Stability of Viscous Shock Wave under Periodic Perturbation for Compressible Navier-Stokes-Korteweg System

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Abstract. In this paper, we study the stability of a viscous shock wave for the isentropic Navier-Stokes-Korteweg equations under space-periodic perturbation. It is shown that if the initial perturbation around the shock and the amplitude of the shock are small, then the solution of the N-S-K equations tends to the viscous shock.

1. Introduction

In this paper, we consider the following one-dimensional compressible Navier-Stokes-Korteweg system in the Lagrangian coordinates:

$$\begin{cases} v_t - u_x = 0, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u_t + p(v)_x = \mu \left(\dfrac{u_x}{v} \right)_x + \kappa \left(\dfrac{-v_{xx}}{v^5} + \dfrac{5v_x^2}{2v^6} \right)_x, & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \end{cases} \tag{1.1}$$

where $v(t,x) > 0$ is the specific volume, $u(t,x)$ the velocity. The pressure $p(v)$ satisfying

$$p(v) > 0, \quad p'(v) < 0, \quad p''(v) > 0, \quad \forall\ v > 0. \tag{1.2}$$

And $\kappa > 0$ is the capillary coefficient. $\mu > 0$ is the viscosity coefficient.

Notice that when $\kappa = 0$, the system (1.1) is reduced to the compressible Navier-Stokes system, which admits viscous shock wave solution $(U,V)(x-st)$ with shock propagation speed s. The stability of viscous shock for system (1.1) has been extensively studied, see [9, 11, 13, 14, 17, 19–21, 23–25, 27].

Unfortunately, when $\kappa > 0$, the Navier-Stokes-Korteweg system (1.1) not always admits the viscous shock wave solution unless some additional conditions are satisfied, see [28, 29]. Chen-He-Zhao [6] showed that this shock wave is asymptotically stable. We refer to [1–5, 7, 8, 12, 16, 26, 30–32] for more interesting works on Navier-Stokes-Korteweg system.

However, the periodic perturbation problem is more difficult. For the scalar equations and 2×2 systems, Lax [18] and Glimm-Lax [10] proved that the periodic solutions time-asymptotically decay to their constant averages. For more interesting

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works about the general Riemann initial data with periodic perturbations, please refer to [15, 33–35].

In this paper, we consider a initial value problem of (1.1) with the initial data

$$(v,u)(x,0)=(v_0,u_0)(x), \quad x\in\mathbb{R}, \tag{1.3}$$

satisfying

$$(v_0,u_0)(x)\to\begin{cases}(\overline{v}_l,\overline{u}_l)+(\phi_{0l},\psi_{0l})(x) & \text{as } x\to-\infty,\\ (\overline{v}_r,\overline{u}_r)+(\phi_{0r},\psi_{0r})(x) & \text{as } x\to+\infty.\end{cases} \tag{1.4}$$

Here $\overline{v}_l>0, \overline{v}_r>0, \overline{u}_l$ and $\overline{u}_r$ are constants. Oscillation function $(\phi_{0l},\psi_{0l})\in\mathbb{R}^2$ and $(\phi_{0r},\psi_{0r})\in\mathbb{R}^2$ are two periodic functions. We assume $(\phi_{0i},\psi_{0i})\in H^4(0,\pi_i); \quad i=l,r$. The constant states $(\overline{v}_l,\overline{u}_l)$ and $(\overline{v}_r,\overline{u}_r)$ satisfy the Rankine-Hugoniot condition,

$$\begin{cases}-s\,(\overline{v}_r-\overline{v}_l)-(\overline{u}_r-\overline{u}_l)=0,\\ -s\,(\overline{u}_r-\overline{u}_l)+(p(\overline{v}_r)-p(\overline{v}_l))=0.\end{cases} \tag{1.5}$$

Here $s=\sqrt{-\dfrac{p(\overline{v}_r)-p(\overline{v}_l)}{\overline{v}_r-\overline{v}_l}}>0$ is the shock speed. Moreover we assume that the constant states satisfy the Lax's entropy condition, i.e,

$$0<\overline{v}_l<\overline{v}_r, \quad \overline{u}_l>\overline{u}_r, \tag{1.6}$$

and

$$\int_0^{\pi_l}(\phi_{0l},\psi_{0l})(x)dx=0 \quad \text{and} \quad \int_0^{\pi_r}(\phi_{0r},\psi_{0r})(x)dx=0. \tag{1.7}$$

Due to the initial perturbation is space periodic, the standard anti-derivative method fails. Inspired by [33], in order to make the anti-derivative method available, in this paper, we introduce a suitable ansatz (V,U), so that the ansatz and solution (v,u) have the same oscillation in the far field. Thus, the perturbation $(v-V,u-U)$ belongs to $(H^3,H^2)(\mathbb{R})$. In (2.5), we give the exact statement of the ansatz. Through energy estimation, we know that the solution not only exists globally, but also tends to the viscous shock with a shift as time tends to infinity. The shift is determined by the periodic oscillation partially.

The structure of this paper is as follows. In Section 2, we construct ansatz (V,U) and state our main results. In Section 3 , we prove the main results. In Section 4, we give some lengthy proofs for ease of reading.

Notation. The functional $\|\cdot\|_{L^p}$ is defined by $\|f\|_{L^p}=(\int_R|f|^p(x)\,\mathrm{d}\,x)^{\frac{1}{p}}$. We denote for simplicity

$$\|f\|=\left(\int_{-\infty}^{\infty}f^2(x)\,\mathrm{d}\,x\right)^{\frac{1}{2}}$$

as $p=2$. H^m represents the m-th order Sobolev space of functions defined by

$$\|f\|_m = \left(\sum_{k=0}^{m} \|\partial_x^k f\|^2\right)^{\frac{1}{2}}.$$

Moreover, we denote $h_\xi(x), h_\xi^{(k)}(x)$ for simplicity

$$h_\xi(x) := h(x-\xi(t)), \quad h_\xi^{(k)}(x) := h^{(k)}(x-\xi(t)), \quad k \geq 1.$$

2. Preliminaries and Main Result

The viscous shock profile $(v^S,u^S)(\xi) = (v^S,u^S)(x-st)$ is a traveling wave solution to (1.1), satisfying the following euqations,

$$\begin{cases} -s\left(v^S\right)' - \left(u^S\right)' = 0, \\ -s\left(u^S\right)' + \left(p(v^S)\right)' = \mu\left(\dfrac{(u^S)'}{v^S}\right)' + \kappa\left(\dfrac{(-v^S)''}{(v^S)^5} + \dfrac{5[(v^S)']^2}{2(v^S)^6}\right)', \\ \left(v^S,u^S\right) \to (\overline{v}_l,\overline{u}_l) \ \text{(resp. } (\overline{v}_r,\overline{u}_r)) \qquad \text{as } \xi \to -\infty \ \text{(resp. } +\infty), \end{cases} \tag{2.1}$$

where $' = \mathrm{d}/\mathrm{d}\,\xi, \quad \xi = x - st$.

Lemma 2.1. *([6]) Let (1.2) and (1.6) hold. If*

$$\frac{\mu^2 s^2 \overline{v}_l^8}{\kappa} - \left(\frac{10\overline{v}_r}{\overline{v}_l} - 6\right)\overline{v}_r^5\left(p'(\overline{v}_r) + s^2\right) > 0, \tag{2.2}$$

then there exists a monotone viscous shock profile $(v^S,u^S)(x-st)$ to system (2.1), which is unique up to a shift and satisfies $(v^S)' > 0$, $(u^S)' < 0$, and

$$\begin{cases} |v^S(\xi) - \overline{v}_l| \leq C\delta e^{-c_1|\xi|}, \quad |u^S(\xi) - \overline{u}_l| \leq C\delta e^{-c_1|\xi|}, \quad \forall \xi \leq 0, \\ |v^S(\xi) - \overline{v}_r| \leq C\delta e^{-c_1|\xi|}, \quad |u^S(\xi) - \overline{u}_r| \leq C\delta e^{-c_1|\xi|}, \quad \forall \xi \geq 0, \\ \left|\dfrac{d^k}{d\xi^k} v^S(\xi)\right| + \left|\dfrac{d^k}{d\xi^k} u^S(\xi)\right| \leq C\delta^2 e^{-c_1|\xi|}, \quad \forall\, \xi \in \mathbb{R}, \forall\, k \geq 1, \end{cases} \tag{2.3}$$

where $\delta := \overline{v}_r - \overline{v}_l$, and c_1, C are two positive constants depending only on $\overline{v}_r, \overline{v}_l, s, \mu$ and κ.

Let $(v_{l,r}, u_{l,r})(x,t)$ are the solutions of (1.1) with the periodic initial data

$$(v_{l,r}, u_{l,r})(x,0) = (\overline{v}_{l,r}, \overline{u}_{l,r}) + (\phi_{0l,0r}, \psi_{0l,0r})(x).$$

We first give a lemma for periodic solution to (1.1).

Lemma 2.2. *Assume that $(v_0,u_0)(x) \in H^4(0,\pi)$ is periodic with period $\pi > 0$ and average $(\overline{v},\overline{u})$, Then there exists $\epsilon_0 > 0$ such that if*

$$\epsilon := \|(v_0,u_0) - (\overline{v},\overline{u})\|_{H^4(0,\pi)} \leq \epsilon_0,$$

and the $\overline{v}$ satisfying

$$\overline{v} > \max\left\{\left(\frac{\kappa}{\mu^3}\right)^{\frac{1}{3}}, \left(\frac{\kappa}{4\mu(\mu-1)}\right)^{\frac{1}{3}}\right\} \tag{2.4}$$

the problem (1.1) has a unique periodic solution with initial data (v_0, u_0)

$$(v, u)(x, t) \in C(0, +\infty; H^4(0, \pi)).$$

It has same period and average as (v_0, u_0). Moreover, it holds that

$$\|(v, u) - (\overline{v}, \overline{u})\|_{H^4(0,\pi)}(t) \leq C\epsilon e^{-\alpha t}, \quad t \geq 0,$$

where the constants $C > 0$ and $\alpha > 0$ are independent of ϵ and t.

