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RESEARCH ARTICLE

Dynamics of a class of nonlinear pest-natural enemy discrete model

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The inappropriate use of insecticides may lead to disastrous pest outbreaks. Aiming at avoiding the outbreak of pests by optimizing the control strategies, we propose a novel mathematical model base on the idea of pulse in this study, namely a discrete-time model of pest-natural enemies, which considers the implementation of spraying insecticides within the time interval between two generations of pests. We first investigates the existence and stability of fixed points, and then using the central manifold theory, we proved that the system can exist period-doubling bifurcation. The main theoretical results are verified by numerical simulations. The experiments show that if the insecticidal time is within a certain range, the lower insecticidal rate stimulates the growth of pests and natural enemies, while the larger insecticidal rate can inhibit the growth of pests and natural enemies. In addition, the effects of pest growth rate, timing of pesticide spraying, and pesticide intensity on the system are comprehensively discussed. The main research provides good reference significance to choose an appropriate time to prevent the pest outbreaks.

KEYWORDS:
discrete-time models; fixed points; stability; period-doubling bifurcation

1 | INTRODUCTION

Pest outbreaks have been frequently happening, which poses great threat to agriculture. There are many studies focusing on pest management, both empirically and theoretically [1,2,3,4,5,6]. In order to suppress the continued growth of pests, insecticides are often the first choice, but in some cases they are only effective in the early stages of their introduction and usually leads to larger outbreaks in the later stages. This paradoxical phenomenon is often referred to as pest resurgence in pest control, which is deeply studied in [7,8,9,10,11]. Several other studies shown that small amounts of pesticides can help to increase the fecundity of pests, while large amounts of pesticides can reduce pest populations [12,13], this phenomenon is known as the 'hormetic effects'. The hormetic effects also plays an important role in the recovery of pests [12].

Hormetic effects is not only a challenge for pest control, but also a major challenge for cancer treatment[14,15,16], neurodegenerative diseases[17] and the presentation of nutritional and ecotoxicological[18] management decisions. Based on endogenous regulatory mechanisms, timing of interventions and changes in dose-response properties, the strategic application of agonistic effects shows promise for optimising aspects of agroecosystem management of pesticides[11,19] or harvests[20,21,22]. To better understand the mechanisms of reignition and agonistic effects, mathematical models of pest control were proposed in determining the effectiveness and optimal timing of pesticide application[23,24,25].

In several previous studies, in order to control the growth of pests, people have always believed that the requirements can be met through various forms of physical control, pesticide spraying and biological control. However, most pests have non-overlapping generations and control actions should be applied to each generation, therefore, it is a matter of reflection how to incorporate external perturbations into the modelling of discrete populations. In order to cope with this issue, some interesting modelling approaches have recently been proposed[20,21,22,26,27], with the main results showing that the timing and intensity of external perturbations can significantly affect the dynamical behaviour of the model. This paper implies that the use of pesticides does not necessarily reduce the population of pests immediately. The assumption that the use of pesticides can eventually lead to the reduction of the population of pests may not be appropriate in some cases. The increase of pesticide use may lead to the increase rather than decrease of the number of pests, which is an unpredictable ecological paradox[28]. Therefore, improper use of pesticides may not effectively control pests, but also lead to an increase in the number of pests, resulting in greater outbreaks. Here, we derive a novel pest-natural enemy system with discrete time considering the shifts between the generations of pests. Through theoretical analysis and numerical simulation of the system, we find that the intrinsic growth rate of the pest, the size of the insecticidal rate and the insecticidal time have great influence on the growth of pests, which is a guideline for achieving efficient strategy for pest control.

2 | MODEL FORMULATION

Considering the following equation:

$$\frac{dH(t)}{dt} = rH(t)\left[1 - \frac{H(n)}{K} - \beta P(n)\right],\ t \in [n, n+1], \tag{1}$$

within $t \in [n, n+1]$, and solving Eq.(1), a typical Nicholson-Bailey host-parasitoid model can be obtained in the following form[29]:

$$\begin{aligned} H(n+1) &= H(n)\exp\left[r\left(1 - \frac{H(n)}{K} - \beta P(n)\right)\right], \\ P(n+1) &= H(n)\left(1 - \exp\left(-\alpha P(n)\right)\right), \end{aligned}$$

where $H(n)$ and $P(n)$ are the densities of hosts and parasitoids in generation n, r is the intrinsic growth rate of the host, K is the carrying capacity of the host and β is the probability of the host being parasitised by the parasite. The parameter α is the parasite's own conversion rate, and the term $\exp(-\alpha P(n))$ is the probability that a host individual can escape parasitism. Thus, a particular host individual is parasitized and converted with the probability $(1 - \exp(-\alpha P(n)))$ into parasite individuals[30]. In this paper, we consider the host population as pest and the parasite population as its natural enemy.

We then introduce the harvesting model proposed by Seno[21]. Their model assumes that there is a specific season as a time unit, in which individuals accumulate energy for reproduction. In this paper, we also use this season interchangeably with reproductive season and assume that the length of this particular season is T, beginning at time $t = 0$ and lasting until $t = T$. We assume that there exists a positive constant $\theta \in (0, 1)$, and the chemical control tactic is applied at $n + \theta$, which kills a certain number of pests without affecting the continued growth of their natural enemies (see Figure 1). We assume

q represents the insecticidal rate, and $p = 1 - q$ means the survival rate after the spraying pesticide[31]. Before θ, the density of pests depends on $\theta H(n)$, after θ, the density of pests depends on $(1-q)H(n)$.

As a conclusion, we have that when $t \in [n, n+\theta]$, the analytical solution

$$\frac{dH(t)}{dt} = rH(t)\left[1 - \frac{H(n)}{K} - \beta P(n)\right],\ t \in (n, n+\theta],$$

and at time point $n + \theta$ we have

$$H_\theta \doteq H(n+\theta) = H(n)\exp\left[r\left(1 - \frac{H(n)}{K} - \beta P(n)\right)\theta\right],$$
$$P_\theta \doteq P(n+\theta) = H(n)[1 - \exp(-\alpha P(n)\theta)].$$

When $t \in [n+\theta, n+1]$, the first part $H_\theta^+ = pH_\theta$ means that pesticides cause loss of fertility, where $p = 1-q$, q is the killing rate of pests due to the use of pesticides, which $P_\theta^+ = P_\theta$.

$$\frac{dH(t)}{dt} = rH(t)\left[1 - \frac{H_\theta^+}{K} - \beta P_\theta^+\right],\ t \in (n+\theta, n+1]. \tag{2}$$

Solving Eq.(2), one yields

$$H(t) = H_\theta^+ \exp\left[r\left(1 - \frac{H_\theta^+}{K} - \beta P_\theta^+\right)(t - n - \theta)\right],$$

which indicates that

$$\begin{aligned}
H(n+1) &= H_\theta^+ \exp\left[r\left(1 - \frac{H_\theta^+}{K} - \beta P_\theta^+\right)(1-\theta)\right] \\
&= pH(n)\exp\left[r\left(1 - \frac{H(n)}{K} - \beta P(n)\right)\theta\right] * \\
&\exp\left[r\left(1 - \frac{pH(n)\exp\left[r\left(1 - \frac{H(n)}{K} - \beta P(n)\right)\theta\right]}{K} - \beta H(n)[1 - \exp(-\alpha P(n)\theta)]\right)(1-\theta)\right],
\end{aligned}$$

and

$$P(n+1) = H_\theta^+\left(1 - \exp\left(-\alpha P_\theta^+(1-\theta)\right)\right) + H(n)[1 - \exp(-\alpha P(n)\theta)].$$

