Well-posedness of solutions to a phase-field model for the martensitic phase transformations

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Abstract

We study a phase-field model, which describes the transformations for the austenite-martensite and the multiple twinning in Martensite. The model consists of two nonlinear parabolic equations of second order. We first show the existence of global solutions to an initial-boundary value problem, and then investigate the regularity and uniqueness of the solution.

Keywords: Phase-field model; Coupled parabolic equations; Well-posedness; Banach fixed-point theorem.

1 Introduction

In the processes of martensitic phase transformations (PTs), twinning can be formed which by reducing the energy associated with internal elastic stresses [1]. In fact, twinning is a common mechanism of metal plastic deformation in which a region of the lattice is uniformly shear into a new orientation, and the phenomenon is pronounced at low temperatures and high stress rates [2].

In this article, the phase transformations for the austenite-martensite and the multiple twinning in Martensite, which are developed for lattice rotations and large strains [3]. In fact, the phase field theory of transformations in Martensite is a non-diffusion type phase transition. The processes of transformations between the two crystal structures are completed through shearing, instead of the long-distance migration of atoms, thus they are a solid-state phase change controlled by interface migration [4-5].

For the investigation of phase-filed model, phase-field method is widely used, and it has been regarded as one of the most significant computing methods [6-9]. The spatial and temporal evolution of the variables is governed by the Cahn-Hilliard equation and/or the Allen-Cahn equation [10-11]. The two well-known equations can describe the ordering of atoms during the phase separation, in which the order parameter is conserved or not conserved. The two kinds of order parameter have been presented in numerous models [12-16]. In these models, one of phase-field models about the phase transformations for the austenite-martensite and the multiple twinning in Martensite is investigated in the rest of this article. First the Helmholtz free energy in one space dimension is given [6]

$$\begin{aligned} F[u,\ \phi] &= \int_R f dx \\ &= \int_R \Big(f^c(u,\ \phi) + f^i(u,\ u_x,\ \phi_x) \Big) dx, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} f^c(u,\ \phi) &= \frac{1}{3}\eta_1 q(u) + a_1 m g(\phi) q(u), \\ f^i(u,\ u_x,\ \phi_x) &= \frac{m\beta_u}{2}|u_x|^2 + \frac{m\beta_\phi}{2}|\phi_x|^2 q(u) + a_0 m \eta_0 h(u), \end{aligned}$$

and the Landau-Ginzburg kinetics equations are given

$$\frac{1}{k_u}\frac{\partial u}{\partial t} = -\frac{1}{m}\frac{\partial f}{\partial u} + \Big(\frac{1}{m}\frac{\partial f}{\partial u_x}\Big)_x; \tag{1.2}$$

$$\frac{1}{k_\phi}\frac{\partial \phi}{\partial t} = -\frac{1}{m}\frac{\partial f}{\partial \phi} + \Big(\frac{1}{m}\frac{\partial f}{\partial \phi_x}\Big)_x. \tag{1.3}$$

We can obtain the equations from (1.1)-(1.3)

$$\phi_t = \alpha_2 \big(q(u)\phi_x\big)_x - \rho g'(\phi) q(u), \tag{1.4}$$

$$u_t = \alpha_1 u_{xx} - \rho_1 h'(u) - \alpha_3 q'(u)|\phi_x|^2 - \rho_2 \frac{\partial f^j}{\partial u}, \tag{1.5}$$

for $(t,\ x) \in (0, +\infty) \times \Omega$.

The boundary and initial conditions are

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial \phi}{\partial n} = 0, \quad (t,\ x) \in (0, +\infty) \times \partial\Omega, \tag{1.6}$$

$$u(0,\ x) = u_0(x), \quad \phi(0,\ x) = \phi_0(x), \quad x \in \Omega, \tag{1.7}$$

where $\Omega \subset \mathbb{R}^1$ is a bounded domain, n is the unit normal to $\partial\Omega$. The function $\phi(t, x)$ is the angular order parameter, which describes the twinning transformations. The function $u(t, x) \in [\epsilon, 1]$ is the radial coordinate of the austenite-martensite transformations, where ϵ is a positive constant.

Here, the positive constants β_i, k_i $(i = u, \phi)$, a_j, η_j $(j = 0, 1)$, m are the gradient energy coefficients, kinetic coefficients, barriers for variant-variant transformations, the temperature at the relevant states, the ratio of mass densities respectively. And we note $\alpha_1 = \beta_u k_u$, $\alpha_2 = \beta_\phi k_\phi$, $\alpha_3 = \frac{1}{2}\beta_\phi k_u$, $\rho_1 = a_0\eta_0 k_u$, $\rho_2 = \frac{\eta_1 k_u}{3m}$, $\rho_3 = \frac{3a_1 m}{\eta_1}$, $\rho = a_1 k_\phi$.

The function $\frac{\partial f^j(u,\ \phi)}{\partial u} = q'(u)(1 + \rho_3 g(\phi))$ in (1.5) with the interpolation function

$$q(u) = u^2(3 - 2u),$$

and the double-well potential with minima at $\phi = 0$ and $\phi = 1$

$$g(\phi) = \phi^2(1 - \phi)^2.$$

Otherwise, the double-well potential with minima at $u=0$ and $u=1$ is given by

$$h(u) = u^2(1-u)^2.$$

To simplify the calculation on the constant coefficients, we set $\rho_2 = \alpha_3$ in equation (1.5), then it can be rewritten as

$$u_t = \alpha_1 u_{xx} - \alpha_3 q'(u)[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1-2u)],$$

where $h'(u) = \frac{1}{3}(1-2u)q'(u)$.

Second law of thermodynamics [17]. From the free energy F in (1.1), it is easy to obtain

$$\begin{aligned}
\frac{dF}{dt} &= \int_R \Big(\frac{\partial}{\partial t}(\frac{m\beta_u}{2}|u_x|^2) + \frac{\partial}{\partial t}(\frac{m\beta_\phi}{2}|\phi_x|^2 q(u)) + \frac{\partial}{\partial t}(a_0 m\eta_0 h(u)) + \frac{\partial f^c}{\partial u}u_t + \frac{\partial f^c}{\partial \phi}\phi_t\Big)\, dx \\
&= \int_R \Big(\big(-m\beta_u u_{xx} + \frac{m\beta_\phi}{2}|\phi_x|^2 q'(u) + a_0 m\eta_0 h'(u) + \frac{\partial f^c}{\partial u}\big)u_t \\
&\quad + \big(-m\beta_\phi\big(q(u)\phi_x\big)_x + \frac{\partial f^c}{\partial \phi}\big)\phi_t\Big)\, dx \\
&= \int_R \big(-\frac{m}{k_u}u_t^2 - \frac{m}{k_\phi}\phi_t^2\big)\, dx \\
&\le 0,
\end{aligned}$$

which implies that the second law of thermodynamics holds for problem (1.4)-(1.5).

In this article, we write $Q_T := (0,\ T)\times\Omega$ for any given constant $T>0$, denote $<\cdot,\ \cdot>$ as the duality pairing between $H^1(\Omega)$ and $(H^1(\Omega))'$; $\|\cdot\|$ as the normal $\|\cdot\|_{L^2(\Omega)}$.