The proof of Lemma 2.2 is lengthy. We put it in the appendix.
Ansatz: Now, we construct ansatz (V, U)

$$\begin{aligned} V(x,t) &:= v_l(x,t)\left(1 - g_{st+\mathcal{X}(t)}(x)\right) + v_r(x,t) g_{st+\mathcal{X}(t)}(x), \\ U(x,t) &:= u_l(x,t)\left(1 - g_{st+\mathcal{Y}(t)}(x)\right) + u_r(x,t) g_{st+\mathcal{Y}(t)}(x), \end{aligned} \tag{2.5}$$

where $g(x) = \dfrac{v^S(x) - \overline{v}_l}{\overline{v}_r - \overline{v}_l} = \dfrac{u^S(x) - \overline{u}_l}{\overline{u}_r - \overline{u}_l}$, and $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ will be determine later. With the aid of Lemma 2.1, one gets that $0 < g(x) < 1$ and $g'(x) > 0$ for all $x \in \mathbb{R}$. By direct calculation, we obtain the equation about ansatz (V, U), i.e,

$$\begin{cases} V_t - U_x = -(G_1)_x - g_2, \\ U_t + p(V)_x - \mu\left(\dfrac{U_x}{V}\right)_x - \kappa\left(\dfrac{-V_{xx}}{V^5} + \dfrac{5V_x^2}{2V^6}\right)_x = -(G_3)_x - g_4, \end{cases} \tag{2.6}$$

where

$$\begin{aligned} G_1 &= (u_r - u_l)(g_{st+\mathcal{Y}} - g_{st+\mathcal{X}}), \\ g_2 &= (u_r - u_l) g'_{st+\mathcal{X}} + (v_r - v_l) g'_{st+\mathcal{X}}(s + \mathcal{X}'), \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
G_3 = & -(p(V)-p(v_l))\left(1-g_{st+\mathcal{Y}}\right)-(p(V)-p(v_r))\, g_{st+\mathcal{Y}} \\
& +\mu\left\{(V^{-1})(u_r-u_l)g'_{st+\mathcal{Y}}+\left(V^{-1}-{v_l}^{-1}\right)u_{lx}(1-g_{st+\mathcal{Y}})+\left(V^{-1}-{v_r}^{-1}\right)u_{rx}g_{st+\mathcal{Y}}\right\}, \\
& -\kappa\left\{\left(V^{-5}-v_l^{-5}\right)v_{lxx}(1-g_{st+\mathcal{X}})-\left(V^{-5}-v_r^{-5}\right)v_{rxx}(g_{st+\mathcal{X}})\right\} \\
& -2\kappa\{(u_{lx}-u_{rx})g'_{st+\mathcal{X}}+(u_l-u_r)g''_{st+\mathcal{X}}\}V^{-5} \\
& +\frac{5\kappa}{2}\left\{(V^{-6}-v_l^{-6})(1-g_{st+\mathcal{X}})v_{lx}^2+(V^{-6}-v_r^{-6})(g_{st+\mathcal{X}})v_{rx}^2\right\} \\
& +\frac{5\kappa}{2V^6}\left\{\left[v_l(x,t)-v_r(x,t)\right]g'_{st+\mathcal{X}}\right\}^2 \\
& -\frac{5\kappa}{V^6}\left[v_l(x,t)-v_r(x,t)\right]\left\{v_{lx}(x,t)\left(1-g_{st+\mathcal{X}}\right)+v_{rx}(x,t)\left(g_{st+\mathcal{X}}\right)\right\}g'_{st+\mathcal{X}} \\
& -\frac{5\kappa}{2V^6}\left\{v_{lx}(x,t)-v_{rx}(x,t)\right\}^2\left(g_{st+\mathcal{X}}\right)\left(1-g_{st+\mathcal{X}}\right) \\
g_4 = & (u_r-u_l)g'_{st+\mathcal{X}}(s+\mathcal{Y}')+\mu\left(u_{rx}{v_r}^{-1}-u_{lx}{v_l}^{-1}\right)g'_{st+\mathcal{X}} \\
& -(p(v_r)-p(v_l))\,g'_{st+\mathcal{Y}}+\kappa\left(v_{lxx}v_l^{-5}-v_{rxx}v_r^{-5}-\frac{5}{2}v_{lx}^2v_l^{-6}+\frac{5}{2}v_{rx}^2v_r^{-6}\right)g'_{st+\mathcal{X}}.
\end{aligned}
$$

Using $V-v_l=(v_r-v_l)\,g_{st+\mathcal{X}}$ and $V-v_r=-(v_r-v_l)\left(1-g_{st+\mathcal{X}}\right)$, one gets that

$$
\lim_{|x|\to+\infty} G_i(x,t)=0 \quad \text{for all } t\geq 0, \quad i=1,3.
$$

In order to using the anti-derivative method, we assume that

$$
\int_{\mathbb{R}}(v-V,u-U)(x,t)dx=0 \quad \text{for all } t\geq 0. \tag{2.8}
$$

With the aid of (1.1), (2.6), one gets that

$$
\frac{d}{dt}\int_{\mathbb{R}}(v-V,u-U)(x,t)dx=\int_{\mathbb{R}}(g_2,g_4)(x,t)dx.
$$

Obviously, the following two conditions (2.9), (2.10) could imply (2.8),

$$
\int_{\mathbb{R}}(g_2,g_4)(x,t)dx=0, \quad t>0, \tag{2.9}
$$

and

$$
\int_{\mathbb{R}}\big(v_0(x)-V(x,0),u_0(x)-U(x,0)\big)dx=0. \tag{2.10}
$$

With the aid of (2.9), we can obtain that

$$
\begin{aligned}
\mathcal{X}'(t) &= -s-\frac{\int_{\mathbb{R}}(u_r-u_l)(x,t)g'_{st+\mathcal{X}}(x)dx}{\int_{\mathbb{R}}(v_r-v_l)(x,t)g'_{st+\mathcal{X}}(x)dx}, \\
\mathcal{Y}'(t) &= -s+\frac{\int_{\mathbb{R}}\mathbb{M}(x,t)g'_{st+\mathcal{Y}}(x)+\mathbb{N}(x,t)g'_{st+\mathcal{X}}(x)dx}{\int_{\mathbb{R}}\left(u_r-u_l\right)(x,t)g'_{st+\mathcal{Y}}(x)dx},
\end{aligned}
\tag{2.11}
$$

where

$$\mathbb{M}(x,t)=p(v_r)-p(v_l)-\mu\frac{u_{rx}}{v_r}+\mu\frac{u_{lx}}{v_l},\quad \mathbb{N}(x,t)=-\kappa\left(\frac{v_{lxx}}{v_l^5}-\frac{v_{rxx}}{v_r^5}-\frac{5v_{lx}^2}{2v_l^6}+\frac{5v_{rx}^2}{2v_r^6}\right).$$

Using a similar technique in [33], we know that there exists a unique $(\mathcal{X}_0,\mathcal{Y}_0)$ such that the condition (2.10) holds, provided that $\|\phi_{0l},\phi_{0r},\psi_{0l},\psi_{0r}\|_{L^\infty(\mathbb{R})}$ is small. We can solve the equation (2.11) once initial data $(\mathcal{X}_0,\mathcal{Y}_0)$ is obtained.

Lemma 2.3. *Under the condition (1.7), there exists $\epsilon_0>0$ such that if*

$$\epsilon:=\|\phi_{0l},\psi_{0l}\|_{H^2(0,\pi_l)}+\|\phi_{0r},\psi_{0r}\|_{H^2(0,\pi_r)}\le\epsilon_0,$$

there exists a unique solution $(\mathcal{X},\mathcal{Y})(t)\in C^1[0,+\infty)$ to (2.11) with the initial data $(\mathcal{X}_0,\mathcal{Y}_0)$. Moreover, the solution satisfies that

$$\begin{aligned}&|\mathcal{X}'(t)|+|\mathcal{X}(t)-\mathcal{X}_\infty|\le C\epsilon e^{-\alpha t},\\&|\mathcal{Y}'(t)|+|\mathcal{Y}(t)-\mathcal{Y}_\infty|\le C\epsilon e^{-\alpha t},\end{aligned}\qquad t\ge0,$$

where the constants $C>0$ and $\alpha>0$ are independent of ϵ and t, and the shifts $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$ are

$$\begin{aligned}\mathcal{X}_\infty:=&\mathcal{X}_0+\frac{1}{\overline{v}_r-\overline{v}_l}\Big\{\int_{-\infty}^0(\phi_{0l}-\phi_{0r})g_{\mathcal{X}_0}dx-\int_0^{+\infty}(\phi_{0l}-\phi_{0r})(1-g_{\mathcal{X}_0})dx\Big\}\\&+\frac{1}{\overline{v}_r-\overline{v}_l}\Big\{\frac{1}{\pi_l}\int_0^{\pi_l}\int_0^y\phi_{0l}(x)dxdy-\frac{1}{\pi_r}\int_0^{\pi_r}\int_0^y\phi_{0r}(x)dxdy\Big\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{Y}_\infty:=&\mathcal{Y}_0+\frac{1}{\overline{u}_r-\overline{u}_l}\Big\{\int_{-\infty}^0(\psi_{0l}-\psi_{0r})g_{\mathcal{Y}_0}dx-\int_0^{+\infty}(\psi_{0l}-\psi_{0r})(1-g_{\mathcal{Y}_0})dx\Big\}.\\&+\Big\{\frac{1}{\pi_l}\int_0^{\pi_l}\int_0^y\psi_{0l}(x)dxdy-\int_0^{+\infty}\frac{1}{\pi_l}\int_0^{\pi_l}\left[p(v_l(y,t))-p(\overline{v}_l)\right]dydt\\&-\frac{1}{\pi_r}\int_0^{\pi_r}\int_0^y\psi_{0r}(x)dxdy+\int_0^{+\infty}\frac{1}{\pi_r}\int_0^{\pi_r}\left[p(v_r(y,t))-p(\overline{v}_r)\right]dydt\\&+\mu\ln(\overline{v}_l)-\mu\ln(\overline{v}_r)-\frac{1}{\pi_l}\int_0^{\pi_l}\mu\ln\left(\overline{v}_l+\phi_{0l}(y)\right)dy\\&+\frac{1}{\pi_r}\int_0^{\pi_r}\mu\ln\left(\overline{v}_r+\phi_{0r}(y)\right)dy\Big\}\frac{1}{\overline{u}_r-\overline{u}_l}.\end{aligned}$$

Now we skip the proof of Lemma 2.3 for easy reading. And we put the tedious proof in Section 4. The ansatz (V,U) in (2.5) tends to $\big(v^S(x-st-\mathcal{X}_\infty),u^S(x-st-\mathcal{Y}_\infty)\big)$

as $t \to +\infty$. In this way, we have to add a new condition $\mathcal{X}_\infty = \mathcal{Y}_\infty$, i.e.,

$$
\begin{aligned}
& s\Big\{\int_{-\infty}^{0}\left(v_0-v^S-\phi_{0l}\right)(x)dx+\int_0^{+\infty}\left(v_0-v^S-\phi_{0r}\right)(x)dx \\
&\quad -\frac{1}{\pi_l}\int_0^{\pi_l}\int_0^x \phi_{0l}(y)dydx+\frac{1}{\pi_r}\int_0^{\pi_r}\int_0^x \phi_{0r}(y)dydx\Big\} \\
&=-\int_{-\infty}^{0}\left(u_0-u^S-\psi_{0l}\right)(x)dx-\int_0^{+\infty}\left(u_0-u^S-\psi_{0r}\right)(x)dx \\
&\quad +\frac{1}{\pi_l}\int_0^{\pi_l}\int_0^x \psi_{0l}(y)dydx-\int_0^{+\infty}\frac{1}{\pi_l}\int_0^{\pi_l}\left[p(v_l(x,t))-p(\overline{v}_l)\right]dxdt \\
&\quad -\frac{1}{\pi_r}\int_0^{\pi_r}\int_0^x \psi_{0r}(y)dydx+\int_0^{+\infty}\frac{1}{\pi_r}\int_0^{\pi_r}\left[p(v_r(x,t))-p(\overline{v}_r)\right]dxdt \\
&\quad +\mu\ln(\overline{v}_l)-\mu\ln(\overline{v}_r)-\frac{1}{\pi_l}\int_0^{\pi_l}\mu\ln(\overline{v}_l+\phi_{0l}(x))dx \\
&\quad +\frac{1}{\pi_r}\int_0^{\pi_r}\mu\ln(\overline{v}_r+\phi_{0r}(x))dx.
\end{aligned} \tag{2.12}
$$

The condition (2.12) is called as a zero-mass type condition. Now, we define the anti-derivative,

$$
\Phi_0(x):=\int_{-\infty}^{x}\left(v_0(y)-V(y,0)\right)dy, \quad \Psi_0(x):=\int_{-\infty}^{x}\left(u_0(y)-U(y,0)\right)dy,
$$

Moreover, we set

$$
E_0:=\|\phi_{0l},\psi_{0l}\|_{H^4(0,\pi_l)}+\|\phi_{0r},\psi_{0r}\|_{H^4(0,\pi_r)}+\|\Phi_0\|_{H^3(\mathbb{R})}+\|\Psi_0\|_{H^2(\mathbb{R})}+\delta.
$$

Now we are ready to give the main result of this paper.

