LIEB'S AND LIONS' TYPE THEOREMS ON HEISENBERG GROUP AND APPLICATIONS

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Abstract. In this paper, we make an effort to establish Lieb's and Lions' type theorems on Heisenberg Group, and then apply them to study the existence of solution for variational problem on Heisenberg group.

1. Introduction

Let $\xi := (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ with $N \geqslant 1$. The Heisenberg group denoted by $\mathbb{H} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$ equipped with the following group operation:

$$\xi \circ \xi' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y')),$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^N$, $(0, 0, 0)$ is the identity element, and $(-x, -y, -t)$ is the inverse element of (x, y, t).

The distance between ξ and η on $\mathbb{H}$ defined by

$$d(\xi, \eta) := d(\eta^{-1} \circ \xi, 0).$$

The Heisenberg ball of center η and radius r is defined by $B_{\mathbb{H}}(\eta, r) := \{\xi \in \mathbb{H} | d(\xi, \eta) < r\}$, and it satisfies

$$|B_{\mathbb{H}}(\eta, r)| := |B_{\mathbb{H}}(0, r)| = r^Q |B_{\mathbb{H}}(0, 1)|, \tag{1.1}$$

where $|\cdot|$ is the $(2N + 1)$ dimensional Lebesgue measure on $\mathbb{H}$, and $Q = 2N + 2$ is the homogeneous dimension of the group.

The Lie algebra of $\mathbb{H}$ is generated by the left invariant vector fields

$$T = \frac{\partial}{\partial t}, \;\; X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \;\; Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

the commutation relations

$$[X_i, Y_j] = -4\delta_{ij} T, \;\; [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_j, T] = 0.$$

The Heisenberg Laplacian is

$$\Delta_{\mathbb{H}} := \sum_{i=1}^{N} \left(X_i^2 + Y_i^2 \right),$$

and we use the notation

$$\nabla_{\mathbb{H}} u := (X_1 u, \ldots, X_N u, Y_1 u, \ldots, Y_N u).$$

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The $D^{1,2}(\mathbb{H})$ is defined as the completion of $C_0^\infty(\mathbb{H})$ with the semi-norm

$$\|u\|_{D^{1,2}(\mathbb{H})} := \left(\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

The norm of $L^p(\mathbb{H})$ $(p>1)$ is given by

$$\|u\|_{L^p(\mathbb{H})} := \left(\int_{\mathbb{H}} |u|^p \mathrm{d}\xi\right)^{\frac{1}{p}}.$$

The Folland-Stein Sobolev type space $S^{1,2}(\mathbb{H})$ [9] is given by

$$S^{1,2}(\mathbb{H}) := \left\{ u \in D^{1,2}(\mathbb{H}) \,\middle|\, \int_{\mathbb{H}} |u|^2 \mathrm{d}\xi < \infty \right\}$$

with the norm

$$\|u\|_{S^{1,2}(\mathbb{H})} := \left(\int_{\mathbb{H}} |\nabla_{\mathbb{H}} u|^2 + |u|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}.$$

Proposition 1.1. *[9] Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$. Then the embedding $S^{1,2}(\mathbb{H}) \hookrightarrow L^r(\mathbb{H})$ is continuous, where $r \in [2, 2^*]$ and $2^* = \frac{2Q}{Q-2}$.*

From Proposition 1.1, we know that the following embeddings are not compactness

$$S^{1,2}(\mathbb{H}) \hookrightarrow L^r(\mathbb{H}), \quad r \in [2, 2^*].$$

Hence, it is difficult to show that a bounded sequence has a convergence subsequence when we seek the weak solution of mathematical physics equation. Lieb [12] established the famous Lieb's translation theorem to overcome the lack of compactness on $\mathbb{R}^N$. Lions [13,14] investigated the famous Lions' vanishing theorem to overcome the lack of compactness on $\mathbb{R}^N$. Their results is not avilable to our problem on Heisenberg group. To overcome it, we study two different methods.

First, we establish a Lieb's translation theorem on Heisenberg group as follows.

Theorem 1.1. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, and $\{u_n\}$ be a bounded sequence in $S^{1,2}(\mathbb{H})$ satisfing* **Condition A***: $\lim\limits_{n\to\infty} \int_{\mathbb{H}} |u_n|^q \mathrm{d}\xi > 0$, where $q \in (2, 2^*)$. Then there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ convergence strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$.*

Second, we investigate the Lions' vanishing theorem on Heisenberg group as follows.

Theorem 1.2. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, and $\{u_n\}$ be a bounded sequence in $S^{1,2}(\mathbb{H})$ satisfing* **Condition B***: $\sup\limits_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi \to 0$, where $q \in [2, 2^*)$. Then $u_n \to 0$ in $L^t(\mathbb{H})$ for $t \in (2, 2^*)$.*

Pucci-Temperini [18] define the fractional Sobolev space $S^{s,p}(\mathbb{H})$ as the completion of $C_0^\infty(\mathbb{H})$ with respect to the norm, for $s \in (0,1)$, $p \in (1, \infty)$ and $sp < Q$,

$$\|u\|_{S^{s,p}(\mathbb{H})} := \|u\|_{D^{s,p}(\mathbb{H})} + \|u\|_{L^p(\mathbb{H})},$$

where

$$\|u\|_{D^{s,p}(\mathbb{H})} := \left(\int_{\mathbb{H}}\int_{\mathbb{H}} \frac{|u(\xi)-u(\eta)|^p}{d^{Q+sp}(\eta^{-1}\circ\xi)}\mathrm{d}\xi\mathrm{d}\xi\right)^{\frac{1}{p}}.$$