Before stating our main results, we give the definition of a weak solution to problem (1.4)-(1.7).

Definition 1.1. Assume that $\phi_0 \in L^2(\Omega)$, $u_0 \in L^2(\Omega)$. We say a function $(\phi,\ u)$ with

$$\phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{1.8}$$

$$\phi_t \in L^2(0,T;L^2(\Omega)), \tag{1.9}$$

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^4(Q_T) \cap L^2(0,T;H^1(\Omega)), \tag{1.10}$$

$$u_t \in L^2(0,T;(H^1(\Omega))'), \tag{1.11}$$

is a weak solution to problem (1.4)-(1.7) provided

$$(\phi,\varphi_t)_{Q_T} - \alpha_2(q(u)\phi_x,\varphi_x)_{Q_T} - \rho(g'(\phi)q(u),\varphi)_{Q_T} + (\phi_0,\varphi_0)_\Omega = 0, \tag{1.12}$$

$$(u,\varphi_t)_{Q_T} - \alpha_1(u_x,\varphi_x)_{Q_T} - \rho_1(h'(u),\varphi)_{Q_T} - \alpha_3(q'(u)|\phi_x|^2,\varphi)_{Q_T} - \rho_2(\frac{\partial f^j}{\partial u},\varphi)_{Q_T} + (u_0,\varphi_0)_\Omega = 0, \tag{1.13}$$

for each $\varphi \in C_0^\infty((-\infty,T)\times\Omega)$ where (1.5), (1.6) are satisfied weakly.

The main results of this article is the following:

Theorem 1.1 *For all $\phi_0 \in H^1(\Omega)$, $u_0 \in L^2(\Omega)$ with $\epsilon \le u_0 \le 1$ a.e. $x \in \Omega$. There exists a weak solution $(\phi,\ u)$ to problem (1.4)-(1.7) in the sense of Definition 1.1, and for all $t \in (0,T)$, $\epsilon \le u \le 1$ a.e. $x \in \Omega$.*

The rest of this paper is organized as follows: In Section 2, we prove the existence of local solution by employing Banach fixed-point theorem for problem (1.4)-(1.7). Then we show the solution is global by establishing asymptotic estimates. In Section 3, we investigate its regularity and uniqueness of the solution under the appropriate assumption.

2 Existence of weak solutions

2.1 Existence to the fixed problem

We first introduce a Banach space $\mathcal{X}$, that is

$$\mathcal{X} = \{(\phi,\ u)\ \mid\ \phi \in (L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),$$
$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^4(Q_T) \cap L^2(0,T;H^1(\Omega)),\ \epsilon \le u \le 1 \text{ a.e. in } Q_T\}.$$

With $\hat{u} \in \mathcal{X}$ fixed, we consider the parabolic problem of ϕ

$$\begin{cases} \phi_t - \alpha_2\big(q(\hat{u})\phi_x\big)_x = -\rho g'(\phi)q(\hat{u}) \qquad in\ \ Q_T, \\ \phi(0,\ x) = \phi_0(x), \quad\ x \in \Omega; \quad \frac{\partial \phi}{\partial n} = 0, \quad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.1}$$

For the system (2.1), we have

Lemma 2.1 *If $\phi_0 \in H^1(\Omega)$, then (2.1) has a unique solution $\phi \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, and*

$$\|\phi\|_{L^2(0,T;H^2(\Omega))} + \|\phi\|_{L^\infty(0,T;H^1(\Omega))} \le C, \qquad \|\phi_t\|_{L^2(Q_T)} \le C. \tag{2.2}$$

Proof: From Them.7.1 of [18] and Them.6.7 of [19], it is easy to obtain the proof of this lemma.

For the given function ϕ, we consider the quasilinear parabolic problem of u

$$\begin{cases} u_t - \alpha_1 u_{xx} = -\alpha_3 q'(u)[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1-2u)] \quad in\ \ Q_T, \\ u(0,\ t) = u_0(x), \quad\ x \in \Omega; \quad \frac{\partial u}{\partial n} = 0, \quad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.3}$$

For the system (2.3), we also have

Lemma 2.2 *Let $\epsilon \le u_0 \le 1$ a.e. $x \in \Omega$. If $u_0 \in L^2(\Omega)$, then (2.3) has a unique solution $\epsilon \le u \le 1$ a.e. in Q_T, and*

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^4(Q_T)} + \|u\|_{L^2(0,T;H^1(\Omega))} \le C, \ \ \|u\|_{L^2(0,T;(H^1(\Omega))')} \le C. \tag{2.4}$$

To prove the existence of solutions to (2.3), we consider the modified problem

$$\begin{cases} u_t - \alpha_1 u_{xx} = -\alpha_3 \tilde{q}'(u)\big[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1-2u)\big] \quad in \ \ Q_T, \\ \\ u(0,x) = u_0(x), \qquad x \in \Omega; \qquad \frac{\partial u}{\partial n} = 0, \qquad (t,\ x) \in (0,\ T) \times \partial\Omega, \end{cases} \tag{2.5}$$

where $\phi \in \mathcal{X}$, and

$$\tilde{q}'(u) = \begin{cases} q'(\epsilon), & \text{if } u \in (-\infty,\ \epsilon), \\ q'(u), & \text{if } u \in [\epsilon,\ 1], \\ q'(1), & \text{if } u \in (1,\ \infty). \end{cases} \tag{2.6}$$

Notation: We observe that the existence of solutions to the system (2.3) can be proved immediately if the hypothesis of $\epsilon \le u \le 1$ and the existence of solution u in (2.5) are shown.

Lemma 2.3 *If $u_0 \in L^2(\Omega)$ with $\epsilon \le u_0 \le 1$ a.e. $x \in \Omega$, then there exists a weak solution $\epsilon \le u \le 1$ a.e. in Q_T, and $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ to the system* (2.5).

Proof: The existence of solution u in (2.5) can be proved from Them.6.7 of [19], and

$$\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u\|^2_{L^2(0,T;H^1(\Omega))} \le C.$$

Next, we prove that if $\epsilon \le u_0 \le 1$ a.e. $x \in \Omega$, then $\epsilon \le u \le 1$ a.e. in Q_T.

Multiplying equation (2.5) by $(u-\epsilon)^-$, where $(u-\epsilon)^- = \max(0,\ \epsilon - u)$, and integrating over Ω, we get, for a.e. $t \in (0,T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |(u-\epsilon)^-|^2 dx + \alpha_1 \int_\Omega |(u-\epsilon)^-_x|^2 dx = &- \alpha_3 \int_\Omega \tilde{q}'(u)(u-\epsilon)^-[\frac{\rho_1}{3\alpha_3}(1-2u) + 1 + |\phi_x|^2 \\ &+ \rho_3 g(\phi)]dx. \end{aligned}$$

It follows from the definitions of $(u-\epsilon)^-$ and $\tilde{q}'(u)$ that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u-\epsilon)^-|^2 dx + \alpha_1 \int_\Omega |(u-\epsilon)^-_x|^2 dx = 0. \tag{2.7}$$

Due to the fact that the second term in (2.7) is positive, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u-\epsilon)^-|^2 dx \le 0. \tag{2.8}$$

From the data $(u_0-\epsilon)^- = 0$, it is easy to verify $(u-\epsilon)^- = 0$. Hence, $u \ge \epsilon$ a.e. in Q_T.