Theorem 2.1. *Assume that the periodic functions in (1.4) satisfy (1.7),(2.2), (2.12) and*

$$
\overline{v}_i>\max\left\{\left(\frac{\kappa}{\mu^3}\right)^{\frac{1}{3}},\left(\frac{\kappa}{4\mu(\mu-1)}\right)^{\frac{1}{3}}\right\}, \quad i=l,r.
$$

Then there exists $\epsilon_1>0$ such that, if $E_0\le\epsilon_1$, there exists a unique global solution of (1.1),(1.3), satisfying

$$
\begin{aligned}
v-V &\in C\left([0,+\infty);H^3(\mathbb{R})\right)\cap L^2\left((0,+\infty);H^2(\mathbb{R})\right), \\
u-U &\in C\left([0,+\infty);H^2(\mathbb{R})\right)\cap L^2\left((0,+\infty);H^1(\mathbb{R})\right),
\end{aligned}
$$

and

$$
\|(v,u)(\cdot,t)-(v^S,u^S)(\cdot-st-\mathcal{X}_\infty)\|_{L^\infty(\mathbb{R})}\to 0 \quad \text{as } t\to+\infty. \tag{2.13}
$$

Remark 2.1. It is an interesting problem to remove the condition that δ is suitable small.

3. Proof of Theorem 2.1

We define the perturbation terms

$$\phi(x,t) = (v - V)(x,t), \quad \psi(x,t) = (u - U)(x,t),$$

and the anti-derivative variables

$$\Phi(x,t) := \int_{-\infty}^{x} \phi(x,t)dx, \quad \Psi(x,t) := \int_{-\infty}^{x} \psi(x,t)dx. \tag{3.1}$$

Equalities (1.1), (2.6) and (3.1) give that

$$\begin{cases} \Phi_t - \Psi_x = H_1, \\ \Psi_t + p'(V)\Phi_x - \mu\dfrac{\Psi_{xx}}{V} + \kappa\dfrac{\Phi_{xxx}}{v^5} + \mu\dfrac{U_x\Phi_x}{V^2} = F + H_2, \end{cases} \tag{3.2}$$

where

$$\begin{aligned} F = &- (p(v) - p(V) - p'(V)\Phi_x) - \frac{\mu\Psi_{xx}\Phi_x}{vV} + \kappa V_{xx}\left(\frac{1}{V^5} - \frac{1}{v^5}\right) \\ &+ \frac{5\kappa}{2}\frac{\Phi_{xx}^2 + 2\Phi_{xx}V_x}{v^6} - \frac{5\kappa}{2}V_x^2\left(\frac{1}{V^6} - \frac{1}{v^6}\right) + \frac{\mu U_x\Phi_x^2}{vV^2}, \end{aligned} \tag{3.3}$$

$$H_1 := G_1 + \int_{-\infty}^{x} g_2(x,t)dx, \quad H_2 := G_3 + \int_{-\infty}^{x} g_4(x,t)dx,$$

with initial data

$$(\Phi,\Psi)(x,0) = (\Phi_0,\Psi_0)(x) \in H^3(\mathbb{R}) \times H^2(\mathbb{R}). \tag{3.4}$$

We aim to seek the solution (Φ,Ψ) in the following functional space $\mathcal{B}$ for $T \in [0,\infty)$

$$\mathcal{B}(0,T) = \left\{ (\Phi,\Psi)(x,t) \;\middle|\; \begin{array}{l} \Phi(x,t) \in C(0,T;H^3(\mathbb{R})), \Phi_x(x,t) \in L^2(0,T;H^3(\mathbb{R})), \\ \Psi(x,t) \in C(0,T;H^2(\mathbb{R})), \Psi_x(x,t) \in L^2(0,T;H^2(\mathbb{R})). \end{array} \right\}$$

Theorem 3.1. *Under the same assumptions of Theorem 2.1, there exists $\epsilon_1 > 0$, if $E_0 \le \epsilon_1$, then the initial value problem (3.2), (3.4) has a unique solution $(\Phi,\Psi) \in \mathcal{B}(0,+\infty)$ satisfying*

$$\sup_{t>0}(\|\Phi\|_3, \|\Psi\|_2)^2 + \int_0^{+\infty} \left(\|\phi\|_3^2 + \|\psi\|_2^2\right) dt \le C\big(\{\|\Phi_0\|_3, \|\Psi_0\|_2\}^2 + \epsilon\big),$$

Moreover, it holds that

$$\|\phi,\psi\|_{L^\infty(\mathbb{R})}(t) \to 0 \quad \text{as } t \to +\infty. \tag{3.5}$$

To prove Theorem 3.1, we list the decay properties of the error term in (3.2), the proof of which we place in Section 4.2.

Lemma 3.1. *Under the assumptions of Theorem 2.1, it holds that*

$$\|H_1\|_3, \|H_2\|_1 \le C\epsilon e^{-\alpha t}, \quad t \ge 0,$$

where $C > 0$ is independent of ϵ and t.

Proposition 3.1 (A priori estimates). *We assume that that for any $T > 0$, $(\Phi, \Psi) \in \mathcal{B}(0,T)$ is a solution of (3.2),(3.4). Then there exist three positive constants ϵ_0, δ_0 and ε_0, independent of T, such that if*

$$\epsilon \leq \epsilon_0, \quad \delta \leq \delta_0 \quad \textit{and} \quad \varepsilon := \sup_{t\in[0,T]} \{\|\Phi(t)\|_3 + \|\Psi(t)\|_2\} \leq \varepsilon_0 \tag{3.6}$$

then

$$\sup_{t\in(0,T)} (\|\Phi\|_3, \|\Psi\|_2)^2 + \int_0^T \left(\|\phi\|_3^2 + \|\psi\|_2^2\right) dt \leq C_0\big(\{\|\Phi_0\|_3, \|\Psi_0\|_2\}^2 + \epsilon\big),$$

where $C_0 > 0$ is independent of ϵ, ε and T.

3.1. **Proof of Proposition 3.1.** Let $(\Phi, \Psi)(x,t) \in \mathcal{B}(0,T)$ be a solution of the initial value problem (3.3)-(3.4) for some positive constants T. Then for $\forall\,(x,t) \in [0,T] \times \mathbb{R}$, one gets that

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \left|V(x,t) - v^S\left(x - st - \mathcal{X}_\infty\right)\right| \\
&\quad \leq \sup_{x\in\mathbb{R}} \left|\big(v^S(x - st - \mathcal{X}(t)) - v^S(x - st - \mathcal{X}_\infty)\big)\right| \\
&\qquad + \sup_{x\in\mathbb{R}} \left|(v_l - \overline{v}_l)\left(1 - g_{\mathcal{X}}\right) + (v_r - \overline{v}_r)\, g_{\mathcal{X}}\right| \\
&\quad \leq C\left|\mathcal{X}(t) - \mathcal{X}_\infty\right| + \sup_{x\in\mathbb{R}} |v_l - \overline{v}_l| + \sup_{x\in\mathbb{R}} |v_r - \overline{v}_r| \leq C\epsilon e^{-\alpha t}.
\end{aligned} \tag{3.7}$$

Moreover, if ε, ϵ are small, we have

$$\begin{aligned}
v(x,t) &= V(x,t) + \Phi_x(x,t) \leq \overline{v}_r + \epsilon + \|\Phi_x\|_{L^\infty} \\
&\leq \overline{v}_r + \|\Phi_x\|_1 + \epsilon \leq \overline{v}_r + \varepsilon + \epsilon \leq \frac{3}{2}\overline{v}_r, \\
v(x,t) &= V(x,t) + \Phi_x(x,t) \geq \overline{v}_l - \epsilon - \|\Phi_x\|_{L^\infty} \\
&\geq \overline{v}_l - \|\Phi_x(t)\|_1 - \epsilon \geq \overline{v}_l - \varepsilon - \epsilon \geq \frac{\overline{v}_l}{2},
\end{aligned} \tag{3.8}$$

where we have used (3.7), Lemma 2.1 and the Sobolev inequality.

Lemma 3.2. *Under the assumptions of Proposition 3.1, there exists a positive constant $C > 0$ which is independent of T such that for $0 \leq t \leq T$,*

$$\begin{aligned}
&\|(\Phi, \Psi, \Phi_x)(t)\|^2 + \int_0^t \left\|\left(\sqrt{(v^S_{\mathcal{Y}})'}\,\Psi, \Psi_x\right)(t)\right\|^2 dt \\
\leq & C\left(\|(\Phi_0, \Psi_0, \Phi_{0x})\|^2 + (\delta + \varepsilon)\int_0^t \|(\Phi_x, \Phi_{xx}, \Psi_{xx})(t)\|^2 dt + \epsilon\right)
\end{aligned} \tag{3.9}$$

holds provided that ϵ, ε and δ are suitably small.

Proof. Multiplying $(3.2)_1$ by Φ, $(3.2)_2$ by $-\frac{1}{p'(V)}\Psi$, then adding these equalities and integrating the resulting equation over $[0,T]\times\mathbb{R}$, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{\Phi^2}{2}+\frac{\Psi^2}{2(-p'(V))}\right)dx+\int_0^t\int_{-\infty}^{\infty}\frac{sp''(V)\Psi^2}{2(p'(V))^2}\left(v_{\mathcal{Y}}^S\right)'-\mu\frac{\Psi_x^2}{p'(V)V}dxdt\\
=&\int_{-\infty}^{\infty}\left(\frac{\Phi^2}{2}+\frac{\Psi^2}{2(-p'(V))}\right)dx\bigg|_{t=0}\\
&+\mu\int_0^t\int_{-\infty}^{\infty}\left[\left(\frac{1}{p'(V)V}\right)_x\Psi_x\Psi+\frac{U_x\Phi_x\Psi}{p'(V)V^2}\right]dxdt\\
&+\kappa\int_0^t\int_{-\infty}^{\infty}\frac{\Phi_{xxx}\Psi}{p'(V)v^5}dxdt-\int_0^t\int_{-\infty}^{\infty}\frac{F\Psi}{p'(V)}dxdt\\
&+\int_0^t\int_{-\infty}^{\infty}\frac{p''(V)}{2(p'(V))^2}\left(U-u_{\mathcal{Y}}^S-H_1\right)_x\Psi^2dxdt+\int_0^t\int_{-\infty}^{\infty}H_1\Phi-\frac{H_2\Psi}{p'(V)}dxdt\\
=&\int_{-\infty}^{\infty}\left(\frac{\Phi^2}{2}+\frac{\Psi^2}{2(-p'(V))}\right)dx\bigg|_{t=0}+\sum_{i=1}^{5}I_i.
\end{aligned}
\tag{3.10}
$$

