer from lemma 3.7 and lemma 3.8 that for $\epsilon \to 0$

$$(\phi^\epsilon, \varphi_t)_{Q_T} \to (\phi, \varphi_t)_{Q_T},$$

$$(u^\epsilon, \varphi_t)_{Q_T} \to (u, \varphi_t)_{Q_T},$$

$$(u_x^\epsilon, \varphi_x)_{Q_T} \to (u_x, \varphi_x)_{Q_T},$$

$$(\phi_0^\epsilon, \varphi(0))_\Omega \to (\phi_0, \varphi(0))_\Omega,$$

$$(u_0^\epsilon, \varphi(0))_\Omega \to (u_0, \varphi(0))_\Omega.$$

Otherwise, the nonlinear terms are also obtained from lemma 3.8 that

$$(q(\epsilon_u)\phi_x^\epsilon, \varphi_x)_{Q_T} \to (q(u)\phi_x, \varphi_x)_{Q_T}$$

$$(g'(\phi^\epsilon)q(u^\epsilon), \varphi)_{Q_T} \to (g'(\phi)q(u), \varphi)_{Q_T},$$

$$(h'(u^\epsilon), \varphi)_{Q_T} \to (h'(u), \varphi)_{Q_T},$$

$$(q'(u^\epsilon)|\phi_x^\epsilon|^2, \varphi)_{Q_T} \to (q'(u)|\phi_x|^2, \varphi)_{Q_T},$$

$$(q'(u^\epsilon), \varphi)_{Q_T} \to (q'(u), \varphi)_{Q_T},$$

$$(q'(u^\epsilon) g(\phi^\epsilon), \varphi)_{Q_T} \to (q'(u) g(\phi), \varphi)_{Q_T}.$$

Thus equation (1.4) and (1.5) follow from the relations (3.20) and (3.21) as $\epsilon \to 0$. The proof of Theorem 1.1 is completed.

3.2 Existence of global solutions

In this subsection, we prove the solution $(\phi,\ u)$ obtained is global.

Theorem 3.4 *Let* $(\phi,\ u)$ *is a weak local solution to problem* (1.4)-(1.7) *in the sense of Definition 1.1. If for any* T*, there is a constant* $C = C(\|\phi_0\|_{L^2(\Omega)},\ \|u_0\|_{H^1(\Omega)},\ T)$ *satisfies*

$$\|u\|_{L^\infty(0,T;H^1(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} + \|u\|_{L^4(Q_T)} \le C, \quad \|\phi\|_{L^\infty(0,T;H^1(\Omega))} \le C, \tag{3.22}$$

then the solution $(\phi,\ u)$ *is global.*

Proof: We multiply (1.5) by u and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha_1 \|u_x\|^2 + \rho_1 \|u\|^4_{L^4(\Omega)} + 6\alpha_3 \int_\Omega u^2(1-u)[1 + |\phi_x|^2 + \rho_3 g(\phi)]dx \le \rho_1 \|u\|^2.$$

Using Gronwall's lemma, we get

$$\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_x\|^2_{L^2(0,T;L^2(\Omega))} + \|u\|^4_{L^4(Q_T)} \le C(T,\ \|u_0\|_{L^2(\Omega)}). \tag{3.23}$$

From (3.6), (3.9) and (3.12), it is easy to calculate

$$\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))} \le C,\ \|u_x\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_{xx}\|^2_{L^2(Q_T)} \le C. \tag{3.24}$$

These relations (3.23) and (3.24) verify the estimate (3.22).

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3.4 Availability of data and materials

Data sharing is not applicable to this article as no data sets were created or analyzed during the current study.

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