We next multiply equation (2.5) by $(u-1)^+$, integrate over Ω, and obtain, for a.e. $t \in (0,T)$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |(u-1)^+|^2 dx + \alpha_1 \int_\Omega |(u-1)^+_x|^2 dx = &- \alpha_3 \int_\Omega \tilde{q}'(u)(u-1)^+[\frac{\rho_1}{3\alpha_3}(1-2u) + 1 + |\phi_x|^2 \\ &+ \rho_3 g(\phi)]dx. \end{aligned}$$

Making use of the definitions of $\tilde{q}'(u)$ and $(u-1)^+$, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |(u-1)^+|^2 dx + \alpha_1 \int_\Omega |(u-1)^+_x|^2 dx = 0. \tag{2.9}$$

Similarly, we deduce $u \leq 1$ a.e. in Q_T. The proof of Lemma 2.3 is complete.

Now, we start to complete the proof of Lemma 2.2. We multiply equation (2.3) by $\eta \in L^2(0,T;H^1(\Omega))$ and use the boundedness of u, the result is

$$\begin{aligned}
|\int_0^t \langle u_t, \eta\rangle ds| &\leq C\big(\|u\|_{L^2(0,T;H^1(\Omega))}\|\eta\|_{L^2(0,T;H^1(\Omega))} + \|\eta\|_{L^2(0,T;H^1(\Omega))}\|\phi\|^2_{L^2(0,T;H^2(\Omega))} \\
&\quad + \|\eta\|_{L^2(0,T;H^1(\Omega))}\|\phi\|^4_{L^\infty(0,T;H^1(\Omega))}\big) \\
&\leq C\|\eta\|_{L^2(0,T;H^1(\Omega))}.
\end{aligned}$$

This implies

$$\|u_t\|_{L^2(0,T;(H^1(\Omega))')} \leq C. \tag{2.10}$$

For the estimate $\|u\|^4_{L^4(Q_T)}$, we have

$$\begin{aligned}
\|u\|^4_{L^4(Q_T)} &= \int_0^t \int_\Omega |u|^4 dxds \\
&\leq C\|u\|^2_{L^\infty(0,T;L^2(\Omega))}\|u\|^2_{L^2(0,T;H^1(\Omega))} \\
&\leq C.
\end{aligned}$$

For the uniqueness, we consider u_1 and u_2 two solutions to the system (2.3). We note $U = u_1 - u_2$, and it satisfies

$$\begin{cases} U_t - \alpha_1 U_{xx} = -\rho_1\big(h'(u_1) - h'(u_2)\big) - \alpha_3\big(q'(u_1) - q'(u_2)\big)\big[1 + |\phi_x|^2 + \rho_3 g(\phi)\big], \\ U(0,\ x) = 0, \quad x \in \Omega; \quad \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0, \quad (t,\ x) \in (0,\ T) \times \partial\Omega. \end{cases} \tag{2.11}$$

Multiplying equation (2.11) by U, integrating over Ω, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|U\|^2 + \alpha_1\|U_x\|^2 &= -\int_\Omega \big\{\rho_1\big(h'(u_1) - h'(u_2)\big) + \alpha_3(q'(u_1) - q'(u_2))[1 + |\phi_x|^2 + \rho_3 g(\phi)\big\}U dx \\
&\leq C\big(1 + \|\phi_x\|^2_{L^\infty(\Omega)} + \|g(\phi)\|_{L^\infty(\Omega)}\big)\|U\|^2 \\
&\leq C\big(1 + \|\phi_{xx}\|^{\frac{3}{2}}\|\phi\|^{\frac{1}{2}} + \|\phi\|^4_{H^1(\Omega)}\big)\|U\|^2.
\end{aligned}$$

From Gronwall's lemma [20], it is easy to obtain

$$\|U\|^2 \leq e^{2C\int_0^t \left(1 + \|\phi_{xx}\|^{\frac{3}{2}}\|\phi\|^{\frac{1}{2}} + \|\phi\|^4_{H^1(\Omega)}\right)ds}\|U(0,\ x)\|^2 = 0,$$

which ensures the uniqueness. The proof of Lemma 2.2 is complete.

2.2 The Banach fixed-point method

In this subsection, we show the existence of local solution to problem (1.4)-(1.7) by employing a Banach fixed-point theorem [18]. Below we introduce the Banach fixed-point theorem, that is

Theorem 2.2 (Banach Fixed-Point Theorem) *$\mathcal{A}$ is a Banach space. Assume $\tau : \mathcal{A} \to \mathcal{A}$ is a nonlinear mapping, and suppose that*

$$\| \tau(u_1) - \tau(u_2) \| \leq \gamma \| u_1 - u_2 \|, \ (u_1, \ u_2 \in \mathcal{A}) \tag{2.12}$$

for some constant $0 \leq \gamma < 1$. Then τ has a unique fixed point.

Now, we begin to show the existence of local solution to problem (1.4)-(1.7). First, from the proof of Lemma 2.1 and Lemma 2.2, we can define a map $A : \mathcal{X} \to \mathcal{X}$ by setting $A(\hat{\phi}, \ \hat{u}) = (\phi, \ u)$, where $(\phi, \ u)$ and $(\hat{\phi}, \ \hat{u})$ satisfy

$$\begin{cases} \phi_t - \alpha_2\big(q(\hat{u})\phi_x\big)_x = -\rho g'(\hat{\phi})q(\hat{u}) \qquad in \ \ Q_T, \\ \\ u_t - \alpha_1 u_{xx} = -\alpha_3 q'(u)[1 + |\phi_x|^2 + \rho_3 g(\phi) + \frac{\rho_1}{3\alpha_3}(1-2u)] \quad in \ \ Q_T, \\ \\ \phi(0,x) = \phi_0(x), \ u(0,x) = u_0(x), \ x \in \Omega; \ \ \frac{\partial \phi}{\partial n} = 0, \frac{\partial U}{\partial n} = 0, \ (t, \ x) \in (0, \ T) \times \partial\Omega. \end{cases} \tag{2.13}$$

Next we need to claim that A is a compressed mapping if $T > 0$ is small enough. To prove this, we choose $(\hat{\phi}_i, \ \hat{u}_i) \in \mathcal{X}$ so that $A(\hat{\phi}_i, \ \hat{u}_i) = (\phi_i, \ u_i)$ with the norm

$$\begin{aligned} \|(\hat{\phi}_i, \ \hat{u}_i)\|_{\mathcal{X}} = \|\hat{\phi}_i\|_{L^\infty(0,T;H^1(\Omega))} + \|\hat{\phi}_i\|_{L^2(0,T;H^2(\Omega))} + \|\hat{u}_i\|_{L^\infty(0,T;H^1(\Omega))} + \|\hat{u}_i\|_{L^2(0,T;H^1(\Omega))} \\ + \|\hat{u}_i\|_{L^4(Q_T)}, \end{aligned}$$

where $(\phi_i, \ u_i), \ i = 1, 2$ are the solutions to the syestem (2.13).