Now, we estimate the right hand side of (3.10)

$$
\begin{aligned}
|I_1|\leq&C\int_0^t\int_{-\infty}^{\infty}(|V_x\Psi\Psi_x|+|U_x\Phi_x\Psi|)dxdt\\
\leq&C\int_0^t\int_{-\infty}^{\infty}(|(v_{\mathcal{Y}}^S)'\Psi\Psi_x|+|(u_{\mathcal{Y}}^S)'\Phi_x\Psi|)dxdt\\
&+C\int_0^t\int_{-\infty}^{\infty}(|(V-v_{\mathcal{Y}}^S)_x\Psi\Psi_x|+|(U-u_{\mathcal{Y}}^S)_x\Phi_x\Psi|)dxdt\\
:=&|I_{1,1}|+|I_{1,2}|.
\end{aligned}
\tag{3.11}
$$

It follows from Lemma 2.1, (3.8), and the Cauchy inequality that

$$
\begin{aligned}
|I_{1,1}|\leq&\frac{1}{8}\int_0^t\int_{-\infty}^{\infty}\frac{sp''(V)(v_{\mathcal{Y}}^S)'\Psi^2}{(p'(V))^2}dxdt\\
&+C\int_0^t\|(v_{\mathcal{Y}}^S)'\|_{L^\infty}\int_{-\infty}^{\infty}(\Psi_x^2+\Phi_x^2)dxdt\\
\leq&\frac{1}{8}\int_0^t\int_{-\infty}^{\infty}\frac{sp''(V)(v_{\mathcal{Y}}^S)'\Psi^2}{(p'(V))^2}dxdt+C\delta\int_0^t\|\Psi_x\|^2+\|\Phi_x\|^2dt,
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
I_2 &= \kappa \int_0^t \int_{-\infty}^{\infty} \left\{ \left(\frac{\Phi_{xx}\Psi}{p'(V)v^5} \right)_x - \left(\frac{\Psi}{p'(V)v^5} \right)_x \Phi_{xx} \right\} dxdt \\
&= -\kappa \int_0^t \int_{-\infty}^{\infty} \frac{\Psi_x \Phi_{xx}}{p'(V)v^5} - \frac{(p'(V)v^5)_x}{(p'(V)v^5)^2} \Psi \Phi_{xx} dxdt, \\
&= -\kappa \int_0^t \int_{-\infty}^{\infty} \frac{\Phi_t \Phi_{xx}}{p'(V)v^5} - \frac{H_1 \Phi_{xx}}{p'(V)v^5} - \frac{(p'(V)v^5)_x}{(p'(V)v^5)^2} \Psi \Phi_{xx} dxdt.
\end{aligned}
$$

With the aid of $(3.2)_1$, we have

$$
\begin{aligned}
& - \kappa \int_0^t \int_{-\infty}^{\infty} \frac{\Phi_t \Phi_{xx}}{p'(V)v^5} dxdt \\
=& \kappa \int_{-\infty}^{\infty} \frac{\Phi_x^2}{2p'(V)v^5} dx - \kappa \int_{-\infty}^{\infty} \frac{\Phi_x^2}{2p'(V)v^5} dx \Big|_{t=0} \\
& + \frac{\kappa}{2} \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)(U - H_1 - u^S_{\mathcal{Y}})_x}{(p'(V))^2 v^5} \Phi_x^2 dxdt + \frac{5\kappa}{2} \int_0^t \int_{-\infty}^{\infty} \frac{(U - u^S_{\mathcal{Y}})_x}{p'(V)v^6} \Phi_x^2 dxdt \\
& + \frac{\kappa}{2} \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)(u^S_{\mathcal{Y}})_x}{(p'(V))^2 v^5} \Phi_x^2 dxdt + \frac{5\kappa}{2} \int_0^t \int_{-\infty}^{\infty} \frac{(u^S_{\mathcal{Y}})_x + \Psi_{xx}}{p'(V)v^6} \Phi_x^2 dxdt \\
& - \kappa \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)(V - v^S_{\mathcal{Y}})_x v^5 + 5p'(V)v^4 (V - v^S_{\mathcal{Y}})_x}{(p'(V)v^5)^2} \Psi_x \Phi_x dxdt \\
& - \kappa \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)(v^S_{\mathcal{Y}})_x v^5 + 5p'(V)v^4 (v^S_{\mathcal{Y}} + \Phi_x)_x}{(p'(V)v^5)^2} \Psi_x \Phi_x dxdt \\
& - \kappa \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)V_x v^5 + 5p'(V)v^4 (V + \Phi_x)_x}{(p'(V)v^5)^2} H_1 \Phi_x dxdt \\
:=& \kappa \int_{-\infty}^{\infty} \frac{\Phi_x^2}{2p'(V)v^5} dx - \kappa \int_{-\infty}^{\infty} \frac{\Phi_x^2}{2p'(V)v^5} dx \Big|_{t=0} + \sum_{i=1}^{7} I_{2,i},
\end{aligned}
$$

and

$$
\begin{aligned}
& \kappa \int_0^t \int_{-\infty}^{\infty} \frac{H_1 \Phi_{xx}}{p'(V)v^5} + \frac{(p'(V)v^5)_x}{(p'(V)v^5)^2} \Psi \Phi_{xx} dxdt, \\
=& \kappa \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)[V_x - (v^S_{\mathcal{Y}})_x] v^5 + 5p'(V)v^4 (V - v^S_{\mathcal{Y}})_x}{(p'(V)v^5)^2} \Psi \Phi_{xx} dxdt \\
& + \kappa \int_0^t \int_{-\infty}^{\infty} \frac{p''(V)(v^S_{\mathcal{Y}})_x v^5 + 5p'(V)v^4 (v^S_{\mathcal{Y}} + \Phi_x)_x}{(p'(V)v^5)^2} \Psi \Phi_{xx} dxdt \\
& + \kappa \int_0^t \int_{-\infty}^{\infty} \frac{H_1 \Phi_{xx}}{p'(V)v^5} dxdt = \sum_{i=8}^{10} I_{2,i}.
\end{aligned}
$$

Using Lemma 2.1, the Sobolev inequality, the Cauchy inequality, one gets that

$$
\begin{aligned}
I_{2,3}+I_{2,4} &\le C\int_0^t \left(\|(v^S_{\mathcal{Y}})_x,(u^S_{\mathcal{Y}})_x\|_{L^\infty}\|\Phi_x^2\|_{L^1}+\|\Phi_x(t)\|_{L^\infty}\|\Phi_x\Psi_{xx}\|_{L^1}\right)dt\\
&\le C\delta\int_0^t\|\Phi_x\|^2dt+C\varepsilon\int_0^t\left(\|\Phi_x\|^2+\|\Psi_{xx}\|^2\right)dt\\
&\le C(\delta+\varepsilon)\int_0^t\left(\|\Phi_x\|^2+\|\Psi_{xx}\|^2\right)dt,
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
I_{2,9} &\le \int_0^t\int_{-\infty}^{\infty}\eta(v^S_{\mathcal{Y}})_x\Psi^2+C_\eta(v^S_{\mathcal{Y}})_x\Phi_{xx}^2dxdt+C\int_0^t\|\Psi(t)\|_{L^\infty}\|\Phi_{xx}^2\|_{L^1}dt\\
&\le \eta\int_0^t\int_{-\infty}^{\infty}(v^S_{\mathcal{Y}})_x\Psi^2dxdt+C_\eta(\delta+\varepsilon)\int_0^t\int_{-\infty}^{\infty}\Phi_{xx}^2dxdt,
\end{aligned}
\tag{3.14}
$$

where we have used (3.6) and (3.8). Here and hereafter, η is a small positive constant and C_η is a positive constant which depends only on η. Similarly, one gets that

$$
I_{2,6}\le C(\delta+\varepsilon)\int_0^t\int_{-\infty}^{\infty}\left(\Psi_x^2+\Phi_x^2+\Phi_{xx}^2\right)dxdt. \tag{3.15}
$$

We rewrite the error term F as:

$$
\begin{aligned}
F=&-\left(p(v)-p(V)-p'(V)\Phi_x\right)-\frac{\mu\Psi_{xx}\Phi_x}{vV}+\kappa(v^S_{\mathcal{Y}})_{xx}\left(\frac{1}{V^5}-\frac{1}{v^5}\right)\\
&+\frac{5\kappa}{2}\frac{\Phi_{xx}^2+2\Phi_{xx}(v^S_{\mathcal{Y}})_x}{v^6}+\frac{5\kappa}{2}((v^S_{\mathcal{Y}})_x)^2\left(\frac{1}{V^6}-\frac{1}{v^6}\right)+\frac{\mu(u^S_{\mathcal{Y}})_x\Phi_x^2}{vV^2}.\\
&+\kappa(V-v^S_{\mathcal{Y}})_{xx}\left(\frac{1}{V^5}-\frac{1}{v^5}\right)+\frac{5\kappa}{v^6}\Phi_{xx}(V-v^S_{\mathcal{Y}})_x\\
&+\frac{5\kappa}{2}[V_x^2-((v^S_{\mathcal{Y}})_x)^2]\left(\frac{1}{V^6}-\frac{1}{v^6}\right)+\frac{\mu(U-u^S_{\mathcal{Y}})_x\Phi_x^2}{vV^2}=\sum_{i=1}^{10}F_i.
\end{aligned}
$$

Thus I_3 can be rewrite as

$$
I_3=-\int_0^t\int_{-\infty}^{\infty}\frac{F\Psi}{p'(V)}dxdt=\left(\sum_{i=1}^{6}+\sum_{i=7}^{10}\right)\int_0^t\int_{-\infty}^{\infty}\frac{-F_i\Psi}{p'(V)}dxdt:=I_{3,1}+I_{3,2}.
$$

Using the similar method in [7], we have

$$
\begin{aligned}
\sum_{i=1}^{6}F_i&=O(1)(|\Phi_x^2|+|\Phi_{xx}^2|+|\Psi_{xx}\Phi_x|)+\left(\left|\frac{(v^S_{\mathcal{Y}})_{xx}}{(v^S_{\mathcal{Y}})_x}\right|+|(v^S_{\mathcal{Y}})_x|\right)\left(|(v^S_{\mathcal{Y}})_x\Phi_x|+|(v^S_{\mathcal{Y}})_x\Phi_{xx}|\right)\\
&=O(1)\left(|\Phi_x^2|+|\Phi_{xx}^2|+|\Psi_{xx}\Phi_x|+|(v^S_{\mathcal{Y}})_x\Phi_x|+|(v^S_{\mathcal{Y}})_x\Phi_{xx}|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{3,1}| \leq & C\int_0^t\int_{-\infty}^{\infty}(|\Phi_x^2|+|\Psi_{xx}\Phi_x|+|\Phi_{xx}^2|+|(v_{\mathcal{Y}}^S)_x\Phi_x|+|(v_{\mathcal{Y}}^S)_x\Phi_{xx}|)|\Psi|dxdt \\
\leq & \frac{1}{8}\int_0^t\int_{-\infty}^{\infty}\frac{sp''(V)(v_{\mathcal{Y}}^S)_x\Psi^2}{(p'(V))^2}dxdt \\
& + C(\delta+\varepsilon)\int_0^t\int_{-\infty}^{\infty}(\Phi_x^2+\Phi_{xx}^2+\Psi_{xx}^2)dxdt.
\end{aligned}
\tag{3.16}
$$

