Fresnel integration & diffraction amplitude

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Abstract. The Fresnel integral is used in diffraction of wave phenomena. It is demonstrated that the ordinary Fresnel integral has additional yet unkown ± 1 multivaluedness. This result gives new insight in diffraction.

1. Introduction

Fresnel integrals are used in wave diffraction theory. A textbook case is the "chirp" function [*Goodman*, 1996, pp.17]. In electromagnetic wave theory, the diffraction of the electromagnetic field by an edge and/or in the vicinity of a shadow boundary, is associated to a Fresnel integral [*Capolini and Maci*, 1995] and [*Legendre, Marsault, Velé and Cueff*, 2011]. In this paper we found the possibility of a not yet reported result which affects Fresnel uniform asymptotic expansion [*Vaudon and Jecko*, 1993] and diffraction amplitude [*Tabakcioglu and Kara*, 2009]. This mathematical result supplements the physical optics approach to radio waves [*Vesnik*, 2014, pp. 948] and is associated to signal scattering [*Costa and Basu*, 2002, eq. (10)].

The reader, perhaps, may think that the concepts of the derivation are quite elementary. Nevertheless an explicit presentation is thought to be necessary in order to provide the required complete overview of the derivation. The presented mathematics reflects a possible additional "degree of freedom" in atmospheric measurements where diffraction of electromagnetic waves is present and also in e.g. water waves [*Tamura, Kawaguchi and Fujiki* , 2019].

2. Fresnel integral

As is well known [*Kleinert*, 2009, p. 86, eq 1B.6], [*Olds*, 1968], the $(0,\infty)$ Fresnel integral is

$$\int_0^\infty e^{iax^2}dx = e^{i\pi/4}\sqrt{\frac{\pi}{4|a|}} \tag{1}$$

and $a \in \mathbb{R}$ with $|a| \neq 0$. Let us subsequently define the integral equation

$$F = \int_0^\infty d\xi \int_{-\xi}^\infty dx e^{i(x+\xi)^2} e^{2i\xi^2} \tag{2}$$

Because in the x integral we can perform the substitution $z = x + \xi$, it follows from (1) that

$$\int_{-\xi}^\infty dx e^{i(x+\xi)^2} = \int_0^\infty dz e^{iz^2} = e^{i\pi/4}\sqrt{\frac{\pi}{4}} \tag{3}$$

Therefore, with the ξ integral and $a = 2$, the F in (2) is equal to

$$F = \left(e^{i\pi/4}\sqrt{\frac{\pi}{4 \times 2}}\right) \times \left(e^{i\pi/4}\sqrt{\frac{\pi}{4}}\right) = \frac{i\pi}{4\sqrt{2}} \tag{4}$$

We will continue to rewrite (2) and make use of (4) to obtain Fresnel integral forms.

Define the transformation

$$u = x + \xi \tag{5}$$

$$v = x - \xi$$

The jacobian determinant of the transformation (5) is

$$||J\xi, x; u, v)|| = \left|\left| \begin{matrix} \frac{\partial \xi}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial \xi}{\partial v} & \frac{\partial x}{\partial u} \end{matrix} \right|\right| = \tag{6}$$

$$= \left| \left(\frac{1}{2} \times \frac{1}{2}\right) - \left(\frac{1}{2} \times \frac{-1}{2}\right) \right| = \frac{1}{2}$$

Because $u = 0$ when $x = -\xi$ we have in the first place $0 \leq u < \infty$. In the second place, because $\xi \geq 0$ and $u - v = 2\xi \geq 0$ we must have $-\infty < v \leq u$. Hence,

$$F = \frac{1}{2}\int_0^\infty du \int_{-\infty}^u dv e^{iu^2} e^{\frac{i}{2}(u-v)^2} \tag{7}$$

If we transform $v \to -v$ then

$$F = \frac{1}{2}\int_0^\infty du \int_{-u}^\infty dv e^{iu^2} e^{\frac{i}{2}(u+v)^2} \tag{8}$$

Subsequently, for (8) the following crucial transformation is used.

$$\alpha = -uv \tag{9}$$
$$\beta = \frac{1}{2}\left(u^2 + v^2\right)$$

The jacobian determinant of (9) is

$$||J(u,v;\alpha,\beta)|| = \left|\left| \begin{matrix} \frac{\partial u}{\partial \alpha} \frac{\partial u}{\partial \beta} \\ \frac{\partial v}{\partial \alpha} \frac{\partial v}{\partial \beta} \end{matrix} \right|\right| \tag{10}$$