Adimurthi-Mallick [1] established the fractional Sobolev inequality,

$$S_{p_s^*}\|u\|^p_{L^{p_s^*}(\mathbb{H})} \leqslant \|u\|^p_{D^{s,p}(\mathbb{H})}, \text{ for } u \in C_0^\infty(\mathbb{H}), \ \ p_s^* = \frac{Np}{N-sp}. \tag{1.2}$$

For more results about inequalities on Heisenberg group, we refer to Ruzhansky-Suragan [21, 22] and Roncal-Thangavelu [20]. It is easy to get the following embedding result by (1.2) and Hölder's inequality.

Proposition 1.2. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, $s \in (0,1)$, $p \in (1,\infty)$ and $sp < Q$. Then the embedding $S^{s,p}(\mathbb{H}) \hookrightarrow L^r(\mathbb{H})$ is continuous, where $r \in [p, p_s^*]$.*

From Proposition 1.2, we know that the following embeddings are not compactness

$$S^{s,p}(\mathbb{H}) \hookrightarrow L^r(\mathbb{H}), \quad r \in [p, p^*].$$

By the principle of symmetric criticality, Balogh-Kristaly [2] and Bisci-Repovs [4] studied the uncompactness problem for $r \in (p, p^*)$. Pucci-Temperini [18] proved the concentration-compactness principle, which is a useful tool to above problem. In the following, we also extended the Lieb's translation theorem and Lions' vanishing theorem to the fractional version without proof.

Theorem 1.3. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, $s \in (0,1)$, $p \in (1,\infty)$, $sp < Q$ and $\{u_n\}$ be a bounded sequence in $S^{s,p}(\mathbb{H})$ satisfing* **Condition A***: $\lim\limits_{n\to\infty} \int_{\mathbb{H}} |u_n|^q \mathrm{d}\xi > 0$, where $q \in (p, p_s^*)$. Then there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ convergence strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$.*

Theorem 1.4. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, $s \in (0,1)$, $p \in (1,\infty)$, $sp < Q$ and $\{u_n\}$ be a bounded sequence in $S^{s,p}(\mathbb{H})$ satisfing* **Condition B***: $\sup\limits_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi \to 0$, where $q \in [p, p_s^*)$. Then $u_n \to 0$ in $L^t(\mathbb{H})$ for $t \in (p, p_s^*)$.*

As applications of Theorems 1.1-1.4, we consider the following minimizing problem

$$S_t = \inf_{u\in S^{s,p}(\mathbb{H})\setminus\{0\}} \frac{\|u\|^p_{S^{s,p}(\mathbb{H})}}{\left(\int_{\mathbb{H}}|u|^t\mathrm{d}\xi\right)^{\frac{p}{t}}}. \tag{S_t}$$

We have

Theorem 1.5. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, $s \in (0,1)$, $p \in (1,\infty)$, $sp < Q$ and $t \in (p, p_s^*)$. Then problem (S_t) has a minimizer.*

Furthermore, we study the following equation

$$(-\Delta)^s_{\mathbb{H}} u + u = \gamma|u|^{r-2}u + |u|^{2_s^*-2}u, \ \ \xi \in \mathbb{H}. \tag{S_*}$$

The integral representation for the fractional operator $(-\Delta)^s_{\mathbb{H}}$ is defined by, $u \in C_0^\infty(\mathbb{H})$,

$$(-\Delta)^s_{\mathbb{H}} u(\xi) = c_{Q,s}\int_{\mathbb{H}} \frac{u(\xi)-u(\eta)}{d^{Q+2s}(\eta^{-1}\circ\xi)}\mathrm{d}\eta$$

where $c_{Q,s}$ is a positive constant, see [20, Proposition 4.1]. We have

Theorem 1.6. *Let $\mathbb{H}$ be a Heisenberg group with $Q \geqslant 4$, $s \in (0,1)$, $2s < Q$ and $r \in (2, 2_s^*)$. Then there exists $\gamma_0 > 0$ such that for every $\gamma \in (\gamma_0, \infty)$ equation (S_*) has a non-trivial solution.*

Remark 1.1. *The critical Schrödinger equation on Heisenberg group have been extensively investigated. Kristaly [11] studied the existence of nodal solutions for the fractional Yamabe problem. For more results, we refer to [3, 5, 7, 15–17, 19] and the references therein.*

2. Lieb's Translation Theorem on Heisenberg group

In this section, we present Lieb's translation theorem on Heisenberg group.

Lemma 2.1. *Let $Q \geqslant 4$ and $q \in (2, 2^*)$. Then the following inequality holds*

$$\int_{\mathbb{H}} |u|^q \mathrm{d}\xi \leqslant 2C(N+1)^2 \left(\sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{q-2}{q}} \|u\|^2_{S^{1,2}(\mathbb{H})},$$

for all $u \in S^{1,2}(\mathbb{H})$.