So we obtain the following model:

$$\begin{cases} H(n+1) = pH(n)e^{[r(1-\frac{H(n)}{K}(\theta+(1-\theta)(pe^{r(1-\frac{H(n)}{K}-\beta P(n))\theta}+\beta K(1-e^{-\alpha P(n)\theta})))-\beta\theta P(n))]}, \\ P(n+1) = pH(n)e^{r(1-\frac{H(n)}{K}-\beta P(n))\theta}(1-e^{-\alpha H(n)(1-e^{-\alpha P(n)\theta})(1-\theta)}) + H(n)(1-e^{-\alpha P(n)\theta}), \end{cases} \tag{3}$$

where $H(n)$ represents the density of the pest population at generation n, $P(n)$ represents the density of its natural enemy population at generation n, r is the intrinsic growth rate of the pest, K is the carrying capacity, β is the predation rate of the natural enemy, and α is the self-transformation rate of the natural enemy. In this study, we focus on discussing the existence and stability of the fixed point of model (3), then analyzing the bifurcation phenomenon to investigate the effect of insecticidal rate q on pest and natural enemy densities, and finally exploring the complex dynamic behaviors of the model (3) through numerical simulations.

3 | EXISTENCE AND STABILITY ANALYSIS OF FIXED POINTS

3.1 | Existence of fixed points

For the system (3), it is easy to see that there is a trivial fixed point $E_0(0,0)$. Let $H(n+1) = H(n) = H^*$ and $P(n+1) = P(n) = P^*$, then the nontrivial fixed point $E^*(H^*, P^*)$ of the system must satisfy the

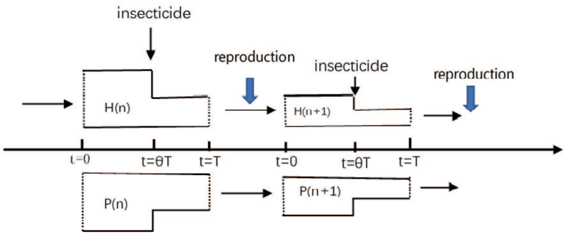


FIGURE 1 The scenario for pest-natural enemy dynamics under insecticidal effects. $H(n)$ is the pest population density and $P(n)$ is the natural enemy population density.

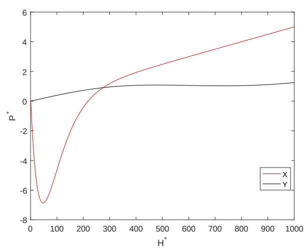
following equations:

$$\begin{cases} H^* = pH^* e^{[r(1-\frac{H^*}{K}(\theta+(1-\theta)(pe^{r(1-\frac{H^*}{K}-\beta P^*)\theta}+\beta K(1-e^{-\alpha P^*\theta})))-\beta\theta P^*)]}, \\ P^* = pH^* e^{r(1-\frac{H^*}{K}-\beta P^*)\theta}(1-e^{-\alpha H^*(1-e^{-\alpha P^*\theta})(1-\theta)}) + H^*(1-e^{-\alpha P^*\theta}). \end{cases}$$

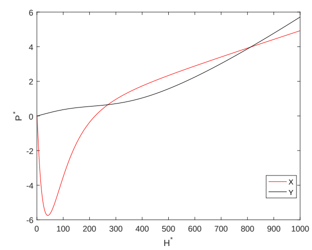
Due to the complexity and high non-linearity of the equations, it's very hard to obtain the analytical solution of $E^*\,(H^*, P^*)$, alternatively we utilize numerical simulation to verify the existence of $E^*\,(H^*, P^*)$. Let

$$\begin{cases} X = H^* - pH^* e^{[r(1-\frac{H^*}{K}(\theta+(1-\theta)(pe^{r(1-\frac{H^*}{K}-\beta P^*)\theta}+\beta K(1-e^{-\alpha P^*\theta})))-\beta\theta P^*)]}, \\ Y = P^* - pH^* e^{r(1-\frac{H^*}{K}-\beta P^*)\theta}(1-e^{-\alpha H^*(1-e^{-\alpha P^*\theta})(1-\theta)}) + H^*(1-e^{-\alpha P^*\theta}), \end{cases}$$

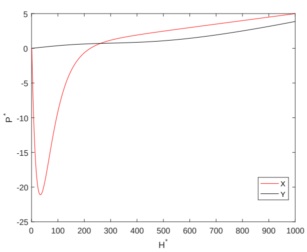
then if the orbits of X and Y have an intersection point, then $E^*\,(H^*, P^*)$ exists, as shown in Figure 2.



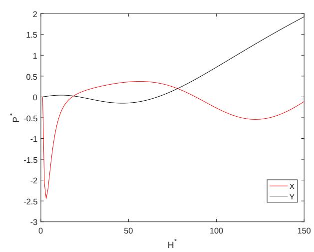
(a) $\theta = 0.01$



(b) $\theta = 0.06$



(c) $\theta = 0.03$



(d) $\theta = 0.5$

FIGURE 2 Existence of fixed points $E^*\,(H^*, P^*)$ of system (3). The parameters are fixed as $p = 0.9$, $K = 1$, $\beta = 0.1$, $\alpha = 4$. Here $r = 4.5$ in (a) and (b); $r = 6$ in (c) and (d).

It's easy to see from Figures 2(a) - 2(d) that the existence conditions of the intersection points can be satisfied, therefore system (3) can have two nontrivial fixed points. Furthermore, we choose r and θ as bifurcation parameters and fix all others as given in Figure 3.

Letting r vary from 2 to 6, θ vary from 0 to 1, we observed that the parameter space is divided into three regions. It can been seen that the existence of fixed points highly depends on the values of r and θ, where the area marked as red indicates the existence of one fixed point, the area marked as blue implies that there exist two fixed points, and in the green area there can exist three fixed points.

3.2 | Stability analysis of fixed points

From the last section, we know that system (3) has a trivial fixed point $E_0(0,0)$, and also nontrivial fixed points $E^*(H^*,P^*)$. In this section, we mainly analyze the stability of $E_0(0,0)$ and $E^*(H^*,P^*)$.

The Jacobian matrix for system (3) at each variable is

$$J(H,P) = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned}
E_{11} &= 1 + Hr(-\frac{\theta+(1-\theta)(a_1+\beta K-a_2)}{K} + \frac{H(1-\theta)r\theta}{K^2}a_1),\\
E_{12} &= Hr(-\frac{H(1-\theta)(-a_1 r\beta\theta+\alpha\theta a_2)}{K} - \beta\theta),\\
E_{21} &= a_1a_3 - \frac{r\theta H a_1 a_3}{K} + [a_1H(1-\theta)(1-a_3)+1]\left(1-e^{-\alpha P\theta}\right),\\
E_{22} &= -r\beta\theta H a_1 a_3 + H^2 a_1\alpha^2\theta\frac{a_2}{\beta K}(1-\theta)(1-a_3) + \frac{a_2}{\beta K}H\theta\alpha,
\end{aligned}$$

$$a_1 = pe^{r(1-\frac{H}{K}-\beta P)\theta}\ ,\ a_2 = \beta K e^{-\alpha P\theta}\ ,\ a_3 = 1-e^{-\alpha H(1-e^{-\alpha P\theta})(1-\theta)}.$$

The characteristic equation of the Jacobian matrix can be written as

$$\lambda^2 - T\lambda + D = 0, \tag{5}$$

where T is the trace of the matrix and D is the determinant of the matrix with

$$\begin{aligned}
T &= 1 + HrA_1 - r\beta\theta H a_1 a_3 + B_1,\\
D &= (1+HrA_1)(-r\beta\theta H a_1 a_3 + B_1) - [Hr(C_1-\beta\theta)](a_1a_3+D_1),
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= -\frac{\theta+(1-\theta)(a_1+\beta K-a_2)}{K} + \frac{H(1-\theta)r\theta}{K^2}a_1,\\
B_1 &= H^2a_1\alpha^2\theta\frac{a_2}{\beta K}(1-\theta)(1-a_3) + \frac{a_2}{\beta K}H\theta\alpha,\\
C_1 &= -\frac{H(1-\theta)(-a_1r\beta\theta+\alpha\theta a_2)}{K},\\
D_1 &= -\frac{r\theta H a_1 a_3}{K} + [a_1H(1-\theta)(1-a_3)+1]\left(1-e^{-\alpha P\theta}\right).
\end{aligned}$$

In order to study the stability of the fixed points of system (3), the following results can be easily obtained utilizing the relationships between the roots and coefficients of the quadratic equation [32,33,34].