Lemma 2.4 *Assume $u_0 \in L^2(\Omega)$, $\phi_0 \in H^1(\Omega)$, and $\|(\hat{\phi}_i, \ \hat{u}_i)\|_{\mathcal{X}} \leq C$. Suppose $A : \ \mathcal{X} \to \mathcal{X}$ is a nonlinear mapping, if the mapping satisfies*

$$\|A(\hat{\phi}_1, \ \hat{u}_1) - A(\hat{\phi}_2, \ \hat{u}_2)\|_{\mathcal{X}} \leq \gamma \|(\hat{\phi}_1 - \hat{\phi}_2, \ \hat{u}_1 - \hat{u}_2)\|_{\mathcal{X}},$$

for some constant $0 \leq \gamma < 1$. Then A is a contraction.

Proof: We set $\phi = \phi_1 - \phi_2$, $U = u_1 - u_2$, then (ϕ, U) satisfy

$$\begin{cases} \phi_t - \alpha_2\big(q(\hat{u}_1)\phi_{1x} - q(\hat{u}_2)\phi_{2x}\big)_x = -\rho g'(\hat{\phi}_1)q(\hat{u}_1) + \rho g'(\hat{\phi}_2)q(\hat{u}_2) \quad in \ \ Q_T, \\ \\ U_t - \alpha_1 U_{xx} = -\alpha_3 q'(u_1)\big(1 + |\phi_{1x}|^2 + \rho_3 g(\phi_1)\big) + \alpha_3 q'(u_2)\big(1 + |\phi_{2x}|^2 + \rho_3 g(\phi_2)\big) \\ \qquad\qquad - \rho_1\big(h'(u_1) - h'(u_2)\big) \qquad\qquad\qquad in \ \ Q_T, \\ \\ \phi(0,x) = 0, \ U(0,x) = 0, \quad x \in \Omega; \quad \frac{\partial \phi}{\partial n} = 0, \frac{\partial U}{\partial n} = 0, \quad (t, \ x) \in (0, \ T) \times \partial\Omega. \end{cases} \tag{2.14}$$

Multiplying the first equation in (2.14) by ϕ, integrating over Ω, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \alpha_2 m\|\phi_x\|^2 \leq \alpha_2 \int_\Omega |q(\hat{u}_1) - q(\hat{u}_2)||\phi_{2x}||\phi_x|dx + \rho\|q(\hat{u}_1)\|_{L^\infty(\Omega)} \int_\Omega |g'(\hat{\phi}_1) - g'(\hat{\phi}_2)||\phi|dx \\ + \rho \int_\Omega |g'(\hat{\phi}_2)||q(\hat{u}_1) - q(\hat{u}_2)||\phi|dx \\ = I_1 + I_2 + I_3 \end{aligned} \tag{2.15}$$

where we set $0 < m \leq q(\hat{u}_i) \leq M,\ i = 1,\ 2$, and

$$
\begin{aligned}
I_1 &= \alpha_2 \int_\Omega |q(\hat{u}_1) - q(\hat{u}_2)||\phi_{2x}||\phi_x|dx \\
&\leq \frac{\alpha_2 m}{2}\|\phi_x\|^2 + C\|\phi_{2x}\|^2 \\
&\leq \frac{\alpha_2 m}{2}\|\phi_x\|^2 + C\|\phi_2\|^2_{H^1(\Omega)}, \qquad (2.16)
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \rho\|q(\hat{u}_1)\|_{L^\infty(\Omega)} \int_\Omega |g'(\hat{\phi}_1) - g'(\hat{\phi}_2)||\phi|dx \\
&\leq C|\phi||\hat{\phi}|(|\hat{\phi}_1|^2 + |\hat{\phi}_2|^2) \\
&\leq \frac{1}{2}\|\phi\|^2(\|\hat{\phi}_1\|^4_{L^\infty(\Omega)} + \|\hat{\phi}_2\|^4_{L^\infty(\Omega)}) + C\|\hat{\phi}\|^2, \qquad (2.17)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \rho \int_\Omega |g'(\hat{\phi}_2)||q(\hat{u}_1) - q(\hat{u}_2)||\phi|dx \\
&\leq C \int_\Omega |\hat{U}||\phi||g'(\hat{\phi}_2)|dx \\
&\leq \frac{1}{2}\|\phi\|^2\|g'(\hat{\phi}_2)\|^2_{L^\infty(\Omega)} + C\|\hat{U}\|^2 \\
&\leq \frac{1}{2}\|\phi\|^2\|\hat{\phi}_2\|^6_{H^1(\Omega)} + C\|\hat{U}\|^2. \qquad (2.18)
\end{aligned}
$$

The estimates (2.15)-(2.18) yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \frac{\alpha_2 m}{2}\|\phi_x\|^2 \\
&\leq \frac{1}{2}\big(\|\hat{\phi}_1\|^4_{H^1(\Omega)} + \|\hat{\phi}_2\|^4_{H^1(\Omega)} + \|\hat{\phi}_2\|^6_{H^1(\Omega)}\big)\|\phi\|^2 + C\big\{\|\phi_2\|^2_{H^1(\Omega)} + \|\hat{\phi}\|^2 + \|\hat{U}\|^2\big\}.
\end{aligned}
$$

We get from Gronwall's lemma and Hölder inequality that

$$
\begin{aligned}
\|\phi\|^2 &\leq C(1 + \int_0^t \|\hat{U}\|^2 ds + \int_0^t \|\hat{\phi}\|^2_{H^1(\Omega)}\ ds) \\
&\leq C(\|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))})(\int_0^t ds) \\
&\leq CT(\|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))}). \qquad (2.19)
\end{aligned}
$$

The estimates (2.15)-(2.19) together yield

$$
\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi\|^2_{L^2(0,T;H^1(\Omega))} \leq CT(\|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))}).
$$

We multiply the first equation in (2.14) by $-\phi_{xx}$, integrate over Ω, and obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2 + \alpha_2 m\|\phi_{xx}\|^2 \\
&\leq \alpha_2 \int_\Omega |q'(\hat{u}_1)\hat{u}_{1x}\phi_{1x} - q'(\hat{u}_2)\hat{u}_{2x}\phi_{2x}||\phi_{xx}|dx + \rho \int_\Omega |q(\hat{u}_1)||g'(\hat{\phi}_1) - g'(\hat{\phi}_2)||\phi_{xx}|dx \\
&\quad + \rho \int_\Omega |g'(\hat{\phi}_2)(q(\hat{u}_1) - q(\hat{u}_2))||\phi_{xx}|dx \\
&= J_1 + J_2 + J_3, \qquad (2.20)
\end{aligned}
$$

where we set $a = \max(q'(\hat{u}_1),\ q'(\hat{u}_2))$, and

$$\begin{aligned} J_1 &= \alpha_2 \int_\Omega |q'(\hat{u}_1)\hat{u}_{1x}\phi_{1x} - q'(\hat{u}_2)\hat{u}_{2x}\phi_{2x}||\phi_{xx}|dx \\ &\le \alpha_2 m \int_\Omega |\hat{U}_x||\phi_{1x}||\phi_{xx}|dx + a\alpha_2 \int_\Omega |\hat{u}_{2x}||\phi_x||\phi_{xx}|dx \\ &\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 + C\big(\|\hat{U}_x\|^2\|\phi_{1x}\|^2_{L^\infty(\Omega)} + \|\hat{u}_{2x}\|^2_{L^\infty(\Omega)}\|\phi_x\|^2\big), \end{aligned} \tag{2.21}$$