Proof. Let $u \in S^{1,2}(\mathbb{H})$ and $q \in (2, 2^*)$. From Hölder's inequality and Proposition 1.1, we have

$$\begin{aligned}
&\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \\
\leqslant &\left(\int_{B_{\mathbb{H}}(z,1)} |u|^2 \mathrm{d}\xi \right)^{\frac{2^*-q}{2^*-2}} \left(\int_{B_{\mathbb{H}}(z,1)} |u|^{2^*} \mathrm{d}\xi \right)^{\frac{q-2}{2^*-2}} \\
\leqslant &C \left(\int_{B_{\mathbb{H}}(z,1)} |\nabla u|^2 \mathrm{d}\xi + \int_{B_{\mathbb{H}}(z,1)} |u|^2 \mathrm{d}\xi \right)^{\frac{2^*-q}{2^*-2}} \left(\int_{B_{\mathbb{H}}(z,1)} |\nabla u|^2 \mathrm{d}\xi \right)^{\frac{2^*}{2}\cdot\frac{q-2}{2^*-2}} \\
= &C \left(\int_{B_{\mathbb{H}}(z,1)} |\nabla u|^2 \mathrm{d}\xi + \int_{B_{\mathbb{H}}(z,1)} |u|^2 \mathrm{d}\xi \right)^{\frac{q}{2}}.
\end{aligned} \tag{2.1}$$

Applying (2.1), we know

$$\begin{aligned}
&\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \\
= &\left(\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{2}{q}} \left(\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{q-2}{q}} \\
\leqslant &C \left(\int_{B_{\mathbb{H}}(z,1)} |\nabla u|^2 \mathrm{d}\xi + \int_{B_{\mathbb{H}}(z,1)} |u|^2 \mathrm{d}\xi \right) \left(\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{q-2}{q}}.
\end{aligned}$$

Covering $\mathbb{R}^N$ by balls of radius 1, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls. Note that $\xi = (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. Covering $\mathbb{H}$

by balls of radius 1, in such a way that each point of $\mathbb{H}$ is contained in at most $(N+1)\times(N+1)\times 2$ balls, we find

$$\int_{\mathbb{H}} |u|^q \mathrm{d}\xi \leqslant 2C(N+1)^2 \left(\sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi\right)^{\frac{q-2}{q}} \|u\|^2_{S^{1,2}(\mathbb{H})}.$$

□

As the application of Lemma 2.1, we present the Lieb's translation theorem on Heisenberg group.

Proof of Theorem 1.1. Note that $\{u_n\}$ is a bounded sequence in $S^{1,2}(\mathbb{H})$. Up to a subsequence, we assume

$$u_n \rightharpoonup u \text{ in } S^{1,2}(\mathbb{H}),\ u_n \to u \text{ a.e. in } \mathbb{H}, u_n \to u \text{ in } L^q_{loc}(\mathbb{H}).$$

Applying Lemma 2.1 and **Condition A**, there exists $C>0$ such that

$$\sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi \geqslant C > 0.$$

Note that $\{u_n\}$ is bounded in $S^{1,2}(\mathbb{H})$ and $S^{1,2}(\mathbb{H}) \hookrightarrow L^q(\mathbb{H})$, we have

$$\sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi \leqslant \int_{\mathbb{H}} |u_n|^q \mathrm{d}\xi \leqslant C.$$

Hence, there exists C_0 such that

$$C_0 \leqslant \sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi \leqslant C_0^{-1}.$$

From above inequality, there exists $z_n \in \mathbb{H}$ such that

$$\int_{B_{\mathbb{H}}(z_n,1)} |u_n|^q \mathrm{d}\xi \geqslant \sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi - \frac{C}{2n} \geqslant C_1 > 0.$$

Set $\bar{u}_n := u_n(x+z_n)$. Then $\|\bar{u}_n\|_{S^{1,2}(\mathbb{H})} = \|u_n\|_{S^{1,2}(\mathbb{H})}$ and

$$\int_{B_{\mathbb{H}}(0,1)} |\bar{u}_n|^q \mathrm{d}\xi \geqslant C_1 > 0.$$

Up to a subsequence, there exists $\bar{u}$ such that

$$\bar{u}_n \rightharpoonup \bar{u} \text{ in } S^{1,2}(\mathbb{H}),\ \ \bar{u}_n \to \bar{u} \text{ a.e. in } \mathbb{H}.$$

Applying the embedding $S^{1,2}(\mathbb{H}) \hookrightarrow L^q_{loc}(\mathbb{H})$ is compact, we deduce that $\bar{u} \not\equiv 0$. □

The proof of Theorem 1.3 is similar to Theorem 1.1. So we omit it.

3. Lions' Vanishing Theorem on Heisenberg group

We establish the following refined Sobolev inequality.

Lemma 3.1. *Let $Q \geqslant 4$, $q \in [2, 2^*)$ and $r = \frac{2(2^*-q)+2^*q}{2^*}$. Then the following inequality holds*

$$\int_{\mathbb{H}} |u|^r \mathrm{d}\xi \leqslant C \left(\sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{2^*-r}{2^*-q}} \|u\|^2_{S^{1,2}(\mathbb{H})},$$

for all $u \in S^{1,2}(\mathbb{H})$.