Lemma 1. Let $F(x) = x^2 - bx + c$. Assume that $F(1) > 0$, x_1 and x_2 are the two roots of $F(x) = 0$. Then:

(i) $|x_1| < 1$ and $|x_2| < 1$ if and only if $F(-1) > 0$ and $c < 1$;
(ii) $|x_1| < 1$ and $|x_2| > 1$ (or $|x_1| > 1$ and $|x_2| < 1$) if and only if $F(-1) > 0$;
(iii) $|x_1| > 1$ and $|x_2| > 1$ if and only if $F(-1) > 0$ and $c > 1$;
(iv) $|x_1| = -1$ and $x_2 \neq 1$ if and only if $F(-1) = 0$ and $c \neq 0, 2$;
(v) x_1 and x_2 are complex and $|x_1| = |x_2| = 1$ if and only if $b^2 - 4c < 0$ and $c = 1$.

Suppose λ_1 and λ_2 are the two eigenvalues of the characteristic equation of the Jacobian matrix calculated at the fixed point E^*. If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, then the fixed point E^* is called a sink, which is a locally asymptotically stable fixed point; if $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then E^* is called a source, which

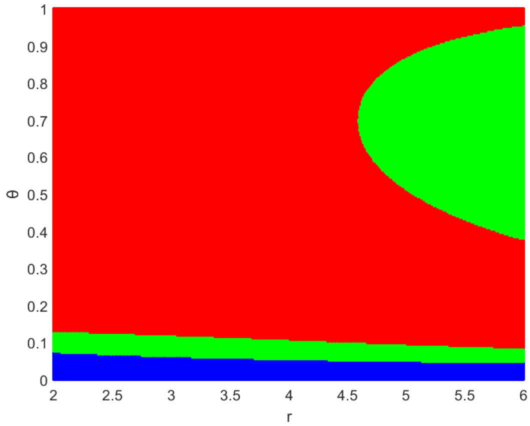


FIGURE 3 Parameters bifurcation diagram for the existence of fixed points of system (3). Here the other parameters are fixed as follows: $p = 0.9$, $K = 1$, $\beta = 0.1$, $\alpha = 4$.

is unstable; if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$), then E^* is called a saddle point, which is always unstable; if $|\lambda_1| = 1$ or $|\lambda_2| = 1$, then the fixed point E^* is non-hyperbolic[35,36]. We therefore have the following conclusion.

Theorem 1. The fixed point $E_0(0,0)$ is non-hyperbolic.

Proof: The Jacobian matrix at $E_0(0,0)$ is

$$J(E_0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalue of the characteristic equation of the Jacobian matrix are 1 and 0. From Lemma 1, we know that $E_0(0,0)$ is non-hyperbolic.

We then introduce Lemma 2, which can help to provide a necessary and sufficient condition on the absolute value of all roots of a quadratic equation to compare its magnitude with 1[36].

Lemma 2. [Jury condition]: Consider the quadratic polynomial equation

$$\lambda^2 + a\lambda + b = 0, \tag{6}$$

where a and b are real numbers, and assume that λ_1 and λ_2 are the two roots of equation (6). Then the following statements are true:

(i) A necessary and sufficient condition for $|\lambda_1| < 1$ and $|\lambda_2| < 1$ is

$$a < 1 + b < 2.$$

(ii) A necessary and sufficient condition for $|\lambda_1| < 1$ and $|\lambda_2| > 1$ or $|\lambda_1| > 1$ and $|\lambda_2| < 1$ is

$$a^2 - 4b > 0, \quad and \quad a > b + 1.$$

(iii) A necessary and sufficient condition for $|\lambda_1| > 1$ and $|\lambda_2| > 1$ is

$$b > 1, \quad and \quad a < b + 1.$$

Theorem 2. 1. If the fixed point $E^*(H^*, P^*)$ is stable, it meets the following criteria:

(i) $1 - Tr(J(E^*)) + Det(J(E^*)) > 0$;

(ii) $1 + Tr(J(E^*)) + Det(J(E^*)) > 0$;

(iii) $1 - Det(J(E^*)) > 0$; where

$$J(E^*) = \begin{pmatrix} E^*_{11} & E^*_{12} \\ E^*_{21} & E^*_{22} \end{pmatrix},$$

$$
\begin{aligned}
E_{11}^* &= 1 + H^* r(-\frac{\theta + (1-\theta)(a_1^* + \beta K - a_2^*)}{K} + \frac{H^*(1-\theta) r\theta}{K^2} a_1^*),\\
E_{12}^* &= H^* r(-\frac{H^*(1-\theta)(-a_1^* r\beta\theta + \alpha\theta a_2^*)}{K} - \beta\theta),\\
E_{21}^* &= a_1^* a_3^* - \frac{r\theta H^* a_1^* a_3^*}{K} + [H^* a_1^*(1-\theta)(1-a_3^*) + 1]\left(1 - e^{-\alpha P^*\theta}\right),\\
E_{22}^* &= -r\beta\theta H^* a_1^* a_3^* + {H^*}^2 a_1^* \alpha^2\theta \frac{a_2^*}{\beta K}(1-\theta)(1-a_3^*) + \frac{a_2^*}{\beta K} H^*\theta\alpha,
\end{aligned}
$$

$$a_1^* = pe^{r(1-\frac{H^*}{K}-\beta P^*)\theta},\ a_2^* = \beta K e^{-\alpha P^*\theta},\ a_3^* = 1 - e^{-\alpha H^*(1-e^{-\alpha P^*\theta})(1-\theta)}.$$

Tr is the trace of the Jacobian matrix $J(E^*)$, and Det is the determinant of the Jacobian matrix $J(E^*)$, thus, we have

$$
\begin{aligned}
Tr(J(E^*)) &= 1 + H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*,\\
Det(J(E^*)) &= (1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r(C_1^* - \beta\theta,)(a_1^* a_3^* + D_1^*),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1^* &= -\frac{\theta + (1-\theta)(a_1^* + \beta K - a_2^*)}{K} + \frac{H^*(1-\theta) r\theta}{K^2} a_1^*,\\
B_1^* &= {H^*}^2 a_1^* \alpha^2\theta \frac{a_2^*}{\beta K}(1-\theta)(1-a_3^*) + \frac{a_2^*}{\beta K} H^*\theta\alpha,\\
C_1^* &= -\frac{H^*(1-\theta)(-a_1^* r\beta\theta + \alpha\theta a_2^*)}{K},\\
D_1^* &= -\frac{r\theta H^* a_1^* a_3^*}{K} + [H^* a_1^*(1-\theta)(1-a_3^*) + 1]\left(1 - e^{-\alpha P^*\theta}\right).
\end{aligned}
$$