Similar like (3.7), we have

$$
\begin{aligned}
& \|U_x-\left(u_{\mathcal{Y}}^S\right)'\|_{L^\infty(\mathbb{R})} \\
\leq & C\left(\|u_{lx}\|_{L^\infty(\mathbb{R})}+\|u_{rx}\|_{L^\infty(\mathbb{R})}+\|u_l-\overline{u}_l\|_{L^\infty(\mathbb{R})}+\|u_r-\overline{u}_r\|_{L^\infty(\mathbb{R})}\right) \\
\leq & C\epsilon e^{-\alpha t},
\end{aligned}
\tag{3.17}
$$

and

$$
\|U_{xx}-\left(u_{\mathcal{Y}}^S\right)''\|_{L^\infty(\mathbb{R})},\|V_x-\left(v_{\mathcal{Y}}^S\right)'\|_{L^\infty(\mathbb{R})},\|V_{xx}-\left(v_{\mathcal{Y}}^S\right)''\|_{L^\infty(\mathbb{R})}\leq C\epsilon e^{-\alpha t}. \tag{3.18}
$$

With the aid of Lemma 2.3, (3.17), (3.18), Hölder inequality, Sobolev inequality, we have

$$
\begin{aligned}
& I_{1,2}+I_{2,1}+I_{2,2}+I_{2,5}+I_{2,7}+I_{2,8}+I_{2,10}+I_{3,2}+I_4+I_5 \\
\leq & C\epsilon\int_0^t e^{-\alpha t}\int_{-\infty}^{\infty}\Phi_x^2+\Phi_{xx}^2+\Psi^2+\Psi_x^2dxdt \\
& + C\int_0^t(\|\Phi\|_2+\|\Psi\|)(\|H_1\|_1+\|H_2\|)dt \\
\leq & C\epsilon\varepsilon_0^2+C\epsilon\varepsilon\leq C\epsilon.
\end{aligned}
\tag{3.19}
$$

Inserting (3.11)-(3.16), (3.19) into (3.10), and using the smallness of η,ε and δ, we can get (3.9). Thus we obtain the proof of Lemma 3.2.

Lemma 3.3. *Under the assumptions of Proposition 3.1, there exists a positive constant $C>0$ which is independent of T such that for $0\leq t\leq T$, it holds that*

$$
\|\Phi_x(t)\|^2+\int_0^t\|\Phi_x(t)\|_1^2dt\leq C\left(\|(\Phi_{0x},\Psi_0,\Phi_0)\|^2+(\delta+\varepsilon)\int_0^t\|\Psi_{xx}(t)\|^2dt+\epsilon\right)
$$

provided that ϵ,ε and δ are suitably small.

Proof. Multiplying $(3.2)_2$ by $-\Phi_x$, and integrating the resulting equation with respect to x over $[0,T]\times\mathbb{R}$, using (2.6) yields

$$\begin{aligned}
&\frac{\mu}{2}\int_{-\infty}^{\infty}\frac{\Phi_x^2}{V}dx+\int_0^t\int_{-\infty}^{\infty}-p'(V)\Phi_x^2dxdt+\kappa\int_0^t\int_{-\infty}^{\infty}\frac{\Phi_{xx}^2}{v^5}dxdt\\
=&\frac{\mu}{2}\int_{-\infty}^{\infty}\frac{\Phi_x^2}{V}dx\Big|_{t=0}+\int_0^t\int_{-\infty}^{\infty}\Psi_t\Phi_xdxdt+\mu\int_0^t\int_{-\infty}^{\infty}\frac{H_{1x}}{2V^2}\Phi_x^2dxdt\\
&+\mu\int_0^t\int_{-\infty}^{\infty}\frac{H_{1x}}{V}\Phi_xdxdt+\mu\int_0^t\int_{-\infty}^{\infty}\frac{U_x\Phi_x^2}{2V^2}dxdt-\int_0^t\int_{-\infty}^{\infty}F\Phi_xdxdt\\
&-\int_0^t\int_{-\infty}^{\infty}H_2\Phi_xdxdt+5\kappa\int_0^t\int_{-\infty}^{\infty}\frac{\Phi_{xx}(\Phi_x+V_x)\Phi_x}{v^6}dxdt\\
=&\frac{\mu}{2}\int_{-\infty}^{\infty}\frac{\Phi_x^2}{V}dx\Big|_{t=0}+\sum_{i=1}^{7}A_i.
\end{aligned}\tag{3.20}$$

With the aid of Lemma 2.1, (3.8), we obtain

$$\begin{aligned}
A_1&=\int_0^t\int_{-\infty}^{\infty}[(\Psi\Phi_x)_t-\Psi\Phi_{xt}]dxdt\\
&=\int_{-\infty}^{\infty}\Psi\Phi_xdx-\int_{-\infty}^{\infty}\Psi\Phi_xdx|_{t=0}-\int_0^t\int_{-\infty}^{\infty}[(\Psi\Psi_x)_x-\Psi_x^2-H_1\Psi_x]dxdt\\
&\le\int_{-\infty}^{\infty}\Psi\Phi_xdx-\int_{-\infty}^{\infty}\Psi\Phi_xdx|_{t=0}+\int_0^t\frac{3}{2}\|\Psi_x\|^2+\frac{1}{2}\|H_1\|^2dt,\\
&<\frac{\mu}{4}\int_{-\infty}^{\infty}\frac{\Phi_x^2}{V}dx+C\|\Psi\|^2+\|\Psi_0\|^2+\|\Phi_{0x}\|^2+\frac{3}{2}\int_0^t\|\Psi_x\|^2+C\epsilon.
\end{aligned}\tag{3.21}$$

Inequality (3.17) gives that

$$A_4\le C\int_0^t\|U_x\|_{L^\infty(\mathbb{R})}\int_{-\infty}^{\infty}\Phi_x^2dxdt\le C(\epsilon+\delta)\int_0^t\|\Phi_x\|^2dt.\tag{3.22}$$

Similarly, one gets that

$$\begin{aligned}
A_5&\le C\int_0^t\int_{-\infty}^{\infty}(|\Phi_x^2|+|\Psi_{xx}\Phi_x|+|\Phi_{xx}^2|+|V_{xx}\Phi_x|+|V_x\Phi_{xx}|)|\Phi_x|dxdx\\
&\le\varepsilon\int_0^t\int_{-\infty}^{\infty}(\Phi_x^2+\Psi_{xx}^2+\Phi_{xx}^2)dxdt+C(\epsilon+\delta)\int_0^t\int_{-\infty}^{\infty}(\Phi_x^2+\Phi_{xx}^2)dxdt,
\end{aligned}\tag{3.23}$$

and

$$A_7\le C(\delta+\epsilon+\varepsilon)\int_0^t\int_{-\infty}^{\infty}(\Phi_{xx}^2+\Phi_x^2)dxdt,\tag{3.24}$$

where we have used the Cauchy inequality. By Lemma 2.3, (3.17), (3.18), and Cauchy inequality, we have

$$
\begin{aligned}
A_2 + A_3 + A_6 \leq & \frac{-p'(V)}{2} \int_0^t \|\Phi_x\|^2 dt + C \int_0^t \|H_1\|_1^2 + \|H_2\|^2 dt \\
\leq & \frac{-p'(V)}{2} \int_0^t \|\Phi_x\|^2 dt + C\epsilon.
\end{aligned} \tag{3.25}
$$

Inserting (3.21)-(3.25) into (3.20), we obtain

$$
\begin{aligned}
& \frac{\mu}{4} \int_{-\infty}^{\infty} \frac{\Phi_x^2}{V} dx + \int_0^t \int_{-\infty}^{\infty} \left(\frac{-p'(V)}{2} \Phi_x^2 + \kappa \frac{\Phi_{xx}^2}{v^5} \right) dxdt \\
\leq & C(\|(\Phi_{0x}, \Psi_0)\|^2 + \|\Psi(t)\|^2) + \frac{3}{2} \int_0^t \|\Psi_x(t)\|^2 dt + C\epsilon.
\end{aligned} \tag{3.26}
$$

It follows (3.26) and Lemma 3.2, we obtain the proof of Lemma 3.3.

Lemma 3.4. *Under the assumptions of Proposition 3.1, there exists a positive constant $C > 0$ which is independent of T such that for $0 \leq t \leq T$, it holds that*

$$
\begin{aligned}
& \|(\Psi_x, \Phi_{xx})(t)\|^2 + \int_0^t \|\Psi_{xx}(t)\|^2 dt \\
& \leq C \left(\|\Phi_0\|_2^2 + \|\Psi_0\|_1^2 + (\delta + \epsilon + \varepsilon) \int_0^t \|\Phi_{xxx}(t)\|^2 dt + \epsilon \right)
\end{aligned} \tag{3.27}
$$

provided that ϵ, ε and δ are suitably small.

Proof. Multiplying $(3.2)_2$ by $-\Psi_{xx}$, and integrating the resulting equation in x over $[0, t] \times \mathbb{R}$, we obtain

$$
\begin{aligned}
& \frac{1}{2} \int_{-\infty}^{\infty} \Psi_x^2 dxdt + \int_0^t \int_{-\infty}^{\infty} \frac{\mu}{V} \Psi_{xx}^2 dxdt \\
= & \frac{1}{2} \int_{-\infty}^{\infty} \Psi_{0x}^2 dx + \int_0^t \int_{-\infty}^{\infty} p'(V) \Phi_x \Psi_{xx} dxdt + \kappa \int_0^t \int_{-\infty}^{\infty} \frac{\Phi_{xxx} \Phi_{tx}}{v^5} dxdt \\
& + \mu \int_0^t \int_{-\infty}^{\infty} \frac{U_x \Phi_x \Psi_{xx}}{V^2} dxdt - \int_0^t \int_{-\infty}^{\infty} F \Psi_{xx} dxdt - \int_0^t \int_{-\infty}^{\infty} H_2 \Psi_{xx} dxdt \\
& - \kappa \int_0^t \int_{-\infty}^{\infty} \frac{\Phi_{xxx} H_{1x}}{v^5} dxdt := \frac{1}{2} \int_{-\infty}^{\infty} \Psi_{0x}^2 dx + \sum_{i=1}^{6} B_i.
\end{aligned} \tag{3.28}
$$

With the help of $(3.2)_1$, we have

$$\begin{aligned}
B_2 &= \kappa \int_0^t \int_{-\infty}^{\infty} [(\Phi_{xx} \frac{\Phi_{tx}}{v^5})_x - \Phi_{xx}(\frac{\Phi_{tx}}{v^5})_x] dxdt \\
&= -\frac{\kappa}{2} \int_{-\infty}^{\infty} \frac{\Phi_{xx}^2}{v^5} dx + \frac{\kappa}{2} \int_{-\infty}^{\infty} \frac{\Phi_{xx}^2}{v^5}|_{t=0} dx - \frac{5\kappa}{2} \int_0^t \int_{-\infty}^{\infty} \Phi_{xx}^2 \frac{\Psi_{xx} + U_x}{v^6} dxdt \\
&\quad + 5\kappa \int_0^t \int_{-\infty}^{\infty} \Phi_{xx} \Psi_{xx} \frac{\Phi_{xx} + V_x}{v^6} dxdt + 5\kappa \int_0^t \int_{-\infty}^{\infty} \Phi_{xx} H_{1x} \frac{\Phi_{xx} + V_x}{v^6} dxdt \\
&= -\frac{\kappa}{2} \int_{-\infty}^{\infty} \frac{\Phi_{xx}^2}{v^5} dx + \frac{\kappa}{2} \int_{-\infty}^{\infty} \frac{\Phi_{xx}^2}{v^5}|_{t=0} dx + \sum_{i=1}^{3} B_{2,i}.
\end{aligned}$$