Proof. Let $u \in S^{1,2}(\mathbb{H})$ and $r \in (q, 2^*)$. From Hölder's and Sobolev's inequalities, we have

$$\begin{aligned}\int_{B_{\mathbb{H}}(z,1)} |u|^r \mathrm{d}\xi \leqslant& \left(\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{2^*-r}{2^*-q}} \left(\int_{B_{\mathbb{H}}(z,1)} |u|^{2^*} \mathrm{d}\xi \right)^{\frac{r-q}{2^*-q}} \\ \leqslant& C \left(\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{2^*-r}{2^*-q}} \left(\int_{B_{\mathbb{H}}(z,1)} |\nabla u|^2 \mathrm{d}\xi \right)^{\frac{2^*}{2}\cdot\frac{r-q}{2^*-q}}.\end{aligned}$$

Choosing $r = \frac{2(2^*-q)+2^*q}{2^*}$. Then $\frac{2^*}{2} \cdot \frac{r-q}{2^*-q} = 1$ and

$$\int_{B_{\mathbb{H}}(z,1)} |u|^r \mathrm{d}\xi \leqslant C \left(\int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}\xi \right)^{\frac{2^*-r}{2^*-q}} \int_{B_{\mathbb{H}}(z,1)} |\nabla u|^2 \mathrm{d}\xi.$$

Covering $\mathbb{R}^N$ by balls of radius 1, in such a way that each point of $\mathbb{R}^N$ is contained in at most $N+1$ balls. Note that $\xi = (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$. Covering $\mathbb{H}$ by balls of radius 1, in such a way that each point of $\mathbb{H}$ is contained in at most $(N+1) \times (N+1) \times 2$ balls, we find

$$\int_{\mathbb{H}} |u|^r \mathrm{d}x \leqslant 2C(N+1)^2 \left(\sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u|^q \mathrm{d}x \right)^{\frac{2^*-r}{2^*-q}} \int_{\mathbb{H}} |\nabla u|^2 \mathrm{d}\xi.$$

□

Proof of Theorem 1.2. For $q \in [2, 2^*)$, if $\sup_{z\in\mathbb{R}^N} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}x \to 0$, then by Lemma 3.1, we have $\int_{\mathbb{H}} |u_n|^r \mathrm{d}x \to 0$, where $r = \frac{2(2^*-q)+2^*q}{2^*}$.

For any $r_1 \in (r, 2^*)$, it follows from Hölder's and Sobolev's inequalities that

$$\int_{\mathbb{H}} |u_n|^{r_1} \mathrm{d}x \leqslant \left(\int_{\mathbb{H}} |u_n|^r \mathrm{d}x \right)^{\frac{2^*-r_1}{2^*-r}} \left(\int_{\mathbb{H}} |u_n|^{2^*} \mathrm{d}x \right)^{\frac{r_1-r}{2^*-r}} \to 0.$$

For any $r_2 \in (2, r)$, it follows from Hölder's and Sobolev's inequalities that

$$\int_{\mathbb{H}} |u_n|^{r_2} \mathrm{d}x \leqslant \left(\int_{\mathbb{H}} |u_n|^2 \mathrm{d}x \right)^{\frac{r-r_2}{r-2}} \left(\int_{\mathbb{H}} |u_n|^r \mathrm{d}x \right)^{\frac{r_2-2}{r-2}} \to 0.$$

□

Remark 3.1. *The proof of Theorem 1.4 is similar to Theorem 1.2. So we omit it.*

4. Proof of Theorem 1.5

First, we give the proof of Theorem 1.5 via Lieb's translation theorem.

Proof of Theorem 1.5 (Method 1). Let $\{u_n\}$ be a minimizing sequence of (S_t). That is

$$\|u_n\|_{S^{s,p}(\mathbb{H})} \to S_t \quad \text{and} \quad \int_{\mathbb{H}} |u_n|^t \mathrm{d}\xi = 1 \quad \text{as} \quad n \to \infty.$$

It is easy to get the following results:

(1) $\{u_n\}$ is bounded in $S^{s,p}(\mathbb{H})$;

(2) $\int_{\mathbb{H}} |u_n|^t \mathrm{d}\xi = 1 > 0$.

From Theorem 1.3, we know that there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^t_{loc}(\mathbb{H})$ for all $t \in (p, p_s^*)$. Moreover, $\{\bar{u}_n\}$ is also a bounded minimizing sequence of (S_t).

Using the Brézis-Lieb lemma [6], one can deduce

$$\begin{aligned}
1 =& \lim_{n\to\infty} \int_{\mathbb{H}} |\bar{u}_n|^t \mathrm{d}\xi \\
=& \lim_{n\to\infty} \int_{\mathbb{H}} |\bar{u}_n - \bar{u}|^t \mathrm{d}\xi + \int_{\mathbb{H}} |\bar{u}|^t \mathrm{d}\xi \\
\leqslant& S_t^{-t} \lim_{n\to\infty} \|\bar{u}_n - \bar{u}\|^t_{S^{s,p}(\mathbb{H})} + S_t^{-t} \|\bar{u}\|^t_{S^{s,p}(\mathbb{H})} \\
\leqslant& S_t^{-t} \left(\lim_{n\to\infty} \|\bar{u}_n - \bar{u}\|_{S^{s,p}(\mathbb{H})} + \|\bar{u}\|_{S^{s,p}(\mathbb{H})} \right)^t \\
=& S_t^{-t} \lim_{n\to\infty} \|\bar{u}_n\|^t_{S^{s,p}(\mathbb{H})} \\
=& 1,
\end{aligned} \tag{4.1}$$

which implies $\|\bar{u}\|_{S^{s,p}(\mathbb{H})} = S_t$ and $\int_{\mathbb{H}} |\bar{u}|^t \mathrm{d}\xi = 1$. The proof is completed. □

We give another proof of Theorem 1.5 via Lions' vanishing theorem.