2. If the fixed point E^* is a source, the condition must be satisfied:

$$|(1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r(C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*)| > 1,$$

and

$$|H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*| < |(1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r(C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*)|.$$

3. If E^* is a saddle point, the condition must be satisfied:

$$(1 + H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*)^2 > 4\left((1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r(C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*)\right),$$

and

$$|H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*| > |(1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r(C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*)|.$$

4. If E^* is non-hyperbolic, the condition is satisfied:

$$|H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*| = |(1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r(C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*)|,$$

and

$$|H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*| \le 1.$$

4 | PERIOD-DOUBLING BIFURCATION (PDB) ANALYSIS

In this section, we focus on studing the period-doubling bifurcation of system (3). To this end, we seek to make system (3) has a non-hyperbolic fixed point under certain parameter conditions, and the characteristic equation calculated at the fixed point $E^*(H^*, P^*)$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 \neq 1$. The characteristic polynomial of system (3) computed at $E^*(H^*, P^*)$ can be rewritten as

$$F(\lambda) = \lambda^2 - A\lambda + B, \tag{7}$$

where

$$\begin{aligned}A &= 1 + H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*,\\ B &= (1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r (C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*).\end{aligned}$$

Assume that $A^2 > 4B$, we can get

$$(1 + H^* r A_1^* - r\beta\theta H^* a_1^* a_3^* + B_1^*)^2 > 4\left[(1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r (C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*)\right].$$

The root of the characteristic equation is -1, from the relationship between the root and the coefficient, there is $A + B = -1$, hence we can get an expression of p.

The roots of the equation $F(\lambda) = 0$ are $\lambda_1 = -1$ and $\lambda_2 = -B$, and the condition $\lambda_2 \neq 1$ indicates

$$-(1 + H^* r A_1^*)(-r\beta\theta H^* a_1^* a_3^* + B_1^*) - H^* r (C_1^* - \beta\theta)(a_1^* a_3^* + D_1^*) \neq \pm 1.$$

Let

$$\Omega_{PD} = \{(p, r, K, \beta, \theta) : A^2 > 4B, A + B = -1, B \neq \pm 1, p, r, K, \beta > 0, 0 < \theta < 1\}.$$

When the parameter p varying in the small neighborhood of Ω_{PD}, the fixed point $E^*(H^*, P^*)$ of system (3) experiences the PDB. Now let $p_1 = p$, taking any parameters $(p, r, K, \beta, \theta) \in \Omega_{PD}$, system (3) can be expressed by the following two-dimensional mapping:

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} p_1 H e^{[r(1-\frac{H}{K}(\theta+(1-\theta)(p_1 e^{r(1-\frac{H}{K}-\beta P)\theta}+\beta K(1-e^{-\alpha P\theta})))-\beta\theta P)]} \\ p_1 H e^{r(1-\frac{H}{K}-\beta P)\theta}(1 - e^{-\alpha H(1-e^{\alpha P\theta})(1-\theta)}) + H(1 - e^{-\alpha P\theta}) \end{pmatrix}. \tag{8}$$

Now, choosing p_1 as the minor bifurcation parameter, the perturbation of the mapping (8) takes the following form :

$$\begin{pmatrix} H \\ P \end{pmatrix} \to \begin{pmatrix} (p_1 + \tilde{p}) H e^{[r(1-\frac{H}{K}(\theta+(1-\theta)((p_1+\tilde{p}) e^{r(1-\frac{H}{K}-\beta P)\theta}+\beta K(1-e^{-\alpha P\theta})))-\beta\theta P)]} \\ (p_1 + \tilde{p}) H e^{r(1-\frac{H}{K}-\beta P)\theta}(1 - e^{-\alpha H(1-e^{-\alpha P\theta})(1-\theta)}) + H(1 - e^{-\alpha P\theta}) \end{pmatrix}, \tag{9}$$

where $\tilde{p} \ll 1$, which is a small perturbation parameter. Let $x = H - H^*$ and $y = P - P^*$, transforming the fixed point $E^*(H^*, P^*)$ to the origin , then, the mapping (9) becomes:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x, y, \tilde{p}) \\ g(x, y, \tilde{p}) \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} f(x, y, \tilde{p}) &= a_{13}x^2 + a_{14}xy + a_{15}y^2 + e_1\tilde{p}x + e_2\tilde{p}y + e_3\tilde{p}xy + O((|x| + |y| + |\tilde{p}|)^3),\\ g(x, y, \tilde{p}) &= a_{23}x^2 + a_{24}xy + a_{25}y^2 + g_1\tilde{p}x + g_2\tilde{p}y + g_3\tilde{p}xy + O((|x| + |y| + |\tilde{p}|)^3), \end{aligned}$$

$$\begin{aligned}
a_{11} &= 1 + H^* r(-\tfrac{\theta+(1-\theta)(a_1^*+\beta K-a_2^*)}{K} + \tfrac{H^*(1-\theta)r\theta}{K^2}a_1^*),\\
a_{12} &= H^* r(-\tfrac{H^*(1-\theta)(-a_1^* r\beta\theta+\alpha\theta a_2^*)}{K} - \beta\theta),\\
a_{13} &= r\phi + H^* r\tfrac{a_1^* r\theta(1-\theta)(2K-H^* r\theta)}{2K^3} + \tfrac{H^* r^2\phi^2}{2}, \text{ where } \phi = \tfrac{H^*(1-\theta)r\theta a_1^*}{K^2} - \tfrac{\theta+(1-\theta)(a_1^*+\beta K-a_2^*)}{K},\\
a_{14} &= r(\psi - \beta\theta) + r(\psi - \tfrac{(1-\theta)H^* r^2\theta^2\beta {a^*}_1}{K^2}) + H^* r^2\phi(\psi - \beta\theta), \text{ where } \psi = -\tfrac{H^*(1-\theta)(-r\beta\theta a_1^*+\alpha\theta a_2^*)}{K},\\
a_{15} &= -\tfrac{(H^*)^2 r(1-\theta)(r^2\theta^2\beta^2 a_1^*-\theta^2\alpha^2 a_2^*)}{2K} + \tfrac{1}{2}H^* r^2(\psi - \beta\theta)^2,\\
e_1 &= \tfrac{1}{p_1}(1 + H^* r\phi) + \tfrac{H^* r a_1^*(1-\theta)}{p_1 K}(\tfrac{H^* r\theta}{K} - 2 - H^* r\phi),\\
e_2 &= \tfrac{H^* r}{p_1}(\psi - \beta\theta) + \tfrac{(H^*)^2 r^2(1-\theta){a_1}^*}{p_1 K}(2\beta\theta - \psi),\\
e_3 &= \tfrac{2r}{p_1}\left(\psi - \gamma - \tfrac{\beta\theta}{2}\right) + \tfrac{r^2 H^* a_1^*(1-\theta)}{p_1 K}\left[H^* r\phi\,(2\beta\theta - \psi) + 3\beta\theta - 2\psi + \gamma\right]\\
&\quad + \tfrac{H^* r^2}{p_1}(\psi - \beta\theta)\left(\phi + \tfrac{H^* a_1^* r\theta(1-\theta)}{K^2} - \tfrac{(1-\theta)a_1^*}{K}\right), \text{ where } \gamma = \tfrac{(H^*)^2 a_1^* r^2\theta^2\beta(1-\theta)}{K^2},\\
a_{21} &= a_1^* a_3^* - \tfrac{r\theta H^* a_1^* a_3^*}{K} + \left[H^* a_1^*(1 - \theta)(1 - a_3^*) + 1\right]\left(1 - e^{-\alpha P^*\theta}\right),\\
a_{22} &= -r\beta\theta H^* a_1^* a_3^* + {H^*}^2 a_1^*\alpha^2\theta\tfrac{a_2^*}{\beta K}(1 - \theta)(1 - a_3^*) + \tfrac{a_2^*}{\beta K}H^*\theta\alpha,\\
a_{23} &= \tfrac{a_1^* a_3^* r\theta}{2K^2}(H^* r\theta - 2K) + \tfrac{a_1^*\omega(1-a_3^*)}{K}(K - H^* r\theta - \tfrac{KH^*\omega}{2}), \text{ where } \omega = \alpha(1 - e^{-\alpha P^*\theta})(1 - \theta),\\
a_{24} &= -\beta r\theta a_1^* a_3^* + 2a_1^*\zeta + \tfrac{H^* r\theta a_1^*}{K}(\beta r\theta a_3^* - \zeta) - H^* a_1^*\omega\left[\beta r\theta\,(1 - a_3^*) + \zeta\right] + \alpha\theta e^{-\alpha P^*\theta},
\end{aligned}$$