To compute (10), firstly

$$\beta - \alpha = \frac{1}{2}(u+v)^2 \geq 0 \tag{11}$$
$$\beta + \alpha = \frac{1}{2}(u-v)^2 \geq 0$$

Secondly, because, $v \geq -u$ and $u \geq 0$, we have $u + v \geq 0$. With $-u \leq v \leq u$ it follows, $u - v \geq 0$ and for $u \leq v < \infty$ we have $u - v \leq 0$. Therefore, there is a $\eta_0 = \pm 1$ such that $u - v = \eta_0 \sqrt{2(\beta+\alpha)}$.

To find the entries of the jacobian (10) we employ $\partial/\partial\alpha$ and $\partial/\partial\beta$ respectively on (11). The two differential quotients entries with $\partial/\partial\alpha$ are

$$\frac{\partial u}{\partial \alpha} = \frac{v}{u^2 - v^2} \tag{12}$$
$$\frac{\partial v}{\partial \alpha} = \frac{-u}{u^2 - v^2}$$

Employing $\partial/\partial\beta$ on the equations in (11) gives

$$\frac{\partial u}{\partial \beta} = \frac{u}{u^2 - v^2} \tag{13}$$
$$\frac{\partial v}{\partial \beta} = \frac{-v}{u^2 - v^2}$$

The determinant follows from (10), (12) and (13). It is

$$||J(\alpha,\beta;u,v)|| = \left|\left(\frac{\partial u}{\partial\alpha}\frac{\partial v}{\partial\beta}\right) - \left(\frac{\partial u}{\partial\beta}\frac{\partial v}{\partial\alpha}\right)\right| = \tag{14}$$

$$= \left|\frac{v}{u^2-v^2}\frac{-v}{u^2-v^2} - \frac{u}{u^2-v^2}\frac{-u}{u^2-v^2}\right| =$$

$$= \left|\frac{u^2-v^2}{(u^2-v^2)^2}\right| = \left|\frac{1}{u^2-v^2}\right| = \frac{1/2}{\sqrt{\beta^2-\alpha^2}}$$

Here, $|u-v| = \sqrt{2\,(\beta+\alpha)}$. Thirdly, please look at $\exp\left[\frac{i}{2}(u+v)^2\right]$. From Euler's and the DeMoivre's rule [*Deshpande*, 1986], it follows

$$e^{i(u+v)^2} = \left\{e^{\frac{i}{2}(u+v)^2}\right\}^2 \tag{15}$$

With $\eta_1 = \pm 1$, we arrive at

$$\eta_1^2 e^{2i(\beta-\alpha)} = e^{i(u+v)^2} = \left\{e^{\frac{i}{2}(u+v)^2}\right\}^2 \tag{16}$$

Then $\eta_1^2 = 1$ and, $2(\beta-\alpha) = (u+v)^2$. This implies for the integral in (8)

$$e^{\frac{i}{2}(u+v)^2} = \eta_1 e^{i(\beta-\alpha)} \tag{17}$$

Fourthly, the u^2 exponent in (8) must be given in terms of α and β. We have, $u^2 = 2\beta - v^2$. If it is noted that $v = -\frac{\alpha}{u}$ we can have, $0 < \epsilon \leq u \leq \infty$, and, $0 < \epsilon \to 0$, then via $u^2 = 2\beta - \left(\frac{\alpha^2}{u^2}\right)$ the following u polynomial is obtained

$$u^4 = 2\beta u^2 - \alpha^2 \tag{18}$$

Writing $y = u^2$ we have, $y(\eta_2) = \beta + \eta_2\sqrt{\beta^2-\alpha^2} \geq 0$ and $\eta_2 = \pm 1$. Note that $y(\eta_2 = -1) \geq 0$. Hence, two forms $\eta_2 = \pm 1$, apply

$$e^{iu^2} = e^{i\left(\beta+\eta_2\sqrt{\beta^2-\alpha^2}\right)} \tag{19}$$

With the transformation (9)-(19) equation (8) can be written

$$F = \frac{\eta_1}{2^2} \int_{-\infty}^{+\infty} d\alpha \times \tag{20}$$

$$\times \int_{|\alpha|}^{+\infty} d\beta \frac{e^{i(\beta-\alpha)} e^{i\left(\beta+\eta_2\sqrt{\beta^2-\alpha^2}\right)}}{\sqrt{\beta^2-\alpha^2}}$$

with $|\alpha| = \max\{\alpha, -\alpha\}$.