Proof of Theorem 1.5 (Method 2). We show $\lim\limits_{n\to\infty} \sup\limits_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi > 0$, for all $q \in [p, p_s^*)$. Suppose on the contrary that there exists $q \in [p, p_s^*)$ such that

$$\lim_{n\to\infty} \sup_{z\in\mathbb{H}} \int_{B_{\mathbb{H}}(z,1)} |u_n|^q \mathrm{d}\xi = 0.$$

From Theorem 1.4, we know $u_n \to 0$ in $L^r(\mathbb{H})$ for $r \in (p, p_s^*)$. This is a contradiction with $\int_{\mathbb{H}} |u_n|^t \mathrm{d}\xi = 1$. Then similar to the above proof, we know that there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ is also a bounded minimizing sequence of (S_t). Moreover, $\{\bar{u}_n\}$ converges strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$ for all $q \in (p, p_s^*)$. Then similar to (4.1), one has $\|\bar{u}\|_{S^{s,p}(\mathbb{H})} = S_t$ and $\int_{\mathbb{H}} |\bar{u}|^t \mathrm{d}\xi = 1$. □

5. Proof of Theorem 1.6

Equation (S_*) is variational and its solutions are the critical points of the functional defined in $S^{s,2}(\mathbb{H})$ by

$$I(u) =\frac{1}{2}\|u\|^2_{S^{s,2}(\mathbb{H})} - \frac{\gamma}{r}\int_{\mathbb{H}}|u|^r \mathrm{d}\xi - \frac{1}{2^*_s}\int_{\mathbb{H}}|u|^{2^*_s}\mathrm{d}\xi,$$

From Proposition 1.2, we know $I \in C^1(S^{s,2}(\mathbb{H}),\mathbb{R})$. It is easy to see that if $u \in S^{s,2}(\mathbb{H})$ is a critical point of I, i.e.

$$\begin{aligned}0 = \langle I'(u), v\rangle = &\int_{\mathbb{H}}\int_{\mathbb{H}}\frac{[u(\xi)-u(\eta)][v(\xi)-v(\eta)]}{d^{Q+2s}(\eta^{-1}\circ\xi)}\mathrm{d}\eta\mathrm{d}\xi\\ &-\gamma\int_{\mathbb{H}}|u|^{r-2}uv\mathrm{d}\xi - \int_{\mathbb{H}}|u|^{2^*_s-2}uv\mathrm{d}\xi,\end{aligned}$$

for $\varphi \in S^{s,2}(\mathbb{H})$, then u is a weak solution of equation (S_*). We denote the Nehari manifold as follows:

$$\mathcal{N} := \left\{u \in S^{s,2}(\mathbb{H})\backslash\{0\}\middle|\langle I'(u), u\rangle = 0\right\}.$$

It is easy to see the following lemma.

Lemma 5.1. *Assume that all conditions described in Theorem 1.6 are satisfied. Then the following statements hold true:*

(1) I has mountain pass geometry structure. There exists a bounded Palais-Smale sequence $\{u_n\} \subset S^{s,2}(\mathbb{H})$ such that

$$I(u_n) \to c \quad \text{and} \quad \|I'(u_n)\|_{S^{-s,2}(\mathbb{H})} \to 0, \quad \text{as } n \to \infty,$$

where

$$c = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I(\gamma(t)) > 0$$

and

$$\Gamma = \left\{\gamma \in C\left([0,1], S^{s,2}(\mathbb{H})\right)\middle|\gamma(0) = 0, I(\gamma(1)) < 0\right\}.$$

(2) For each $u \in S^{s,2}(\mathbb{H})\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $I(t_u u) = \max_{t>0} I(tu)$.

(3) $c = \bar{c} = \bar{\bar{c}} > 0$, where

$$\bar{c} := \inf_{u\in\mathcal{N}} I(u) \quad \text{and} \quad \bar{\bar{c}} := \inf_{u\in S^{s,2}(\mathbb{H})\backslash\{0\}}\sup_{t\geqslant 0} I(tu)$$

(4) For $u \in \mathcal{N}$, we have $\Psi'(u) \neq 0$, where

$$\begin{aligned}\Psi(u) =&\langle I'(u), u\rangle\\ =&\int_{\mathbb{H}}(\nabla_{\mathbb{H}}u\nabla_{\mathbb{H}}v + uv)\mathrm{d}\xi - \gamma\int_{\mathbb{H}}|u|^{r-2}uv\mathrm{d}\xi - \int_{\mathbb{H}}|u|^{2^*_s-2}uv\mathrm{d}\xi.\end{aligned} \tag{5.1}$$

Then we have

$$\langle\Psi'(u), u\rangle =2\|u\|^2_{S^{s,2}(\mathbb{H})} - r\gamma\int_{\mathbb{H}}|u|^r\mathrm{d}\xi - 2^*_s\int_{\mathbb{H}}|u|^{2^*_s}\mathrm{d}\xi. \tag{5.2}$$

Moreover, if $u \in \mathcal{N}$ and $J(u) = \bar{c}$, then u is a ground state solution of equation (S_*).