Using (2.3), (3.17) and the Cauchy inequality, one gets

$$\begin{aligned}
B_1 \leq & \eta \int_0^t \|\Psi_{xx}\|^2 dt + C_\eta \int_0^t \int_{-\infty}^{\infty} (p'(V))^2 \Phi_x^2 dxdt \\
\leq & \eta \int_0^t \|\Psi_{xx}\|^2 dt + C_\eta \int_0^t \|\Phi_x\|^2 dt,
\end{aligned} \tag{3.29}$$

and

$$B_3 \leq \eta \int_0^t \|\Psi_{xx}\|^2 dt + C_\eta (\delta + \epsilon)^2 \int_0^t \|\Phi_x\|^2 dt.$$

Using Lemma 2.1, the Sobolev inequality, the Young inequality, the Cauchy inequality, (3.6) and (3.8), we have

$$\begin{aligned}
B_{21} &\leq C \int_0^t \|\Phi_{xx}(t)\|_{L^\infty} \|\Phi_{xx}\Psi_{xx}\|_{L^1} dt + C \int_0^t \|U_x\|_{L^\infty} \|\Phi_{xx}\|^2 dt \\
&\leq C \int_0^t (\delta + \epsilon) \|\Phi_{xx}(t)\|^2 dt + C \int_0^t \|\Phi_{xx}(t)\|^{\frac{3}{2}} \|\Phi_{xxx}(t)\|^{\frac{1}{2}} \|\Psi_{xx}(t)\| dt, \\
&\leq C(\delta + \epsilon + \sup_t \|\Psi_{xx}\|) \int_0^t \|(\Phi_{xx}, \Phi_{xxx})(t)\|^2 dt \\
&\leq C(\delta + \epsilon + \varepsilon) \int_0^t \|(\Phi_{xx}, \Phi_{xxx})(t)\|^2 dt,
\end{aligned}$$

and

$$B_{22} \leq C(\delta + \epsilon + \varepsilon) \int_0^t \|(\Phi_{xx}, \Psi_{xx})(t)\|^2 dt.$$

Using (2.3), (3.17), (3.18), we estimate the error term F as:

$$F \leq C \left(|\Phi_x| + |\Phi_{xx}| + |V_x| + |V_{xx}| \right) \left(|\Phi_x| + |\Psi_{xx}| + |\Phi_{xx}| \right). \tag{3.30}$$

Using (3.30), B_4 can be controlled as follows:

$$\begin{aligned}
B_4 &\leq \eta \int_0^t \|\Psi_{xx}\|^2 dt + C_\eta \int_0^t \int_{-\infty}^{\infty} |F|^2 dx dt \\
&\leq \eta \int_0^t \|\Psi_{xx}\|^2 dt + C_\eta \int_0^t \left(\|\Phi_x, \Phi_{xx}, V_x, V_{xx}\|_{L^\infty}^2 \right) \int_{-\infty}^{\infty} \left(\Phi_x^2 + \Psi_{xx}^2 + \Phi_{xx}^2 \right) dx dt \\
&\leq \eta \int_0^t \|\Psi_{xx}(t)\|^2 dt + C_\eta (\delta + \epsilon + \varepsilon)^2 \|(\Phi_x, \Psi_{xx}, \Phi_{xx})(t)\|^2 dt,
\end{aligned}$$

where we have used (3.18) in last inequality. With the aid of (3.6), (3.18) and Hölder inequality, one gets

$$\begin{aligned}
B_{23} + B_5 + B_6 \leq & C \int_0^t \int_{-\infty}^{\infty} |H_2 \Psi_{xx}| + |H_{1x}|(|\Phi_{xx}| + |\Phi_{xxx}|) dx dt \\
\leq & C \int_0^t (\|H_2\| + \|H_{1x}\|)(\|\Psi_{xx}\| + \|\Phi_{xx}\|_1) dt \leq C\epsilon.
\end{aligned} \tag{3.31}$$

Inserting (3.29)-(3.31) into (3.28), choosing η, ϵ, δ and ε suitable small, one gets

$$\begin{aligned}
& \|\Psi_x(t)\|^2 + \|\Phi_{xx}(t)\|^2 + \int_0^t \|\Psi_{xx}(t)\|^2 dt \\
\leq & C \left(\|(\Psi_{0x}, \Phi_{0xx})\|^2 + \int_0^t \|\Phi_x(t)\|_1^2 dt \right) \\
& + C(\delta + \varepsilon + \epsilon) \int_0^t \|\Phi_{xxx}(t)\|^2 dt + C\epsilon.
\end{aligned} \tag{3.32}$$

Taking ε, ϵ and δ small enough in (3.32), we can obtain (3.27). Thus the proof of Lemma 3.4 is accomplished.

Lemma 3.5. *Under the assumptions of Proposition 3.1, there exists a positive constant $C > 0$ which is independent of T such that for $0 \leq t \leq T$,*

$$\|\Phi_{xx}(t)\|^2 + \int_0^t \|\Phi_{xx}(t)\|_1^2 dt \leq C \left(\|\Phi_0\|_2^2 + \|\Psi_0\|_1^2 + \epsilon \right) \tag{3.33}$$

holds provided that ϵ, ε and δ are suitably small.

Proof. Multiplying $(3.2)_2$ by Φ_{xxx} and using $(3.2)_1$, integrating the result over $[0,t]\times\mathbb{R}$, one gets that

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mu\Phi_{xx}^2}{V}dx+\int_0^t\int_{-\infty}^{\infty}\left(-p'(V)\Phi_{xx}^2+\frac{\kappa\Phi_{xxx}^2}{v^5}\right)dxdt\\
=&\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mu\Phi_{xx}^2}{V}dx\Big|_{t=0}-\int_{-\infty}^{\infty}\Psi_x\Phi_{xx}dx\Big|_{t=0}+\int_0^t\int_{-\infty}^{\infty}\frac{\mu}{V}H_{1xx}\Phi_{xx}dxdt\\
&-\int_0^t\int_{-\infty}^{\infty}\left(\frac{\mu}{V}\right)_x\Phi_{xx}\Psi_{xx}dxdt+\int_0^t\int_{-\infty}^{\infty}\left(\frac{\mu}{V}\right)_t\frac{\Phi_{xx}^2}{2}dxdt\\
&+\int_0^t\int_{-\infty}^{\infty}p''(V)V_x\Phi_x\Phi_{xx}dxdt+\int_0^t\int_{-\infty}^{\infty}\Psi_{xx}^2dxdt\\
&-\mu\int_0^t\int_{-\infty}^{\infty}\frac{U_x\Phi_x}{V^2}\Phi_{xxx}dxdt+\int_0^t\int_{-\infty}^{\infty}F\Phi_{xxx}dxdt+\int_{-\infty}^{\infty}\Psi_x\Phi_{xx}dx\\
&+\int_0^t\int_{-\infty}^{\infty}\Phi_{xxx}H_2dxdt-\int_0^t\int_{-\infty}^{\infty}H_{1x}\Psi_xdxdt\\
:=&\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mu\Phi_{xx}^2}{V}dx\Big|_{t=0}-\int_{-\infty}^{\infty}\Psi_x\Phi_{xx}dx\Big|_{t=0}+\sum_{i=1}^{10}M_i.
\end{aligned}
\tag{3.34}
$$

Similar like (3.31), it follows that

$$
\begin{aligned}
&M_1+M_9+M_{10}\\
\le&C\int_0^t\int_{-\infty}^{\infty}(|H_{1x}+H_{1xx}+H_2|)(|\Phi_{xx}|+|\Phi_{xxx}|+|\Psi_x|)dxdt\\
\le&C\int_0^t(\|H_1\|_2+\|H_2\|)(\|\Phi_x\|_2+\|\Psi_x\|)dt\le C\epsilon\varepsilon.
\end{aligned}
\tag{3.35}
$$

Using the Cauchy inequality, we obtain

$$
\begin{aligned}
M_2+M_3+M_4\le&\eta\int_{\mathbb{R}}\frac{\mu\Phi_{xx}^2}{V}dx+C_\eta\int_0^t\|V_x,U_x,H_1x\|_{L^\infty}\int_{\mathbb{R}}|\Psi_{xx}^2+\Phi_{xx}^2+\Phi_x^2|dxdt,\\
\le&\eta\int_{\mathbb{R}}\frac{\mu\Phi_{xx}^2}{V}dx+C_\eta(\delta+\epsilon)\int_0^t\|(\Phi_{xx},\Psi_{xx},\Phi_x)(t)\|^2dt.
\end{aligned}
$$

By Lemma 2.1, the Sobolev inequality, the Cauchy inequality, (3.6) and (3.30), we have

$$
\begin{aligned}
M_6+M_7\le&C\int_0^t\int_{\mathbb{R}}|U_x||\Phi_x\Phi_{xxx}|dxdt\\
&+C\int_0^t\int_{\mathbb{R}}(|\Phi_x|+|\Psi_{xx}|+|V_x|+|V_{xx}|)\left(|\Psi_{xx}|+|\Phi_x|+|\Phi_{xx}|\right)|\Phi_{xxx}|dxdt\\
\le&C(\delta+\epsilon+\varepsilon)\int_0^t\|(\Phi_x,\Psi_{xx},\Phi_{xx},\Phi_{xxx})(t)\|^2dt,
\end{aligned}
$$

and

$$M_5 + M_8 \leq \eta \int_{\mathbb{R}} \frac{\mu \Phi_{xx}^2}{V} dx + C_\eta \|\Psi_x(t)\|^2 + \int_0^t \|\Psi_{xx}(t)\|^2 dt. \quad (3.36)$$

Combining (3.34)-(3.36), and the smallness of $\delta, \varepsilon, \epsilon$, one gets (3.33). Thus the proof of Lemma 3.5 is obtained. We can get the higher order estimates similarly and the proof is omitted for brevity.

Lemma 3.6. *Under the assumptions of Proposition 3.1, there exists a positive constant $C > 0$ which is independent of T such that for $0 \leq t \leq T$,*

$$\|(\Phi_{xxx}, \Psi_{xx})(t)\|^2 + \int_0^t \|\Psi_{xxx}(t)\|^2 + \|\Phi_{xxx}(t)\|_1^2 dt \leq C \left(\|\Phi_0\|_3^2 + \|\Psi_0\|_2^2 + \epsilon\right) \quad (3.37)$$

holds provided that ε, ϵ and δ are suitably small.

Combining Lemma 3.2-Lemma 3.6, we obtain Proposition 3.1.

3.2. **Proof of Theorem 3.1 and Theorem 2.1.** By standard argument, one gets that the initial value problem (3.2),(3.4) has a solution on $[0, T_0]$. Choosing $\|\Phi_0\|_2, \|\Psi_0\|_1$ and ϵ sufficiently small, such that $\|\Phi_0\|_2^2 + \|\Psi_0\|_1^2 + \epsilon < \frac{\varepsilon_0^2}{4C_0}$. Using Proposition 3.1, we have

$$\sup_{t\in[0,T]} \{\|\Phi(t)\|_3 + \|\Psi(t)\|_2\} \leq 2\sqrt{C_0(\|\Phi_0\|_3^2 + \|\Psi_0\|_2^2 + \epsilon)} = \varepsilon_0^2.$$

Moreover, by standard continuation argument, it follows that

$$\int_0^{+\infty} \left(\|\phi(t)\|_2^2 + \|\psi(t)\|_1^2 + \left| \frac{d}{dt} \left(\|\phi(t)\|_2^2 + \|\psi(t)\|_1^2\right) \right| \right) dt < \infty.$$

It follows that $\|\phi(t)\|_2 + \|\psi(t)\|_1 \to 0$ as $t \to +\infty$. Using Sobolev inequality, one gets that

$$\|\phi(t)\|_{L^\infty} \leq \|\phi(t)\|^{\frac{1}{2}} \|\phi_x(t)\|^{\frac{1}{2}} \to 0, \text{ as } t \to +\infty. \quad (3.38)$$

and

$$\|\psi(t)\|_{L^\infty} \leq \|\psi(t)\|^{\frac{1}{2}} \|\psi_x(t)\|^{\frac{1}{2}} \to 0, \text{ as } t \to +\infty.$$

Thus Theorem 3.1 is proved. With the help of (3.7) and (3.38), we can obtain (2.13) for v. And the estimate for u in (2.13) can be proved similarly. Therefore Theorem 2.1 is proved.