$$\begin{aligned} J_2 &= \rho \int_\Omega |q(\hat{u}_1)||g'(\hat{\phi}_1) - g'(\hat{\phi}_2)||\phi_{xx}|dx \\ &\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 + \|q(\hat{u}_1)\|^2_{L^\infty(\Omega)}\|g'(\hat{\phi}_1) - g'(\hat{\phi}_2)\|^2 \\ &\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 + C(\|\hat{\phi}_1\|^4_{L^\infty(\Omega)} + \|\hat{\phi}_2\|^4_{L^\infty(\Omega)})\|\hat{\phi}\|^2, \end{aligned} \tag{2.22}$$

$$\begin{aligned} J_3 &= \rho \int_\Omega |g'(\hat{\phi}_2)(q(\hat{u}_1) - q(\hat{u}_2))||\phi_{xx}|dx \\ &\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 + C\int_\Omega |g'(\hat{\phi}_2)|^2|\hat{U}|^2dx \\ &\le \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 + C\|\hat{U}\|^2_{L^2(\Omega)}\|\hat{\phi}_2\|^6_{H^1(\Omega)}. \end{aligned} \tag{2.23}$$

It follows from (2.20)–(2.23) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2 + \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 \\ &\le C\|\hat{u}_{2x}\|^2_{L^\infty(\Omega)}\|\phi_x\|^2 + C_1\big(\|\hat{U}_x\|^2\|\phi_{1x}\|^2_{L^\infty(\Omega)} + (\|\hat{\phi}_1\|^4_{L^\infty(\Omega)} + \|\hat{\phi}_2\|^4_{L^\infty(\Omega)})\|\hat{\phi}\|^2 \\ &\quad + \|\hat{U}\|^2\|\hat{\phi}_2\|^6_{H^1(\Omega)}\big). \end{aligned} \tag{2.24}$$

Applying Gronwall's lemma, the estimates (2.20) and (2.24) imply

$$\begin{aligned} \|\phi_x\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi_{xx}\|^2_{L^2(0,T;L^2(\Omega))} &\le C(\|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))})(\int_0^t ds) \\ &\le CT(\|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))}). \end{aligned} \tag{2.25}$$

Multiplying the second equation in (2.14) by U and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \alpha_1\|U_x\|^2 \le C_1(1 + \|\phi_{1x}\|^2_{L^\infty(\Omega)} + \|\phi_{2x}\|^2_{L^\infty(\Omega)})\|U\|^2 + C_2\int_\Omega \|g(\phi_1) - g(\phi_2)\|^2dx.$$

Applying Gronwall's lemma, we get

$$\|U\|^2_{L^\infty(0,T;L^2(\Omega))} + \|U\|^2_{L^2(0,T;H^1(\Omega))} \le CT(\|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))}). \tag{2.26}$$

For the estimate $\|U\|^4_{L^4(Q_T)}$, we infer

$$\begin{aligned} \int_0^t\int_\Omega |U|^4dxds &\le C\int_0^t \|U\|^2_{L^\infty(\Omega)}\|U\|^2_{L^2(\Omega)}ds \\ &\le C\|U\|^2_{L^\infty(0,T;L^2(\Omega))}\|U\|^2_{L^2(0,T;H^1(\Omega))} \\ &\le (CT)^2(\|\hat{\phi}\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\hat{U}\|^2_{L^\infty(0,T;L^2(\Omega))})^2. \end{aligned} \tag{2.27}$$

Then, thanks to (2.15)-(2.27) we have

$$\|(\phi, U)\|_{\mathcal{X}} \leq (CT)^{\frac{1}{2}} \|(\hat{\phi},\ \hat{U})\|_{\mathcal{X}}. \tag{2.28}$$

Thus, A is a contractive operator if $T > 0$ is so small that $(CT)^{\frac{1}{2}} < 1$. Then we apply Banach fixed-point theorem to obtain a solution $(\phi,\ u)$ to problem (1.4)-(1.7) on the interval $[0,\ T_1]$, where $0 < T_1 < T$ so small that $(CT_1)^{\frac{1}{2}} < 1$.

2.3 Existence of global solution

In this section, we prove the solution $(\phi,\ u)$ obtained is global.

Theorem 2.3 *Let* $(\phi,\ u)$ *be a weak local solution to problem* (1.4)-(1.7) *in the sense of Definition 1.1. If for any* T*, there is a constant* $C = C(\|\phi_0\|_{H^1(\Omega)},\ \|u_0\|_{L^2(\Omega)},\ T)$ *satisfies*

$$\|(\phi,\ u)\|_{\mathcal{X}} \leq C, \tag{2.29}$$

then the solution $(\phi,\ u)$ *is global.*

Proof: We multiply (1.5) by u and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha_1\|u_x\|^2 + \rho_1 \int_\Omega h'(u)u\ dx + \alpha_3 \int_\Omega q'(u)u[1 + |\phi_x|^2 + \rho_3 g(\phi)]dx = 0. \tag{2.30}$$

Simplifying (2.30) as

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha_1\|u_x\|^2 + \rho_1\|u\|^4_{L^4(\Omega)} + 6\alpha_3 \int_\Omega u^2(1-u)[1 + |\phi_x|^2 + \rho_3 g(\phi)]dx \leq \rho_1\|u\|^2.$$

Using Gronwall's lemma, we get

$$\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \|u_x\|^2_{L^2(0,T;L^2(\Omega))} + \|u\|^4_{L^4(Q_T)} \leq C(T,\ \|u_0\|_{L^2(\Omega)}). \tag{2.31}$$

Multiplying (1.4) by ϕ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \alpha_2 \int_\Omega q(u)|\phi_x|^2 dx + \rho \int_\Omega g'(\phi)\phi q(u)dx = 0.$$

We obtain the estimate from Hölder inequality and the boundedness of $q(u)$ that

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2 + \alpha_2 m\|\phi_x\|^2 + \rho m\|\phi\|^4_{L^4(\Omega)} \leq C\|\phi\|^2. \tag{2.32}$$

Applying Gronwall's lemma, the estimate (2.32) implies

$$\|\phi\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi_x\|^2_{L^2(0,T;L^2(\Omega))} \leq C(T,\ \|\phi_0\|_{L^2(\Omega)}). \tag{2.33}$$

Multiplying (1.4) by $-\phi_{xx}$, integrating over Ω, and using Gagliardo-Nirenberg inequality, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2 + \alpha_2 m\|\phi_{xx}\|^2 &\leq \rho \int_\Omega |g'(\phi)q(u)\phi_{xx}|dx + \alpha_2 \int_\Omega |q'(u)u_x\phi_x\phi_{xx}|dx \\
&\leq \frac{\alpha_2 m}{4}\|\phi_{xx}\|^2 + C\big(\|g'(\phi)\|^2 + \|\phi_x\|^2_{L^\infty(\Omega)}\|u_x\|^2_{L^2(\Omega)}\big) \\
&\leq \frac{\alpha_2 m}{2}\|\phi_{xx}\|^2 + C\big(\|\phi\|^6_{H^1(\Omega)} + \|\phi\|^2\|u\|^8_{H^1(\Omega)}\big).
\end{aligned} \tag{2.34}$$

It follows from Gronwall's lemma and (2.34) that

$$\|\phi_x\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi_{xx}\|^2_{L^2(Q_T)} \le C(T,\ \|\phi_0\|_{H^1(\Omega)}). \tag{2.35}$$

These relations (2.31)-(2.35) verify the estimate (2.29).