Proof. (1). In terms of Proposition 1.2, we get

$$I(u) \geqslant \frac{1}{2}\|u\|^2_{S^{s,2}(\mathbb{H})} - C\|u\|^r_{S^{s,2}(\mathbb{H})} - C\|u\|^{2^*_s}_{S^{s,2}(\mathbb{H})}.$$

By $2^*_s > r > 2$, for $\rho > 0$ small enough, there has

$$\varsigma := \inf_{\|u\|_{S^{s,2}(\mathbb{H})}=\rho} I(u) > 0 = I(0).$$

For $u \in S^{s,2}(\mathbb{H})\backslash\{0\}$, one has

$$I(tu) = \frac{t^2}{2}\|u\|^2_{S^{s,2}(\mathbb{H})} - \gamma\frac{t^r}{r}\int_{\mathbb{H}} |u|^r \mathrm{d}\xi - \frac{t^{2^*_s}}{2^*_s}\int_{\mathbb{H}} |u|^{2^*_s}\mathrm{d}\xi.$$

From the above expression, we can deduce that $J(tu) < 0$ for some $t > 0$ large enough. By the mountain pass theorem, there exists a (PS) sequence $\{u_n\} \subset S^{s,2}(\mathbb{H})$ such that

$$I(u_n) \to c \ \text{ and } \ \|I'(u_n)\|_{S^{-1,2}(\mathbb{H})} \to 0, \ \text{ as } n \to \infty.$$

Moreover, we have

$$c + o_n(1) = I(u_n) - \frac{1}{r}\langle I'(u_n), u_n\rangle \geqslant \left(\frac{1}{2} - \frac{1}{r}\right)\|u_n\|^2_{S^{s,2}(\mathbb{H})},$$

which implies $\{u_n\}$ is bounded in $S^{s,2}(\mathbb{H})$.

(3). For each $u \in S^{s,2}(\mathbb{H})\backslash\{0\}$ and $t > 0$, set $f(t) := I(tu)$. Then

$$f'(t) = t\|u\|^2_{S^{s,2}(\mathbb{H})} - \gamma t^{r-1}\int_{\mathbb{H}} |u|^r \mathrm{d}\xi - t^{2^*_s-1}\int_{\mathbb{H}} |u|^{2^*_s}\mathrm{d}\xi.$$

By $2^*_s > r > 2$, it is standard to check there exists a unique $t_u \in (0, \infty)$ such that $f'(t_u) = 0$ holds. This implies $t_u u \in \mathcal{N}$. Moreover, we know that the unique critical point t_u on $(0, \infty)$ is a maximum point of $I(tu)$.

(4). Let $u \in \mathcal{N}$. By $\langle I'(u), u\rangle = 0$ and Proposition 1.1, we have

$$0 = \langle J'(u), u\rangle \geqslant \|u\|^2_{S^{s,2}(\mathbb{H})} - C\|u\|^r_{S^{s,2}(\mathbb{H})} - C\|u\|^{2^*_s}_{S^{s,2}(\mathbb{H})},$$

which implies $\|u\|_{S^{s,2}(\mathbb{H})} \geqslant C$. Hence I is bounded from below on $\mathcal{N}$ and $\bar{c} > 0$.

From the above arguments, it is easy to see that $\bar{c} = \bar{\bar{c}}$. Notice that for any $u \in S^{s,2}(\mathbb{H})\backslash\{0\}$, there exists a large $\bar{t} > 0$ such that $I(\bar{t}u) < 0$. Define a path $\bar{\gamma} : [0, 1] \to S^{s,2}(\mathbb{H})$ by $\bar{\gamma}(t) = t\bar{t}u$. Clearly, $\bar{\gamma} \in \Gamma$ and $c \leqslant \bar{\bar{c}}$.

For all path $\gamma \in \Gamma$, set $h(t) := \langle I'(\gamma(t)), \gamma(t)\rangle$. Then $h(0) = 0$ and $h(t) > 0$ for $t > 0$ small enough. One has

$$I(\gamma(1)) - \frac{1}{r}\langle I'(\gamma(1)), \gamma(1)\rangle \geqslant \left(\frac{1}{2} - \frac{1}{r}\right)\|\gamma(1)\|^2_{S^{s,2}(\mathbb{H})} \geqslant 0.$$

which implies

$$\langle I'(\gamma(1)), \gamma(1)\rangle \leqslant r \cdot I(\gamma(1)) < 0.$$

Thus, there exists $\bar{\bar{t}} \in (0, 1)$ such that $h(\bar{\bar{t}}) = 0$, i.e. $\gamma(\bar{\bar{t}}) \in \mathcal{N}$. So, we get $c \geqslant \bar{c}$.