where $\zeta = \alpha^2 H^*\theta e^{-\alpha P^*\theta}(1 - \theta)(1 - a_3^*)$,

$a_{25} = a_{25} = \frac{1}{2}[H^*\theta^2\left(\beta^2 r^2 a_1^* a_3^* - \alpha^2 e^{-\alpha P^*\theta}\right) - H^* a_1^*\theta\zeta\left(2r\beta - \alpha\right) - \frac{H^* a_1^* \zeta^2}{1-a_3^*}],$
$g_1 = \frac{a_1^* a_3^*}{p_1 K}\left(K - H^* r\theta\right) + \frac{H^*}{p_1} a_1^*\omega\left(1 - a_3^*\right),$
$g_2 = \frac{H^* a_1^*}{p_1}\left(\zeta - r\beta\theta a_3^*\right),$
$g_3 = \frac{H^* r\beta\theta a_1^*}{p_1 K}\left(r\theta a_3^* - \frac{K a_3^*}{H^*} - K\omega\left(1 - a_3^*\right) - \frac{\zeta}{\beta}\right) + \frac{a_1^*\zeta}{p_1}\left(2 - H^*\omega\right).$

Construct an invertible matrix

$$T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}$$

transforming the system (10)

$$\begin{pmatrix} x \\ y \end{pmatrix} = T\begin{pmatrix} u \\ v \end{pmatrix}, \tag{11}$$

then with the transformation of (11), the system (10) can be transformed as:

$$\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,\tilde{p}) \\ g(u,v,\tilde{p}) \end{pmatrix}, \tag{12}$$

where

$$\begin{aligned} f(u,v,\tilde{p}) =& \left(\frac{a_{13}(\lambda_2 - a_{11})}{a_{12}(1+\lambda_2)} - \frac{a_{23}}{1+\lambda_2}\right)x^2 + \left(\frac{a_{14}(\lambda_2 - a_{11})}{a_{12}(1+\lambda_2)} - \frac{a_{24}}{1+\lambda_2}\right)xy + \left(\frac{a_{15}(\lambda_2 - a_{11})}{a_{12}(1+\lambda_2)} - \frac{a_{25}}{1+\lambda_2}\right)y^2 \\ &+ \left(\frac{e_1(\lambda_2 - a_{11})}{a_{12}(1+\lambda_2)} - \frac{g_1}{1+\lambda_2}\right)\tilde{p}x + \left(\frac{e_2(\lambda_2 - a_{11})}{a_{12}(1+\lambda_2)} - \frac{g_2}{1+\lambda_2}\right)\tilde{p}y + \left(\frac{e_3(\lambda_2 - a_{11})}{a_{12}(1+\lambda_2)} - \frac{g_3}{1+\lambda_2}\right)\tilde{p}xy \\ &+ O((|x| + |y| + |\tilde{p}|)^3), \\ g(u,v,\tilde{p}) =& \left(\frac{a_{13}(1+a_{11})}{a_{12}(1+\lambda_2)} + \frac{a_{23}}{1+\lambda_2}\right)x^2 + \left(\frac{a_{14}(1+a_{11})}{a_{12}(1+\lambda_2)} + \frac{a_{24}}{1+\lambda_2}\right)xy + \left(\frac{a_{15}(1+a_{11})}{a_{12}(1+\lambda_2)} + \frac{a_{25}}{1+\lambda_2}\right)y^2 \\ &+ \left(\frac{e_1(1+a_{11})}{a_{12}(1+v\lambda_2)} + v\frac{g_1}{1+\lambda_2}\right)\tilde{p}x + \left(\frac{e_2(1+a_{11})}{a_{12}(1+\lambda_2)} + \frac{g_2}{1+\lambda_2}\right)\tilde{p}y + \left(\frac{e_3(1+a_{11})}{a_{12}(1+\lambda_2)} + \frac{g_3}{1+\lambda_2}\right)\tilde{p}xy \\ &+ O((|x| + |y| + |\tilde{p}|)^3), \end{aligned}$$

where $x = a_{12}(u+v)$ and $y = -(1+a_{11})u + (\lambda_2 - a_{11})$.

We approximate the central manifold at the origin with respect to the parameter p_1, using the central manifold theorem (CMT) of system (12). According to the CMT, for parameter values near $\tilde{p} = 0$, the dynamics of system (10) around the fixed point $(0,0)$ can be analysed using the behaviour of a mapping of a family of parameters on the central manifold [37].

Let $W^c(0,0,0)$ be the center manifold of (12) evaluated at $(0,0)$ in a small neighborhood of $\tilde{p} = 0$; then, $W^c(0,0,0)$ can be written as:

$$W^c = \left\{(u,v,\tilde{p}) \in IR^3 : v = h_2\left(u,\tilde{p}\right), h_2\left(0,0\right) = Dh_2\left(0,0\right) = 0\right\}$$

The conditions $h_2\left(0,0\right) = Dh_2\left(0,0\right) = 0$ show that the center manifold is tangent to $v = 0$ axis. Assume that $h_2\left(u,\tilde{p}\right)$ has the following form:

$$h_2\left(u,\tilde{p}\right) = m_1 u^2 + m_2 u\tilde{p} + m_3\tilde{p}^2 + O\left(\left(u+\tilde{p}\right)^3\right). \tag{13}$$

The center manifold (13) must satisfy the following equation:

$$N\left(h_2\left(u,\tilde{p}\right)\right) = h_2\left(-u + f\left(u, h_2\left(u,\tilde{p}\right),\tilde{p}\right)\right) = \lambda_2 h_2\left(u,\tilde{p}\right) + g\left(u, h_2\left(u,\tilde{p}\right),\tilde{p}\right). \tag{14}$$

Substituting (12) and (13) into (14) and comparing the coefficients, we obtain

$$\begin{aligned}
m_1 &= \frac{1}{a_{12}\left(1-\lambda_2{}^2\right)}[{a_{12}}^2 a_{13}(1+a_{11}) + a_{23}{a_{12}}^3 - a_{14}a_{12}(1+a_{11})^2 - {a_{12}}^2 a_{24}(1+a_{11}) \\
&\quad + a_{15}(1+a_{11})^3 + a_{25}(1+a_{11})^2], \\
m_2 &= \frac{1}{a_{12}\left(1+\lambda_2{}^2\right)}[-a_{12}e_1\left(1+a_{11}\right) - g_1{a_{12}}^2 + e_2(1+a_{11})^2 + g_2\left(1+a_{11}\right)], \\
m_3 &= 0.
\end{aligned}$$