4. Proof of Lemmas 2.3 and Lemma 3.1

4.1. **Proof of Lemma 2.3.** By (1.5) and Lemma 2.2 , we have $|\mathcal{X}'(t)|, |\mathcal{Y}'(t)| \leq C\epsilon e^{-\alpha t}$ for all $t > 0$. Thus $\lim_{t\to+\infty} \mathcal{X}(t)$ and $\lim_{t\to+\infty} \mathcal{Y}(t)$ are all exist. In the following part of this subsection, we compute the two limits.

Motivated by [15], we first compute $\lim_{t\to+\infty} \mathcal{Y}(t)$. We define the domain

$$\begin{cases}\Omega^N_{(x,t)} := \big\{(y,\tau) : 0<\tau<t, \quad \Gamma^N_l(\tau)<y<\Gamma^N_r(t)\big\}, x\in[0,1],\\ \Gamma^N_l(\tau) := s\tau+\mathcal{Y}(t)+(-N+x)\pi_l, N\in N^*,\\ \Gamma^N_r(\tau) := s\tau+\mathcal{Y}(t)+(N+x)\pi_r, N\in N^*, .\end{cases} \tag{4.1}$$

Using $(2.6)_2$, we have

$$\begin{aligned}&\int_{\Gamma^N_l(0)}^{\Gamma^N_r(0)} U(y,0)dy+\int_0^t\left[-p(V)+\mu\frac{U_x}{V}+\kappa\left(\frac{-V_{xx}}{V^5}+\frac{5V_x^2}{2V^6}\right)+(s+\mathcal{Y}')\,U\right]\big(\Gamma^N_r(\tau),\tau\big)\,d\tau\\ &-\int_{\Gamma^N_l(t)}^{\Gamma^N_r(t)} U(y,t)dy-\int_0^t\left[-p(V)+\mu\frac{U_x}{V}+\kappa\left(\frac{-V_{xx}}{V^5}+\frac{5V_x^2}{2V^6}\right)+(s+\mathcal{Y}')\,U\right]\big(\Gamma^N_l(\tau),\tau\big)\,d\tau\\ =&-\iint_{\Omega^N_{(x,t)}}\left((G_3)_y+g_4\right)dyd\tau\end{aligned} \tag{4.2}$$

With the aid of U in (2.5), one has that

$$\begin{aligned}&\int_{\Gamma^N_l(0)}^{\Gamma^N_r(0)} U(y,0)dy-\int_{\Gamma^N_l(t)}^{\Gamma^N_r(t)} U(y,t)dy\\ =&\int_0^{\Gamma^N_r(0)}\left(\psi_{0l}-\psi_{0r}\right)(1-g_{\mathcal{Y}_0})dy+\int_0^{\mathcal{Y}_0+x\pi_r}\psi_{0r}(y)dy\\ &+\int_{\Gamma^N_l(0)}^0(\psi_{0r}-\psi_{0l})g_{\mathcal{Y}_0}dy+\int_{\mathcal{Y}_0+x\pi_l}^0\psi_{0l}(y)dy-R_1,\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}R_1(x,t)=&\int_0^{\Gamma^N_l(t)}\left(\psi_l-\psi_r\right)(1-g_{\mathcal{Y}})dy+\int_0^{st+\mathcal{Y}(t)+x\pi_r}\psi_r(y,t)dy\\ &+\int_{\Gamma^N_l(t)}^0(\psi_r-\psi_l)g_{\mathcal{Y}}dy+\int_{st+\mathcal{Y}(t)+x\pi_l}^0\psi_l(y,t)dy.\end{aligned}$$

Moreover, by [15], we have $\|R_1\|_{L^\infty(\mathbb{R})}\le C\epsilon e^{-\alpha t}$. The second part on the left hand side of (4.2) satisfies

$$\begin{aligned}&\int_0^t\left[-p(V)+\mu\frac{U_x}{V}+\kappa\left(\frac{-V_{xx}}{V^5}+\frac{5V_x^2}{2V^6}\right)+(s+\mathcal{Y}')\,U\right]\big(\Gamma^N_r(\tau),\tau\big)\,d\tau\\ =&\int_0^t\left[-p(v_r)+u_r\,(s+\mathcal{Y}')+\kappa\left(\frac{-v_{rxx}}{v_r^5}+\frac{5v_{rx}^2}{2v_{rx}^6}\right)\right](s\tau+\mathcal{Y}(\tau)+x\pi_r,\tau)\,d\tau\\ &+\mu\{\ln\big(v_r(st+\mathcal{Y}(t)+x\pi_r,t)\big)-\ln\big(\overline{v}_r+\phi_{0r}(\mathcal{Y}_0+x\pi_r)\big)\}+R_2,\end{aligned} \tag{4.4}$$

where

$$R_2 = \int_0^t \Big\{ -[p(V)-p(v_r)] + \mu\left[\frac{U_x}{V} - \frac{u_{rx}}{v_r}\right] - (s+\mathcal{Y}')(U-u_r) \\ + \kappa\left[\left(\frac{-V_{xx}}{V^5} + \frac{5V_x^2}{2V^6}\right) - \left(\frac{-v_{rxx}}{v_r^5} + \frac{5v_{rx}^2}{2v_{rx}^6}\right)\right]\Big\} \left(\Gamma_r^N(\tau),\tau\right) d\tau.$$

By tedious calculation, we have

$$\mu\frac{U_x}{V} - \mu\frac{u_{rx}}{v_r} = O(1)g'_{st+\mathcal{Y}} + O(1)(1-g_{st+\mathcal{Y}}) + O(1)(1-g_{st+\mathcal{X}}),$$

and

$$\kappa\left[\left(\frac{-V_{xx}}{V^5} + \frac{5V_x^2}{2V^6}\right) - \left(\frac{-v_{rxx}}{v_r^5} + \frac{5v_{rx}^2}{2v_{rx}^6}\right)\right] = O(1)g'_{st+\mathcal{X}} + O(1)(1-g_{st+\mathcal{Y}}) + O(1)(1-g_{st+\mathcal{X}}).$$

Thus, $|R_2|$ tends to zero as $N \to +\infty$ for fixed $t > 0$. Similarly, we obtain

$$\int_0^t \left[-p(V) + \mu\frac{U_x}{V} + \kappa\left(\frac{-V_{xx}}{V^5} + \frac{5V_x^2}{2V^6}\right)(s+\mathcal{Y}')\,U\right]\left(\Gamma_l^N(t),t\right) dt \\ = \int_0^t \left[-p(v_l) + u_l\,(s+\mathcal{Y}')\right](st+\mathcal{Y}(t)+y\pi_l,t)\,dt \\ + \mu\left\{\ln\left(v_l(st+\mathcal{Y}(t)+y\pi_l,t)\right) - \ln\left(\overline{v}_l + \phi_{0l}(\mathcal{Y}_0+y\pi_l)\right)\right\} + R_3. \tag{4.5}$$

And the remaining term $|R_3|$ tends to zero as $N \to +\infty$.

With the aid of (4.3),(4.4),(4.5), integrate (4.2) with respective to x over $[0,1]$. Then letting $N \to +\infty$, we obtain

$$0 = \int_0^{+\infty} (\psi_{0l}-\psi_{0r})(1-g_{\mathcal{Y}_0})dy + \frac{1}{\pi_r}\int_0^{\pi_r}\int_0^x \psi_{0r}(y)dydx - \int_{-\infty}^0 (\psi_{0l}-\psi_{0r})g_{\mathcal{Y}_0}dy \\ - \frac{1}{\pi_l}\int_0^{\pi_l}\int_0^x \psi_{0l}(y)dydx - \int_0^t \frac{1}{\pi_r}\int_0^{\pi_r}[p(v_r(x,\tau)) - p(\overline{v}_r)]\,dxd\tau - p(\overline{v}_r)t \\ + \overline{u}_r\,(st+\mathcal{Y}(t)-\mathcal{Y}_0) + \mu\ln(\overline{v}_r) - \frac{1}{\pi_r}\int_0^{\pi_r}\mu\ln(\overline{v}_r+\phi_{0r}(x))dx \\ + \int_0^t \frac{1}{\pi_l}\int_0^{\pi_l}[p(v_l(x,\tau)) - p(\overline{v}_l)]\,dxd\tau + p(\overline{v}_l)t - \overline{u}_l\,(st+\mathcal{Y}(t)-\mathcal{Y}_0) \\ - \mu\ln(\overline{v}_l) + \frac{1}{\pi_l}\int_0^{\pi_l}\mu\ln(\overline{v}_l+\phi_{0l}(x))dx + O(1)\epsilon e^{-\alpha t} \\ = (\overline{u}_r - \overline{u}_l)(\mathcal{Y}(t)-\mathcal{Y}_\infty) + O(1)\epsilon e^{-\alpha t}.$$

Using the same method, we can get $\lim\limits_{t\to+\infty}\mathcal{X}(t) = \mathcal{X}_\infty$. The proof of Lemma 2.3 is obtained.