Subsequently a linear transformation is applied. It is

$$\lambda = \beta + \alpha \geq 0 \tag{21}$$

$$\zeta = \beta - \alpha \geq 0$$

Recall that, $\lambda = \beta + \alpha = \frac{1}{2}(u+v)^2 \geq 0$, and, $\zeta = \beta - \alpha = \frac{1}{2}(u-v)^2 \geq 0$. The jacobian here is $||J(\lambda, \zeta; \alpha, \beta)|| = 1/2$. This transforms the equation (20) to

$$F = \frac{\eta_1}{2^3} \int_0^{+\infty} d\lambda \int_0^{+\infty} d\zeta \frac{1}{\sqrt{\lambda\zeta}} e^{i\zeta} e^{i\left(\frac{\lambda+\zeta}{2}+\eta_2\sqrt{\lambda\zeta}\right)} \tag{22}$$

Continuing

$$p = \sqrt{\lambda} \geq 0 \tag{23}$$

$$q = \sqrt{\zeta} \geq 0$$

The jacobian is: $||J(p, q; \lambda, \zeta)|| = 4pq$. Equation (22) therefore is

$$F = \frac{\eta_1}{2} \int_0^{+\infty} dq \int_0^{+\infty} dp e^{iq^2} e^{i\left\{\frac{1}{2}\left(p^2+q^2\right)+\eta_2 pq\right\}} \tag{24}$$

Then, $\frac{1}{2}(p+\eta_2 q)^2 - \frac{1}{2}q^2 = \frac{1}{2}p^2 + \eta_2 pq$. Hence,

$$F = \frac{\eta_1}{2} \int_0^{+\infty} dq e^{iq^2} \int_0^{+\infty} dp e^{\frac{i}{2}(p+\eta_2 q)^2} \tag{25}$$

Using $r = p + \eta_2 q$, $dp = dr$ and $r(p=0) = \eta_2 q$

$$F = \frac{\eta_1}{2} \int_0^{+\infty} dq e^{iq^2} \int_{\eta_2 q}^{+\infty} dr e^{ir^2/2} \tag{26}$$

This gives

$$\int_{\eta_2 q}^{+\infty} dr e^{ir^2/2} = e^{i\pi/4}\sqrt{\frac{\pi}{2}} - \eta_2 \int_0^q dr e^{ir^2/2} \tag{27}$$

If (27) goes in (26) it follows, using (1)

$$F = \frac{\eta_1}{2}\left(e^{i\pi/4}\sqrt{\frac{\pi}{4}} \times e^{i\pi/4}\sqrt{\frac{\pi}{2}}\right) + \tag{28}$$
$$-\frac{\eta_1\eta_2}{2}\int_0^{+\infty} dq e^{iq^2}\int_0^q dr e^{ir^2/2}$$

With F in (4) and

$$\int_0^{\infty} e^{ir^2/2} dr = \int_0^q e^{ir^2/2} dr + \int_q^{\infty} e^{ir^2/2} dr$$

then,

$$I = \lim_{0<\epsilon'\to 0}\int_{\epsilon'}^{+\infty} dq e^{iq^2}\int_q^{\infty} dr e^{ir^2/2} = \tag{29}$$
$$= \frac{i\pi}{2\sqrt{2}}\left(1-\eta_2+\eta_1\eta_2\right)$$

with $I = I(\eta_1, \eta_2)$.

3. Discussion

To harmonize (29) with (1) we claim the proportionality

$$\int_q^{\infty} dr e^{ir^2/2} \propto \tag{30}$$
$$e^{i\pi/4}\sqrt{\left(\frac{\pi}{2}\right)} \lim_{0<\epsilon\to 0}\left\{ \left(1-\eta_2+\eta_1\eta_2\right)^{\ 1-\theta(\epsilon-q)}\right.$$
$$\left. + f_\epsilon(q)\right\}$$

with $\theta(x) = 1$ when $x \geq 0$ and $\theta(x) = 0$ when $x < 0$, $\forall(x \in \mathbb{R})$. Further, $f_\epsilon(q) \to 0$ for, $0 \leq q \leq \epsilon$, with, $0 < \epsilon \to 0$. Therefore,

$$\int_0^{\infty} dr e^{ir^2/2} = e^{i\pi/4}\sqrt{\left(\frac{\pi}{2}\right)} \tag{31}$$