Restricting mapping (12) to the central manifold $W^c(0,0,0)$, we can obtain

$$F: u \to -u + f_1u^2 + f_2u\tilde{p} + O\left((u+\tilde{p})^3\right),$$

where

$$\begin{aligned}
f_1 &= \frac{1}{a_{12}\left(1+\lambda_2\right)}[\left(a_{13}\left(\lambda_2-a_{11}\right)-a_{12}a_{23}\right){a_{12}}^2 - a_{12}\left(1+a_{11}\right)\left(a_{14}\left(\lambda_2-a_{11}\right)-a_{12}a_{24}\right) \\
&\quad + \left(1+a_{11}\right)^2\left(a_{15}\left(\lambda_2-a_{11}\right)-a_{12}a_{25}\right)], \\
f_2 &= \frac{1}{a_{12}\left(1+\lambda_2\right)}[a_{12}e_1\left(\lambda_2-a_{11}\right) - a_{12}g_1 - \left(1+a_{11}\right)e_2\left(\lambda_2-a_{11}\right) + a_{12}g_2\left(1+a_{11}\right)].
\end{aligned}$$

Let

$$\begin{aligned}
\alpha_1 &= \left.\left(\frac{\partial^2 f}{\partial u\partial\tilde{p}} + \frac{1}{2}\frac{\partial f}{\partial\tilde{p}}\frac{\partial^2 f}{\partial^2 u}\right)\right|_{(0,0)}, \\
\alpha_2 &= \left.\left(\frac{1}{6}\frac{\partial^3 f}{\partial^3 u} + \left(\frac{1}{2}\frac{\partial^2 f}{\partial^2 u}\right)^2\right)\right|_{(0,0)}.
\end{aligned}$$

From the above analysis, we have the following results regarding the existence of the PDB of system (3).

Theorem 3. If $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, the original system bifurcates at the immobile point (H^*, P^*) when the parameters $\tilde{p}$ vary in a sufficiently small neighbourhood of the origin, and if $\alpha_2 > 0\,(\alpha_2 < 0)$, then a stable (unstable) 2-cycle track bifurcates from the immobile point.

5 | NUMERICAL SIMULATIONS

In this section, we will give some numerical simulations to verify the main theoretical results analyzed above, and also discuss more complex dynamic behaviors of system (3). In order to better observe the stability of system (3), a mean line (red curve) is drawn in the bifurcation diagram of this paper (see Figure 4, Figure 7 - Figure 10). When the curve in the bifurcation diagram coincides with the mean line, the system is stable at this time.

Example 1. In Figure 4(a), choose q as the bifurcation parameter, and the other parameter values are fixed as: $r = 4.5$, $K = 1$, $\beta = 0.1$, $\theta = 0.1$, $\alpha = r$. It can be seen from the Figure 4(a) that when the insecticidal rate satisfies $0 < q < 0.3023$, the period-doubling bifurcation occurs in system (3), and the period decreases gradually. When $0.3023 < q < 0.8826$, the number of pests increases as the insecticidal rate increases. In this scenario, the paradox occurs. When $0.8826 < q < 1$, both of the pest population and their natural enemy decreases gradually as the increase of insecticidal rate. As we continuously increase the insecticidal rate q, the number of natural enemies first becomes chaotic, then increases, and gradually decreases when it reaches the local maximum. From the biological point of view, an increasing of the pest population can stimulate the growth of natural enemies, if the number of pests are small, which consequently cannot provide sufficient food sources to the enemy, as a result, the number of natural enemies will decrease as well. In Figure 4(b), we choose $\beta = 3$ and the other parameters as the same as those in Figure 4(a). It can be seen that when $0 < q < 0.4184$, system (3) is in a state of double periodic bifurcation. When $0.4184 < q < 0.9690$, the number of pests increases with the increase of q, while the number of natural enemies decreases with the increase of q. When $0.9690 < q < 1$, the number of pests and natural enemies decrease as q increases. In Figure 4(c), we set the parameter $\theta = 0.9$, and the other parameters are chosen as the same as those in Figure 4(a). It can be seen from the Figure 4(c) that system (3) does not suffer from paradoxes, and when $0.7912 < q < 1$, the number of pests and their natural enemies decrease with the increase of insecticidal rate q. In Figure 4(d), we choose the parameters $r = 5$, $K = 1$, $\beta = 4$, $\theta = 0.7$ and $\alpha = r$. From the Figure 4(d), we can see that system (3) have very complex dynamic behavior, and there are a large number of attractors. In Figure 4(e), if we fix the parameters as $r = 5$, $K = 1$, $\beta = 3$, $\theta = 0.4$ and $\alpha = r$, system (3) is chaotic when $0 < q < 0.78$, and system (3) becomes stable when $0.78 < q < 1$.

It can be seen from figures 4(a) - 4(e) that when different parameter values are selected, system (3) can show various complex dynamic behaviors. When the growth rate of the pest r is small, choosing the later stage to spray large doses of pesticides, the growth of pests will be effectively suppress. In what follows, we take the growth rate r and the time θ of insecticide application as the key parameters to explore the impact of them on pest density, aiming at achieving the efficient pest control strategies and successfully prevent pest outbreak.

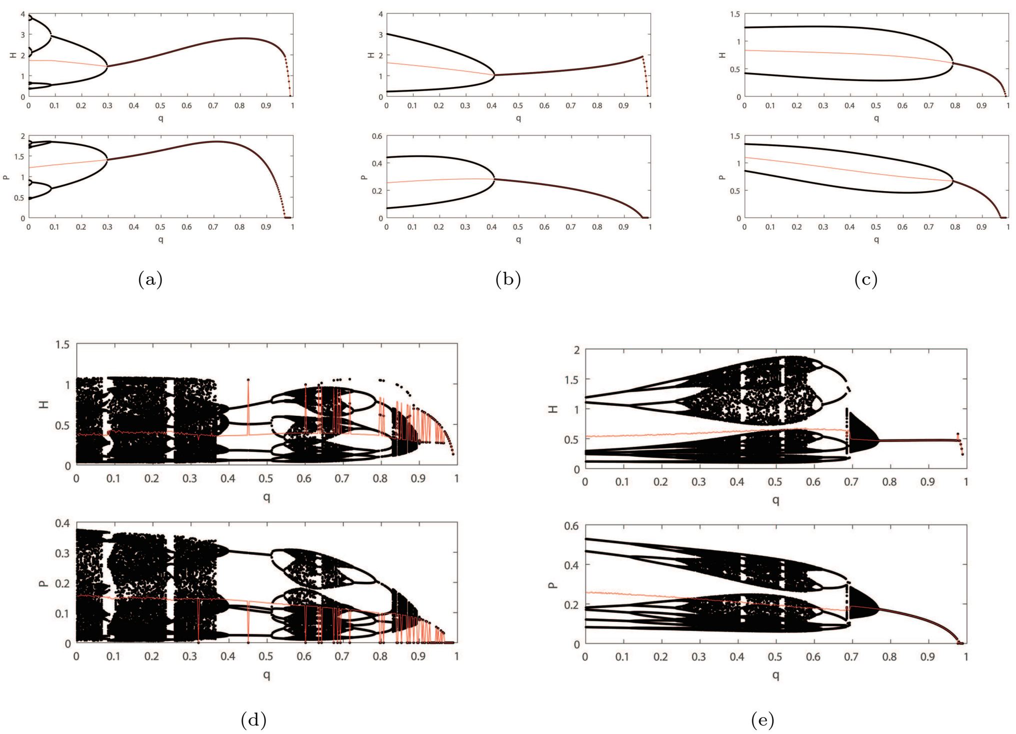
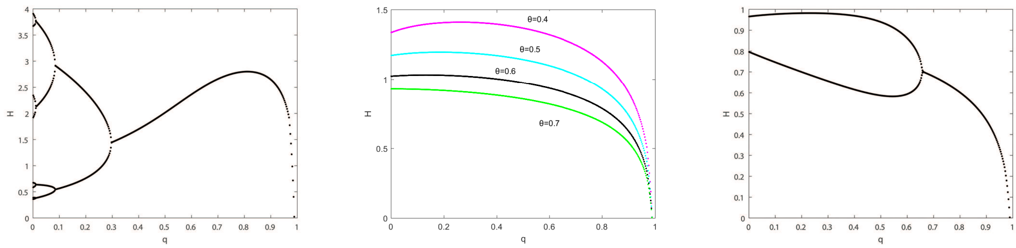


FIGURE 4 Bifurcation diagram of system (3) with respect to q.