3 Regularity and uniqueness

In this section, we investigate the regularity and uniqueness of the solutions obtained in Section 2. To begin with, we first introduce the Gagliardo-Nirenberg inequalities, which will be utilised later on. That is

Lemma 3.5 *Let $\Omega \in \mathbb{R}^1$ be a bounded and open set with a smooth boundary. Then the following Gagliardo-Nirenberg inequalities hold:*

$$\begin{aligned}
&\|u_x\|_{L^4(\Omega)} \le C_1\|u_{xx}\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{L^\infty(\Omega)} \quad for \quad all \quad u \in H^2(\Omega);\\
&\|u_x\|_{L^\infty(\Omega)} \le C_2\|u_{xx}\|^{\frac{3}{4}}_{L^2(\Omega)}\|u\|^{\frac{1}{4}}_{L^2(\Omega)} \quad for \quad all \quad u \in H^2(\Omega);\\
&\|u_x\|_{L^\infty(\Omega)} \le C_3\|u_{xxx}\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{L^2(\Omega)} \quad for \quad all \quad u \in H^3(\Omega);
\end{aligned}$$

where C_1, C_2, C_3 are positive constants.

Theorem 3.4 *Let $\phi_0 \in H^2(\Omega)$ such that $\frac{\partial \phi_0}{\partial n} = 0$ and $u_0 \in H^1(\Omega)$, then for any $T > 0$, there exists a unique solution $(\phi,\ u)$ to problem (1.4)-(1.7), which satisfies*

$$\phi \in L^2(0,T;H^3(\Omega)) \cap H^1(0,T;H^1(\Omega)); \tag{3.1}$$

$$u \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)),\ u_t \in L^2(0,T;L^2(\Omega)). \tag{3.2}$$

The proof of regularity: We multiply (1.5) by u_t and integrate over Ω to calculate

$$\begin{aligned}
\|u_t\|^2 + \frac{\alpha_1}{2}\frac{d}{dt}\|u_x\|^2 + \rho_1\int_\Omega h'(u)u_t dx &= -\alpha_3\int_\Omega q'(u)u_t[1 + |\phi_x|^2 + \rho_3 g(\phi)]dx\\
&\le \frac{1}{2}\|u_t\|^2 + C(\|\phi_x\|^4_{L^4(\Omega)} + \|g(\phi)\|^2_{L^\infty(\Omega)} + 1)\\
&\le \frac{1}{2}\|u_t\|^2 + C(\|\phi_{xx}\|^2\|\phi\|^2_{L^\infty(\Omega)} + \|\phi\|^8_{H^1(\Omega)} + 1).
\end{aligned} \tag{3.3}$$

Integrating (3.3) with respect to t, we obtain

$$\begin{aligned}
\|u_t\|^2_{L^2(Q_T)} + \alpha_1\|u_x\|^2 + 2\rho_1\int_\Omega h(u(t))dx &\le C(\|u_x(0,x)\|^2 + \|h(u_0)\|_{L^1(\Omega)} + \|\phi\|^8_{L^\infty(0,\ T;\ H^1(\Omega))}\\
&\quad + \|\phi_{xx}\|^2_{L^2(Q_T)}\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))} + 1)\\
&\le C.
\end{aligned}$$

Hence $u \in L^\infty(0,T;H^1(\Omega)),\ u_t \in L^2(0,T;L^2(\Omega))$.

We multiply (1.5) by $-u_{xx}$ and integrate over Ω to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_x\|^2+\alpha_1\|u_{xx}\|^2 &= \rho_1\int_\Omega h'(u)u_{xx}dx+\alpha_3\int_\Omega q'(u)[1+|\phi_x|^2+\rho_3 g(\phi)]u_{xx}dx\\ &\le \frac{\alpha_1}{2}\|u_{xx}\|^2+C\big(1+\|h'(u)\|^2+\|\phi_x\|^4_{L^4(\Omega)}+\|g(\phi)\|^2_{L^\infty(\Omega)}\big). \quad (3.4)\end{aligned}$$

It follows from Gronwall's lemma and (3.4) that

$$\|u_x\|^2_{L^\infty(0,T;L^2(\Omega))}+\|u_{xx}\|^2_{L^2(Q_T)}\le C.$$

Hence (3.2) follows.

Next, multiplying (1.4) by ϕ_{xxxx}, integrating over Ω, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi_{xx}\|^2 &= \alpha_2\int_\Omega (q(u)\phi_x)_x\phi_{xxxx}dx-\rho\int_\Omega g'(\phi)q(u)\phi_{xxxx}dx\\ &= -\alpha_2\int_\Omega\big(q''(u)u_x^2\phi_x+q'(u)u_{xx}\phi_x+2q'(u)u_x\phi_{xx}+q(u)\phi_{xxx}\big)\phi_{xxx}dx\\ &\quad+\rho\int_\Omega\big(g''(\phi)\phi_x q(u)+g'(\phi)q'(u)u_x\big)\phi_{xxx}dx. \quad (3.5)\end{aligned}$$

We make use of the assumption $0<m\le q(u)\le M$ and (3.5) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi_{xx}\|^2+\alpha_2 m\|\phi_{xxx}\|^2 &\le C\Big(\int_\Omega |u_x|^2|\phi_x||\phi_{xxx}|dx+\int_\Omega |u_{xx}||\phi_x||\phi_{xxx}|dx+\int_\Omega |u_x||\phi_{xx}||\phi_{xxx}|dx\\ &\quad+\int_\Omega |g''(\phi)||\phi_x||\phi_{xxx}|dx+\int_\Omega |g'(\phi)||u_x||\phi_{xxx}|dx\Big)\\ &= C(K_1+K_2+K_3+K_4+K_5), \quad (3.6)\end{aligned}$$

where

$$\begin{aligned}K_1 &= \int_\Omega |u_x|^2|\phi_x||\phi_{xxx}|dx\\ &\le \|\phi_x\|_{L^\infty(\Omega)}\|u_x\|^2_{L^4(\Omega)}\|\phi_{xxx}\|\\ &\le \|\phi_{xxx}\|^{\frac{1}{2}}\|\phi\|^{\frac{1}{2}}\|u_{xx}\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{L^\infty(\Omega)}\|\phi_{xxx}\|\\ &\le \frac{\alpha_2 m}{6}\|\phi_{xxx}\|^2+C\|\phi\|^2\|u_{xx}\|^2\|u\|^2_{H^1(\Omega)}; \quad (3.7)\end{aligned}$$