(5). For $u \in \mathcal{N}$, it follows from (5.1) and (5.2) that

$$\langle \Psi'(u), u\rangle = \langle \Psi'(u), u\rangle - 2\Psi(u) < 0,$$

which indicates $\Psi'(u) \neq 0$ for $u \in \mathcal{N}$. If $u \in \mathcal{N}$ and $I(u) = \bar{c}$, then there exists $\lambda \in \mathbb{R}$ such that $I'(u) = \lambda\Psi'(u)$. One has

$$\langle \lambda\Psi'(u), u\rangle = \langle I'(u), u\rangle = \Psi(u) = 0.$$

This showes $\lambda = 0$ and $I'(u) = 0$. □

Lemma 5.2. *Assume that all conditions described in Theorem 1.6 are satisfied. Then we have*

$$0 < c < c^* := \left(\frac{1}{2} - \frac{1}{2_s^*}\right) S_{2_s^*}^{-\frac{2_s^*}{2_s^*-2}}.$$

Proof. We choose $v \in S^{s,2}(\mathbb{H})$ such that

$$\|v\|^2_{S^{s,2}(\mathbb{H})} = 1, \quad \int_{\mathbb{H}} |v|^q \mathrm{d}\xi > 0 \quad \text{and} \quad \lim_{t\to\infty} I(tv) = -\infty.$$

There exists $t_{v,\gamma} > 0$ such that

$$\sup_{t\geqslant 0} I(tv) = I(t_{v,\gamma} v).$$

Hence, $t_{v,\gamma} > 0$ satisfies

$$t_{v,\gamma}\|v\|^2_{S^{s,2}(\mathbb{H})} = \gamma t_{v,\gamma}^{r-1} \int_{\mathbb{H}} |v|^r \mathrm{d}\xi + t_{v,\gamma}^{2_s^*-1} \int_{\mathbb{H}} |v|^{2_s^*} \mathrm{d}\xi,$$

which gives

$$t_{v,\gamma}\|v\|^2_{S^{s,2}(\mathbb{H})} \geqslant t_{v,\gamma}^{2_s^*-1} \int_{\mathbb{H}} |v|^{2_s^*} \mathrm{d}\xi.$$

This shows that $\{t_{v,\gamma}\}_\gamma$ is bounded.

We next prove $t_{v,\gamma} \to 0$ as $\gamma \to \infty$. Suppose on the contrary that $t_{v,\gamma} \not\to 0$ as $\gamma_n \to \infty$. Then there exist $\hat{t} > 0$ and a sequence $\{\gamma_n\}$ with $\gamma_n \to \infty$ as $n \to \infty$, such that $t_{v,\gamma_n} \to \hat{t}$ as $n \to \infty$. Passing the limit as $n \to \infty$, we can deduce

$$\gamma_n t_{v,\gamma_n}^{r-1} \int_{\mathbb{H}} |v|^r \mathrm{d}\xi \to \infty,$$

which gives

$$\begin{aligned}
\infty &= \lim_{n\to\infty} \gamma_n t_{v,\gamma_n}^{r-1} \int_{\mathbb{H}} |v|^r \mathrm{d}\xi \\
&\leqslant \lim_{n\to\infty} \left[\gamma_n t_{v,\gamma_n}^{r-1} \int_{\mathbb{H}} |v|^r \mathrm{d}\xi + t_{v,\gamma_n}^{2_s^*-1} \int_{\mathbb{H}} |v|^{2_s^*} \mathrm{d}\xi\right] \\
&= \lim_{n\to\infty} t_{v,\gamma_n} \|v\|^2_{S^{s,2}(\mathbb{H})} \\
&= \hat{t}\|v\|^2_{S^{s,2}(\mathbb{H})}.
\end{aligned}$$

This shows $\|v\|^2_{S^{s,2}(\mathbb{H})} = \infty$, which is a contradiction with $\|v\|^2_{S^{s,2}(\mathbb{H})} = 1$. Thus, we have $t_{v,\lambda} \to 0$ as $\lambda \to \infty$. Then, we arrive at

$$\lim_{\gamma\to\infty} \sup_{t\geqslant 0} I(tv) = \lim_{\gamma\to\infty} I(t_{v,\gamma} v) = 0.$$

Hence, there exists $0 < \gamma_0 < \infty$, such that for any $\gamma > \gamma_0$, one has

$$\sup_{t\geqslant 0} I(tv) < c^*.$$

The proof is completed. □

Lemma 5.3. *Assume that all conditions described in Theorem 1.4 hold. Let $\{u_n\}$ be a bounded $(PS)_c$ sequence of with $c \in (0, c^*)$. Then there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ is also a bounded $(PS)_c$ sequence of with $c \in (0, c^*)$. Moreover, $\{\bar{u}_n\}$ converges strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$ for all $q \in (2, 2^*_s)$.*

Proof. We show $\lim\limits_{n\to\infty} \int_{\mathbb{H}} |u_n|^q \mathrm{d}\xi > 0$ for all $q \in (2, 2^*_s)$. Otherwise, we suppose that there exists $q \in (2, 2^*_s)$ such that

$$\lim_{n\to\infty} \int_{\mathbb{H}} |u_n|^q \mathrm{d}\xi = 0.$$

By Hölder's inequality, one has

$$\lim_{n\to\infty} \int_{\mathbb{H}} |u_n|^r \mathrm{d}\xi = 0.$$

Then

$$c + o_n(1) = \frac{1}{2}\|u_n\|^2_{S^{s,2}(\mathbb{H})} - \frac{1}{2^*_s}\int_{\mathbb{H}} |u_n|^{2^*_s} \mathrm{d}\xi$$

and

$$o_n(1) = \|u_n\|^2_{S^{s,2}(\mathbb{H})} - \int_{\mathbb{H}} |u_n|^{2^*_s} \mathrm{d}\xi, \tag{5.3}$$

which gives

$$c + o_n(1) \geqslant \left(\frac{1}{2} - \frac{1}{2^*_s}\right) \|u_n\|^2_{S^{s,2}(\mathbb{H})}. \tag{5.4}$$