Example 2. Set $r = 4.5$, $K = 1$, $\beta = 0.1$, $\alpha = r$. When $\theta = 0.1$, the density of pests increases as the increase of insecticidal rate q, that is, there is the paradox phenomenon, when a local maximum is reached, the density of pests will decrease with the increasing of insecticidal rate q, as shown in Figure 5(a). It can be seen from Figure 5(b) that for intermediate value of θ, the larger θ is, the less time the pest system produces paradox with the increase of insecticidal rate q; however, comparing Figures 5(a) and 5(c), it can be found that the pest system will produce period-doubling bifurcation when θ is large or small. In this situation, the control of pests can be very difficult, and it is crucial to choose the right time to spray the insecticide. In other words, the earlier use of pesticide does not mean a better outcomes in terms of the control of the growth of the pests.

In addition, to understand the impact of key parameters on pest density, we drew regional plots of $q-\theta$ with different r.



(a) (b) (c)

FIGURE 5 Relation curve between insecticidal rate q and pest density H with different θ. (a) $\theta = 0.1$, (c) $\theta = 0.85$. In particular, four reaction curves are shown in (b): magenta ($\theta = 0.4$); blue ($\theta = 0.5$); black ($\theta = 0.6$); green ($\theta = 0.7$).

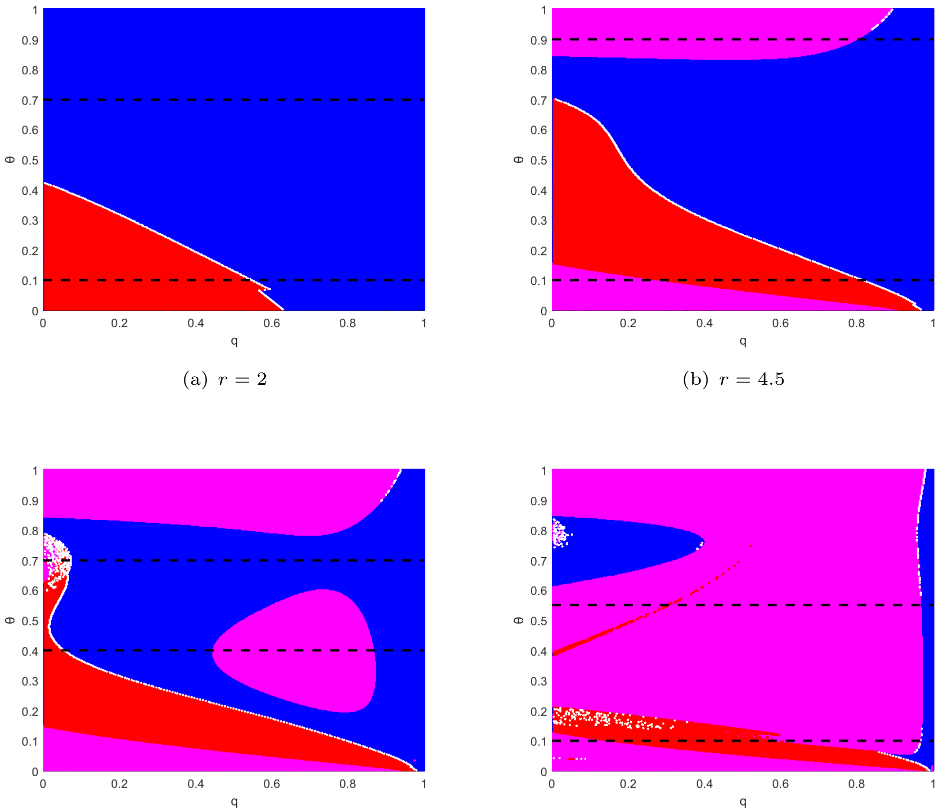
Example 3. Selecting parameters $K = 1$, $\beta = 0.1$, $\alpha = 4$. By observing Figure 6, it can be found that with the continuous increase of pest growth rate r, both the regions with paradoxes of stability and the regions without paradoxes of stability are gradually decrease and the unstable region gradually increases. When r is small, there will be no chaos phenomenon in the pest system, but there will be paradoxes at this time, that is to say, with the increase of the insecticidal rate, the density of pests will increase at the beginning, and when it reaches a critical value, the density of pests will decrease, as shown in Figure 6(a). Once r gradually increases, the pest system will generate chaos, and the ecosystem will become more complex, so the number of pests will not be easily controlled, as shown in Figures 6(b), 6(c) and 6(d). In Figure 6(d), the white dots represent multiple attractors. Next we will draw some bifurcation diagrams to analyze Figure 6 in detail.

Example 4. When $\theta = 0.1$, $q \in (0, 0.54)$, the pest system produces paradox phenomenon, that is, the number of pests increases with q, which corresponds to the red region in Figure 6(a), and for $q \in (0.54, 1)$, the number of pests decreases with q, which corresponds to the blue region in Figure 6(a); when $\theta = 0.7$, the pest system is generally stable without paradox, corresponding to the blue region. Combining the dynamic behavior shown in Figures 7(a) and 7(b), it can be seen that the dynamic behavior is consistent with that shown in Figure 6(a) above. The following sub-bifurcation diagrams are all similar analyses and will not be detailed later.

Example 5. At $\theta = 0.1$, it can be seen that the pest system is first chaotic, then paradoxical, and finally stable and paradox-free; at $\theta = 0.9$, there are only two states of the pest system, first chaotic and then stable and paradox-free, in line with the region division in Figure 6(b).

Example 6. In Figure 9(a), when q is small, the pest system experiences paradox in a small range, and when $q \in (0.48, 0.86)$, the pest system occurs period-doubling bifurcation; in Figure 9(b), for $0 < q < 0.9$, the pest system has multiple attractors and paradox phenomenon, while for $0.9 < q < 1$, the number of pests decreases as q increases and there are no paradox phenomenon. The overall trend of the combined Figures 9(a) and 9(b) are consistent with our regional division of Figure 6(c) above.

Example 7. When $\theta = 0.1$, the pest system first undergoes chaos, then stabilizes to produce paradoxes, then chaos again, and the pest system tends to stabilize when q is large enough; when $\theta = 0.55$, the pest system is in a chaotic state on a large scale, and there are multiple attractors with a small range of paradoxes produced, and the pest system tends to stabilize only when q is large enough. Analysis of the overall trend of Figures 10(a) and 10(b) are consistent with the division of the region in Figure 6(d).