The q integration procedure, equivalent (29), is

$$\lim_{0<\epsilon\to 0}\lim_{0<\epsilon'\to 0:(\epsilon<\epsilon')} \tag{32}$$
$$\left\{\int_{\epsilon'}^{\infty} dq\,\left(1-\eta_2+\eta_1\eta_2\right)^{\,1-\theta(\epsilon-q)} e^{iq^2}\right.$$
$$\left.\int_{\epsilon'}^{\infty} dq + f_\epsilon(q)e^{iq^2}\right\} =$$
$$= e^{i\pi/4}\sqrt{\left(\frac{\pi}{4}\right)}\left(1-\eta_2+\eta_1\eta_2\right)$$

Only if $q = 0$; $(0 \le q \le \epsilon)$, we have

$$\lim_{0<\epsilon\to 0}\left(1-\eta_2+\eta_1\eta_2\right)^{\,1-\theta(\epsilon-q)} = 1.$$

In (32), $\left(1-\eta_2+\eta_1\eta_2\right)$, is treated as a constant. Hence, $\forall(\epsilon < \epsilon' \le q)\theta(\epsilon-q) = 0$. To obtain (29) from (30) we need to have

$$\lim_{0<\epsilon\to 0}\lim_{0<\epsilon'\to 0:(\epsilon<\epsilon')}\int_{\epsilon'}^{\infty} dq f_\epsilon(q)e^{iq^2} = 0 \tag{33}$$

As an example of the function f_ϵ

$$f_\epsilon(q) = 2q\sin(q^2/\epsilon) + 2q\cos(q^2/\epsilon) \tag{34}$$

with, $f_\epsilon(0) = 0$. Equation (33) gives, with $y = q^2$ and $y \ge 0$ and $\epsilon < \epsilon'$ together with $\epsilon'^2 < \epsilon$, (e.g. $\epsilon' = \sqrt{2}\,\epsilon$).

$$I_{\epsilon,\epsilon'} = \int_{\epsilon'^2}^{\infty} dy\left\{e^{(i-\epsilon)y}\sin(y/\epsilon)\right\} \tag{35}$$

For practical purposes $\epsilon'^2 \approx 0$:

$$I_{\epsilon,0} = I_\epsilon \approx \frac{\frac{1/\epsilon}{(i-\epsilon)^2}}{1+\frac{1/\epsilon^2}{(i-\epsilon)^2}} = \frac{\epsilon^2(i-\epsilon)^2}{\epsilon^2(i-\epsilon)^2}\frac{\frac{1/\epsilon}{(i-\epsilon)^2}}{1+\frac{1/\epsilon^2}{(i-\epsilon)^2}} \tag{36}$$
$$= \frac{\epsilon}{1+\epsilon^2(i-\epsilon)}$$

This implies $\lim_{0<\epsilon\to 0} I_\epsilon = 0$. Subsequently, cos in (34) gives

$$\lim_{0<\epsilon\to 0} \lim_{0<\epsilon'\to 0:(\epsilon<\epsilon')} \int_{\epsilon'^2}^{\infty} dy \left\{ e^{(i-\epsilon)y} \cos(y/\epsilon) \right\} \tag{37}$$

$$= -\lim_{0<\epsilon\to 0} \frac{1}{i-\epsilon} + \lim_{0<\epsilon\to 0} \frac{1/\epsilon}{i-\epsilon} I_\epsilon = 0$$

With $\lim_{0<\epsilon\to 0} \lim_{0<\epsilon'\to 0:(\epsilon<\epsilon')}$, (33) is observed. Therefore, (30) is correct viz. (34).

4. Conclusion

We conclude that the Fresnel integral used by *Capolini and Maci* [1995]

$$F(q) = e^{iq^2} \int_q^{\infty} dr e^{ir^2/2}$$

depends on, previously unknown, ± 1 variables. The crucial transformation is (9). This multivaluedness was not accounted for previously, [*Capolini and Maci*, 1995] - [*Legendre, Marsault, Velé and Cueff*, 2011]. It affects the use of generalized Fresnel integrals [*Albeverio and Mazzuchi*, 2005] and leads to a multivalued amplitude diffraction coefficient, viz. [*Tabakcioglu and Kara*, 2009] in general. More specific, it may add to insight into bending angle errors [*Hordyniec, Norman, Rohm, Huang and Le Marshall*, 2019] and supplement methodology.

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