It follows from (5.3) and Sobolev's inequality that

$$\begin{aligned} \|u_n\|^2_{S^{s,2}(\mathbb{H})} &= \int_{\mathbb{H}} |u_n|^{2^*_s} \mathrm{d}\xi \\ &\leqslant S_{2^*_s}^{\frac{2^*_s}{2}} \|u_n\|^{2^*_s}_{S^{s,2}(\mathbb{H})}, \end{aligned}$$

which showes

$$S_{2^*_s}^{-\frac{2^*_s}{2^*_s - 2}} \leqslant \|u_n\|^2_{S^{s,2}(\mathbb{H})}. \tag{5.5}$$

In view of (5.4) and (5.5), we have

$$\begin{aligned} c + o_n(1) &\geqslant \left(\frac{1}{2} - \frac{1}{2^*_s}\right) \|u_n\|^2_{S^{s,2}(\mathbb{H})} \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2^*_s}\right) S_{2^*_s}^{-\frac{2^*_s}{2^*_s - 2}}. \end{aligned}$$

This is a contradiction with Lemma 5.2.

Applying Theorem 1.3, there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ convergence strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$.

We now show $\{\bar{u}_n := u_n(\xi + z_n)\}$ is also a bounded $(PS)_c$ sequence of with $c \in (0, c^*)$. Clearly,

$$c = I(u_n) = I(\bar{u}_n).$$

For all $\varphi \in S^{s,2}(\mathbb{H})$, we obtain

$$\begin{aligned}|\langle I'(\bar{u}_n), \varphi\rangle| &= |\langle I'(u_n), \bar{\varphi}_n\rangle| \\ &\leqslant \|I'(u_n)\|_{S^{-s,2}(\mathbb{H})}\|\bar{\varphi}_n\|_{S^{s,2}(\mathbb{H})} \\ &= o(1)\|\bar{\varphi}_n\|_{S^{s,2}(\mathbb{H})},\end{aligned}$$

where $\bar{\varphi}_n = \varphi(\xi - z_n)$. Since $\|\bar{\varphi}_n\|_{S^{s,2}(\mathbb{H})} = \|\varphi\|_{S^{s,2}(\mathbb{H})}$, we get

$$I'(\bar{u}_n) \to 0 \text{ as } n \to \infty.$$

□

Now we are in a position to give the proof of Theorem 1.6 via Lieb's translation theorem.

Proof of Theorem 1.6. From Lemma 5.3, we know that there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ is also a bounded $(PS)_c$ sequence of with $c \in (0, c^*)$. Moreover, $\{\bar{u}_n\}$ converges strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$ for all $q \in (2, 2^*_s)$. Using the Brézis-Lieb lemma [6], one can deduce

$$\begin{aligned}\bar{c} \leqslant & I(\bar{u}) \\ =& I(\bar{u}) - \frac{1}{r}\langle I'(\bar{u}), \bar{u}\rangle \\ =& \left(\frac{1}{2} - \frac{1}{q}\right)\|\bar{u}\|^2_{S^{s,2}(\mathbb{H})} + \left(\frac{1}{r} - \frac{1}{2^*_s}\right)\int_{\mathbb{H}} |\bar{u}|^{2^*_s} \mathrm{d}\xi \\ \leqslant & \lim_{n\to\infty}\left[\left(\frac{1}{2} - \frac{1}{r}\right)\|\bar{u}_n\|^2_{S^{s,2}(\mathbb{H})} + \left(\frac{1}{r} - \frac{1}{2^*_s}\right)\int_{\mathbb{H}} |\bar{u}_n|^{2^*_s} \mathrm{d}\xi\right] \\ =& \lim_{n\to\infty}\left[I(\bar{u}_n) - \frac{1}{r}\langle I'(\bar{u}_n), \bar{u}_n\rangle\right] \\ =& \lim_{n\to\infty} I(\bar{u}_n) = \bar{c},\end{aligned} \tag{5.6}$$

which implies $I(\bar{u}) = \bar{c}$. The proof is completed. □

We give another proof of Theorem 1.6 via Lions' vanishing theorem.

Proof of Theorem 1.6. We show $\lim\limits_{n\to\infty}\sup\limits_{z\in\mathbb{H}}\int_{B_{\mathbb{H}}(z,1)}|u_n|^q\mathrm{d}\xi > 0$, for all $q \in [2, 2^*_s)$. Suppose on the contrary that there exists $q \in [2, 2^*_s)$ such that

$$\lim_{n\to\infty}\sup_{z\in\mathbb{H}}\int_{B_{\mathbb{H}}(z,1)}|u_n|^q\mathrm{d}\xi = 0.$$

From Theorem 1.4, we know $u_n \to 0$ in $L^t(\mathbb{H})$ for $t \in (2, 2^*_s)$. Repeatting the proof of Lemma 5.3, we know that there exists $\{z_n\} \subset \mathbb{H}$ such that $\{\bar{u}_n := u_n(\xi + z_n)\}$ is also a bounded $(PS)_c$ sequence of with $c \in (0, c^*)$. Moreover, $\{\bar{u}_n\}$ converges strongly and a.e. to $\bar{u} \not\equiv 0$ in $L^q_{loc}(\mathbb{H})$ for all $q \in (2, 2^*_s)$. Then similar to (5.6), one has $I(\bar{u}) = \bar{c}$. □

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