(a) $r = 2$ (b) $r = 4.5$

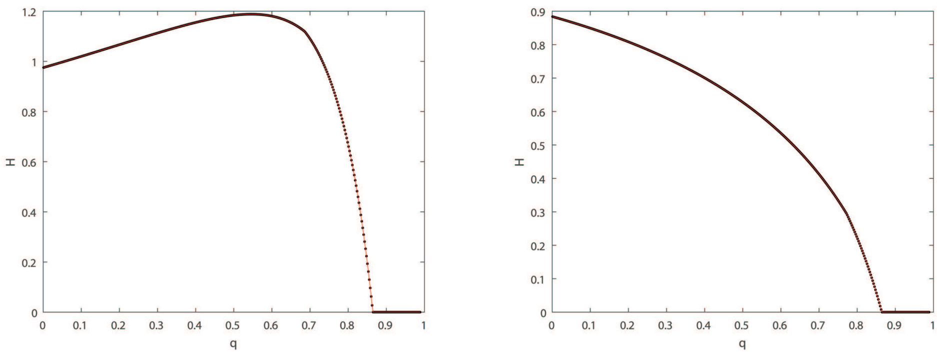
(c) $r = 5$ (d) $r = 6$

FIGURE 6 The effect of timing of insecticide delivery and intensity of kill rate (θ, q) on pest density $H(n)$ with different r. Among them, blue indicates stable without paradox, red indicates stable with paradox, magenta indicates unstable, white indicates paradox line and the region with white dots indicates the presence of attractor.

6 | CONCLUSION AND DISCUSSION

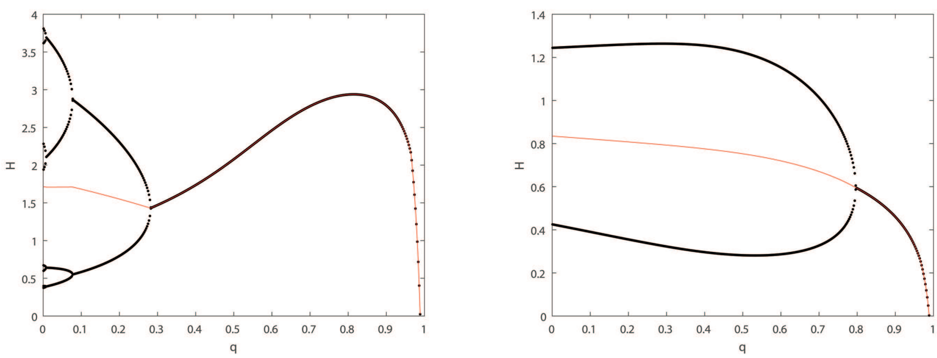
Existing studies shown that the contradictory phenomena in pest control have brought significant challenges to the modeling and prediction of the pest growth process. We extend the Nicholson-Bailey model by considering the pesticide delivery is implemented within the time interval between two generations of pests, and construct a discrete-time model of two populations to describe the impact of pesticides on the growth of pests and natural enemies, and further explores the impact of pesticide delivery's time and intensity on pests' growth. In contrast to continuous-time models, the proposed model can not only provides a natural description of the growth of pests and natural enemies and the insecticidal control in arbitrary time within two generations, but also shows complex dynamics of the development of pests and natural enemies and the relationship between the growth of pests and the insecticidal rate. In addition, it can be found that the model has a paradox phenomenon under certain parameter conditions (see Figure 4(a)), in this case, if we spray a lot of pesticides, the number of pests can be effectively controlled.

In conclusion, we first theoretically analyzed the existence of trivial fixed point, and verified the existence of the positive immobile point by numerical simulation. We then analyzed the stability of the fixed points and provided the corresponding conditions. Finally, using the central manifold theory and bifurcation theory, we studied the influence of the use of insecticide on pest density and natural enemy density, and observed the existence of the period-doubling bifurcation. The numerical analysis reveals that pest growth is sensitive to the timing and intensity of insecticide application, which poses a great challenge



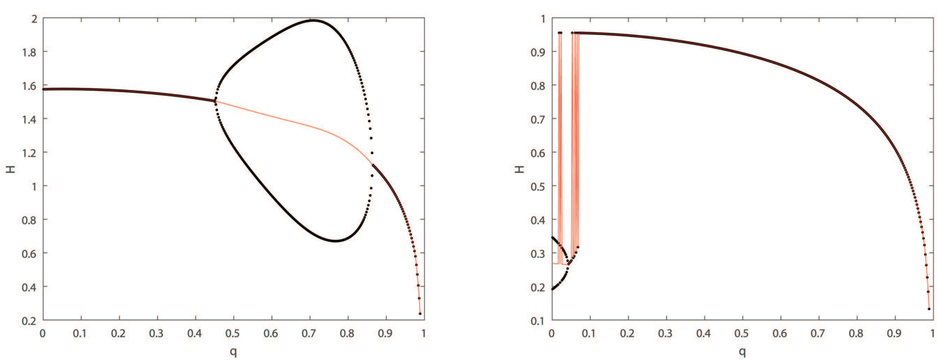
(a) $\theta = 0.1$ (b) $\theta = 0.7$

FIGURE 7 Bifurcation of H with respect to q under different θ (Sub-bifurcation diagram for Figure 6(a)). The rest of the parameters are the same as in Figure 6(a).



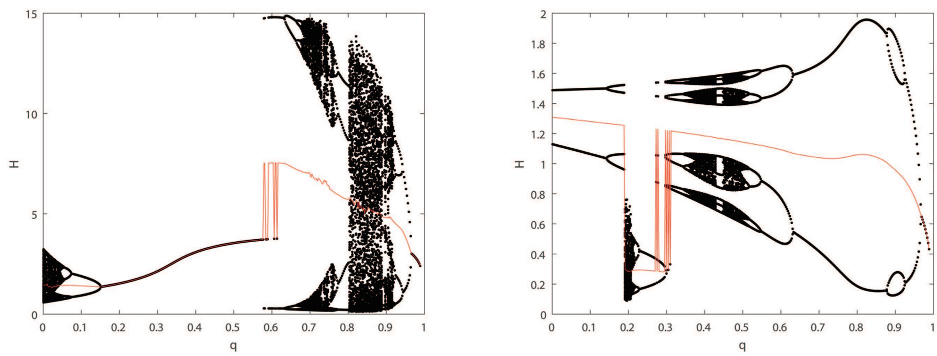
(a) $\theta = 0.1$ (b) $\theta = 0.9$

FIGURE 8 Bifurcation of H with respect to q under different θ (Sub-bifurcation diagram for Figure 6(b)). The rest of the parameters are the same as in Figure 6(b).



(a) $\theta = 0.4$ (b) $\theta = 0.7$

FIGURE 9 Bifurcation of H with respect to q under different θ (Sub-bifurcation diagram for Figure 6(c)). The rest of the parameters are the same as in Figure 6(c)



(a) $\theta = 0.1$ (b) $\theta = 0.55$

FIGURE 10 Bifurcation of H with respect to q under different θ (Sub-bifurcation diagram for Figure 6(d)). The rest of the parameters are the same as in Figure 6(d).

for pest control. A massive amounts of numerical studies have shown that for some specific values of θ and r, low-intensity insecticides can promote the growth of pests, while high-intensity insecticides inhibit the growth of pests, namely, the hormetic effects occur, as shown in Figures 5-10. In addition, it was also found that internal growth rate r, insecticidal rate q and insecticidal time θ have significant effects on the growth of pests and natural enemies. The fine-tuning of these three parameters may lead to an imbalance in system homeostasis, cause oscillations and complex dynamic processes of pests and natural enemies (see Figures 4(d) and 4(e)), making the study of pest system challenging. Therefore, the timing of the implementation of insecticides spraying, the intensity of insecticides and the growth rate of pests are the three key factors to be considered when dealing with pest managment, that is, choosing a reasonable timing and intensity of insecticides can not only reduce the complexity of the system, but also can successfully prevent pest outbreaks.

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