$$\begin{aligned}K_2 &= \int_\Omega |u_{xx}||\phi_x||\phi_{xxx}|dx\\ &\le \|\phi_x\|_{L^\infty(\Omega)}\|u_{xx}\|\|\phi_{xxx}\|\\ &\le \|\phi_{xx}\|^{\frac{3}{4}}\|\phi\|^{\frac{1}{4}}\|u_{xx}\|\|\phi_{xxx}\|\\ &\le \frac{\alpha_2 m}{6}\|\phi_{xxx}\|^2+C\|\phi_{xx}\|^2\|\phi\|^{\frac{1}{2}}\|u_{xx}\|^2; \quad (3.8)\end{aligned}$$

$$\begin{aligned}K_3 &= \int_\Omega |u_x||\phi_{xx}||\phi_{xxx}|dx\\ &\le \|u_x\|_{L^\infty(\Omega)}\|\phi_{xx}\|\|\phi_{xxx}\|\\ &\le \|u\|_{H^2(\Omega)}\|\phi_{xx}\|\|\phi_{xxx}\|\\ &\le \frac{\alpha_2 m}{6}\|\phi_{xxx}\|^2+C\|u\|^2_{H^2(\Omega)}\|\phi_{xx}\|^2; \quad (3.9)\end{aligned}$$

$$
\begin{aligned}
K_4 &= \int_\Omega |g''(\phi)||\phi_x||\phi_{xxx}|dx \\
&\le \|\phi\|^2_{L^\infty(\Omega)}\|\phi_x\|\|\phi_{xxx}\| \\
&\le \|\phi\|^3_{H^1(\Omega)}\|\phi_{xxx}\| \\
&\le \frac{\alpha_2 m}{6}\|\phi_{xxx}\|^2 + C\|\phi\|^6_{H^1(\Omega)}; \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
K_5 &= \int_\Omega |g'(\phi)||u_x||\phi_{xxx}|dx \\
&\le \|\phi\|^3_{L^\infty(\Omega)}\|u_x\|\|\phi_{xxx}\| \\
&\le \|\phi\|^3_{H^1(\Omega)}\|u\|_{H^1(\Omega)}\|\phi_{xxx}\| \\
&\le \frac{\alpha_2 m}{6}\|\phi_{xxx}\|^2 + C\|\phi\|^6_{H^1(\Omega)}\|u\|^2_{H^1(\Omega)}. \qquad (3.11)
\end{aligned}
$$

Combination with (3.6)-(3.11) and Gronwall's lemma yield

$$
\begin{aligned}
\|\phi_{xx}\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\phi_{xxx}\|^2_{L^2(0,T;L^2(\Omega))} &\le C\big(1 + \|\phi_{xx}(0,x)\|^2_{L^2(\Omega)}\big) \\
&\le C. \qquad (3.12)
\end{aligned}
$$

We conclude from (3.12) that $\phi \in L^\infty(0,T;H^2(\Omega)) \cap L^2(0,T;H^3(\Omega))$.

To get the estimate of $(\phi_t)_x$, we multiply (1.4) by $-(\phi_{xx})_t$ and integrate over Ω to obtain

$$
\|(\phi_t)_x\|^2 = -\alpha_2 \int_\Omega (q(u)\phi_x)_x(\phi_{xx})_t dx + \rho \int_\Omega g'(\phi)q(u)(\phi_{xx})_t dx. \qquad (3.13)
$$

Using integration by parts, and the assumption $0 < m \le q(u) \le M$, we have

$$
\begin{aligned}
\|(\phi_t)_x\|^2 + \frac{\alpha_2 m}{2}\frac{d}{dt}\|\phi_{xx}\|^2 &= \alpha_2 \int_\Omega \big(q''(u)u_x^2\phi_x + q'(u)u_{xx}\phi_x + q'(u)u_x\phi_{xx}\big)(\phi_t)_x dx \\
&\quad -\rho \int_\Omega \big(g''(\phi)\phi_x q(u) + g'(\phi)q'(u)u_x\big)(\phi_t)_x dx \\
&\le C\Big(\int_\Omega |u_x|^2|\phi_x||(\phi_t)_x|dx + \int_\Omega |u_{xx}||\phi_x||(\phi_t)_x|dx + \int_\Omega |u_x||\phi_{xx}||(\phi_t)_x|dx \\
&\quad + \int_\Omega |g''(\phi)||\phi_x||(\phi_t)_x|dx + \int_\Omega |g'(\phi)||u_x||(\phi_t)_x|dx\Big) \\
&= C(L_1 + L_2 + L_3 + L_4 + L_5), \qquad (3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
L_1 &= \int_\Omega |u_x|^2|\phi_x||(\phi_t)_x|dx \\
&\le \|\phi_x\|_{L^\infty(\Omega)}\|u_x\|^2_{L^4(\Omega)}\|(\phi_t)_x\| \\
&\le \|\phi_{xx}\|^{\frac{2}{3}}\|\phi\|^{\frac{1}{3}}_{L^\infty(\Omega)}\|u_{xx}\|\|u\|_{L^\infty(\Omega)}\|(\phi_t)_x\| \\
&\le \frac{1}{5}\|(\phi_t)_x\|^2 + C\|\phi_{xx}\|^2\|\phi\|^{\frac{2}{3}}_{H^1(\Omega)}\|u_{xx}\|^2\|u\|^2_{H^1(\Omega)}; \qquad (3.15)
\end{aligned}
$$

$$
\begin{aligned}
L_2 &= \int_\Omega |u_{xx}||\phi_x||(\phi_t)_x|dx \\
&\le \|\phi_x\|_{L^\infty(\Omega)}\|u_{xx}\|\|(\phi_t)_x\| \\
&\le \|\phi_{xx}\|^{\frac{2}{3}}\|\phi\|^{\frac{1}{3}}_{L^\infty(\Omega)}\|u_{xx}\|\|(\phi_t)_x\| \\
&\le \frac{1}{5}\|(\phi_t)_x\|^2 + C\|\phi_{xx}\|^2\|\phi\|^{\frac{2}{3}}_{H^1(\Omega)}\|u_{xx}\|^2; \qquad (3.16)
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= \int_\Omega |u_x||\phi_{xx}||(\phi_t)_x|dx \\
&\le \|u_x\|_{L^\infty(\Omega)}\|\phi_{xx}\|\|(\phi_t)_x\| \\
&\le \frac{1}{5}\|(\phi_t)_x\|^2 + C\|u\|^2_{H^2(\Omega)}\|\phi_{xx}\|^2; \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
L_4 &= \int_\Omega |g''(\phi)||\phi_x||(\phi_t)_x|dx \\
&\le \|\phi\|^2_{L^\infty(\Omega)}\|\phi_x\|\|(\phi_t)_x\| \\
&\le \frac{1}{5}\|(\phi_t)_x\|^2 + C\|\phi\|^6_{H^1(\Omega)}; \qquad (3.18)
\end{aligned}
$$

$$
\begin{aligned}
L_5 &= \int_\Omega |g'(\phi)||u_x||(\phi_t)_x|dx \\
&\le \|\phi\|^3_{L^\infty(\Omega)}\|u_x\|\|(\phi_t)_x\| \\
&\le \frac{1}{5}\|(\phi_t)_x\|^2 + C\|\phi\|^6_{H^1(\Omega)}\|u\|^2_{H^1(\Omega)}. \qquad (3.19)
\end{aligned}
$$

Here we use the embedding theorem $H^2(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ and Lemma 3.5 in (3.14)-(3.19) respectively. Then it follows from (3.14)-(3.19) and Gronwall's lemma that

$$
\begin{aligned}
\|(\phi_t)_x\|^2_{L^2(0,T;L^2(\Omega))} + \|\phi_{xx}\|^2_{L^\infty(0,T;L^2(\Omega))} &\le C\big(1 + \|\phi_{xx}(0)\|^2_{L^2(\Omega)}\big) \\
&\le C,
\end{aligned}
$$

which implies $\phi \in H^1(0, T; H^1(\Omega))$. The regularity is proved.