4.2. **Proof of Lemma 3.1.** With the help of (2.7), we rewrite H_1 as

$$H_1(x,t) = G_1(x,t) + \int_{-\infty}^{x} g_2(x,t)dx := D_{1,1}^{-}(x,t) + D_{1,2}^{-}(x,t), \quad x<0,$$

where

$$\begin{aligned} D_{1,1}^{-}(x,t) :=& \left(\overline{u}_r - \overline{u}_l\right)\left(g_{st+\mathcal{Y}} - g_{st+\mathcal{X}}\right)(x) + (\overline{u}_r - \overline{u}_l)g_{st+\mathcal{X}}(x) + (s+\mathcal{X}')(\overline{v}_r - \overline{v}_l)g_{st+\mathcal{X}}(x) \\ =&(\overline{v}_r - \overline{v}_l)\left[\mathcal{X}'(t)g_{st+\mathcal{X}}(x) + s\left(g_{st+\mathcal{X}} - g_{st+\mathcal{Y}}\right)(x)\right], \\ D_{1,2}^{-}(x,t) :=& \left(\phi_r - \phi_l\right)(x,t)\left(g_{st+\mathcal{Y}} - g_{st+\mathcal{X}}\right)(x) + \int_{-\infty}^{x} (\psi_r - \psi_l)(x,t)g'_{st+\mathcal{X}}(y)dx \\ &+ (s+\mathcal{X}')\int_{-\infty}^{x} (\phi_r - \phi_l)(x,t)g'_{st+\mathcal{X}}(x)dx. \end{aligned}$$

Lemma 2.2 gives that

$$\sum_{k=0}^{3} \|\partial_x^k\left(\phi_l,\psi_l,\phi_r,\psi_r\right)\|_{L^\infty(\mathbb{R})} \le C\epsilon e^{-\alpha t}, \quad t>0.$$

By

$$\left(g_{st+\mathcal{X}} - g_{st+\mathcal{Y}}\right)(x) = \int_0^1 g'\left(x - st - \mathcal{X} - \theta\left(\mathcal{Y}-\mathcal{X}\right)\right)d\theta\left(\mathcal{Y}-\mathcal{X}\right)$$

and $|\mathcal{Y}-\mathcal{X}|(t) \le C\epsilon e^{-\alpha t}$, we have

$$\begin{aligned} \sum_{k=0}^{3}\int_{-\infty}^{0}\left|\partial_x^k H_1\right|^2 dx \le& \sum_{k=0}^{3}\int_{-\infty}^{0}\left(\left|\partial_x^k D_{1,1}^{-}\right|^2 + \left|\partial_x^k D_{1,2}^{-}\right|^2\right)dx \\ &\le C\epsilon^2 e^{-2\alpha t}\sum_{k=0}^{4}\int_{-\infty}^{M_0}\left|\frac{d^k}{dx^k}g(x)\right|^2 dx \le C\epsilon^2 e^{-2\alpha t}. \end{aligned}$$

Here M_0 is $\sup_{t\ge 0}\left(|\mathcal{X}| + |\mathcal{Y}|\right)(t) + st$. Similar, For $x>0$, we can obtain

$$\sum_{k=0}^{3}\int_{0}^{+\infty}\left|\partial_x^k H_1\right|^2 dx \le C\epsilon^2 e^{-2\alpha t}.$$

H_2 can be estimate by the same method and thus we omit it. Thus the proof of Lemma 3.1 is accomplished .

Appendix A. Proof of Lemma 2.2

In this section, we denote $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{T})}, \|\cdot\|_k = \|\cdot\|_{H^k(\mathbb{T})}, \phi := v - \overline{v}, \psi := u - \overline{u}$ where $\mathbb{T} = [0,\pi]$. Then using (1.1), we have

$$\begin{cases} \phi_t = \psi_x, \\ \psi_t + p(v)_x = \mu\left(\dfrac{\psi_x}{v}\right)_x + \kappa\left(\dfrac{-\phi_{xx}}{v^5} + \dfrac{5\phi_x^2}{2v^6}\right)_x. \end{cases} \tag{A.1}$$

Sobolev inequality and the a priori assumption gives that

$$\sum_{l=0}^{k-1} \|\partial_x^l\left(\phi, \psi\right)\|_{L^\infty(\mathbb{T})} \leq C\delta. \tag{A.2}$$

Multiplying ψ on $(\mathrm{A.1})_2$, we have

$$\left(\frac{\psi^2}{2}\right)_t + \frac{\mu}{v}\psi_x^2 = (\cdots)_x + p(v)\psi_x + \kappa\frac{\phi_{xx}}{v^5}\psi_x - \kappa\frac{5\phi_x^2}{2v^6}\psi_x.$$

So we have

$$\begin{aligned} &\frac{d}{dt}\left\{\int_0^\pi \Big(\frac{\psi^2}{2} - \int_{\overline{v}}^{\overline{v}+\phi} p(s)ds + p(\overline{v})\phi\Big)dx\right\} + \int_0^\pi \mu\frac{\psi_x^2}{v} - \kappa\frac{\phi_{xx}\psi_x}{v^5}dx \\ =& -\int_0^\pi \kappa\frac{5\phi_x^2}{2v^6}\psi_x dx \leq C_1\delta\int_0^\pi \mu\frac{\psi_x^2}{v}dx + C_1\delta\int_0^\pi \frac{|p'(v)|}{v}\phi_x^2 dx, \end{aligned} \tag{A.3}$$

where we have used (A.2) and

$$\begin{aligned} p(v)\psi_x = p(v)\phi_t &= \Big(\int_{\overline{v}}^{\overline{v}+\phi} p(s)ds - p(\overline{v})\phi\Big)_t + p(\overline{v})\phi_t \\ &= \Big(\int_{\overline{v}}^{\overline{v}+\phi} p(s)ds - p(\overline{v})\phi\Big)_t + \left(p(\overline{v})u\right)_x. \end{aligned}$$

Equation (A.1) gives that

$$\psi_t + p(v)_x = \mu\left(\frac{\phi_x}{v}\right)_t + \kappa\left(\frac{-\phi_{xx}}{v^5} + \frac{5\phi_x^2}{2v^6}\right)_x. \tag{A.4}$$

Then multiplying (A.4) by $\mu\dfrac{\phi_x}{v}$, we obtain

$$\begin{aligned} \mu\frac{\psi_t\phi_x}{v} &= \mu\left(\frac{\psi\phi_x}{v}\right)_t + (...)_x + \mu\frac{\psi_x^2}{v}, \\ -\kappa\mu\left(\frac{\phi_{xx}}{v^5}\right)_x\frac{\phi_x}{v} &= (...)_x + \kappa\mu\frac{\phi_{xx}^2}{v^6} + \kappa\mu\frac{\phi_{xx}\phi_x^2}{v^7}, \\ -\kappa\mu\left(\frac{5\phi_x^2}{2v^6}\right)_x\frac{\phi_x}{v} &= 5\mu\kappa\frac{\phi_{xx}\phi_x^2}{v^7} - 15\mu\kappa\frac{\phi_x^4}{v^8}. \end{aligned}$$

Using the Cauchy inequality, one gets

$$\begin{aligned} &\frac{d}{dt}\int_0^\pi \Big[\frac{1}{2}\left(\frac{\mu\phi_x}{v}\right)^2 - \mu\frac{\phi_x\psi}{v}\Big]dx + \mu\int_0^\pi \frac{|p'(v)|}{v}\phi_x^2 dx + \kappa\mu\int_0^\pi \frac{\phi_{xx}^2}{v^6}dx \\ =&\mu\int_0^\pi \frac{\psi_x^2}{v}dx - 4\mu\kappa\int_0^\pi \frac{\phi_{xx}\phi_x^2}{v^7}dx + 15\mu\kappa\int_0^\pi \frac{\phi_x^4}{v^8}dx \\ \leq&\mu\int_0^\pi \frac{\psi_x^2}{v}dx + \eta\int_0^\pi \frac{\phi_{xx}^2}{v^6}dx + C_2\delta^2\int_0^\pi \frac{|p'(v)|}{v}\phi_x^2 dx, \end{aligned}$$

for some constant C_2 .Thus

$$\begin{aligned}&\frac{d}{dt}\int_0^\pi\left[\frac{1}{2}\left(\frac{\mu}{v}\phi_x\right)^2-\frac{\mu}{v}\phi_x\psi\right]dx+\int_0^\pi(\mu-C_2\delta^2)\frac{|p'(v)|}{v}\phi_x^2dx+(\kappa\mu-\eta)\int_0^\pi\frac{\phi_{xx}^2}{v^6}dx\\&\leq\mu\int_0^\pi\frac{\psi_x^2}{v}dx.\end{aligned}\tag{A.5}$$

Multiplying $(\mathrm{A}.1)_2$ by $-\psi_{xx}$ yield that

$$\begin{aligned}&\frac{1}{2}\left(\kappa\frac{\phi_{xx}^2}{v^5}+\psi_x^2\right)_t+\mu\frac{\psi_{xx}^2}{v}\\&=-5\kappa\frac{\phi_x\phi_{xx}\psi_{xx}}{v^6}+15\kappa\frac{\phi_x^3\psi_{xx}}{v^7}+p'(v)\phi_x\psi_{xx}+\mu\frac{\phi_x\psi_x\psi_{xx}}{v^2}-\frac{5}{2}\kappa\frac{\psi_x\phi_{xx}^2}{v^6},\end{aligned}$$

which implies that

$$\begin{aligned}&\frac{d}{dt}\int_0^\pi\left(\kappa\frac{\phi_{xx}^2}{v^5}+\psi_x^2\right)dx+\int_0^\pi\mu\frac{\psi_{xx}^2}{v}-p'(v)\phi_x\psi_{xx}dx\\&\leq\eta\mu\int_0^\pi\frac{\psi_{xx}^2}{v}dx+C_3\delta\left(\int_0^\pi\frac{|p'(v)|}{v}\phi_x^2dx+\int_0^\pi\frac{\phi_{xx}^2}{v^6}dx+\int_0^\pi\mu\frac{{\psi_x}^2}{v}dx\right),\end{aligned}\tag{A.6}$$

for some constant $C_3>0$. Choose suitable constants $M_1,M_2>0$, then $M_2\cdot$ (A.3) $+M_1\cdot$ (A.5) + (A.6) gives that

$$\begin{aligned}&\frac{d}{dt}\int_0^\pi\left[\frac{M_2}{2}\psi^2+\frac{M_1}{2}\left(\frac{\mu}{v}\phi_x\right)^2-M_1\frac{\mu}{v}\phi_x\psi\right]dx\\&+\frac{d}{dt}\int_0^\pi\left[M_2\Big(-\int_{\overline{v}}^{\overline{v}+\phi}p(s)ds+p(\overline{v})\phi\Big)+\psi_x^2+\frac{\kappa\phi_{xx}^2}{v^5}\right]dx\\&+\int_0^\pi\left\{\mu[(1-C_1\delta)M_2-M_1-C_3\delta]\frac{\psi_x^2}{v}-\kappa M_2\frac{\phi_{xx}\psi_x}{v^5}+[M_1(\kappa\mu-\eta)-C_3\delta]\frac{\phi_{xx}^2}{v^6}\right\}dx\\&+\int_0^\pi\left\{[(\mu-C_2\delta^2)M_1-C_3\delta-C_1\delta M_2]\frac{|p'(v)|}{v}\phi_x^2-p'(v)\phi_x\psi_{xx}+\mu(1-\eta)\frac{\psi_{xx}^2}{v}\right\}dx<0.\end{aligned}\tag{A.7}$$

By directly calculate, if (2.4) holds, choosing δ suitably small, one gets that there exist five constants a_1,a_2,a_3,M_1,M_2, such that

$$\begin{aligned}&\frac{M_2}{2}\psi^2+\frac{\mu^2M_1}{2v^2}\phi_x^2-M_1\frac{\mu}{v}\phi_x\psi>a_1(\psi^2+\phi_x^2),\\&(\mu M_2-M_1)\frac{\psi_x^2}{v}-\kappa M_2\frac{\phi_{xx}\psi_x}{v^5}+M_1\kappa\mu\frac{\phi_{xx}^2}{v^6}>a_2(\psi_x^2+\phi_{xx}^2),\\&\mu M_1\frac{|p'(v)|}{v}\phi_x^2-p'(v)\phi_x\psi_{xx}+\mu\frac{\psi_{xx}^2}{v}>a_3(\psi_{xx}^2+\phi_x^2).\end{aligned}\tag{A.8}$$

Thus it follows from (A.7), (A.8) and Poincaré inequality, choosing δ, η suitably small, that for some $\alpha > 0$, one has that

$$(\|\phi\|_2, \|\psi\|_1)(t) \leq C(\|\phi_0\|_2, \|\psi_0\|_1)e^{-\alpha t} \leq C\epsilon e^{-\alpha t}.$$

The estimate of the higher order derivatives $\partial_x^l(\phi, \psi)$ with $l = 2, 3, 4$, is similar and thus omitted.

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