The proof of uniqueness: We consider (ϕ_1, u_1) and (ϕ_2, u_2) two solutions to problem (1.4)-(1.7). And we set $\phi = \phi_1 - \phi_2$, $u = u_1 - u_2$, then (ϕ, u) satisfies

$$
\begin{cases}
\phi_t - \alpha_2\big(q(u_1)\phi_{1x} - q(u_2)\phi_{2x}\big)_x = -\rho g'(\phi_1)q(u_1) + \rho g'(\phi_2)q(u_2) \quad in \;\; Q_T, \\[2ex]
u_t - \alpha_1 u_{xx} = -\alpha_3 q'(u_1)\big(1 + |\phi_{1x}|^2 + \rho_3 g(\phi_1)\big) + \alpha_3 q'(u_2)\big(1 + |\phi_{2x}|^2 + \rho_3 g(\phi_2)\big) \\
\qquad\qquad - \rho_1\big(h'(u_1) - h'(u_2)\big) \qquad\qquad\qquad\qquad in \;\; Q_T, \\[2ex]
\phi(0,x) = 0, \;\; u(0,x) = 0, \qquad x \in \Omega; \qquad \frac{\partial \phi}{\partial n} = 0, \;\; \frac{\partial u}{\partial n} = 0, \qquad (t, \; x) \in (0, \; T) \times \partial\Omega.
\end{cases} \qquad (3.20)
$$

We multiply the first equation in (3.20) by ϕ, integrate over Ω, then we get after integrating by parts

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\phi\|^2+\alpha_2 m\|\phi_x\|^2 &\le \alpha_2\int_\Omega |q(u_2)-q(u_1)||\phi_{2x}||\phi_x|dx+\rho\|q(u_1)\|_{L^\infty(\Omega)}\int_\Omega |g'(\phi_1)-g'(\phi_2)||\phi|dx\\ &\quad+\rho\int_\Omega |g'(\phi_2)||q(u_1)-q(u_2)||\phi|dx\\ &\le C\big(\|\phi_x\|\|\phi_{2x}\|+(\|\phi_1\|^2_{L^\infty(\Omega)}+\|\phi_2\|^2_{L^\infty(\Omega)})\|\phi\|^2+\|\phi_2\|^3_{L^\infty(\Omega)}\|\phi\|\|u\|\big),\end{aligned}$$

where we set $0<m\le q(u_i)\le M,\ i=1,\ 2$. Owing to the continuous embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ and the fact $\phi_i\in L^\infty(0,T;H^1(\Omega)),\ i=1,\ 2$, we obtain

$$\frac{d}{dt}\|\phi\|^2+\alpha_2 m\|\phi_x\|^2\le C(\|\phi\|^2+\|u\|^2). \tag{3.21}$$

We multiply the first equation in (3.20) by $-\phi_{xx}$, integrate over Ω, and obtain after integrating by parts,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2+\|\phi_{xx}\|^2\\ \le\ &\alpha_2\int_\Omega |q'(u_1)u_{1x}\phi_{1x}-q'(u_2)u_{2x}\phi_{2x}||\phi_{xx}|dx+\rho\int_\Omega |q(u_1)||g'(\phi_1)-g'(\phi_2)||\phi_{xx}|dx\\ &+\rho\int_\Omega |g'(\phi_2)||q(u_1)-q(u_2)||\phi_{xx}|dx\\ \le\ &C\int_\Omega(|u_{2x}||\phi_x||\phi_{xx}|+|u_x||\phi_{1x}||\phi_{xx}|+|\phi_1^2+\phi_2^2||\phi||\phi_{xx}|+|\phi_2|^3|u||\phi_{xx}|)dx\\ \le\ &\frac{4}{5}\|\phi_{xx}\|^2+C\big(\|u_{2x}\|^2_{L^\infty(\Omega)}\|\phi_x\|^2+\|\phi_{1x}\|^2_{L^\infty(\Omega)}\|u_x\|^2+\|\phi_1+\phi_2\|^4_{L^\infty(\Omega)}\|\phi\|^2+\|\phi_2\|^6_{L^\infty(\Omega)}\|u\|^2\big)\\ \le\ &\frac{4}{5}\|\phi_{xx}\|^2+C\big(\|u_{2xx}\|^{\frac{3}{2}}\|u_2\|^{\frac{1}{2}}\|\phi_x\|^2+\|\phi_{1xx}\|^{\frac{3}{2}}\|\phi_1\|^{\frac{1}{2}}\|u_x\|^2+\|\phi_1+\phi_2\|^4_{H^1(\Omega)}\|\phi\|^2\\ &+\|\phi_2\|^6_{H^1(\Omega)}\|u\|^2\big),\end{aligned} \tag{3.22}$$

where we set $a=\max(|q'(u_1)|,\ |q'(u_2)|)$, $\frac{1}{\alpha_2}=|q(u_1)-q(u_2)|$ and using the continuous embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$. Combination with the fact $\phi_i,u_i\in L^\infty(0,T;H^1(\Omega))(i=1,\ 2)$ and (3.22) yield

$$\frac{1}{2}\frac{d}{dt}\|\phi_x\|^2+\frac{1}{5}\|\phi_{xx}\|^2\le C\big(\|u_{2xx}\|^{\frac{3}{2}}\|\phi_x\|^2+\|\phi_{1xx}\|^{\frac{3}{2}}\|u_x\|^2+\|\phi\|^2+\|u\|^2\big). \tag{3.23}$$

Since the estimates of u is similar to the calculation of ϕ, we make use of the continuous embedding $H^1(\Omega)\hookrightarrow L^\infty(\Omega)$ and the fact $\phi_i\in L^\infty(0,T;H^2(\Omega))(i=1,\ 2)$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2+\alpha_1\|u_x\|^2\le C_1(1+\|u_{2xx}\|^2)\|u\|^2+C_2\|\phi_x\|^2+C_3\|\phi\|^2, \tag{3.24}$$

where $C_1,\ C_2,\ C_3$ are positive constants.

With the help of (3.21)-(3.24), we have

$$\frac{d}{dt}(\|\phi\|^2_{H^1(\Omega)}+\|u\|^2)\le C_1\|\phi\|^2_{H^1(\Omega)}+C_2(1+\|u_{2xx}\|^2)\|u\|^2. \tag{3.25}$$

From Gronwall's lemma and (3.25), it easy to know

$$\int_\Omega (\|\phi\|^2_{H^1(\Omega)} + \|u\|^2) dx = 0.$$

The uniqueness is proved. Thus the proof of Theorem 3.4 is complete.

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